ZEROS OF DIRICHLET L-FUNCTIONS AND ONE-SIDED ESTIMATES

Abstract. We study a one-sided criterion for the Generalized Riemann Hypothesis for Dirichlet L-functions. For each modulus the criterion involves a smoothed count of prime powers in a progression weighted by a quadratic character. It suffices to bound this remainder at squares. A comparison of covolumes of families of automorphic forms would supply the bound. We construct short rows and compute their Gram forms under theta lifts. The signed lower height on the same rows, including its integral indices and local Hecke labels, remains open. Dirichlet GRH is the target conjecture.

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1. The main conjecture

Definition 1.1 (Characters and their series)**.** A Dirichlet character modulo an integer $q \geq 1$ is a homomorphism $(\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$, extended to a periodic function on $\mathbb{Z}$ by zero on nonunits. The principal character is one on the units. A character is primitive if its restriction to the units does not factor through the units of any proper positive divisor of q. For $q = 1$ the unique character has value one on every integer. For $\operatorname{Re} s > 1$ write $L(s, \chi) = \sum_{n \geq 1} \chi(n) n^{-s}$, where $u^s = \exp(s \log u)$ for $u > 0$.

For $\operatorname{Re} s > 0$ we use the meromorphic continuation constructed in Proposition 3.3. Write $\varphi(q) = |(\mathbb{Z}/q\mathbb{Z})^\times|$.

Conjecture 1 (Dirichlet GRH)**.** Let $q \geq 1$ be an integer and let χ be any Dirichlet character modulo q, primitive or imprimitive. If $L(\rho, \chi) = 0$ and $0 < \operatorname{Re} \rho < 1$, then $\operatorname{Re} \rho = 1/2$.

Proposition 1.2 (Equivalent half-plane formulation)**.** *Conjecture 1 is equivalent to the following assertion. For every $q \geq 1$ and every character $\chi \bmod q$, $L(s, \chi)$ is holomorphic and nonzero at every s with $\operatorname{Re} s > 1/2$, with the sole exception of $s = 1$ for a principal character. At that point it has a simple pole of residue $\varphi(q)/q$.*

Proof. Propositions 3.3 and 7.4 settle $\operatorname{Re} s \geq 1$, including the residue. In $1/2 < \operatorname{Re} s < 1$ the conjecture excludes every zero. Conversely, the half-plane assertion excludes in particular all primitive zeros in this half-strip. Theorem 4.5 then gives the conjecture by reflection and the finite Euler factors for induction of characters. □

2. Strategy of proof

The analytic argument detects a zero through a weighted count of prime powers. The arithmetic argument seeks an upper bound for that count by comparing determinants of families of automorphic forms. Figure 1 shows how the two arguments meet.

Definition 2.1 (The prime remainder)**.** Let $M \geq 1$ and let $d \nmid M$ be an odd prime. Let η_d take the values $0, 1, -1$ on multiples of d, nonzero squares and nonsquares modulo d, respectively. Put $\Lambda(p^j) = \log p$ for primes p and $j \geq 1$, and $\Lambda(k) = 0$ otherwise. Write $\lambda(k) = \Lambda(k)(1 - \eta_d(k))\mathbf{1}_{k \equiv 1 \bmod M}$. Set

$$\kappa_{d,M} = \lim_{\sigma \downarrow 1} \left(\sum_{k \geq 1} \lambda(k) k^{-\sigma} - \frac{\sigma}{\varphi(M)(\sigma - 1)} \right),$$

$$R_{d,M}(x) = \sum_{k \leq x} \lambda(k)(1/k - 1/x) - \frac{\log x}{\varphi(M)} - \kappa_{d,M} \qquad (x \geq 1).$$

The identity for the Dirichlet series in the proof of Proposition 8.2 identifies the expression in parentheses with $H_{d,M}(\sigma)$ of Definition 8.1, which is holomorphic at 1. This proves existence of the limit and identifies the two normalizations.

Theorem 2.2 (A one-sided test at squares)**.** *For each $q \geq 1$ put $M = 18480q$. Suppose that for every prime $d \equiv M/2 - 1 \bmod M$ and every $0 < \delta < 1/2$ there is a constant $C = C(d, M, \delta) > 0$ such that, for all sufficiently large positive integers j,*

$$R_{d,M}(j^2) \leq C j^{-1+2\delta}.$$

If this holds for every q, the half-plane assertion of Proposition 1.2 and Conjecture 1 follow.

Proof. Each summand $\lambda(k) \max(1/k - 1/x, 0)$ is nondecreasing in x. For $j = \lceil \sqrt{x} \rceil$ the finite sums in Definition 2.1 therefore give

$$R_{d,M}(x) \leq R_{d,M}(j^2) + \frac{\log(j^2/x)}{\varphi(M)} \leq C x^{-1/2+\delta} + \frac{2}{\varphi(M)\sqrt{x}}.$$

Here $j/\sqrt{x} \le 1 + 1/\sqrt{x}$ and $\log(1+u) = \int_0^u (1+t)^{-1}\,\mathrm{d}t \le u$ for $u \ge 0$. Given $1/2 < \beta < 1$, take $0 < \delta < \beta - 1/2$. The displayed bound is then at most $x^{\beta-1}$ eventually. Corollary 9.8 proves the assertion. In that corollary Theorem 9.7 supplies a quadratic twist free of zeros in the rectangle being tested, and Theorem 8.4 uses the sign of the untwisted zero multiplicities in (4). □

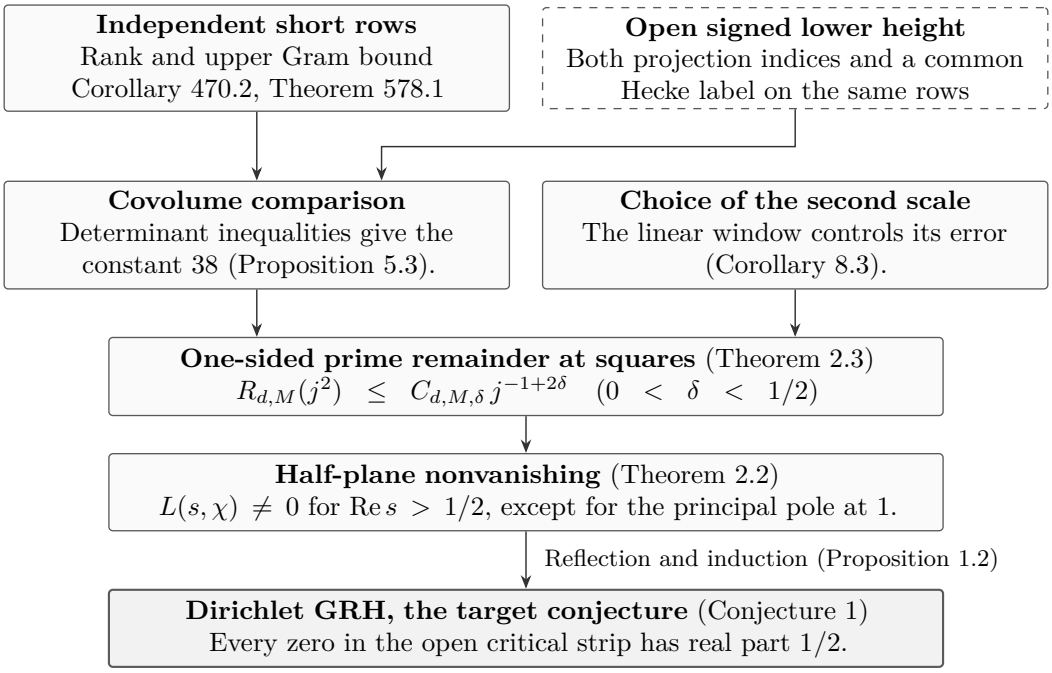

FIGURE 1. The route from short forms to the critical line. The dashed box marks the open arithmetic step. The covolume comparison and the choice of the second scale must apply at the same point.

Theorem 2.3 (A covolume bound sufficient at square scales)**.** *Fix M, d as in Theorem 2.2. There is $B_0 = B_0(d, M) > 1$ with the following property. Fix $0 < \epsilon < 1/100$ and put $x = j^2$, $t = 10^6 \log x$, $\ell = x^{\epsilon}$. For the cyclotomic places in Definition 10.1 let f be their number, $U = f/\varphi(M)$, and let S be the size of the subset of Proposition 10.5. Take m, a from Theorem 10.8. Write $\mathcal{R}_x(v)$ for the weighted sum of Definition 10.6 at $x_1 = \exp v$. Suppose that for all sufficiently large j,*

$$m\mathcal{R}_x(v) \le 38 f a \ell^4 \qquad (t \le v \le B_0 t).$$

Then $R_{d,M}(x) \le 38\varphi(M) x^{-1/2+5\epsilon} + x^{-10^6}$ at every sufficiently large square x. The same conclusion holds if at each such square the covolume bound holds at just one point $v \in [t, B_0 t]$ satisfying $R_{d,M}(\exp v) - (U/S) R_{d,1}(\exp v) \le \exp(-t)$. If either hypothesis holds for each such M, d and arbitrarily small fixed $\epsilon > 0$, Conjecture 1 follows.

Proof. Propositions 10.3 and 10.5 give $m \ge \sqrt{x}$ and $0 < U/S \le \ell \le \exp t$. Corollary 8.3 chooses $v \in [t, B_0 t]$ such that $R_{d,M}(\exp v) - (U/S) R_{d,1}(\exp v) \le \exp(-t)$. Since $B_0 t \le t^2$ eventually, the scales are admissible by Theorem 10.8. At this very v, Proposition 10.7 gives

$$R_{d,M}(x) = \frac{m\mathcal{R}_x(v)}{maS} + R_{d,M}(\exp v) - \frac{U}{S} R_{d,1}(\exp v).$$

Substitute the hypothesis and $f/S \le \varphi(M)\ell$. Under the one-point hypothesis use its specified v in the same identity. For any δ in Theorem 2.2, choose an available ϵ with $5\epsilon < \delta$. The estimate at $x = j^2$ satisfies the test of that theorem. □

Proposition 5.3 gives the constant 38 from a rank bound and two determinant inequalities. Corollary 470.2 constructs the independent short rows, and Theorem 578.1 proves that their

images stay independent under the theta lifts. To obtain the required signed lower height on those same rows, it remains to bound the integral indices of the two projections and to establish the common local Hecke label needed in Theorem 583.2. Consequently the upper bound for $m\mathcal{R}_x(v)$ in Theorem 2.3 is still a hypothesis.

3. Induction and continuation

Definition 3.1 (Foundational conventions)**.** We use classical logic, the positive integers, the complete ordered real field, and $\mathbb{C} = \mathbb{R}[i]$. We take $\exp z = \sum_{k\ge 0} z^k/k!$, and use the real logarithm for powers of positive numbers. A holomorphic function is complex differentiable on an open set. Meromorphic functions are locally quotients of holomorphic functions with denominator not identically zero. Integrals on real intervals use length measure. An integral against a counting measure is a sum. For a positive function g, $f = O(g)$ means $|f| \le Cg$ throughout the stated range, or eventually if the argument tends to infinity. Subscripts on O indicate permitted dependence of C.

Lemma 3.2. *Every character $\chi \bmod q$ is induced by a primitive character $\psi \bmod f$ for some $f \mid q$. For $\operatorname{Re} s > 1$,*

$$L(s,\chi) = L(s,\psi) \prod_{\substack{p\mid q\\ p\nmid f}} (1 - \psi(p)p^{-s}).$$

Every factor in the finite product is nonzero for $\operatorname{Re} s > 0$.

Proof. For $f \mid q$, reduction on units is onto. Indeed, given a prime to f, let P be the product of the primes dividing q but not f. Since $(f, P) = 1$, choose k with $fk \equiv 1 - a \bmod P$. Then $a + fk$ is prime to q and lifts a. Among divisors through whose unit group χ factors, choose the least. The induced character there is primitive by minimality.

Values on the units are roots of unity, so the defining series converges absolutely for $\operatorname{Re} s > 1$. Expanding finite products of geometric series and using unique factorisation sums exactly the terms with prime factors in the chosen finite set. Absolute convergence permits its increase to all primes. Thus $L(s,\psi) = \prod_p (1 - \psi(p)p^{-s})^{-1}$. Lemma A.1 justifies the divisibility steps. Deleting the factors at newly excluded primes gives the displayed identity. If $\operatorname{Re} s > 0$, then $|\psi(p)p^{-s}| \le p^{-\operatorname{Re} s} < 1$. □

Proposition 3.3. *The series $L(s,\chi)$ continues meromorphically to $\operatorname{Re} s > 0$. Its only possible pole there is at $s = 1$, a simple pole of residue $\varphi(q)/q$ for the principal character. Lemma 3.2's identity holds throughout this half-plane as a meromorphic identity.*

Proof. Let $A(x) = \sum_{1\le n\le x} \chi(n)$ and $\mu = q^{-1} \sum_{a=1}^{q} \chi(a)$. For a nonprincipal character, multiplying the argument of the complete sum by a unit on which $\chi \ne 1$ shows that $\mu = 0$. For the principal character $\mu = \varphi(q)/q$. Here $\varphi(q) = |(\mathbb{Z}/q\mathbb{Z})^\times|$. Complete periods cancel in $B(x) = A(x) - \mu x$, so $|B(x)| \le 2q$. Integrating the series of step functions, for $\operatorname{Re} s > 1$ one obtains

$$L(s,\chi) = s \int_1^\infty A(x) x^{-s-1}\,\mathrm{d}x = \frac{\mu s}{s-1} + s \int_1^\infty B(x) x^{-s-1}\,\mathrm{d}x.$$

In the last integral, on $\operatorname{Re} s \ge \delta > 0$, the integrands before and after one differentiation in s are dominated by a constant times $(1 + \log x)x^{-1-\delta}$, an integrable function. Lemmas A.2 and A.3 give the continuation, and the explicit term gives the residue. The two sides of the earlier identity now agree by analytic continuation. □

4. Reflection in the critical strip

Lemma 4.1. *Let $\psi \bmod f$ be primitive, $e(u) = \exp(2\pi i u)$, and $\tau = \sum_{a \bmod f} \psi(a) e(a/f)$. For every integer k,*

$$\sum_{a \bmod f} \psi(a) e(ak/f) = \bar{\psi}(k)\tau, \qquad |\tau| = \sqrt{f}.$$

Proof. For a unit k, change variables $a \mapsto ak$. Otherwise $h = f/(k,f) < f$. There is a unit $u \equiv 1 \bmod h$ with $\psi(u) \neq 1$. Indeed, triviality on the kernel of the surjection of unit groups proved in Lemma 3.2 would give a factorisation through h. Multiplication by u fixes the additive factor $e(ak/f)$ and multiplies the sum by $\psi(u)$, so the sum is zero. Summing the squared modulus over $k \bmod f$ gives $f\sum_{a \bmod f} |\psi(a)|^2 = f\varphi(f)$, since $\sum_{k \bmod f} e(k(a-b)/f)$ equals f for $a = b \bmod f$ and zero otherwise by the finite geometric sum. The first identity gives $\varphi(f)|\tau|^2$ for the same sum. For $f = 1$ both identities read $1 = 1$. □

Lemma 4.2. *Let $\psi \bmod f$ be primitive and let $a \in \{0,1\}$ satisfy $\psi(-1) = (-1)^a$. For $t > 0$ put*

$$\Theta_\psi(t) = \sum_{n\in\mathbb{Z}} \psi(n) n^a \mathrm{e}^{-\pi n^2 t/f} \quad (0^0 = 1), \qquad \omega_\psi = (-i)^a \tau/\sqrt{f}.$$

Then $|\omega_\psi| = 1$ and $\Theta_\psi(t) = \omega_\psi t^{-a-1/2}\Theta_{\bar\psi}(1/t)$.

Proof. Use the Fourier transform $\widehat{g}(y) = \int_{\mathbb{R}} g(x)e(-xy)\,\mathrm{d}x$. For $\alpha > 0$ the transform of $\exp(-\pi\alpha x^2)$ is $\alpha^{-1/2}\exp(-\pi y^2/\alpha)$. At $y = 0$ this follows by squaring the Gaussian integral and using polar coordinates. Differentiation in y and integration by parts give the differential equation $\widehat{g}'(y) = -(2\pi/\alpha)y\widehat{g}(y)$. Multiplying g by x differentiates its transform and multiplies the result by $-1/(2\pi i)$. Consequently, for $g(x) = x^a\mathrm{e}^{-\pi t x^2/f}$,

$$\widehat{g}(k/f) = (-i)^a (k/t)^a \sqrt{f/t}\,\mathrm{e}^{-\pi k^2/(ft)}.$$

The periodisation $\sum_{j\in\mathbb{Z}} g(u+jf)$ is continuous and its k-th Fourier coefficient is $f^{-1}\widehat{g}(k/f)$, by integrating one period. These coefficients are absolutely summable. Their Fourier series recovers the periodisation. Indeed, the continuous difference has all coefficients zero, so all its convolutions with the Fejér kernels vanish. In the variable u/f, these kernels are $\frac{1}{N}|\sum_{j=0}^{N-1} e(jv)|^2$. Their integrals on a period are one and their suprema off any neighbourhood of the integers tend to zero by the geometric sum formula. Uniform continuity gives convergence of the convolutions to the difference. Evaluating at each residue u, multiplying by $\psi(u)$ and summing now proves the formula by Lemma 4.1. □

Lemma 4.3. *For $\operatorname{Re} z > 0$ the integral $\Gamma(z) = \int_0^\infty \mathrm{e}^{-u}u^{z-1}\,\mathrm{d}u$ is holomorphic and nonzero.*

Proof. On compact subsets of this half-plane, integrable bounds at zero and infinity persist after insertion of $\log u$, giving holomorphy. Integrating by parts n times gives $\int_0^1 t^{z-1}(1-t)^n\,\mathrm{d}t = n!/[z(z+1)\cdots(z+n)]$. After multiplication by n^z and substitution $u = nt$, dominated convergence applies by $(1-u/n)^n \le \mathrm{e}^{-u}$ on $[0,n]$. The resulting limit is therefore $\Gamma(z)$. The logarithm of the absolute value of the finite expression equals

$$-\log|z| + (\operatorname{Re} z)\log n - \sum_{k=1}^{n} \log|1 + z/k|.$$

It has a finite limit since $\log|1+z/k| = (\operatorname{Re} z)/k + O_z(k^{-2})$ and $\sum_{k\le n} 1/k - \log n$ converges by integral comparison. The modulus of the limit is positive. □

Proposition 4.4. *For every primitive character ψ and every s with $0 < \operatorname{Re} s < 1$, $L(s,\psi) = 0$ if and only if $L(1-s,\bar\psi) = 0$.*

Proof. Write $\Lambda_\psi(s) = (f/\pi)^{(s+a)/2}\Gamma((s+a)/2)L(s,\psi)$. If $f > 1$, then $\psi(0) = 0$. Termwise integration in $\operatorname{Re} s > 1$, using parity to pair n and $-n$, gives

$$2\Lambda_\psi(s) = \int_0^\infty \Theta_\psi(t) t^{(s+a)/2}\frac{\mathrm{d}t}{t}.$$

Absolute convergence for the interchange follows from the corresponding sum of $n^{-\operatorname{Re} s}$ times the real gamma integral. At infinity the theta function decreases exponentially. At zero Lemma 4.2 bounds it by a power of $1/t$ times an exponentially decreasing function

of $1/t$. The integral thus continues to an entire function of s. Substituting $t = 1/u$ gives $\Lambda_\psi(s) = \omega_\psi \Lambda_{\bar\psi}(1-s)$ in the strip, using Proposition 3.3 to identify the continued functions.

For $f = 1$, write $\Theta(t) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi n^2 t}$ and $I(s) = \int_1^\infty (\Theta(t) - 1) t^{s/2}\,\mathrm{d}t/t$, which is entire. Using $\Theta(t) = t^{-1/2}\Theta(1/t)$ in the integral over $(0,1)$ gives, initially in $\operatorname{Re} s > 1$,

$$2\Lambda_1(s) = I(s) + I(1-s) + \frac{2}{s-1} - \frac{2}{s}.$$

Here the two rational terms are the integrals of $(t^{-1/2} - 1)t^{s/2}\,\mathrm{d}t/t$ over $(0,1)$. Continuation gives the same identity in the strip, where the right side is invariant under $s \mapsto 1 - s$. All completion factors on both sides in both cases are finite and nonzero there by Lemma 4.3, proving the assertion. □

Theorem 4.5. *Dirichlet GRH is equivalent to the assertion that, for every primitive Dirichlet character ψ, $L(s,\psi) \neq 0$ whenever $1/2 < \operatorname{Re} s < 1$.*

Proof. The forward implication follows by restriction. Conversely, suppose the right half of the open strip is zero-free for all primitive characters. For a zero ρ in the open strip of any $L(s,\chi)$, take ψ as in Lemma 3.2. The finite Euler factors have nonzero product there, so $L(\rho,\psi) = 0$. We have $\operatorname{Re}\rho \le 1/2$. If the inequality were strict, Proposition 4.4 would give a zero at $1-\rho$ for $\bar\psi$ in the excluded half-strip. The conjugate is primitive because any factorisation through a smaller modulus would, on conjugation, give the same for ψ. Hence $\operatorname{Re}\rho = 1/2$. □

5. Covolume and the missing dimensions

Lemma 5.1. *Let $v_1, \dots, v_n$ lie in a finite-dimensional complex inner product space. Write $G = (\langle v_i, v_j\rangle)_{i,j}$. If $\|v_i\|^2 \le \mathrm{e}^h$ for every i, with $h \in \mathbb{R}$, then $0 \le \det G \le \mathrm{e}^{nh}$.*

Proof. If the vectors are dependent, their Gram determinant vanishes. Otherwise subtract from each vector its orthogonal projection onto the span of its predecessors. The change of basis is triangular with diagonal entries one. It preserves the Gram determinant and changes the Gram matrix to a diagonal matrix with diagonal entries the squared lengths of the orthogonal components. Each entry is at most the squared length of its original vector. Their product gives both inequalities. The empty family has determinant one. □

Lemma 5.2 (Loss of rank)**.** *Let $r > 0$ and $n, H, T, E, \delta, u, j \in \mathbb{R}$. Suppose*

$$r(H - 3E) \le n(H - T + E) + 2\delta,$$
$$(r-n)(-H + T - E) \le ru, \qquad 2\delta \le rj.$$

Then $T \le 4E + u + j$.

Proof. Writing $h = H - T + E$ gives $nh = rh + (r-n)(-h)$. The three inequalities imply $r(H - 3E) \le r(H - T + E + u + j)$. Cancel rH and divide by $r > 0$. □

Proposition 5.3 (A logarithm in the covolume estimate)**.** *Let $x \ge 2$, $b \ge x^{99}$, $f \ge 1$, $a \ge b$, $r > 0$. Put $\ell = \log b$, $B = fa\ell^3$, $E = \ell B$. Suppose $H, T, \delta, n \in \mathbb{R}$ satisfy*

$$(1-4/b^2)r \le n \le r, \qquad |H| + |T| \le b^2 x^{99}, \tag{1}$$

$$\delta \le 5r\log(1+b^2)B, \qquad r(H-3E) \le n(H-T+E) + 2\delta. \tag{2}$$

Then

$$T \le 38 fa(\log b)^4. \tag{3}$$

Proof. Both B and E are positive. For $h = H - T + E$, $-h \le |H| + |T|$. Thus

$$(r-n)(-h) \le (r-n)b^2x^{99} \le 4rx^{99}.$$

The first multiplication uses $r-n \geq 0$, which is essential when h changes sign. By Lemma 5.2,

$$T \leq 4E + 4x^{99} + 10\log(1+b^2)B.$$

Now $\ell > 1$ and $E = fa\ell^4 \geq b \geq x^{99}$. Also $1 + b^2 \leq b^3$ for $b \geq 2$, whence $\log(1+b^2) \leq 3\ell$. The last display is at most $(4+4+30)E$, proving (3). □

Corollary 5.4. *In Proposition 5.3, let $m \geq \sqrt{x}$, let $U, S > 0$, and suppose $f = \varphi(M)U$ and $S \geq U/\ell$ for a positive integer M. If real numbers R, R', R_1' satisfy*

$$T = ma\{S(R - R') + UR_1'\},$$

then

$$R \leq R' - (U/S)R_1' + 38\varphi(M)(\log b)^5/\sqrt{x}.$$

Proof. Divide (3) by $maS > 0$. The quotient f/S is at most $\varphi(M)\ell$. Solving the displayed identity for R gives the assertion. □

6. A linear window for two Laplace transforms

Lemma 6.1. *Let $F : [0,\infty) \to \mathbb{R}$ be measurable with $|F(v)| \leq C\mathrm{e}^v$. Suppose the function*

$$\Phi_F(s) = \int_0^\infty F(v)\mathrm{e}^{(1-s)v}\,\mathrm{d}v, \qquad \operatorname{Re} s > 2,$$

extends holomorphically to a neighbourhood of $|s-4| \leq 3$. Then there exist $R > 3$ and $K < \infty$ such that for every integer $k \geq 0$,

$$\left|\int_0^\infty F(v)v^k\mathrm{e}^{-3v}\,\mathrm{d}v\right| \leq Kk!R^{-k}.$$

For two such functions the same R, K can be used.

Proof. By Lemma A.3 and compactness the continuation is holomorphic on a neighbourhood of a larger closed disk with radius $R > 3$. On its boundary let K bound the modulus. The k-th Taylor coefficient at 4 has modulus at most K/R^k, by integrating the function times $(s-4)^{-k-1}/(2\pi i)$ around that circle. Differentiating the defining integral at 4 gives $(-1)^k$ times the displayed moment. The interchange is justified in $|s-4| < 1/2$ by the integrable majorant $Cv^k\mathrm{e}^{-3v/2}$. Take a common smaller radius and a larger boundary bound for two functions. □

Theorem 6.2 (A linear window)**.** *Let F, G each satisfy Lemma 6.1, including its continuation hypothesis. There is $B_0 > 1$, depending only on F, G, such that for all sufficiently large real t and all $0 \leq X \leq \mathrm{e}^t$ there exists $v \in [t, B_0t]$ with*

$$F(v) - XG(v) \leq \mathrm{e}^{-t}.$$

Proof. Fix common R, K as in Lemma 6.1 and choose $C \geq 1$ so that $|F(v)| + |G(v)| \leq C\mathrm{e}^v$. Choose positive A, B_0 satisfying

$$A\log(R/3) > 4, \qquad A\log(4/3) > 6, \qquad B_0 > A\log 3 + 5.$$

For $t \geq 1$ set $k = \lceil At \rceil$. Give Y on $[0,\infty)$ density $3^{k+1}v^k\mathrm{e}^{-3v}/k!$. Integration by parts gives $\int_0^\infty v^k\mathrm{e}^{-cv}\,\mathrm{d}v = k!/c^{k+1}$ for $c > 0$, first for $k = 0$ and then by induction. The density therefore has mass one and $\mathbb{E}[\mathrm{e}^{sY}] = (3/(3-s))^{k+1}$ for real $s < 3$. Set $D_X = F - XG$. The moment bound gives, uniformly in X,

$$|\mathbb{E}D_X(Y)| \leq 3K(1+\mathrm{e}^t)(3/R)^k = O(\mathrm{e}^{-3t}).$$

Write $J = [t, B_0t]$. On $Y < t$ we have $\mathrm{e}^Y \leq \mathrm{e}^{2t-Y}$. On $Y > B_0t$ we have $\mathrm{e}^Y \leq \mathrm{e}^{-B_0t+2Y}$. The two exponential moments just computed imply

$$\mathbb{E}[\mathrm{e}^Y\mathbf{1}_{Y\notin J}] \leq \mathrm{e}^{2t}(3/4)^{k+1} + \mathrm{e}^{-B_0t}3^{k+1} = O(\mathrm{e}^{-4t}).$$

This also bounds $\mathbb{P}(Y \notin J)$ since $\mathrm{e}^Y \geq 1$. Furthermore $\mathbb{E}[|D_X(Y)|\mathbf{1}_{Y\notin J}] = O(\mathrm{e}^{-3t})$ by the growth bound. All constants are independent of X, t. If $D_X(v) > \mathrm{e}^{-t}$ throughout J, then

$\mathbb{E}D_X(Y) \geq \mathrm{e}^{-t}(1-O(\mathrm{e}^{-4t}))-O(\mathrm{e}^{-3t})$, contrary to the first moment bound for all sufficiently large t. □

7. Characters and the boundary of the Euler product

Lemma 7.1. *Let G be a finite abelian group and let $\widehat{G}$ denote its complex characters. Then $|\widehat{G}| = |G|$ and, for $u, v \in G$,*

$$|G|^{-1} \sum_{\xi \in \widehat{G}} \xi(u)\overline{\xi(v)} = \mathbf{1}_{u=v}.$$

For characters modulo M, with v a fixed unit, this formula also holds with u any residue and the equality interpreted modulo M.

Proof. If $H \leq G$ and the coset of g has order r in G/H, extend a character λ on H to $H\langle g\rangle$ by $\lambda(hg^j) = \lambda(h)z^j$, where $z^r = \lambda(g^r)$. Two representations differ by a multiple of r in the exponent, so the formula is well defined. The r roots give precisely the r possible extensions. Starting with the trivial subgroup gives the count by multiplication of indices. If $w \neq 1$, the character on $\langle w\rangle$ sending w to $\exp(2\pi i/\operatorname{ord}(w))$ extends to G. Multiplication of every summand in $\sum_\xi \xi(w)$ by this character's value at w permutes the summands, forcing the sum to vanish. For $w = 1$ the count just proved gives the assertion. Take $w = uv^{-1}$. On a nonunit all Dirichlet characters vanish by Definition 1.1. □

Lemma 7.2 (The real singularity of a positive Laplace transform)**.** *Let ν be a positive measure on $[0,\infty)$ such that $\int \mathrm{e}^{-bv}\,\mathrm{d}\nu(v) < \infty$ for some $b \in \mathbb{R}$. Put $\alpha = \inf\{u \in \mathbb{R} : \int \mathrm{e}^{-uv}\,\mathrm{d}\nu(v) < \infty\}$. Then $J(s) = \int \mathrm{e}^{-sv}\,\mathrm{d}\nu(v)$ is holomorphic on $\operatorname{Re} s > \alpha$. If α is finite, J has no holomorphic continuation to a neighbourhood of the real point α.*

Proof. The set of convergence on the real line is upward closed. On each compact subdisk of $\operatorname{Re} s > \alpha$, choose a real convergence point strictly to its left. Its exponential dominates the integrand and, up to a constant depending on k, its k-th derivative there. Lemmas A.2 and A.3 give holomorphy and the moment formula for derivatives. If continuation existed on $|s-\alpha| < 4h$, $h > 0$, the Taylor series about $b_0 = \alpha + h$ would converge at $b_0 - 2h$. There it is the series of nonnegative terms

$$\sum_{k\geq 0} \frac{(2h)^k}{k!} \int v^k \mathrm{e}^{-b_0 v}\,\mathrm{d}\nu(v).$$

By increasing convergence in Lemma A.2, its sum is $\int \mathrm{e}^{-(\alpha-h)v}\,\mathrm{d}\nu(v)$, contradicting the infimum. □

Definition 7.3. Write $\zeta(s) = L(s, 1 \bmod 1)$. For a character $\chi \bmod q$ let $L^\circ(s,\chi) = L(s,\chi)$ for nonprincipal χ, and $L^\circ(s,\chi) = (s-1)L(s,\chi)$ for principal χ, with $L^\circ(1,\chi) = \varphi(q)/q$ in the latter case. These functions are holomorphic for $\operatorname{Re} s > 0$ by Proposition 3.3. Put $\ell_\chi = (L^\circ)'/L^\circ$. Write $o_\chi(\rho)$ for the vanishing order of $L^\circ(\cdot,\chi)$ at ρ, zero at a nonzero value.

Proposition 7.4. *For every Dirichlet character χ, $L^\circ(s,\chi) \neq 0$ on $\operatorname{Re} s \geq 1$.*

Proof. For $\sigma = \operatorname{Re} s > 1$ the Euler product of Lemma 3.2 is the exponential of the series

$$\sum_{p \text{ prime}} \sum_{k\geq 1} \chi(p)^k p^{-ks}/k.$$

This series converges absolutely, since $\sum_{n\geq 2} n^{-\sigma}/(1-2^{-\sigma}) < \infty$. The logarithm formula follows by integrating the geometric series near zero, or by exponentiating its power series. Hence L has no zeros there. For real t and $\sigma > 1$ the Euler product gives

$$\zeta(\sigma)^3\,|L(\sigma+it,\chi)|^4 |L(\sigma+2it,\chi^2)| \;\geq\; 1.$$

Indeed each prime-power contribution to the logarithm, after removing the positive factor $p^{-k\sigma}/k$, is 3 if $p \mid q$ and otherwise $3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2$, where $\exp(i\theta) =$

$\chi(p)^k p^{-ikt}$. A zero at $1+it$ would contribute a factor $O((\sigma-1)^4)$; $\zeta(\sigma)^3 = O((\sigma-1)^{-3})$ by Proposition 3.3. The last factor is bounded unless $t=0$ and χ^2 is principal. In the bounded case the displayed inequality gives a contradiction.

The exceptional nonprincipal χ takes only $0, \pm 1$ as values. The product $\zeta(s)L(s,\chi)$ in $\operatorname{Re} s > 1$ has coefficients $a_n = \sum_{h|n} \chi(h)$. By prime factorization, a_n is the product over $p^j \| n$ of $1+\chi(p)+\cdots+\chi(p)^j$. Each factor is nonnegative and is at least one for even j. Hence $a_{m^2} \geq 1$. The convergence abscissa of $\sum a_n n^{-s}$ is at least $1/2$, by the divergent harmonic sum, and at most one. A zero of $L(1,\chi)$ would make $\zeta L(\cdot,\chi)$ holomorphic in $\operatorname{Re} s > 0$, including at this abscissa. Uniqueness in Lemma A.3 and Lemma 7.2, for the measure with masses a_n at $\log n$, give a contradiction. For a principal character its regularized value at one is positive by Definition 7.3. □

Corollary 7.5. *For every $M \geq 1$ and every unit $c \bmod M$, $\sum_{p \equiv c \bmod M} 1/p = \infty$.*

Proof. For $\sigma > 1$ use Lemma 7.1 to express the prime sum in the form

$$\sum_{p \equiv c \bmod M} p^{-\sigma} = \frac{1}{\varphi(M)} \sum_{\xi \bmod M} \overline{\xi(c)} \sum_p \xi(p) p^{-\sigma}.$$

In each inner sum replacing it by the Euler logarithm in Proposition 7.4 changes it by $O(1)$ uniformly as $\sigma \downarrow 1$, since $\sum_p \sum_{k\geq 2} p^{-k\sigma} \leq 2 \sum_{n \geq 2} n^{-2}$, with $k \geq 2$ and p prime. For a nonprincipal character this logarithm stays bounded. On a disk about one on which $L/L(1)$ is close to one, the convergent series for $\log(1+z)$ gives a bounded branch after adding a logarithm of $L(1)$. The Euler logarithm differs on the real interval by a constant multiple of $2\pi i$, by continuity. For the principal character the real logarithm is $\log(1/(\sigma-1)) + O_M(1)$ by Proposition 3.3. The prime sum therefore tends to infinity, while its summands are at most $1/p$. □

8. A transform of the weighted prime sum

Definition 8.1. Let $M \geq 1$ and let d be an odd prime with $d \nmid M$. Write $\eta_d(n) = (n/d)$, equal to zero on multiples of d, one on nonzero squares modulo d, and minus one on the other residues. Squaring on the units has kernel $\{1,-1\}$, since a field has no zero divisors. Its image has index two, so η_d is a nonprincipal character. Let $\Lambda(p^j) = \log p$ for prime p and $j \geq 1$, and set $\Lambda(n) = 0$ otherwise. Define

$$\lambda_{d,M}(n) = \Lambda(n)(1-\eta_d(n))\mathbf{1}_{n \equiv 1 \bmod M}, \qquad H_{d,M}(s) = \frac{\sum_{\xi \bmod M} (\ell_{\xi\eta_d}(s) - \ell_\xi(s)) - 1}{\varphi(M)}.$$

Here $\xi\eta_d$ is the pointwise product modulo Md. It is nonprincipal by simultaneous congruences from Lemma A.1: choose a residue equal to 1 mod M and a nonsquare unit modulo d. Proposition 7.4 makes $H_{d,M}$ holomorphic near 1. Write $\kappa_{d,M} = H_{d,M}(1)$. This is real since character conjugation permutes the summands and commutes with their evaluation and differentiation at real arguments, first for $\operatorname{Re} s > 1$ and then by continuation. For $x \geq 1$ set

$$R_{d,M}(x) = \sum_{n \leq x} \lambda_{d,M}(n)(1/n - 1/x) - \frac{\log x}{\varphi(M)} - \kappa_{d,M}.$$

All sums over n in this section start at 1.

Proposition 8.2. *For $v \geq 0$, $F_{d,M}(v) = R_{d,M}(\mathrm{e}^v)$ is real and continuous and $|F_{d,M}(v)| = O_{d,M}((1+v)^2)$. For $\operatorname{Re} s > 1$,*

$$\Phi_{d,M}(s) = \int_0^\infty F_{d,M}(v) \mathrm{e}^{(1-s)v} \, \mathrm{d}v = \frac{H_{d,M}(s)/s - \kappa_{d,M}}{s-1}.$$

The right side extends holomorphically near every point of $\operatorname{Re} s \geq 1$ and meromorphically throughout $\operatorname{Re} s > 0$, with at most simple poles. For $0 < \operatorname{Re} \rho < 1$ the residue, interpreted as

zero at a regular point, is

$$\frac{\sum_{\xi \bmod M} \left(o_\xi(\rho) - o_{\xi\eta_d}(\rho)\right)}{\varphi(M)\rho(1-\rho)}. \tag{4}$$

Proof. Each summand enters with weight zero at $x = n$. The nonnegative finite sum is at most $2\log x \sum_{n\le x} 1/n \le 2\log x(1+\log x)$, giving continuity, growth and absolute convergence of the integral. For $\operatorname{Re} s > 1$, termwise differentiation of the Euler logarithm from Proposition 7.4 gives $-L'(s,\chi)/L(s,\chi) = \sum_n \chi(n)\Lambda(n)n^{-s}$. Uniform absolute convergence of the derivatives on compact subdisks follows by comparison with $\sum_n (\log n)n^{-1-\delta}$ for some $\delta > 0$. Apply Lemma 7.1 at the identity and subtract the corresponding twisted sums. Only the untwisted principal character acquires a term $1/(s-1)$ when passing from L'/L to ℓ. It follows that

$$\sum_{n\ge 1} \lambda_{d,M}(n)n^{-s} = H_{d,M}(s) + \frac{s}{\varphi(M)(s-1)}.$$

With $x = \mathrm{e}^v$ the transform uses $x^{-s}\,\mathrm{d}x$. Integrating the summand $1/n - 1/x$ over $x \ge n$ against this measure gives $n^{-s}/[s(s-1)]$. Interchange is justified by the same formula with $\operatorname{Re} s$ and the nonnegative weights, using Lemma A.2. The transforms of $\log x$ and 1 are $(s-1)^{-2}$ and $(s-1)^{-1}$, by elementary integration. Substitution gives the formula. Its numerator vanishes at 1 by Definition 8.1. All logarithmic derivatives are regular on the closed right half-plane of Proposition 7.4. At an interior zero use Lemma A.3 to take their residues. The minus sign from $1/(\rho-1)$ reverses the order of subtraction and gives (4). □

Corollary 8.3. *For each d, M of Definition 8.1 there is $B_0 > 1$ such that for sufficiently large t and every $0 \le X \le \mathrm{e}^t$ there is $v \in [t, B_0 t]$ satisfying $R_{d,M}(\mathrm{e}^v) - X R_{d,1}(\mathrm{e}^v) \le \mathrm{e}^{-t}$.*

Proof. The disk $|s-4| \le 3$ lies in $\operatorname{Re} s \ge 1$. Proposition 8.2 supplies a holomorphic neighbourhood of that compact disk for both transforms and its polynomial bound implies the exponential bound in Lemma 6.1. Apply Theorem 6.2. □

Theorem 8.4 (A one-sided criterion at a prescribed height)**.** *Let d, M be as in Definition 8.1. Fix $1/2 < \beta < 1$ and $Y \ge 0$. Suppose $R_{d,M}(x) \le x^{\beta-1}$ for all sufficiently large x, and $L^\circ(s, \xi\eta_d) \ne 0$ for every $\xi \bmod M$ when $\operatorname{Re} s \ge \beta$ and $|\operatorname{Im} s| \le Y$. Then $L^\circ(s,\xi) \ne 0$ for every $\xi \bmod M$ when $\operatorname{Re} s > \beta$ and $|\operatorname{Im} s| \le Y$.*

Proof. By the assumed upper bound and continuity in Proposition 8.2 there is $C \ge 1$ with $\mathrm{e}^v F_{d,M}(v) \le C\mathrm{e}^{\beta v}$ for all $v \ge 0$. Give $[0,\infty)$ the positive measure of density $C\mathrm{e}^{\beta v} - \mathrm{e}^v F_{d,M}(v)$. In $\operatorname{Re} s > 1$ its transform equals $J(s) = C/(s-\beta) - \Phi_{d,M}(s)$. Its abscissa α is at most one by the growth estimate. If $\alpha > \beta$, the real point α is a singularity by Lemma 7.2. The meromorphic continuation is regular at 1 and has at most simple poles, so $\beta < \alpha < 1$ and this must be a simple pole. Its residue is strictly positive: along real $\sigma \downarrow \alpha$, $(\sigma-\alpha)J(\sigma) \ge 0$, and the residue of a simple pole is nonzero. On the other hand all twisted orders at α vanish. Formula (4) gives J a nonpositive residue there since $\alpha(1-\alpha) > 0$. This contradiction shows $\alpha \le \beta$, also allowing $\alpha = -\infty$.

The positive measure's integral is consequently holomorphic throughout $\operatorname{Re} s > \beta$. By uniqueness in Lemma A.3, $\Phi_{d,M}$ is holomorphic there as well. At any point ρ of the asserted region with $\operatorname{Re}\rho < 1$, the residue in (4) is zero and all twisted orders vanish. Its numerator is therefore a sum of nonnegative integers equal to zero. Each untwisted order is zero. Proposition 7.4 handles $\operatorname{Re}\rho \ge 1$. □

9. A simultaneous choice of quadratic twist

Lemma 9.1. *For $n \ge 1$ let m be its odd part, obtained by removing its power of two. For any integer v set $J_v(n) = \prod_{p^j \| m} (-v/p)^j$, the empty product being 1. Here $(\cdot/p)$ is the symbol of Definition 8.1. Then J_v is completely multiplicative in n. If d is prime and $d \equiv 7 \bmod 8$, then $J_d(n) = (n/d)$ for all n.*

Proof. Multiplicativity follows by adding exponents. We give the reciprocity calculation. For an odd prime p, put $h = (p-1)/2$. Multiplication by a unit b permutes all the units, so multiplying their values gives $b^{p-1} = 1$ in the residue field. The h nonzero squares are roots of $X^h - 1$ and exhaust its roots, since division by a linear factor lowers degree by one. Consequently $b^h = (b/p)$ in that field. The absolute values of the least signed residues of $b, 2b, \dots, hb$ permute $1, \dots, h$, since $bi \equiv \pm bj$ forces $i = j$ in this range. Multiplying shows $(b/p) = (-1)^N$, where N counts negative signed residues. In particular $(-1/p) = (-1)^h$. For $b = 2$, $N = h - \lfloor p/4 \rfloor$, even when $p \equiv 7 \bmod 8$.

If b is positive and odd and prime to p, let $r_i \in [1, p-1]$ be the positive residue of bi. The sum of the r_i is congruent modulo two to $\sum_{i=1}^h i + N$: each negative signed representative replaces its absolute value j by $p - j$. Summing $bi = p\lfloor bi/p \rfloor + r_i$ gives $N \equiv \sum_{i=1}^h \lfloor bi/p \rfloor \bmod 2$. For distinct odd primes b, p, this last floor sum and its counterpart with b, p interchanged count complementary points in $1 \le i \le (p-1)/2$, $1 \le j \le (b-1)/2$, according as $pj < bi$ or $pj > bi$. Equality is impossible. Hence $(b/p)(p/b) = (-1)^{(p-1)(b-1)/4}$. For $b = d \equiv 7 \bmod 8$ this gives $(p/d) = (-d/p)$ for odd $p \ne d$, and both sides vanish at $p = d$. The factor at 2 is one by the calculation above. Multiplication proves the formula. □

Lemma 9.2. *There is an absolute C such that, if the odd part of $u \ge 1$ is not a square and $X \ge 1$, then*

$$\left| \sum_{1 \le v \le X} J_v(u) \right| \le C\sqrt{u}(1 + \log u).$$

Proof. First let $\lambda \bmod f$ be primitive and nonprincipal. By Lemma 4.1 and Fourier inversion on the residue classes, $\lambda(n)$ is $1/f$ times a Fourier sum whose nonconstant coefficients have modulus at most $\sqrt{f}$, and whose constant coefficient vanishes. Inversion follows by summing the finite geometric sum used in that lemma. The modulus of a sum of $\exp(2\pi ikn/f)$ over an initial interval is at most $1/|\sin(\pi k/f)| \le f/[2\min(k, f-k)]$. Here $\sin t \ge 2t/\pi$ on $[0, \pi/2]$ follows from concavity. Summing over $1 \le k < f$ gives $O(\sqrt{f}(1 + \log f))$, uniformly in the interval length.

Write m for the odd part of u. Up to the constant sign $\prod_{p^j \| m}(-1/p)^j$, the symbol is a character in $v \bmod m$. Choose a prime with odd exponent in m; a nonsquare there and ones at the other primes give a unit of character value -1 by Lemma A.1. Its primitive inducing character $\lambda \bmod f$ from Lemma 3.2 is therefore nonprincipal. Let P be the product of the primes dividing m but not f. Expanding $\mathbf{1}_{(v,P)=1} = \prod_{p|P}(1 - \mathbf{1}_{p|v})$ expresses the interval sum as at most $2^{\#\{p:p|P\}}$ sums of $\lambda(k)$ over $k \le X/h$, with coefficients of modulus at most one and $h \mid P$. For each prime at least 5, $2 \le \sqrt{p}$, and for the primes $2, 3$ their combined excess factor is less than two. Thus $2^{\#\{p:p|P\}} \le 2\sqrt{P}$. Since $fP \mid m$, the preceding bound proves the claim. □

Definition 9.3. For $j \in \{0, 1\}$, $t > 0$, $T \ge 1$ and any complex sequence c of modulus at most one, set

$$g_j(t) = t^j \mathrm{e}^{-\pi t^2}, \quad \theta_j(c, t) = \sum_{n \ge 1} c(n) g_j(nt), \quad I_{j,T}(c) = \int_{1/T}^{\infty} |\theta_j(c, t)| \, \frac{\mathrm{d}t}{\sqrt{t}}.$$

Write $G_j(s) = \int_0^\infty g_j(t) t^{s-1} \, \mathrm{d}t$ for $\operatorname{Re} s > 0$. Define $\mu(1) = 1$, $\mu(n) = 0$ if a prime square divides n, and otherwise $\mu(n) = (-1)^k$ for n a product of k distinct primes.

Lemma 9.4. *For fixed j, every $\epsilon > 0$, $T \ge 1$, $1 \le X \le T^2$, and $|c(n)| \le 1$,*

$$\sum_{v \le X} |\theta_j(cJ_v, t)|^2 \ll_\epsilon T^2 t^{-1-2\epsilon} \quad (t \ge 1/T), \qquad \sum_{v \le X} I_{j,T}(cJ_v) \ll_\epsilon T^{2+\epsilon}.$$

Here and below $\ll$ has the meaning of O in Definition 3.1, and v ranges over positive integers.

Proof. For every $b > 0$, $g_j(t) \ll_b t^{-b}$ because $t^{b+j}\mathrm{e}^{-\pi t^2}$ is bounded on $t > 0$. Expand the squared sum in indices n, m. Absolute convergence follows from the Gaussian decay at each fixed t. The sum in v has kernel $J_v(nm)$ by Lemma 9.1. If the odd part of nm is a square, either nm or $2nm$ is a square. For each such choice $k = 1, 2$, the pair has the form $kn = ab^2,\ m = ah^2$ for some squarefree a and positive b, h, by collecting the primes occurring to odd exponent in kn; these are exactly the primes occurring to odd exponent in m. For $\sigma = 1/2 + \epsilon$ the sum of $(nm)^{-\sigma}$ over these pairs is at most

$$\sum_{k=1,2} k^{\sigma} \sum_{a,b,h\geq 1} (abh)^{-2\sigma} < \infty.$$

Use $|\sum_{v\leq X} J_v(nm)| \leq X$ and $g_j(nt)g_j(mt) \ll_\epsilon t^{-2\sigma}(nm)^{-\sigma}$. These pairs contribute $O_\epsilon(T^2t^{-1-2\epsilon})$.

For the other pairs Lemma 9.2 bounds the kernel sum by $O_\epsilon((nm)^{1/2+\epsilon/2})$, since $(1 + \log u)u^{-\epsilon/2}$ is bounded on $u \geq 1$. Use the Gaussian bound with exponent $3/2 + \epsilon$. The double sum is then at most a constant times $t^{-3-2\epsilon}(\sum_n n^{-1-\epsilon/2})^2$, which is $O_\epsilon(T^2t^{-1-2\epsilon})$ as $t \geq 1/T$. All numerical series here converge by comparison with their power integrals. Finally $\sum_v |z_v| \leq \sqrt{\lfloor X\rfloor \sum_v |z_v|^2}$ by expanding the sum of squares of differences of the $|z_v|$. Integrate the resulting $O_\epsilon(T^2t^{-1/2-\epsilon})$ against $\mathrm{d}t/\sqrt{t}$ to obtain the second assertion. □

Lemma 9.5. *If $a < b$, D is bounded and continuous on $a \leq \operatorname{Re} s \leq b$ and holomorphic inside, and $|D| \leq A, B$ on the left and right boundary respectively, with $A, B > 0$, then at $\operatorname{Re} s = \sigma$*

$$|D(s)| \leq A^{(b-\sigma)/(b-a)}B^{(\sigma-a)/(b-a)}.$$

Proof. Divide $D(s)$ by $A^{(b-s)/(b-a)}B^{(s-a)/(b-a)}$ to obtain a bounded function D_0 of boundary modulus at most one. For any $\delta > 0$ multiply by $\exp(\delta(s-a)^2)$. On the two sides the modulus is at most $\exp(\delta(b-a)^2)$. On horizontal segments of heights $\pm H$ it is at most $\sup|D_0|\exp(\delta((b-a)^2 - H^2))$. The maximum on a closed rectangle is on its boundary: an interior maximum of a nonconstant holomorphic function contradicts the circle mean formula of Lemma A.3, since equality in the bound for the mean forces the function to be constant on a small circle and then inside it by the disk formula. The zero maximum case is immediate. Take H arbitrarily large and then $\delta \downarrow 0$. □

Proposition 9.6. *Fix $0 < \epsilon < 1/18,\ Y \geq 0$. For all sufficiently large T, depending only on ϵ, Y, the following holds. Let χ be a nonprincipal primitive character modulo $Q \leq T^2$, with $\chi(-1) = (-1)^j$. If*

$$I_{j,T}(\chi) + I_{j,T}(\mu\chi) \leq T^{2\epsilon},$$

then $L(s, \chi) \neq 0$ whenever $\operatorname{Re} s \geq 1/2 + 9\epsilon,\ |\operatorname{Im} s| \leq Y$.

Proof. The change of variable $u = \pi t^2$ gives $G_j(s) = \frac{1}{2}\pi^{-(s+j)/2}\Gamma((s+j)/2)$. By Lemma 4.3 this is holomorphic and nonzero for $\operatorname{Re} s > 0$. The integral bounds it uniformly on $1/2 \leq \operatorname{Re} s \leq 2$ by using $t^{-1/2}$ for $t \leq 1$ and t for $t \geq 1$ in the integral. Moreover $g_j(z) \ll \mathrm{e}^{-2z}$ for $z > 0$, whence geometric summation gives $|\theta_j(c, t)| \ll 1/(\mathrm{e}^{2t} - 1) \leq \mathrm{e}^{-t}/(2t)$ up to an absolute constant. The last inequality follows from $\mathrm{e}^t - \mathrm{e}^{-t} \geq 2t$. For $\operatorname{Re} s > 1$, termwise integration of the theta series gives $G_j(s)\sum c(n)n^{-s}$, justified by absolute summability and Lemma A.2. For $c = \chi$, Lemma 4.2 at Qt^2 gives exactly

$$\theta_j(\chi, t) = \frac{\omega_\chi}{\sqrt{Q}\,t}\theta_j(\bar{\chi}, 1/(Qt)).$$

Indeed $2\theta_j(\chi, t) = t^j\Theta_\chi(Qt^2)$. This yields exponential decay at zero, up to powers of t, for each fixed Q. Consequently the full Mellin integral equals $B(s) = G_j(s)L(s, \chi)$ throughout $\operatorname{Re} s > 0$ by Proposition 3.3 and Lemma A.3. Both it and $C_T(s) = \int_{1/T}^{\infty} \theta_j(\mu\chi, t)t^{s-1}\,\mathrm{d}t$ are bounded on $1/2 \leq \operatorname{Re} s \leq 2$ for each fixed T, χ, by the same absolute integrals at its endpoints.

Consider $D(s) = G_j(s)^2 - B(s)C_T(s)$. On the line $\operatorname{Re} s = 1/2$, $|C_T(s)| \le I_{j,T}(\mu\chi)$. The functional equation of the theta series and the substitution $u = 1/(Qt)$ give

$$\int_0^{1/T} |\theta_j(\chi, t)| \, \frac{\mathrm{d}t}{\sqrt{t}} = \int_{T/Q}^{\infty} |\theta_j(\bar{\chi}, u)| \, \frac{\mathrm{d}u}{\sqrt{u}} \le I_{j,T}(\chi).$$

Here $T/Q \ge 1/T$ and the kernel is real. Thus $|B| \le 2I_{j,T}(\chi)$ and $|D| \ll T^{4\epsilon}$. On $\operatorname{Re} s = 2$ the full Mellin integral of $\mu\chi$ equals $G_j(s)/L(s,\chi)$: the absolutely convergent product of its Dirichlet series with L has coefficient $\chi(n)\sum_{h|n}\mu(h) = \mathbf{1}_{n=1}$, by expanding $\prod_{p|n}(1-1)$. The part omitted in C_T has modulus $O(1/T)$ by the preceding bound $|\theta| \ll t^{-1}\mathrm{e}^{-t}$. Also $|B| \ll 1$ there by $\sum_n n^{-2} < \infty$. Thus $|D| \ll T^{-1}$ on this line. The constants on both lines are absolute, with just two choices of j. By Lemma 9.5,

$$|D(s)| \ll T^{4\epsilon(1-u)-u}, \qquad u = (\operatorname{Re} s - 1/2)/(3/2).$$

When $1/2 + 9\epsilon \le \operatorname{Re} s \le 1$ this is $O(T^{-2\epsilon})$. At a zero of L we would have $D = G_j^2$. On the compact rectangle $1/2 \le \operatorname{Re} s \le 1$, $|\operatorname{Im} s| \le Y$, the minimum of $|G_j|^2$ over both parities is positive by Lemma 4.3. This contradicts the last bound for sufficiently large T. Proposition 7.4 handles $\operatorname{Re} s \ge 1$. □

Theorem 9.7. *Let $M \ge 1$, and let c be a unit modulo M such that every prime $d \equiv c \bmod M$ satisfies $d \equiv 7 \bmod 8$. For every $1/2 < \beta < 1$, $Y \ge 0$ there are infinitely many such primes for which, simultaneously for every $\xi \bmod M$,*

$$L^{\circ}(s, \xi\eta_d) \ne 0 \quad (\operatorname{Re} s \ge \beta, \ |\operatorname{Im} s| \le Y).$$

Proof. Choose $0 < \epsilon < 1/18$ with $9\epsilon \le \beta - 1/2$. For a fixed ξ take its inducing primitive $\psi \bmod f$ from Lemma 3.2. For a prime of the specified class we have $d \nmid M$. The character $\psi\eta_d$ modulo fd is primitive. Indeed a divisor of fd has the form hd^e with $h \mid f$ and $e = 0, 1$. If the product factors through it, varying the component modulo d with the component modulo f equal to 1 shows $e = 1$. Similarly, take the component modulo d equal to 1 and vary the component modulo f over the kernel of reduction modulo h. Primitivity and surjectivity in Lemma 3.2 give $h = f$. These independent choices follow from Lemma A.1. The product is nonprincipal and its parity j is fixed as d varies because $(-1/d) = -1$. The extra factors passing from fd to Md are nonzero for $\operatorname{Re} s > 0$, by the proof of Lemma 3.2 and Proposition 3.3.

For X large put $T = \sqrt{fX}$. If $d \le X$ in this class has a zero of the twist in the asserted region, Proposition 9.6 and Lemma 9.1 imply $I_{j,T}(\psi J_d) + I_{j,T}(\mu\psi J_d) > T^{2\epsilon}$. Here $fd \le T^2$. The sum of the left side over all positive integers $d \le X$ is $O_\epsilon(T^{2+\epsilon})$ by Lemma 9.4. So the number of exceptional primes for ξ is $O_{\epsilon,M,Y}(X^{1-\epsilon/2})$. The same estimate holds for their union over the finitely many ξ. The sum of their reciprocal primes in $[2^k, 2^{k+1})$ is $O_{\epsilon,M,Y}(2^{-k\epsilon/2})$, summable in k. Corollary 7.5 gives an infinite reciprocal sum in the whole class. Infinitely many primes survive the exclusion. □

Corollary 9.8. *For every positive integer q, put $M = 18480q$, $c = M/2 - 1$. Assume that for every prime $d \equiv c \bmod M$ and every $1/2 < \beta < 1$ there is $x_0(d, M, \beta)$ such that $R_{d,M}(x) \le x^{\beta-1}$ for $x \ge x_0(d, M, \beta)$. If this holds for every q, then $L(s, \chi) \ne 0$ for every character $\chi \bmod q$ and $\operatorname{Re} s > 1/2$, except at $s = 1$ for the principal character, where it has a simple pole of residue $\varphi(q)/q$. In particular Dirichlet GRH follows.*

Proof. Since $16 \mid M$, $c^2 - 1 = M(M/4 - 1)$ and $c \equiv 7 \bmod 8$. For $1/2 < \operatorname{Re} s < 1$, take $1/2 < \beta < \operatorname{Re} s$, $Y = |\operatorname{Im} s|$. Theorem 9.7 supplies a single prime d for which Theorem 8.4 applies. It excludes all zeros at s for characters modulo M. Inducing a character modulo q to one modulo M multiplies its L-function by nonvanishing finite Euler factors, by the argument of Lemma 3.2. This gives the claim in the half-strip. Proposition 7.4 and Proposition 3.3 give the assertion on $\operatorname{Re} s \ge 1$. Apply Theorem 4.5. □

10. Cyclotomic places and their weights

Definition 10.1. Fix $q \geq 1$, $M = 18480q$, and a prime $d \equiv M/2 - 1 \bmod M$. Let $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$, choosing the square root with positive imaginary part. For $s \in \mathbb{Z}_{\geq 1}$, put

$$c_s = \lfloor \log s / \log 3 \rfloor + 5, \quad U = 3^{c_s} 2^s, \quad N_1 = MU, \quad E = K(\zeta_{N_1}), \qquad \zeta_h = \exp(2\pi i/h).$$

An infinite place of E means a pair of complex-conjugate embeddings $E \to \mathbb{C}$. Denote the set of these pairs by Φ.

Lemma 10.2. *For $h \geq 1$, the polynomial $P_h(X) = \prod_{a \in (\mathbb{Z}/h\mathbb{Z})^\times} (X - \zeta_h^a)$ is irreducible in $\mathbb{Q}[X]$. Every unit $a \bmod h$ acts on $\mathbb{Q}(\zeta_h)$ by the automorphism $\zeta_h \mapsto \zeta_h^a$.*

Proof. The factorization $X^h - 1 = \prod_{k|h} P_k(X)$ groups the roots by their orders. Division by monic polynomials and induction on h give $P_h \in \mathbb{Z}[X]$. Monic rational factors of a monic integral polynomial are integral. Indeed the product of two integral polynomials with relatively prime coefficients within each has the same property, since reduction modulo any prime preserves their nonzero product. Clearing denominators and extracting the gcd of the coefficients in a rational factorization proves the claim by comparing contents and leading coefficients.

Let A be a monic irreducible factor with root ζ_h, and let α be any root of A. For a prime $p \nmid h$, α^p is again primitive. If it is a root of a different irreducible factor B, then $A \mid B(X^p)$. Reduction modulo p gives $\bar{A} \mid \bar{B}(X)^p$. Thus $\bar{A}, \bar{B}$ share an irreducible factor, and since $AB \mid X^h - 1$ the latter has a repeated factor modulo p. This contradicts its being coprime to its derivative hX^{h-1} there. Existence of irreducible factors and the divisibility property used here follow from polynomial division over a field, by the least-degree gcd argument of Lemma A.1. Repeatedly applying this conclusion to prime factors of a positive integer coprime to h shows that A contains every primitive h-th root. Thus $A = P_h$. Evaluation at ζ_h^a in $\mathbb{Q}[X]/(P_h)$ gives the asserted automorphism, whose inverse uses the inverse exponent. For $h = 1$ the polynomial is $X - 1$. □

Proposition 10.3. *We have $d \nmid N_1$, and $\eta_d(p) = 1$ for every prime $p \mid N_1$. The infinite places form a group identified with $G = (\mathbb{Z}/N_1\mathbb{Z})^\times$ by assigning to $a \in G$ the place of the embedding fixing K and sending ζ_{N_1} to $\zeta_{N_1}^a$. The subgroup $A = \ker(G \to (\mathbb{Z}/M\mathbb{Z})^\times)$ has size U, and $f = |G| = \varphi(M)U$.*

Proof. All prime divisors of N_1 divide M, since $6 \mid M$. The congruence on d gives $d \equiv 7 \bmod 8$, and at each odd prime $p \mid M$ it gives $d \equiv -1 \bmod p$. Thus $d \nmid N_1$ and $\eta_d(p) = 1$, including $p = 2$, by Lemma 9.1. Put $\tau = \sum_{a \bmod d} \eta_d(a)\zeta_d^a$. The Legendre character is primitive since d is prime and it is nonprincipal. Lemma 4.1 gives $|\tau|^2 = d$; $\bar{\tau} = \eta_d(-1)\tau = -\tau$, so $\tau^2 = -d$. Consequently $K \subset \mathbb{Q}(\zeta_d)$. Both ζ_d and ζ_{N_1} lie in $\mathbb{Q}(\zeta_{dN_1})$. For each $a \bmod N_1$ a unit, choose a unit exponent equal to $a \bmod N_1$ and $1 \bmod d$ by Lemma A.1. The induced automorphism of Lemma 10.2 fixes K and acts as specified on ζ_{N_1}. These are all embeddings extending the chosen inclusion of K, since the image of a primitive root must be primitive. Each conjugate pair contains just one such embedding, the two images of $\sqrt{-d}$ being opposite. Composition transports the group law on G to the places. Each unit class modulo M has exactly U lifts modulo $N_1 = MU$, all units since the prime divisors are the same. Counting proves the last two assertions. □

Lemma 10.4 (A sphere in a cyclic subgroup)**.** *Let G, H be abelian groups, $\pi : G \to H$ a homomorphism, $z, g \in G$, and h the finite order of $\pi(g)$. Let n, L, B be positive integers, $LB \geq 2$ and $(LB)^n \leq h$. There is a set $D \subset z\langle g \rangle$ with $|D| \geq B^n/(nB^2)$ such that if $u, v, w \in D$, $0 < i < j \leq L$, and $\pi(v/u)^j = \pi(w/u)^i$, then $u = v = w$.*

Proof. In $\{0, \dots, B-1\}^n$ some squared Euclidean norm occurs at least $B^n/(nB^2)$ times, since there are at most $1 + n(B-1)^2 \leq nB^2$ possible norms. For a vector a on that sphere put $e(a) = \sum_{\ell=0}^{n-1} a_\ell (LB)^\ell$, and take $zg^{e(a)}$ in D. The exponents are distinct integers in

$[0,h)$, so even their projections after removing z are distinct. For u,v,w corresponding to vectors a,b,c, the group relation gives $je(b)\equiv(j-i)e(a)+ie(c) \bmod h$. The quantities on both sides have base-LB digits at most $j(B-1)<LB$ and lie in $[0,h)$. Equality holds in each digit. Thus $b=(1-i/j)a+(i/j)c$. For $0<t<1$,

$$(1-t)\|a\|^2+t\|c\|^2-\|(1-t)a+tc\|^2=t(1-t)\|a-c\|^2$$

by expansion. Equality of the three norms gives $a=b=c$. □

Proposition 10.5. *Fix $0<\epsilon<1/100$. For every sufficiently large x put*

$$m=2^{\lceil(\log x)/(2\log 2)\rceil},\quad r=3^{\lceil 5\log x/\log 3\rceil},\quad s=22\log m/\log 2,\qquad \ell=x^{\epsilon},\quad b=\exp\ell.$$

Then $\sqrt{x}\le m\le 2\sqrt{x}$, $x^5\le r\le 3x^5$, $s\le 32\log x$. For the places in Definition 10.1 there is a subset $D\subset G$ of size $S\ge U/\ell$, all of whose elements reduce to $-1 \bmod M3^{c_s}$. If $u,v,w\in D$, $0<i<j\le 324s$, and $(v/u)^j=(w/u)^i \bmod 2^s$, then $u=v=w$.

Proof. The ceiling bounds give the stated ranges and $s\le(11/\log 2)\log x+22\le 32\log x$ eventually, since $\log 2=\int_1^2 du/u>1/2$. Put $K_0=M3^{c_s}$, $g=1+K_0$, and a_2 equal to the exponent of 2 in M. The integer g is a unit modulo N_1. For all sufficiently large s its order modulo 2^s is $h=2^{s-a_2}$. Indeed $g-1$ has exact exponent $a_2\ge 4$. Squaring a number congruent to 1 mod 4 raises the exponent of its difference from one by precisely one, by factoring its difference of squares. Raising to an odd positive power preserves that exponent by the geometric sum. This proves the order assertion.

Take an integer $n>\max(2,44/\epsilon)$, $L=324s$, $B=\lfloor h^{1/n}/L\rfloor$. For large x, $h^{1/n}/L\ge 2$, since $s\to\infty$ and the exponential dominates a fixed power of s. Apply Lemma 10.4 to reduction modulo 2^s with $z=-1$. Every chosen element is $-1 \bmod K_0$. Since $B\ge h^{1/n}/(2L)$, the count gives

$$U/S\le n\,3^{c_s}2^{a_2}(2L)^{n-2}h^{2/n}\ll_{M,n}(1+\log x)^{n-1}x^{22/n}.$$

Here $3^{c_s}\le 243s$, $2^s=m^{22}\le 2^{22}x^{11}$. The right side is at most x^{ϵ} eventually. In these comparisons any fixed power of $\log x$ is $o(x^{\delta})$ for $\delta>0$: the exponential series at $\delta\log x$ dominates the term of one higher integer degree. □

Definition 10.6. Use G,A,D,S,U,f of Propositions 10.3 and 10.5. Let $a\ge 1$ be integral and $1\le x,x_1\le a$. Give $g\in D$ weight $w(g)=a/x$ and $g\in G\setminus D$ weight $w(g)=a/x_1$. Write $g(n)=-n \bmod N_1$ if n is a unit, and $g(n)=1$ otherwise. With $y_+=\max(y,0)$ define

$$P(\tau)=\sum_{n\le a}\lambda_{d,1}(n)(a/n-w(g(n)\tau))_+,\qquad \bar w=\frac{1}{f}\sum_{g\in G}\log w(g),$$
$$P_0=a(\log a+\kappa_{d,1}-\bar w),\qquad \mathcal{R}=\sum_{\tau\in A}(P(\tau)-P_0).$$

Proposition 10.7. *For the weights of Definition 10.6,*

$$\mathcal{R}=a\{S(R_{d,M}(x)-R_{d,M}(x_1))+UR_{d,1}(x_1)\}.$$

Proof. If $(n,N_1)>1$ and $\Lambda(n)\ne 0$, then n is a power of a prime dividing N_1 and $\lambda_{d,1}(n)=0$ by Proposition 10.3. For a unit, translation by $g(n)$ maps A bijectively onto the fiber of $-n \bmod M$. It therefore hits all S elements of D precisely when $n\equiv 1 \bmod M$, and otherwise hits none. Hence $\sum_{\tau\in A}P(\tau)/a$ equals

$$U\sum_{n\le x_1}\lambda_{d,1}(n)(1/n-1/x_1)+S\left\{\sum_{n\le x}\lambda_{d,M}(n)(1/n-1/x)-\sum_{n\le x_1}\lambda_{d,M}(n)(1/n-1/x_1)\right\}.$$

For $z=x,x_1$, the positive part truncates the corresponding sum exactly at $n\le z$, because $z\le a$. Substitute Definition 8.1. The constants $\kappa_{d,M}$ cancel. The other terms in addition to the asserted remainders are

$$U(\log x_1+\kappa_{d,1})+S(\log x-\log x_1)/\varphi(M).$$

They equal UP_0/a, since $\bar{w} = \log a - \log x_1 - (S/f)(\log x - \log x_1)$ and $f = \varphi(M)U$ by Proposition 10.3. Subtraction gives the identity. □

Theorem 10.8. *For each M, d, ϵ as in Proposition 10.5, choose the places and scales there and set $a = N_1\lceil xb/N_1\rceil$, $t = 10^6 \log x$. Suppose that for all sufficiently large x and every $v \in [t, t^2]$ the quantity $\mathcal{R}$ with $x_1 = \mathrm{e}^v$ satisfies $m\mathcal{R} \le 38fa\ell^4$. Then*

$$R_{d,M}(x) \le 38\varphi(M)x^{-1/2+5\epsilon} + x^{-10^6} \tag{5}$$

eventually. If the hypothesis holds for every M, d, ϵ in these ranges, then Dirichlet GRH holds.

Proof. As $x \to \infty$, $t^2 \le \ell$, so $1 \le x, x_1 \le a$ throughout the window. Proposition 10.5 gives $0 < U/S \le \ell = x^\epsilon \le \mathrm{e}^t$. By Corollary 8.3 there exists $v \in [t, B_0 t]$ with $R_{d,M}(\mathrm{e}^v) - (U/S)R_{d,1}(\mathrm{e}^v) \le \mathrm{e}^{-t}$; the whole interval lies in $[t, t^2]$ eventually since B_0 depends only on d, M. Divide the hypothesized estimate at this v by maS and use Proposition 10.7. Since $f/(mS) \le \varphi(M)\ell/\sqrt{x}$, this gives (5). For $1/2 < \beta < 1$ choose $0 < \epsilon < 1/100$ so small that $5\epsilon < \beta - 1/2$. The right side of (5) divided by $x^{\beta-1}$ tends to zero. Corollary 9.8 now applies. □

11. Primes for the auxiliary levels

Lemma 11.1. *Fix an integer $B \ge 4$ divisible by four and a positive integer A_0. For all sufficiently small $\delta > 0$, for every $Q = B^k \le \delta^{-A_0}$ with $k \ge 2$, every character $\chi \bmod Q$, and every z with $|z - 1| \le \delta$,*

$$\delta\,|(L^\circ)'(z, \chi)| < |L^\circ(z, \chi)|.$$

Proof. First suppose χ^2 is principal. Every unit $u \equiv 1 \bmod B^2$ is a square modulo B^k. At an odd $p \mid B$ start with the root $1 \bmod p$, then lift $y^2 \equiv u \bmod p^j$ by adding hp^j to y: the new coefficient modulo p can be chosen arbitrarily since $2y$ is invertible. At $p = 2$ start at $1 \bmod 8$. For $j \ge 3$, changing an odd y to $y + 2^{j-1}$ changes its square by $2^j \bmod 2^{j+1}$. These lifts and Lemma A.1 give the assertion (smaller powers follow by reduction). Therefore χ is trivial on the kernel of reduction to B^2. By surjectivity in Lemma 3.2 it comes from a character modulo B^2. The two moduli have the same prime divisors, so the two L°-functions coincide. The finitely many functions so obtained are nonzero at 1 by Proposition 7.4. Continuity of their values and derivatives gives the assertion for all of them on sufficiently small disks.

For a nonprincipal $\psi \bmod Q$, and any fixed $0 < r < 1/8$, we have

$$\sup_{|w-1|\le 2r} |L(w, \psi)| \ll_r Q^{2r}, \qquad \sup_{|w-1|\le r} |L'(w, \psi)| \ll_r Q^{2r}.$$

Indeed truncate the sum at Q. Its initial terms have absolute sum at most $1 + \int_1^Q t^{-1+2r}\,\mathrm{d}t$. All complete character sums vanish by the proof of Proposition 3.3. The remaining partial sums, starting after Q, are at most Q in modulus, so integration as in that proposition bounds the tail by $|w|Q\int_Q^\infty t^{-1-\mathrm{Re}\,w}\,\mathrm{d}t \ll_r Q^{2r}$. The derivative estimate follows from Lemma A.3 on circles of radius r.

Now suppose χ^2 is nonprincipal. Choose r so small that $10rA_0 < 1$, and take $\delta < r$. If the desired inequality failed at z, the preceding derivative estimate there and along the segment to $1 + \delta$ would give $|L(1 + \delta, \chi)| \ll_r \delta Q^{2r}$. The same size estimate on the disk gives $|L(1 + \delta, \chi^2)| \ll_r Q^{2r}$. Since $\zeta(1 + \delta) \le 1 + 1/\delta$, the Euler-product inequality in Proposition 7.4 at $t = 0$ yields $1 \ll_r \delta Q^{10r} \le \delta^{1-10rA_0}$, impossible for sufficiently small δ. □

Theorem 11.2. *Let $4 \mid B$, $B > 0$, and fix a positive integer A_0. For all sufficiently large W, all $k \ge 2$ with $Q = B^k \le W^{A_0}$, and all real $V \in [W, W^{A_0}]$, there is a prime $p \equiv 1 \bmod Q$ with $V \le \log p \le V + W$.*

Proof. Put $r = W^{-1/2}$, $a_n = \Lambda(n)\mathbf{1}_{n\equiv 1 \bmod Q}$, and

$$H(z) = \frac{1 - z\sum_{\chi \bmod Q} \ell_\chi(1+z)}{\varphi(Q)}.$$

The Euler logarithm in Proposition 8.2 and Lemma 7.1 give $H(z)/z = \sum_{n\ge 1} a_n n^{-1-z}$ for $\operatorname{Re} z > 0$. The numerator's 1 accounts for regularization of the sole principal character. Lemma 11.1 at $\delta = r$ with exponent $2A_0$ shows that H is holomorphic on a neighbourhood of $|z| \le r$ and $|H| \le 2$ there on the closed disk, since $|\ell_\chi| < 1/r$ and there are $\varphi(Q)$ summands. Thus for $D = 1/V \le r$, positivity gives $\sum a_n n^{-1-D} \le 2V$.

Suppose there is no such prime. Subtract the initial terms and put

$$P(z) = H(z) - z\sum_{\log n < V} a_n n^{-1-z}, \qquad B_1(z) = 2 - Vz - \tfrac{\sqrt{W}}{4}(1+\sqrt{W}z)^2.$$

On $|z| = r$, with $c = \operatorname{Re} z$, direct expansion gives $\operatorname{Re} B_1(z) = E_1 = 2 - (V+\tau)c$, where $\tau = \frac{W}{2}(1+\sqrt{W}c) \in [0, W]$. If $c \le D$, then $E_1 \ge 0$ and $1 - Vc \le E_1$, because $\tau c \le W/V \le 1$ when $c > 0$. For $\log n < V$, $(D-c)\log n \le 1 - Vc$. Hence

$$|P(z)| \le 2 + |z|\mathrm{e}^{E_1}\sum_n a_n n^{-1-D} \le (2 + 2rV)\mathrm{e}^{E_1}.$$

If $c > D$, the Dirichlet series gives $P(z) = z\sum_{\log n \ge V} a_n n^{-1-z}$. When $\log n > V + W$, $(D-c)\log n \le (V+W)(D-c) \le E_1$. Terms with $V \le \log n \le V + W$ and $a_n \ne 0$ are proper prime powers by the supposition. For them $(-1/4 - c)\log n \le -V/4 - Vc \le E_1$ for large W, since $\tau c \le \sqrt{W} \le 2 + V/4$. The sum of $\Lambda(n)n^{-3/4}$ over proper prime powers is at most $(1-2^{-3/4})^{-1}\sum_{m\ge 2}(\log m)m^{-3/2} < \infty$. It follows that $|P(z)| \le r(2V + O(1))\mathrm{e}^{E_1}$. Thus $|P\exp(-B_1)| \ll 1 + V$ on the whole circle. Its center value is bounded by the same quantity by Lemma A.3. Since $P(0) = 1/\varphi(Q)$, this implies $\exp(\sqrt{W}/4 - 2) \ll \varphi(Q)(1+V) \ll W^{2A_0}$. The exponential series contradicts this for large W, uniformly in k, V. □

Lemma 11.3. *Fix d as in Definition 10.1. There is an integer $H_d \ge 1$ such that among any $H_d + 1$ distinct odd primes $p \equiv 1 \bmod d$ there are two distinct p_0, p_1 and an algebraic integer $\pi \in K$ with $\pi\bar\pi = p_0 p_1$.*

Proof. Put $\omega = (1+\sqrt{-d})/2$ and $O = \mathbb{Z} + \mathbb{Z}\omega$. This is the ring of integers of K. Indeed ω satisfies $X^2 - X + (d+1)/4 = 0$, and every $z \in O$ has integral trace and norm and hence satisfies a monic integral quadratic. Conversely if $z = x + y\sqrt{-d}$, $x, y \in \mathbb{Q}$, $y \ne 0$, is a root of a monic integral polynomial, its monic minimal polynomial $X^2 - 2xX + x^2 + dy^2$ is integral by the factor argument of Lemma 10.2. Thus $a = 2x$ and $d(2y)^2$ are integers. The squared denominator of $2y$ in lowest terms divides the prime d, so $b = 2y$ is integral. The condition $4 \mid a^2 + db^2$ and $d \equiv 3 \bmod 4$ show that a, b have the same parity. Then $z = (a-b)/2 + b\omega \in O$. Rational algebraic integers are integers by the same monic factor argument.

For a prime p in the assertion, $(-d/p) = (p/d) = 1$ by the proof of Lemma 9.1. Thus the polynomial of ω has two distinct roots modulo p. Evaluation at them identifies O/pO with $\mathbf{F}_p \times \mathbf{F}_p$, since a linear polynomial can attain prescribed values at two distinct points. Let $I_p, \bar I_p$ be the two kernels in O, interchanged by conjugation. They have index p, their sum is O, and their intersection is pO. Their product is their intersection: if $a + b = 1$ with $a \in I_p, b \in \bar I_p$, multiply by an element of the intersection to express it as a sum of products. In particular $I_p\bar I_p = pO$.

Two of the $(1+\lceil\sqrt{p}\rceil)^2$ elements $i + j\omega$, $0 \le i, j \le \lceil\sqrt{p}\rceil$, agree modulo I_p. Their difference $z_p \in I_p$ is nonzero and $|z_p| \le 2(1+|\omega|)\sqrt{p}$. Then $J_p = (z_p/p)\bar I_p$ is an ideal contained in O. As a lattice it has index $|z_p|^2/p$ there. To verify the calculation, the index of a full sublattice of $\mathbb{Z}^2$ with basis columns v, w is $|\det(v, w)|$. Integral row operations from the Euclidean algorithm send v to $(h, 0)$; successive reduction of the second and then the first coordinate

gives exactly $|hw_2'|$ cosets, where w' is the transformed second column. In the complex plane, multiplication by u has real determinant $|u|^2$, so areas in a fixed lattice basis are multiplied by that factor. Here $\overline{I}_p$ has a basis given by $p, \omega - b_p$, for the corresponding root b_p.

The indices of the J_p are therefore bounded by $D_d = \lceil 4(1+|\omega|)^2 \rceil$. There are only finitely many such sublattices. An additive subgroup of index h contains hO, since the order of each element of the quotient divides h by counting cosets of its cyclic subgroup. For $h \le D_d$ there are only finitely many subsets of O/hO. Take H_d to be the number of sublattices of these indices. Two distinct primes p_0, p_1 give the same J. Now $JI_{p_i} = z_{p_i}O$, hence $z_{p_1} I_{p_0} = z_{p_0} I_{p_1}$. Multiplying by $\overline{I}_{p_1}$ gives $I_{p_0}\overline{I}_{p_1} = (p_1 z_{p_0}/z_{p_1})O$. Its generator belongs to O. Equality of the two indices $|z_{p_i}|^2/p_i$ shows that this generator has norm $p_0 p_1$. □

Theorem 11.4 (Auxiliary arithmetic data)**.** *With $M, d, \epsilon, \ell, b, x, N_1, s, U, m, r, D$ as in Definition 10.1 and Proposition 10.5, fix any $C > 0$. For every sufficiently large x, put $M_\omega = (Md)^{s+6}$ and $t = 10^6 \log x$. There are distinct primes $g_0, g_1 \in [b^{10}, b^{11}]$ congruent to $1 \bmod M_\omega^2$ and $1 \bmod N_1 d$, and an integer π of K of norm $g_0 g_1$. For every real V satisfying $100\ell \le V \le 2^s$, there is a prime $p \equiv 1 \bmod M_\omega^2$ with $V \le \log p \le V + \sqrt{\ell}$. The same data work for all $v \in [t, t^2]$, and throughout that interval*

$$10^{12} C \le \log \ell, \quad 100 \log \ell \le \log x, \quad 10^6 \log x \le v, \quad 10^6 v \le \ell, \quad 6 \log v \le \log \ell.$$

They also work with $v = x^{\epsilon/6}$, for which the last inequality is an equality.

Proof. Put $W = x^{\epsilon/6}$, so $\ell = W^6$. Since $s \le 32 \log x$, there is an integer $A_0 > 7$, depending only on M, d, ϵ, for which $M_\omega^2 \le W^{A_0}$ and $2^s \le W^{A_0}$ eventually. Indeed the logarithms of the two left sides are $2(s+6)\log(Md)$ and $s \log 2$, and $\log W = (\epsilon/6)\log x$. We have $N_1 d \mid M_\omega$: after removing the initial Md, the factor $2^s 3^{c_s}$ divides $(Md)^{s+5}$ since $6 \mid M$ and $c_s \le s+5$. Apply Theorem 11.2 with base Md and exponent $2(s+6)$. For $V \in [100\ell, 2^s]$, its range hypotheses hold, and the permitted width W is at most $\sqrt{\ell}$. For each $0 \le i \le H_d$ use the same theorem with $V_i = 10\ell + 2iW$, where H_d is from Lemma 11.3. Eventually all intervals $[V_i, V_i + W]$ are disjoint subsets of $[10\ell, 11\ell]$ with $V_i \le W^{A_0}$. Their primes satisfy the two required congruences, and the lemma supplies two and their norm generator. Finally $\log \ell = \epsilon \log x$, $100\epsilon < 1$ and t^2 is a fixed quadratic polynomial in $\log x$. The power comparison proved in Proposition 10.5 gives $10^6 t^2 \le \ell$, $6\log(t^2) \le \log \ell$ and $t^2 \le W$ eventually. For $v = W$ we have $10^6 W \le W^6$ eventually. This proves all the scale conditions. □

12. The level prime and the arithmetic part of the height

Lemma 12.1. *For the field E in Definition 10.1, the ring O_E of algebraic integers is a free abelian group of rank $2f$. Write $\Delta_E = \det(\mathrm{Tr}_{E/\mathbb{Q}}(b_i b_j))$ for the discriminant of any integral basis, and $\mathrm{rd}_E = |\Delta_E|^{1/(2f)}$. Then $1 \le \mathrm{rd}_E \le 2f \max(1, |\omega|)^2$, where $\omega = (1+\sqrt{-d})/2$.*

Proof. The elements $b_{e,j} = \omega^e \zeta_{N_1}^j$, $e = 0, 1$, $0 \le j < f$, are a rational basis. Indeed the minimal polynomial of ζ_{N_1} over K has degree f, since Proposition 10.3 gives f distinct images and Lemma 10.2 gives a polynomial of that degree. Let $\mathcal{B}$ be the integer span of this basis. It is closed under multiplication by the two monic relations; its elements are integral by the characteristic polynomial of their multiplication matrices. More generally sums and products of algebraic integers are integral: the powers of two such integers, reduced by their monic relations, span a finite integral module containing 1 and stable under both. The determinant of $XI - A$ annihilates any element acting by the matrix A on these generators, by multiplying by the adjugate. This argument needs no independence of the generators.

Let D be the matrix with rows the $2f$ embeddings applied to the basis. It is invertible. Group the rows by the image of ω; in each block all f distinct primitive roots occur, so a polynomial of degree less than f vanishing there vanishes identically. The two distinct images of ω then separate the two coefficients of each power. Multiplication by a field element is diagonal in these evaluation coordinates. Its trace is therefore the sum of its images,

and for an algebraic integer this trace is a rational algebraic integer, hence an integer by Lemma 11.3. The trace matrix on $\mathcal{B}$ is $D^{\mathsf{T}}D$, invertible. Thus $\mathcal{B} \subset O_E \subset \{y : \mathrm{Tr}(y\mathcal{B}) \subset \mathbb{Z}\}$, the last group a full lattice obtained by applying the inverse trace matrix to $\mathbb{Z}^{2f}$. Any subgroup of a finite-rank free abelian group is free: project to the first coordinate, lift a generator of its image when nonzero, and apply induction to the kernel. This gives an integral basis of O_E. Changing between integral bases has integer determinant with integer inverse and so preserves the discriminant. The integral inclusion matrix of $\mathcal{B}$ into O_E has nonzero integer determinant, hence $1 \le |\Delta_E| \le |\det D|^2$. All entries of D have modulus at most $\max(1, |\omega|)$; apply Lemma 5.1 to its columns and take the $2f$-th root. □

Proposition 12.2. *Use the data of Theorem 11.4, $v \in [t, t^2]$, $x_1 = \mathrm{e}^v$, $a = N_1\lceil xb/N_1 \rceil$, and Definition 10.6. Write $g = g_0 g_1$,*

$$G_0 = \log\left(\frac{g\ \mathrm{rd}_E}{2\pi}\right), \qquad V_0 = m\bar{w} + (m-1)G_0 - \tfrac{15}{2}\ell.$$

For all sufficiently large x, uniformly in this interval of v, there is a prime $N \equiv 1 \bmod M_\omega^2$ such that

$$V_0 \le \log N \le V_0 + \sqrt{\ell}, \qquad m\ell \le \log N \le 30m\ell.$$

There is $\pi_N \in O_E$ with $\pi_N \bar{\pi}_N = N$.

Proof. By Proposition 10.5, $N_1 \ll_M (1 + \log x)x^{11}$ and $f \le N_1$. Hence $xb \le a \le xb + N_1 \le 2xb$ eventually. Since $t^2 = o(\ell)$ by Proposition 10.5, all weights satisfy $1 \le w(\tau) \le 2b$. The prime sizes in Theorem 11.4 and the bounds in Lemma 12.1 give $19\ell \le G_0 \le 23\ell$ eventually: $\log g \in [20\ell, 22\ell]$ and $0 \le \log \mathrm{rd}_E = O_{M,d}(1 + \log x) = o(\ell)$. It follows from $m \to \infty$ and $0 \le \bar{w} \le 2\ell$ that $m\ell \le V_0 \le 25m\ell$. This lies in $[100\ell, 2^s]$ eventually as $2^s = m^{22}$ and $\ell = x^\epsilon$. Apply Theorem 11.4 and enlarge the upper bound to $30m\ell$.

Here $N \equiv 1 \bmod 4$, so choose $u \in \mathbb{Z}$ with $u^2 \equiv -1 \bmod N$ by Lemma 9.1. The ring $\mathbb{Z}[i]$ admits division with a smaller squared norm as remainder: round both coordinates of a complex quotient to their nearest integers, with squared error at most $1/2$. Take the Euclidean gcd c of N and $u + i$. It is a nonunit, for otherwise the Bezout identity multiplied by $u - i$ would give $N \mid u - i$. It is not associated to N since $N \nmid u + i$. Its positive integer norm is therefore a nontrivial proper divisor of N^2, hence N. Take $\pi_N = c$. Since $4 \mid N_1$, $i = \zeta_{N_1}^{N_1/4} \in O_E$. □

Definition 12.3. Continue with the notation of Proposition 12.2. For $0 \le y \le a$ let $P(y, \tau)$ be the sum $P(\tau)$ of Definition 10.6 with a/n replaced by y/n and all other quantities fixed. Put $k = r + 1 - m$, $h_2 = r^8$, $h = 2h_2$, $t_1 = 5r$. Put $b_1 = N_1\lceil a/(xN_1) \rceil$, the positive multiple of N_1 obtained by rounding a/x up. For integer $j \ge 0$, set $w_j(\tau) = N_1\lceil w(\tau)/N_1 \rceil + jh + h_2$. Write $\gamma = \lim_{n\to\infty}(\sum_{k=1}^n 1/k - \log n)$, whose existence follows by integral comparison as in Lemma 4.3, and

$$\Omega = \frac{2\pi}{\mathrm{rd}_E}\exp(\gamma - L'(1, \eta_d)/L(1, \eta_d)), \qquad J(u) = \log(u!) - u\log(\Omega/g) \quad (u \in \mathbb{Z}_{\ge 0}),$$
$$D_0 = \sum_{\tau \in \Phi}\{J(w_0(\tau)) - P(w(\tau), \tau)\}, \qquad H = mkD_0,$$
$$h_0 = H + m\sum_{\tau \in A}\{J(a) - P(a, \tau)\} - Ua\log N, \qquad B_0' = fa\ell^3.$$

The quotient defining Ω is real and finite by Proposition 7.4, as η_d is real.

Proposition 12.4. *For all sufficiently large x, uniformly in $t \le v \le t^2$,*

$$0 \le w_j(\tau) - w(\tau) \le x^{42}, \qquad w_j(\tau) \le 2b, \qquad 0 \le j \le 2m,$$
$$h_0 \le H - m\mathcal{R} + B_0', \qquad |H| + m|\mathcal{R}| \le b^2 x^{99}.$$

Proof. Rounding adds at most N_1. The largest additional shift is $(4m+1)r^8 \le (8\sqrt{x}+1)(3x^5)^8$, hence the total is at most x^{42} eventually. Also $x_1 \ge x$, so $w(\tau) \le a/x \le b + N_1/x$. Since $b = \exp(x^\epsilon)$, the asserted upper bound follows.

In Definition 8.1 at modulus one, the untwisted regularization is $(s-1)\zeta(s)$. At $s=1$ its value is 1 and its derivative, by the integral in Proposition 3.3, is $1 + \int_1^\infty (\lfloor u \rfloor - u) u^{-2}\,\mathrm{d}u = \gamma$. Indeed integration of $\lfloor u \rfloor = \sum_{k \le u} 1$ up to the integer n gives $\sum_{k \le n} 1/k - 1$ for the integral of $\lfloor u \rfloor / u^2$. Therefore $\kappa_{d,1} = L'(1,\eta_d)/L(1,\eta_d) - \gamma - 1$, and $J(a) = \log(a!) + a(G_0 + 1 + \kappa_{d,1})$. Monotonicity of log gives $\log(a!) \le \log a + \int_1^a \log u\,\mathrm{d}u = a\log a - a + 1 + \log a$, also for $a = 1$. Consequently $J(a) - P_0 \le a(G_0 + \bar{w}) + 1 + \log a$. Substitute $\sum_{\tau \in A} P(a,\tau) = \mathcal{R} + UP_0$. The lower bound $\log N \ge V_0$ now gives

$$h_0 - H + m\mathcal{R} \le U\{m(1+\log a) + a(G_0 + 15\ell/2)\}.$$

We have $1 + \log a \le 3\ell$, $m \le a$, $G_0 \le 23\ell$, and $U \le f$. This is at most $fa\ell^3$ eventually.

For the crude bound, $0 \le P(y,\tau) \le 2y \log a(1+\log a)$ by the harmonic estimate in Proposition 8.2, and $|P_0| \ll_d a(1+\ell)$. For $1 \le u \le 2b$, $|J(u)| \le u(\log u + |\log(\Omega/g)|) \ll_d b(1+\ell)$, since $-\log(\Omega/g) = G_0 + 1 + \kappa_{d,1}$. It follows that $|H| + m|\mathcal{R}| \ll_{M,d} (mkfb + mUa)(1+\ell)^2$. Use $k \le r+1$, $f \le x^{12}$, $m \le 2\sqrt{x}$, $a \le 2xb$, $r \le 3x^5$. The last bound is $O_{M,d}(bx^{18}(1+\ell)^2)$ and is at most b^2x^{99} eventually. □

Proposition 12.5. *For each prime p, $0 \le y \le a$, and $\tau \in G$, define*

$$A_p(y,\tau) = \tfrac{1}{2}\sum_{j \ge 1} (1 - \eta_d(p)^j)(y/p^j - w(g(p^j)\tau))_+.$$

Then $P(y,\tau) = 2\sum_{p<x_1} A_p(y,\tau)\log p$. Define the nonnegative gain

$$G_p = m\left(\sum_{\tau \in A} A_p(a,\tau) + k\sum_{\tau \in \Phi} A_p(w(\tau),\tau)\right).$$

It vanishes unless $p < x_1$ and $\eta_d(p) \ne 1$.

Proof. Since $\min w = a/x_1$, a positive summand requires $p^j < x_1 \le a$. The definition of Λ decomposes $P(y,\tau)$ into exactly the indicated prime-power terms with coefficient $\log p$. At an inert prime $\eta_d(p) = -1$ just the odd exponents contribute, each with coefficient one in A_p. At $p = d$ all exponents have coefficient $1/2$, and at $\eta_d(p) = 1$ every coefficient vanishes. □

Proposition 12.6. *For a prime $p < x_1$ with $\eta_d(p) \ne 1$, set*

$$e_p = 3fa\ell^2\, p^{-1} \begin{cases} m & p \equiv 1 \text{ or } -1 \bmod m, \\ 1 & \text{otherwise.} \end{cases}$$

Set $e_N = 4mfb$, and set $e_p = 0$ at all other primes. Eventually these rules have disjoint ranges, and $\sum_p e_p \log p \le B_0'$. There is a positive rational c with

$$\log c \le B_0' - \sum_p G_p \log p, \qquad p^{G_p - e_p} \le |c|_p < p^{G_p - e_p + 1} \quad \text{for every prime } p.$$

Here $|c|_p = p^{-\operatorname{ord}_p(c)}$, with ord_p the exponent in the numerator minus the exponent in the denominator.

Proof. We have $\log N \ge m\ell > v = \log x_1$ eventually by Proposition 12.2 and Theorem 11.4. The sum of $1/p$ up to x_1 is at most $1 + v$. In the two exceptional classes the primes are of the form $mj \pm 1$ with $1 \le j \le x_1$ for large m. Then $m/p \le 2/j$, so their total weighted sum is at most $4(1+v)$. Bounding $\log p$ by v shows that the contribution from $p < x_1$ is at most $15fa\ell^2 v(1+v)$. By Theorem 11.4, $v^6 \le \ell$, so its ratio to $fa\ell^3$ tends uniformly to zero. The term at N is at most $120m^2fb\ell \le 480fa\ell$, by $m^2 \le 4x$, $a \ge xb$. This proves the sum bound. Put $c = \prod_p p^{\lfloor e_p - G_p \rfloor}$. Both sequences have finite support by Proposition 12.5, so this is a positive rational. The two inequalities follow from $e_p - G_p - 1 < \lfloor e_p - G_p \rfloor \le e_p - G_p$ and the sum bound. □

13. The definite unitary group and its columns

Definition 13.1. For a rational prime p, $\mathbb{Z}_p = \varprojlim_j \mathbb{Z}/p^j\mathbb{Z}$ is the ring of compatible residues and $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$. Put $E_p = E \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $O_p = O_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For the rank $k = r + 1 - m$ of Definition 12.3, use the standard hermitian form $h(u, v) = \sum_{i=1}^{k} \bar{u}_i v_i$. Use the same involution on E_p, acting trivially on $\mathbb{Q}_p$. Put

$$\Gamma = \{g \in \mathrm{GL}_k(E) : \bar{g}^{\mathsf{T}} g = 1\}, \quad G_p^u = \{g \in \mathrm{GL}_k(E_p) : \bar{g}^{\mathsf{T}} g = 1\},$$
$$K_p^0 = \{g \in G_p^u : gO_p^k = O_p^k\}.$$

Let G^u consist of tuples $(g_p)_p$ with $g_p \in G_p^u$ and $g_p \in K_p^0$ for all but finitely many p. Write $K^0 = \prod_p K_p^0$. The diagonal copy of Γ lies in G^u, since the entries of a rational matrix and its inverse in an integral basis have only finitely many denominator primes.

Lemma 13.2. *If a prime $p \equiv 1 \bmod N_1 d$, then O_p is a product of $2f$ copies of $\mathbb{Z}_p$, paired without fixed points by conjugation. Choosing one from each pair identifies G_p^u with $\mathrm{GL}_k(\mathbb{Q}_p)^f$, and K_p^0 with $\mathrm{GL}_k(\mathbb{Z}_p)^f$. For $p = g_0, g_1$ choose the component in each pair where $\bar{\pi}_g$ is divisible by p, with π_g the integer of norm $g_0 g_1$ from Theorem 11.4. For $p = N$ choose the component where π_N is divisible by N. These choices exist uniquely.*

Proof. A finite multiplicative subgroup of a field is cyclic. If its exponent is e, for each prime power exactly dividing e an element of that order is obtained by taking a power of a suitable group element. Multiplication of these elements of relatively prime orders gives an element of order e. All group elements are roots of $X^e - 1$, so the group has at most e elements and is generated by this element. Consequently $\mathbf{F}_p^\times$ contains N_1-th roots of exact order N_1. All the N_1-th roots are distinct since $p \nmid N_1$. The factorization in Lemma 10.2, applied successively to divisors of N_1, shows that P_{N_1} has exactly the primitive ones as roots modulo p. The polynomial of ω also has distinct roots modulo p, as in Lemma 11.3. A simple root modulo p of any integral polynomial F lifts uniquely at each step from p^j to p^{j+1}, because $F(y + p^j b) \equiv F(y) + p^j b F'(y) \bmod p^{j+1}$. The compatible sequence gives a simple root in $\mathbb{Z}_p$. An element of $\mathbb{Z}_p$ nonzero modulo p is a unit, by compatible inversion at each finite level. Every nonzero element is a unit times p^a: shift its digits after the first nonzero one. This also shows that $\mathbb{Q}_p$ is a field.

In the order $\mathcal{B}$ from Lemma 12.1, reduction at p is thus evaluation on the grid consisting of two choices for the root of the polynomial of ω and f for the root of P_{N_1}. This evaluation, also over $\mathbb{Z}_p$, is an isomorphism onto the product: one-variable interpolation uses only differences of roots which are units, and can be applied successively in the two variables. The trace matrix on this product is nondegenerate modulo p since it is the ordinary coordinate pairing in its idempotent basis. Thus p does not divide the trace determinant on $\mathcal{B}$, and hence does not divide its change-of-basis determinant in O_E, by Lemma 12.1. Its matrix is invertible over $\mathbb{Z}_p$, giving the same product for O_p. Conjugation interchanges the two images of ω and inverts the chosen root of unity. A unitary matrix has component $(A^{-1})^{\mathsf{T}}$ at the partner of any component A. This proves the identifications. At $p = g_i$ the valuations of the two conjugate components of $\bar{\pi}_g$ are nonnegative integers summing to 1, since g_0, g_1 are distinct. The identical argument applies to $\pi_N \bar{\pi}_N = N$. □

Definition 13.3. Define $K_c \subset K^0$ as follows. At g_0, g_1 require all the chosen matrices to reduce to the identity. At N require the last row of each chosen matrix to reduce to $(0, \ldots, 0, 1)$. Elsewhere use K_p^0. Write $\mathcal{C}(K) = \Gamma \backslash G^u / K$ for any subgroup K of K^0. At the infinite places $\tau \in A$ take the a-th symmetric power of the conjugate standard representation in rank k, and at the other places take the trivial representation. Write W for their tensor product, with the action ρ of Γ given by applying the chosen embedding at each place to the matrix entries and conjugating. In the symmetric power, multilinear symmetric products of a vectors are represented by homogeneous degree-a polynomials in formal basis vectors,

with the group acting on each basis vector by its matrix. The columns form the complex vector space

$$\mathrm{Col} = \{F : G^u \to W : \ F(\gamma g h) = \rho(\gamma) F(g) \text{ for } \gamma \in \Gamma,\ h \in K_c,\ g \in G^u\}.$$

In the Young diagram convention the rank-k weight uses heights $1, \dots, k-1$, with determinant exponent zero. Its only nonzero multiplicity is a at height one at the places in A.

Lemma 13.4. *There are finitely many double cosets in $\mathcal{C}(K^0)$. For any $g \in G^u$, the stabilizer $\Gamma \cap gK^0g^{-1}$ is finite.*

Proof. Let $L_g = \{x \in E^k : x \in g_p O_p^k \text{ for all } p\}$. For some positive integer b, the local modules lie between bO_p^k and $b^{-1}O_p^k$ for every p, by clearing the finitely many local denominators of g_p, g_p^{-1}. Thus L_g is a full $\mathbb{Z}$-lattice between bO_E^k and $b^{-1}O_E^k$ and is stable under O_E. Its completion at each p is precisely $g_p O_p^k$. Indeed in the quotient of the two bounding lattices, multiplication by b identifies the problem with imposing subgroups of $(\mathbb{Z}/b^2\mathbb{Z})^{2fk}$ independently at its prime divisors, which is possible by Lemma A.1. Thus two such lattices are mapped onto one another by Γ exactly when their tuples are in the same double coset.

On L_g put $B(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}\, h(x,y)$. This is symmetric positive definite, since $B(x,x) = 2\sum_{\tau\in\Phi,i} |\tau(x_i)|^2$. It is integral: over every $\mathbb{Z}_p$, the local isometry g_p^{-1} identifies the pairing with the trace pairing on O_p^k, and a rational number integral at every prime is an integer by Lemma A.1. The Gram determinant in an integral basis of L_g is independent of g. It is a positive integer whose exponent at every p equals that for L_1, since the two local Gram matrices are related by a basis change with determinant in $\mathbb{Z}_p^\times$.

For completeness, positive definite integral lattices of fixed rank n and fixed Gram determinant have only finitely many Gram matrices up to integral basis change. In Euclidean coordinates their covolume V is the square root of that determinant and the shortest nonzero length is at least one. More generally a lattice of rank n and covolume at most V with shortest length at least $c > 0$ has a basis of lengths bounded in terms of n, V, c. A cube of volume greater than V contains two points differing by a nonzero lattice vector. To see this, sum its translated indicators over any fundamental parallelotope; the integral is the cube's volume by disjoint tiling, so at some point the sum exceeds one. A cube of side $2V^{1/n}$ therefore bounds the shortest length from above by $2\sqrt{n}V^{1/n}$. A shortest vector u exists by discreteness and boundedness of a ball in basis coordinates. Its coordinates have gcd one and thus integer elementary row operations extend it to a lattice basis. Projection along u onto $u^\perp$ gives a lattice with covolume at most V/c, by the determinant in that basis. Any nonzero projected vector lifts, by subtracting an integer multiple of u, to one with parallel component of length at most $\|u\|/2$. The original shortest-length property then gives projected length at least $\sqrt{3}\|u\|/2$. Induction on rank gives a bounded basis there, whose reduced lifts together with u give a bounded basis here. The rank-one assertion follows from the covolume directly. Integral Gram matrices in a bounded basis have only finitely many possible entries by Cauchy–Schwarz, which follows by orthogonal projection to a line.

Fix one trace lattice from each isometry class. Under an identification with it, multiplication by each member z of a fixed integral basis of O_E has norm at most $\max_\tau |\tau(z)|$, by the displayed expression for $B(x,x)$. Each basis vector of the trace lattice then has finitely many possible images. Thus there are finitely many possible O_E-actions on these representatives. A trace isometry respecting the action extends rationally to an E-linear isometry of h: varying either argument by a field element detects equality from all traces, since for $y \neq 0$ the trace of $y^{-1}y$ is $2f$. It thus belongs to Γ, proving finiteness of the double cosets. A stabilizer is finite because an isometry of a positive definite lattice has finitely many choices for the image of each basis vector. □

Proposition 13.5. *The column level has index*

$$I_c = [K^0 : K_c] = (N^k - 1)^f \prod_{i=0}^{1} \left(\prod_{j=0}^{k-1} (g_i^k - g_i^j) \right)^f .$$

All stabilizers at the column level are trivial. For representatives g of $\mathcal{C}(K^0)$ put $\mathfrak{m}_k = \sum_{[g]} |\Gamma \cap gK^0 g^{-1}|^{-1}$. Evaluation at representatives of $\mathcal{C}(K_c)$ constructs all the columns, and

$$\dim_{\mathbb{C}} \mathrm{Col} = \mathfrak{m}_k I_c \binom{a+k-1}{k-1}^U > 0.$$

Proof. Reduction of $\mathrm{GL}_k(\mathbb{Z}_p)$ onto $\mathrm{GL}_k(\mathbf{F}_p)$ is onto, since any lifts of the entries of an invertible residue matrix give a unit determinant. Over $\mathbf{F}_p$ successive basis columns have $p^k - p^j$ choices after j independent columns. The stabilizer of a specified nonzero row has index $p^k - 1$, as the group acts transitively on those rows by basis extension. Lemma 13.2 gives the index by multiplication. The primes N, g_0, g_1 are distinct because $\log N \geq m\ell > 11\ell$ eventually.

The congruence kernel $I + pM_k(\mathbb{Z}_p)$ is torsion-free for an odd prime p. For $A = I + p^a B \neq I,\ a \geq 1$, with a unit entry in B, raising to a positive power prime to p preserves the least entry valuation a of $A - I$. Raising to p raises it by exactly one: modulo p^{a+2} the result is $I + p^{a+1}B$, by the binomial expansion and $p \geq 3$. Factoring a putative order into its p-part and its prime-to-p part contradicts these two claims. Now $\Gamma \cap gK_c g^{-1}$ is finite by Lemma 13.4. At g_0 its conjugate lies in a product of such kernels. It must be trivial there, hence also over E, since base extension to E_{g_0} is injective. The same argument works for any level contained in K^0 imposing the identity just at the chosen components over g_0.

Above $[g] \in \mathcal{C}(K^0)$ the stabilizer of gK^0 acts on the I_c finer cosets. It acts freely by the preceding argument, so the number of orbits is $I_c/|\Gamma \cap gK^0 g^{-1}|$. Summation gives $|\mathcal{C}(K_c)| = \mathfrak{m}_k I_c$. For a representative b_c of each finer class, choose any $w_c \in W$. The rule $F(\gamma b_c h) = \rho(\gamma) w_c$ is well defined by triviality of the stabilizer, and recovers every column. A monomial basis of the a-th symmetric power is indexed by nonnegative k-tuples summing to a. Inserting $k-1$ separators among a identical letters gives $\binom{a+k-1}{k-1}$ possibilities. Taking the tensor product over the U places proves the formula. □

Theorem 13.6 (Ramified columns)**.** *For every one of the f chosen components over N, lift a generator of $\mathbf{F}_N^\times$ to a scalar unit there, place the inverse scalar at its partner and take the identity everywhere else. Denote the resulting central element by z_w, and by T_w its right translation on* Col*. Put*

$$L = \prod_w (1 - T_w), \qquad \mathrm{Col}_{\mathrm{ram}} = L(\mathrm{Col}), \qquad c_0 = \dim_{\mathbb{C}} \mathrm{Col}.$$

Then

$$\dim_{\mathbb{C}} \mathrm{Col}_{\mathrm{ram}} = c_0 \left(1 - \frac{1}{N-1}\right)^f .$$

For any subspace V of the columns, $\dim L^{-1}V \leq \dim V + c_0 f/(N-1)$.

Proof. Centrality ensures the translations preserve the level. Their generated subgroup modulo K_c is $C = (\mathbf{F}_N^\times)^f$: the condition on the last row forces a scalar to be 1 mod N in each component. This group acts freely on $\mathcal{C}(K_c)$. Indeed let K' be generated by K_c and all z_w. It still imposes identity over g_0. If $gz = \gamma gh$ with $h \in K_c$ and z in the scalar subgroup, then $\gamma \in \Gamma \cap gK'g^{-1} = \{1\}$ by Proposition 13.5, hence $z \in K_c$.

Choose a class representative at the start of each orbit and take its translates as representatives along that orbit. Changing scalar lifts by an element of K_c does not alter the values. Thus on each orbit the action in the evaluation description of Proposition 13.5 is ordinary translation on W-valued functions on C. For a cyclic group of order $h = N - 1$, the difference between a function and its shift has image the subspace of coordinate sum

zero. Its kernel consists of constants, and the sum-zero condition has codimension one, proving the equality of the two spaces by dimension. Functions on the product are tensor products of the single-coordinate function spaces, as seen from their delta bases. The product of the f differences has image their tensor product of images, hence rank $(h-1)^f$ per scalar coordinate of W per orbit. This gives the equality. For the final inequality, $\dim L^{-1}V = \dim\ker L + \dim(V \cap \operatorname{im} L)$, by extending a basis of the kernel by lifts of an image basis. The elementary bound $1-(1-u)^f \le fu$ for $u \in [0,1]$ follows by induction. □

14. Integral columns of bounded size

Lemma 14.1. *Write $D_* = dN_1$, let L_g and B be as in Lemma 13.4, and put $L_g^\# = \{y : B(y, L_g) \subset \mathbb{Z}\}$ in the real span of L_g. Then $D_* L_g^\# \subset L_g$, and every nonzero vector of $L_g^\#$ has B-length at least $1/D_*$.*

Proof. We first show $D_*\{y \in E : \operatorname{Tr}(y\mathcal{B}) \subset \mathbb{Z}\} \subset \mathcal{B}$ for the order of Lemma 12.1. Write $\zeta = \zeta_{N_1}$, and let y' denote the conjugate under the automorphism fixing ζ and exchanging $\omega, \bar\omega$. It exists because the $2f$ embeddings in that lemma realize both choices for each primitive root and have image E. In the two sums

$$u_0 = \sum_{j=0}^{N_1-1} \operatorname{Tr}(y\zeta^{-j})\zeta^j, \qquad u_1 = \sum_{j=0}^{N_1-1} \operatorname{Tr}(y\omega\zeta^{-j})\zeta^j$$

all coefficients are integers. Upon expansion of the trace the finite geometric sum vanishes except at embeddings fixing ζ, where it equals N_1. Thus $u_0 = N_1(y+y')$, $u_1 = N_1(\omega y + \bar\omega y')$. In particular $N_1(\omega-\bar\omega)y = u_1 - \bar\omega u_0 \in \mathcal{B}$. Multiplication by $-(\omega-\bar\omega)$ gives $D_* y \in \mathcal{B}$. The trace dual of O_E is contained in that of $\mathcal{B}$, and $\mathcal{B} \subset O_E$, so the same inclusion holds for O_E.

In a lattice basis its trace dual has coordinate matrix the inverse of the integral Gram matrix. Consequently the dual commutes with extension of scalars to $\mathbb{Z}_p$, and its elements in the real span have rational coordinates. Conjugation preserves O_E, so the inclusion just proved holds coordinate by coordinate for O_p^k with the hermitian trace pairing. The p-component g_p is an isometry; by Lemma 13.4 the inclusion therefore holds for each completion of L_g. Intersecting over p gives the first claim. A nonzero element of L_g has positive integral squared length, giving the second. □

Lemma 14.2 (Short vectors spanning a lattice over the reals)**.** *Let L be a full lattice in Euclidean n-space, $n \ge 1$, and assume every nonzero vector of its dual lattice $L^\#$ has length at least $c > 0$. Then each point of the space is within $9n/c$ of L, and the vectors of L of length at most $27n/c$ span the space over $\mathbb{R}$.*

Proof. For $s > 0$, covolume V, and $x \in \mathbb{R}^n$ the formula needed is

$$Z_s(x) = \sum_{y\in L} \mathrm{e}^{-\pi\|y-x\|^2/s^2} = \frac{s^n}{V} \sum_{w\in L^\#} \mathrm{e}^{-\pi s^2\|w\|^2} \exp(2\pi i\langle w, x\rangle).$$

To prove it, periodize the Gaussian on a fundamental parallelotope. Absolute uniform convergence there follows by writing the lattice in a basis, in which Euclidean lengths bound a positive multiple of coordinate lengths. Its coefficient at $w \in L^\#$ is the Gaussian integral over the whole space divided by V; its value is the product of the one-dimensional transforms in Lemma 4.2. The dual basis parametrizes all Fourier indices. The Fourier series is absolutely convergent and recovers the periodization: convolving with the product of the one-dimensional Fejér kernels in lattice basis coordinates proves uniqueness by the same concentration argument as in Lemma 4.2.

Disjoint open balls of radius $c/2$ around dual lattice points within radius R fit inside the ball of radius $R + c/2$; scaling volume bounds their number by $(1+2R/c)^n$. Take $s = 3\sqrt{n}/c$. For $j \ge 1$ the shell $jc \le \|w\| < (j+1)c$ contributes at most $(2j+3)^n \exp(-9\pi n j^2)$ to the dual sum in absolute value. Since $2j+3 \le 5^j$, the sum of these bounds is dominated

by $\sum_{j\ge1}(5\mathrm{e}^{-9\pi})^j < 1/2$. Thus $s^n/(2V) \le Z_s(x) \le 2s^n/V$, and the same bounds with the corresponding power hold for any larger s. If every y were at distance greater than $3s\sqrt{n}$ from x, we would have

$$Z_s(x) \le \mathrm{e}^{-9\pi n/2} Z_{\sqrt{2}s}(x) \le \mathrm{e}^{-9\pi n/2} 2^{1+n/2} s^n/V < s^n/(2V).$$

This proves the distance bound. If all vectors of length at most $3R$, $R = 3s\sqrt{n}$, were in a proper subspace, take a point of length $2R$ perpendicular to it. A lattice point within R of that point has length at most $3R$ and lies outside the subspace, a contradiction. □

Definition 14.3. Use coordinates for the symmetric power of Definition 13.3 in which the pure power at $z \in \mathbb{C}^k$ has evaluation vector $b(z) = (z^\alpha)_{|\alpha|=a}$, with $z^\alpha = \prod_i z_i^{\alpha_i}$. Concretely these are the coordinates relative to the formal monomials scaled by the multinomials $\binom{a}{\alpha} = a!/\prod_i \alpha_i!$. Give coordinate vectors the hermitian squared norm $\sum_\alpha \binom{a}{\alpha}|u_\alpha|^2$, and W the tensor product norm. For a matrix A_1 write $S(A_1)$ for the matrix with coefficients determined by $b(A_1 z) = S(A_1)b(z)$, expanded as polynomials. This definition works over every commutative ring, and polynomial substitution gives $S(A_1A_2) = S(A_1)S(A_2)$.

For each $\tau \in A$ let $\theta_\tau : E \to E$ be $z \mapsto \tau(\bar{z})$, extended $\mathbb{Q}_p$-linearly to E_p. This uses the unique embedding at the place fixing K as in Proposition 10.3. On finite matrices write $\rho_p(u) = \bigotimes_{\tau\in A} S(\theta_\tau(u))$. A column with all values in the E-coordinate space is called integral if for every $g \in G^u$ and every prime p the coordinates of $\rho_p(g_p^{-1})F(g)$ belong to O_p. Each such condition gives integral coordinates under every embedding at p, because multiplication by an element of O_p in its integral basis gives a monic equation over $\mathbb{Z}_p$. Write $\sigma F(g)$ for σ applied to the individual coordinates of $F(g)$.

Proposition 14.4. *Let $n = 2fk$, $C_0 = 27nD_*$. The columns are spanned over $\mathbb{C}$ by integral columns whose values satisfy, for every embedding $\sigma : E \to \mathbb{C}$ and every $g \in G^u$,*

$$\|\sigma F(g)\|^2 \le (kaC_0)^{2aU}.$$

Proof. By Lemmas 14.1 and 14.2, for each representative g of a column class there are k vectors $z_1, \dots, z_k \in L_g$ linearly independent over E, each of trace length at most C_0. Indeed all vectors with this length bound span the real vector space, and a proper E-subspace has real dimension at most $2f(k-1) < n$. For every $\tau \in A$, independently choose $z_\tau = \sum_{j=1}^k t_{\tau j} z_j$ with $t_{\tau j} \in \{0, \dots, a\}$. Prescribe at g the value

$$w = \bigotimes_{\tau\in A} b(\theta_\tau(z_\tau)).$$

Here the automorphism acts coordinatewise on each vector. Extend to a column supported on its double coset by Proposition 13.5. The values belong to E. At g, the moved coordinates are products of monomials in $\theta_\tau(g_p^{-1}z_\tau)$. These belong to O_p since $z_\tau \in L_g$, and θ_τ preserves O_p. At γgh with $h \in K_c$, the rational transformation $\rho(\gamma)$ cancels, and $\rho_p(h_p^{-1})$ preserves the integral coordinates. This proves integrality everywhere.

By the multinomial expansion, $\langle b(z), b(y)\rangle = (\sum_i \bar{z}_i y_i)^a$. Thus $b(z)$ has squared norm $\|z\|^{2a}$, and unitary matrices preserve the form on their span. Each embedding of E is the restriction of a power automorphism of $\mathbb{Q}(\zeta_{dN_1})$: to choose its action on K, choose a square or nonsquare exponent modulo d, since such exponents act on the quadratic Gauss sum by their Legendre symbol (Lemma 4.1); independently choose its action modulo N_1. In particular these embeddings commute with complex conjugation. Thus every matrix $\sigma\theta_\tau(\gamma)$ for $\gamma \in \Gamma$ is unitary. Each $\sigma\theta_\tau(z_\tau)$ has Euclidean length at most kaC_0, since the squared trace length bounds the squared length at every embedding. This gives the asserted bound at g and at every rational translate.

For a single tensor factor, a linear functional vanishing on all these vectors gives a homogeneous polynomial of degree a which, after an invertible linear substitution with the vectors $\theta_\tau(z_j)$ as columns, vanishes on $\{0, \dots, a\}^k$. A polynomial of degree at most a in each variable vanishing on this grid is zero, by successive use of the univariate root bound.

Its monomial coefficients therefore vanish. This proves spanning in each factor; independent choices span their tensor product by expansion of elementary tensors. Varying the class gives the entire column space by Proposition 13.5. □

Corollary 14.5. *Eventually* $\mathrm{Col}_{\mathrm{ram}}$ *is spanned over* $\mathbb{C}$ *by integral forms* F *with* $\|\sigma F(g)\|^2 \le \exp(fa\ell^3)$ *for every* g, σ.

Proof. Apply L of Theorem 13.6 to the spanning family of Proposition 14.4. Right translation by z_w preserves integrality, since $\rho_p(g_p^{-1})F(gz_w) = \rho_p((z_w)_p)\rho_p((g_p(z_w)_p)^{-1})F(gz_w)$ and $(z_w)_p$ has integral entries and inverse. The difference of two integral columns is integral. In squared norm the triangle inequality costs at most a factor four for each difference. The logarithm of the resulting bound is at most $f\log 4 + 2aU\log(kaC_0)$. By Proposition 10.5, $k = O(x^5)$, $N_1, f = O_M(x^{12})$, so $\log(kC_0) = O_{M,d}(1+\log x)$. By Proposition 12.2, $\log a \le 2\ell$ eventually. Because $\log x = o(\ell)$ and $U \le f$, the asserted bound follows. These coordinates are algebraic; an arbitrary field automorphism of $\mathbb{C}$ acts on them through its restriction to E, so the same estimate holds for all such automorphisms. □

15. Moments and ranks of projected translates

Definition 15.1. Put $\nu = m/2 - 1$. The rational hermitian space Y_0 has basis

$$e_1, \dots, e_\nu,\ x_1, \dots, x_k,\ z,\ e_1^*, \dots, e_\nu^*.$$

Here x_i denotes a basis vector. Its nonzero pairings are $h(e_i, e_j^*) = h(e_j^*, e_i) = \delta_{ij}$, $h(x_i, x_j) = \delta_{ij}$ and $h(z, z) = -N$. Use the integral lattice with this basis except that x_k is replaced by $v = x_k/\bar{\pi}_N + z/N$. At g_0, g_1 the finite level is the kernel of reduction on the chosen components of the stabilizer of this lattice. The corresponding finite group is written $\mathcal{G}$.

Proposition 15.2. *The hermitian space of Definition 15.1 has rank* r *and signature* $(k+\nu, \nu+1)$ *at each archimedean place. Its specified integral lattice has a Gram matrix of determinant a unit. The level quotient is* $\mathcal{G} \simeq \prod_{i=0}^{1} \mathrm{GL}_r(\mathbf{F}_{g_i})^f$*; in particular*

$$|\mathcal{G}| = \prod_{i=0}^{1}\prod_{j=0}^{r-1}(g_i^r - g_i^j)^f.$$

Proof. Each hyperbolic plane has orthogonal eigenvectors $e_i + e_i^*$ and $e_i - e_i^*$ of opposite squared lengths. The middle block has k positive directions and one negative direction. Since $k + 1 + 2\nu = r$, this gives the signature. The Gram matrix on v, z is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$ because $\pi_N\bar{\pi}_N = N$; all other pairings are unchanged. Its determinant and the determinants of the other blocks are units. At each split pair the matrix on one component may be chosen arbitrarily and the hermitian identity determines the other by inverse transpose relative to the Gram matrix. Unimodularity makes this identification valid on integral stabilizers. Lemma 13.2 and the reduction and counting arguments of Proposition 13.5 give the formula. □

Lemma 15.3. *Let a finite group* G *act unitarily by* ρ *on a finite-dimensional complex Hilbert space* $\mathcal{H}$*. Let* $Q : \mathbb{C}^c \to \mathcal{H}$*,* $c > 0$*, let* R *be an orthogonal projection commuting with* ρ*, and set* $W = R\mathcal{H}$*,* $d_0 = \dim W$*. With standard basis* e_i *of* $\mathbb{C}^c$ *and Hilbert–Schmidt norm* $\|T\|_{\mathrm{HS}}^2 = \sum_i \|Te_i\|^2$ *in an orthonormal domain basis, put*

$$E_1 = \sum_i \|(Q^*Q - I)e_i\| + \|(1-R)Q\|_{\mathrm{HS}}^2,$$
$$E_2 = |G|^{-1}\sum_{s\in G}\|Q^*\rho(s)Q\|_{\mathrm{HS}}^2.$$

If $E_1 < c$, then $d_0 > 0$. If $d_0 > 0$, $E_1 \le \alpha c$ and $E_2 \le (1+\lambda)c^2/d_0$, for $\alpha, \lambda \ge 0$, then for each subspace $F \subset W$ there is $s \in G$ with

$$\dim(R\rho(s)Q)^{-1}(F) \le c\left(\frac{1+\dim F/d_0}{2} + \alpha + \frac{\lambda}{2}\right).$$

Proof. Write $Q_s = \rho(s)Q$, P_A for orthogonal projection to a subspace A, and $D_s = (RQ_s)^{-1}(F)$. The trace of a product is unchanged on cyclic permutation by summing its matrix entries with indices reversed. In particular

$$\left|\operatorname{tr}\left(P_{D_s}(I - Q^*Q)\right)\right| \le \sum_i \|(I-Q^*Q)e_i\|,$$

by expanding the trace in e_i, since each $\|P_{D_s}e_i\| \le 1$. In an orthonormal basis beginning with a basis of D_s, the trace of $P_{D_s}Q^*Q$ is the sum of $\|Q_s u\|^2$ over those first vectors. For each of them $RQ_s u \in F$; hence $\|Q_s u\|^2 = \|P_F Q_s u\|^2 + \|(1-R)Q_s u\|^2$. Enlarging the two sums to the whole basis bounds $\dim D_s$ by $E_1 + \|P_F Q_s\|_{\mathrm{HS}}^2$, using commutation of R and ρ. If $R = 0$, take $F = 0$; this yields $c \le E_1$, proving the first claim. Independence of the Hilbert–Schmidt norm from the basis follows from its expression as $\operatorname{tr}(T^*T)$.

Let $A_0 = |G|^{-1}\sum_s Q_s Q_s^*$. Cyclicity and counting the $|G|$ pairs with $s^{-1}t = g$ give $\operatorname{tr}(A_0^2) = E_2$. The average of $\|P_F Q_s\|_{\mathrm{HS}}^2$ is $\operatorname{tr}(P_F A_0) \le \sqrt{\dim F}\sqrt{E_2}$ by Cauchy–Schwarz on the matrix entries. Some s is at most this average. With $u = \dim F/d_0$, the inequality $\sqrt{(1+\lambda)u} \le (1+\lambda+u)/2$ gives the result. □

Theorem 15.4 (Selection of translates)**.** *Let V, W be complex vector spaces of positive finite dimensions c, d_0. Let $A_s : V \to W$ be a family of linear maps. Let $0 < \eta \le 1/8$ and suppose that, for each subspace $F \subset W$, some s satisfies*

$$\dim A_s^{-1}F \le c\left(\frac{1+\dim F/d_0}{2} + \eta\right).$$

There is a finite set T of indices such that

$$\dim \sum_{s\in T} A_s(V) \ge (1-4\eta)d_0, \qquad |T|c \le 3d_0\log(1+1/\eta).$$

Proof. From $F = 0$ we get $c(1/2 - \eta) \le d_0$ by rank and nullity. Start with $F = 0$. If its codimension k_0 exceeds $4\eta d_0$, choose s as in the hypothesis and add $A_s(V)$. The dimension gain is the rank of $V \to W/F$, so for the new codimension k_1

$$k_0 - k_1 \ge c(k_0/(2d_0) - \eta), \qquad k_1 - 2\eta d_0 \le (1 - c/(2d_0))(k_0 - 2\eta d_0).$$

Every such step adds a new index since its gain is positive. If $c \ge 2d_0$, the first step gives $k_1 \le 2\eta d_0$ and costs $c \le 8d_0/3$, which satisfies the asserted bound. If $c < 2d_0$, after j steps made before stopping the excess codimension is at most $d_0(1-2\eta)\exp(-jc/(2d_0))$ by $1 - u \le \mathrm{e}^{-u}$. The process therefore stops by $j = \lceil (2d_0/c)\log((1-2\eta)/(2\eta))\rceil$. Its cost is at most $2d_0\log(1/(2\eta)) + 2d_0$. For $0 < \eta \le 1/8$ this is at most $3d_0\log(1+1/\eta)$: use $\log(1/\eta) \ge \log 8 > 1$ and $2 - 2\log 2 < 1$. The target codimension is at most $4\eta d_0$. □

Definition 15.5. For a commutative integral domain O, two O-submodules M, P in the same module, and $a \in O$, write $\mathcal{I}(M, P; a)$ if every finite list $y_1, \dots, y_n \in M$ admits a matrix $U = (u_{ij}) \in M_n(O)$ with $\det U \mid a$ in O and $\sum_j u_{ij}y_j \in P$ for every i. An empty list has determinant one.

Lemma 15.6. *The relations $\mathcal{I}(M, P; a)$ and $\mathcal{I}(P, T; b)$ imply $\mathcal{I}(M, T; ab)$. If $f_j : M \to Y$ are O-linear maps, $P \subset Y$ is a submodule and $\mathcal{I}(M, f_j^{-1}P; a_j)$ holds for $1 \le j \le h$, then*

$$\mathcal{I}\left(P + \sum_{j=1}^{h} f_j(M), P; \prod_{j=1}^{h} a_j\right).$$

For an absolute value $|\cdot|$ with $|u| \le 1$ on O, each of the resulting matrices has $|\det U| \ge |\prod_j a_j|$ in the second assertion.

Proof. Apply the second recombination matrix to the vectors furnished by the first. The product of the determinants divides ab. For the sum of images, write elements of the enlarged module as $y_i = y_i' + f_h(x_i)$ with y_i' in the preceding sum including P. Recombine all x_i with one matrix for f_h. The recombined $f_h(x_i)$ belong to P, and the recombined y_i' still lie in the preceding module. Apply induction and the product argument. The initial case uses the identity matrix. If $\prod_j a_j = t \det U$ with $t \in O$, multiplicativity gives the last bound. □

Proposition 15.7. *Use the arithmetic data above and let $Q : \mathrm{Col} \to \mathcal{H}$, $\rho : \mathcal{G} \to U(\mathcal{H})$, R, W, d_0, E_1, E_2 satisfy Lemma 15.3, with a chosen Hilbert structure on* Col*, $c = c_0$. Suppose $E_1 \le \alpha c_0 < c_0$, $E_2 \le (1+\lambda)c_0^2/d_0$ and*

$$\alpha + \lambda/2 + f/(N-1) \le b^{-2}.$$

On $\mathrm{Col}_{\mathrm{ram}}$ let $B_s = c' R\rho(s)Q$ for a common nonzero scalar c'. Then some set $T \subset \mathcal{G}$ with $|T|c_0 \le 3d_0 \log(1+b^2)$ supplies a basis of $\sum_{s\in T} B_s(\mathrm{Col}_{\mathrm{ram}})$, each basis vector of the form $B_s z$ with z from the short integral family of Corollary 14.5. Its length n satisfies $(1-4/b^2)d_0 \le n \le d_0$.

Proof. For each $F \subset W$, choose s as in Lemma 15.3. Multiplication by $c' \neq 0$ preserves F. Pull the inverse image back through the central difference operator L; by Theorem 13.6 this adds at most $c_0 f/(N-1)$ to its dimension. The hypotheses of Theorem 15.4 with $\eta = b^{-2}$ therefore hold for the maps $B_s L$; $b^{-2} \le 1/8$ eventually. Their images equal the images on the ramified columns. The indicated short vectors span those columns, so the union of their images spans the selected sum. Repeatedly choose a vector outside the span of those already chosen until the span is the whole sum; this takes exactly its dimension in steps and produces the stated basis. □

Corollary 15.8. *In Proposition 5.3, retain* (1) *and the second inequality of* (2)*, and assume $\delta \le 3r\log(1+b^2)B$. Then $T \le 26 f a (\log b)^4$. If each of the selected translates in Proposition 15.7 costs at most $c_0 B_0'$ in logarithmic determinant index, the total additive cost is at most $3d_0\log(1+b^2)B_0'$.*

Proof. The proof of Proposition 5.3 through Lemma 5.2 now gives $T \le 4E + 4x^{99} + 6\log(1+b^2)B$. Using $x^{99} \le E$, $\log(1+b^2) \le 3\log b$ gives the asserted constant $4+4+18 = 26$. The second assertion follows by multiplication of the translate count by B_0'; for local determinants the multiplication of divisors that gives additivity after taking logarithms is Lemma 15.6. With $T = m\mathcal{R}$, the new numerical upper bound implies the one required in Theorem 10.8. □

16. Heights of families indexed by finite places

Definition 16.1. We use set theory with the axiom of choice in its maximal-element form: a partially ordered set in which each chain has an upper bound has a maximal element if it is nonempty. For a prime p the ring $\mathbb{Z}_p$ consists of compatible residue classes modulo p^n for $n \ge 1$. Each element has a unique expansion $\sum_{j\ge 0} a_j p^j$ in these residues, with $0 \le a_j < p$. Indeed each lift of a residue modulo p^j specifies precisely its next digit. A nonzero element is p^k times an element with nonzero constant digit, where k is its first nonzero digit position. An element with nonzero constant digit has compatible inverses in the finite residue rings by Lemma A.1. It follows by multiplication that $\mathbb{Z}_p$ is a domain with unique maximal ideal $p\mathbb{Z}_p$. Its fraction field is $\mathbb{Q}_p$. On that field the valuation v_p sends $p^k u$, $u \in \mathbb{Z}_p^\times$, to k, and $|x|_p = p^{-v_p(x)}$ for $x \neq 0$, with $|0|_p = 0$.

Lemma 16.2. *Each $\mathbb{Q}_p$ has an algebraic closure K_p with a multiplicative absolute value extending $|\cdot|_p$ and satisfying $|x+y|_p \le \max(|x|_p, |y|_p)$. There are field isomorphisms $\mathbb{C} \to K_p$. Every embedding of a number field $F \subset \mathbb{C}$ into K_p, or into $\mathbb{C}$, extends to such an isomorphism onto the specified field.*

Proof. A field K admits an algebraic extension splitting all polynomials in $K[X]$: well-order the polynomials, adjoin successively roots of irreducible factors by taking polynomial-ring quotients, and for each polynomial repeat until it splits. At a limit take the union. Repeating this construction for countably many rounds and taking their union gives an algebraic closure, since the coefficients of any polynomial in the union occur in one round. Algebraicity is transitive here: a finite list of algebraic elements generates a finite extension by the power bases, and multiplication of the dimensions in a tower follows by multiplying the elements of the two bases. A field isomorphism extends to an isomorphism of algebraic closures. In fact order its algebraic partial extensions into the target by inclusion and take a maximal one by Definition 16.1. An element outside its domain has a minimal polynomial with a root in the target; evaluation at that root extends the embedding, a contradiction. The image is now algebraically closed and the target is algebraic over it, giving surjectivity.

To extend the absolute value, order local subrings of K_p containing $\mathbb{Z}_p$ by domination: the larger contains the smaller and its maximal ideal contracts to the smaller maximal ideal. Consider only the subrings dominating $\mathbb{Z}_p$. A union over a chain is local with maximal ideal the union of the maximal ideals, since every element outside this union has its inverse already in a member of the chain. Write $S, \mathfrak{m}$ for a maximal such pair. For $u \neq 0$ outside S, one has $\mathfrak{m}S[u] = S[u]$. Otherwise a maximal ideal of $S[u]$ containing $\mathfrak{m}S[u]$ exists by the same maximal-element principle on proper ideals; it contracts to $\mathfrak{m}$ since no unit can belong to it. Inverting all elements outside that ideal would give a strictly larger dominating local subring of K_p. Thus $1 = a_0 + \cdots + a_n u^n$ with $a_i \in \mathfrak{m}$, $n \ge 1$. Division by $u^n(1-a_0)$ gives a monic equation for u^{-1}. Consequently $S[u^{-1}]$ is a finitely generated S-module. Its product with $\mathfrak{m}$ is proper: otherwise a finite generating column v satisfies $v = Cv$ with C a matrix over $\mathfrak{m}$. The determinant of $I - C$ lies in $1 + \mathfrak{m}$, hence is a unit, and multiplying by its adjugate would give $v = 0$, although the module contains 1. Localizing at a maximal ideal as above and using maximality of S proves $u^{-1} \in S$.

Write the group $K_p^\times / S^\times$ additively and denote the class of y by $v(y)$. Order classes by $v(y) \ge v(z)$ if $y/z \in S$. The last paragraph gives a total order, and the ring laws give $v(y+z) \ge \min(v(y), v(z))$ for nonzero sums. The restriction to $\mathbb{Q}_p^\times$ is $v_p(\cdot)v(p)$, with $v(p) > 0$. In a polynomial relation for nonzero y over $\mathbb{Q}_p$, the nonzero terms of least value occur at least twice: division by a unique least term would make 1 the negative of a sum in $\mathfrak{m}$. Thus for two different exponents i, j one has $(i-j)v(y) = kv(p)$ for some integer k. Ordered groups have no torsion, so $v(y)$ is uniquely identified with a rational multiple of $v(p)$, respecting sums and order. Normalizing $v(p) = 1$ and taking $p^{-v(y)}$ now gives the required absolute value.

The complex field is algebraically closed: if a nonconstant polynomial has no zero, its reciprocal is entire and bounded (use the leading term outside a disk and compactness inside). The coefficient bound of Lemma A.3 on disks of arbitrarily large radius makes this reciprocal constant. The cardinality of $\mathbb{Z}_p$ is the continuum by its digit expansion: binary sequences inject into its digit sequences, and each digit can be encoded in a fixed-length binary block. The real numbers have the same cardinality, by binary expansions and by injecting binary sequences into ternary expansions with digits $0, 2$. The field of fractions and the algebraic closure of $\mathbb{Z}_p$'s fraction field have the same cardinality, since a polynomial is specified by a finite list and has finitely many roots. Here finite tuples and countable unions of finite tuples over a set of continuum size can be encoded in binary sequences by interleaving digits and prefixing the length. Both K_p and $\mathbb{C}$ thus have cardinality continuum. For a number field F, maximal algebraically independent sets over F exist by unions of chains. The whole field is algebraic over the rational functions in such a set by maximality.

If its cardinal is κ, the rational functions and their algebraic elements number at most $\max(\aleph_0, \kappa)$: they are specified by finite lists from an infinite alphabet of this size and finitely many root choices. For completeness, an infinite well-ordered cardinal λ satisfies $|\lambda^2| = \lambda$. Order ordinal pairs by their maximum and then lexicographically within each block. By induction on infinite cardinals every proper initial segment ending at a pair of maximum $\alpha < \lambda$ has size $< \lambda$, since it lies in $(\alpha + 1)^2$; for finite α this is finite. Their order ranks therefore inject λ^2 into λ. Finite powers and the countable union follow. It follows that the independent sets on the two sides both have cardinality continuum. Match them by a bijection, extend the given embedding first to the rational function fields and then to their algebraic closures by the first paragraph. □

Definition 16.3. For each prime fix one valued field K_p as in Lemma 16.2. Let $\mathcal{P}$ be the set of pairs $\iota = (p, j)$ with $j : \mathbb{C} \to K_p$ a field isomorphism. For a family $y = (y_\iota)_{\iota \in \mathcal{P}}$ of complex numbers and $C \in \mathbb{R}$, write $\mathrm{ht}(y) \le C$ to mean that there exist $h \in \mathbb{C}^\times$, $A \in \mathbb{R}$, and reals e_p such that

$$\begin{aligned} |\sigma h| &\le \mathrm{e}^{A} \quad (\sigma \in \mathrm{Aut}(\mathbb{C})), \\ |j(hy_\iota)|_p &\le \mathrm{e}^{e_p} \quad (\iota = (p, j) \in \mathcal{P}), \\ A + \sum_{p < M} e_p &\le C \quad \text{for all sufficiently large integers } M. \end{aligned}$$

The last sum runs over primes. This notation denotes the stated relation even when an infimum of possible bounds is unattained.

Proposition 16.4. *Height bounds in Definition 16.3 add under products, and multiply by n under a nonnegative integral power n. If $\mathrm{ht}(y) \le C$, $C \le C'$ and $|jz_\iota|_p \le |jy_\iota|_p$ for every $\iota = (p, j)$, then $\mathrm{ht}(z) \le C'$. For the constant family $\mathbf{1}$ the height bound C holds exactly when $C \ge 0$.*

Proof. Multiply witnesses and add archimedean and finite exponents. Monotonicity uses the original witness. Taking $h = 1, A = e_p = 0$ works for $\mathbf{1}$ at cost zero and gives the zeroth power; induction proves the statement for all powers.

Conversely let h, A, e_p witness a bound for $\mathbf{1}$. The first inequality implies that h is algebraic over $\mathbb{Q}$. Indeed if it were transcendental one could include it in a maximal algebraically independent set over $\mathbb{Q}$. For each positive integer n, translation $h \mapsto h + n$ fixing the other basis elements is an automorphism of the rational function field, with inverse $h \mapsto h - n$. The extension argument of Lemma 16.2 gives an automorphism of $\mathbb{C}$. The values $|h + n|$ are unbounded. For algebraic $h \ne 0$ of degree D, let $a \in \mathbb{Q}^\times$ be the signed constant coefficient $(-1)^D P_h(0)$ of its monic minimal polynomial. It is the product of its D roots, distinct because a characteristic-zero irreducible polynomial is coprime to its nonzero derivative. Each root, in either $\mathbb{C}$ or K_p, defines an embedding of $\mathbb{Q}(h)$ that extends by Lemma 16.2. Multiplying the respective bounds gives

$$D^{-1} \log |a| \le A, \qquad D^{-1} \log |a|_p \le e_p.$$

Once M exceeds all prime divisors in numerator and denominator of a, prime factorization in Lemma A.1 gives $\log |a| + \sum_{p<M} \log |a|_p = 0$. Summing the inequalities and taking such an M in the eventual range gives $0 \le C$. □

17. The determinant of a completed family of rows

Lemma 17.1. *Let S be a positive definite symmetric h-by-h matrix over $\mathbb{Q}$. For an h-by-n complex matrix X define*

$$C_\sigma(X) = \det\left((\sigma X)^* S (\sigma X)\right) \quad (\sigma \in \mathrm{Aut}(\mathbb{C})),$$

where * is conjugate transpose. Write $\sigma' = \mathfrak{c}\sigma\mathfrak{c}$ for complex conjugation $\mathfrak{c}$ on both sides of σ. Then

$$\left|\sigma \det(X^*SX)\right|^2 \le C_{\sigma'}(X) C_\sigma(X).$$

If X is enlarged by u columns, each of squared S-norm at most e^β after application of σ, then C_σ of the enlarged matrix is at most $C_\sigma(X)\mathrm{e}^{u\beta}$.

Proof. For two matrices Y, Z with n columns, the inequality

$$|\det(Y^*SZ)|^2 \le \det(Y^*SY)\det(Z^*SZ)$$

is immediate if either family of columns is dependent. Otherwise carry out Gram–Schmidt for the S-inner product on each family. This writes $Y = Y_0 B$, $Z = Z_0 D$ with Y_0, Z_0 isometries from $\mathbb{C}^n$, and B, D invertible. After canceling $|\det B \det D|^2$ the assertion is $|\det(Y_0^* S Z_0)| \le 1$. Each column has Euclidean length at most one, since its squared length is that of the S-orthogonal projection of a unit vector. Lemma 5.1 bounds the determinant of its Gram matrix by one. Its determinant is the squared absolute determinant of the matrix, as required. Apply this with $Y = \sigma' X$, $Z = \sigma X$, since $\overline{\sigma' x} = \sigma \bar{x}$ and S is rational. For the second assertion apply the Gram–Schmidt product in Lemma 5.1 in the given order, starting with the old columns. Each further squared residual length is at most the original squared length. A dependent family has determinant zero, and remains dependent under enlargement. □

Definition 17.2. For a linear map $u : W \to H$ of finite-dimensional complex Hilbert spaces let $Z(u) = \det(u^*u)$. In an orthonormal basis of W it equals the Gram determinant of the images of that basis. It is real and nonnegative, and is positive exactly when u is injective, by Lemma 5.1 and its proof. For any ordered basis $x = (x_1, \dots, x_D)$ of W, put $G(u, x)_{ij} = \langle u x_i, u x_j \rangle$, conjugate linear in the first argument. If C has the coordinates of x as columns in an orthonormal basis, then $G(u, x) = C^*(u^*u)C$ in that basis. In particular $\det G(u, x) = |\det C|^2 Z(u)$.

Theorem 17.3 (A determinant bound at all finite places)**.** *Let t range over a set with a distinguished member 0; let $u_t : W \to H_t$ be linear maps of finite-dimensional complex Hilbert spaces. Assume u_0 injective and fix an ordered basis x of W, an integer $k' \ge 0$, and real numbers V, δ, g. For each $\iota = (p, j) \in \mathcal{P}$ of Definition 16.3 specify t_ι and $\gamma_\iota \in \mathbb{C}^\times$. Assume $\mathrm{ht}((\bar{\gamma}_\iota \gamma_\iota)^{-1}) \le g$, and*

$$|\sigma \det G(u_0, x)| \le \mathrm{e}^V \quad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Let $e_p \in \mathbb{R}$ satisfy $\sum_{p \in J} e_p \le \delta$ for every finite set J of primes. For each fixed $\iota = (p, j)$ and for each $\varphi \in \{j, j\mathfrak{c}\}$ suppose we have a set of vectors $\mathcal{L}_{\iota,\varphi} \subset H_{t_\iota}$ and a square matrix U_φ of size $\dim W$ over $\mathbb{C}$ such that

$$|j\langle v, w\rangle|_p \le 1 \qquad (v \in \mathcal{L}_{\iota, j\mathfrak{c}},\ w \in \mathcal{L}_{\iota,j}),$$
$$|\varphi(\det U_\varphi)|_p \ge \mathrm{e}^{-e_p} |\varphi(\gamma_\iota)|_p^{k'}, \quad \sum_l (U_\varphi)_{il} u_{t_\iota} x_l \in \mathcal{L}_{\iota,\varphi} \quad (\textit{all } i).$$

Then the squared-volume ratio satisfies

$$\mathrm{ht}\left(\frac{Z(u_{t_\iota})}{Z(u_0)}\right) \le V + 2\delta + k' g.$$

Proof. For fixed $\iota = (p, j)$ write $G = G(u_{t_\iota}, x)$. The matrix of pairings of the two recombined families is $\overline{U_{j\mathfrak{c}}}\, G\, U_j^{\mathsf{T}}$. Under j, all its entries have absolute value at most one. Expanding its determinant as a finite signed sum gives the same bound for its determinant by Lemma 16.2. From the two lower determinant bounds, noting $j(\overline{z}) = (j\mathfrak{c})(z)$, one obtains

$$\left| j\big(\det G\,(\bar{\gamma}_\iota \gamma_\iota)^{k'}\big) \right|_p \le \mathrm{e}^{2e_p}.$$

By Definition 17.2, $h = \det G(u_0, x)$ is nonzero and $h\, Z(u_{t_\iota})/Z(u_0) = \det G$. Thus $h, V, (2e_p)_p$ witness the height bound $V + 2\delta$ for the volume ratio times $(\bar\gamma_\iota\gamma_\iota)^{k'}$. Multiply by the k'-th power of its scalar correction using Proposition 16.4. □

Corollary 17.4. *Suppose independent rows $x_1, \dots, x_n$ in a space W of dimension d_0 are given. Let $u_t, t_\iota, \gamma_\iota$, the pairing bound, and the exponents e_p, δ, g be as in Theorem 17.3. Suppose that, in a coordinate realization of H_0 with positive rational Gram matrix S, the columns $X = (u_0 x_i)_{i\le n}$ satisfy $C_\sigma(X) \le \mathrm{e}^V$ for every σ. For each of the two embeddings at ι assume these n vectors on applying u_{t_ι} can be recombined into $\mathcal{L}_{\iota,\varphi}$ by an n-row matrix of φ-determinant of absolute value at least e^{-e_p}. Suppose W has a spanning set of vectors z such that each $u_0 z$ has squared norm at most e^β after every field automorphism in these coordinates, and $\gamma_\iota u_{t_\iota} z \in \mathcal{L}_{\iota,\varphi}$ for both required embeddings at every ι. Then*

$$\mathrm{ht}\left(\frac{Z(u_{t_\iota})}{Z(u_0)}\right) \le V + 2\delta + (d_0 - n)(\beta + g).$$

Proof. As long as the span of the current rows is proper, some member of the spanning set lies outside it. In precisely $k' = d_0 - n$ steps this gives a basis. Lemma 17.1 gives the archimedean determinant bound for the completed basis with exponent $V + k'\beta$. Use the block diagonal matrix with first block the recombination of the original rows and second block $\gamma_\iota I_{k'}$. Its determinant satisfies the bound of Theorem 17.3; the two diagonal blocks give exactly the stipulated integral vectors. Apply that theorem. □

Lemma 17.5. *Let $u : U \to H$ be injective and $A : T \to H$ linear between finite-dimensional complex Hilbert spaces. Put $L = u(U)$ and let P_L be the orthogonal projection onto L. For $s \in \mathbb{C}$ the endomorphism $M = sP_L AA^*|_L$ satisfies*

$$Z(u) \det M = s^{\dim U} Z(A^* u).$$

If $v : U \to H'$ is another linear map such that for every $x, y \in U$ one has $\langle vx, vy\rangle = a\langle x, Sy\rangle$ for $S \in \mathrm{End}(U)$, $a \in \mathbb{C}$, then $Z(v) = a^{\dim U} \det S$.

Proof. Regard u as an isomorphism $\tilde u : U \to L$. In orthonormal coordinates its squared absolute determinant is $Z(u)$. Since P_L fixes the image of u, $\tilde u^* M \tilde u = s(A^*u)^*(A^*u)$. Take determinants. For the last claim the assumed equality in all matrix coefficients gives $v^*v = aS$, so take determinants again. □

Proposition 17.6 (Cancellation of the two volume ratios)**.** *Use k, m, B_0', D_0, H from Definition 12.3, and put $E_0 = \ell B_0'$. Let $d_0 = \dim W > 0$ and assume the hypotheses of Corollary 17.4. Suppose $n \ge (1 - 4/b^2)d_0$, $\beta + g \le b^3/4$, and every u_{t_ι} is injective. For $t = 0$ and all t_ι suppose there are $c_t \in \mathbb{C}^\times, \lambda_j(t) > 0$ $(1 \le j \le m)$ for which*

$$Z(u_t) = |c_t|^{d_0} \prod_{j=1}^{m} \lambda_j(t),$$

$$\mathrm{ht}(|c_{t_\iota}/c_0|^{-2}) \le 2E_0, \qquad \mathrm{ht}(\lambda_j(0)/\lambda_j(t_\iota)) \le d_0 k \left(\frac{E_0}{mk} - D_0\right) \quad (1 \le j \le m).$$

For the parameter range of Proposition 5.3,

$$d_0(H - 3E_0) \le V + 2\delta.$$

Proof. Write $Z_\iota = Z(u_{t_\iota})/Z(u_0) > 0$. The completion term $(d_0 - n)(\beta + g) \le d_0 b$, since $0 \le d_0 - n \le 4d_0/b^2$ and $b > 0$. By Corollary 17.4, $\mathrm{ht}(Z_\iota) \le V + 2\delta + d_0 b$. The m ratios of λ_j have product of height at most $d_0(E_0 - mkD_0) = d_0(E_0 - H)$ by Proposition 16.4. Square that product and multiply by $|c_{t_\iota}/c_0|^{-2d_0}$. The factorization shows the result is Z_ι^{-2}, so its height is at most $2d_0(2E_0 - H)$. Multiplication by Z_ι^2 gives the constant family 1; the same proposition therefore gives $0 \le 2(V + 2\delta + d_0 b) + 2d_0(2E_0 - H)$. Here $E_0 = fa\ell^4 \ge b$ by the parameter hypotheses. Rearrangement yields the assertion. □

Corollary 17.7. *For all sufficiently large x and every $t \le v \le t^2$ in the range of Theorem 10.8, suppose the hypotheses of Propositions 15.7 and 17.6 hold for the same space of rows and its selected family. Assume each selected translate costs at most $c_0 B_0'$ in logarithmic index and the images of the selected rows under u_0 all have squared norm at most $\exp(H - m\mathcal{R} + E_0)$ in the archimedean coordinates of Corollary 17.4, for every σ. Here the index costs bound the sums of the exponents e_p there by the sum of the costs of the translates. Then*

$$m\mathcal{R} \le 26 f a \ell^4.$$

If the hypotheses hold for all the choices of q, d, ϵ in Theorem 10.8, Dirichlet GRH follows.

Proof. Lemma 5.1 permits $V = n(H - m\mathcal{R} + E_0)$. Proposition 15.7 gives the dimension range. Corollary 15.8 gives $\delta \le 3 d_0 \log(1 + b^2) B_0'$. Proposition 17.6 proves the lower volume inequality, and Proposition 12.4 gives the crude bound on $H, m\mathcal{R}$. Thus all the numerical hypotheses of Corollary 15.8 hold with $r = d_0,\ T = m\mathcal{R}$; this gives the claimed bound. It implies the hypothesis with constant 38 in Theorem 10.8, which in turn gives the assertion through Corollary 9.8. □

18. The different and the additive character for the unitary groups

Lemma 18.1. *Let A be a complete discrete valuation ring with uniformizer p, meaning that every nonzero element has the form $p^j u$, u a unit, and residues modulo p^l determine elements by compatible limits. Let $F = A/pA$. If $P \in A[X]$ is monic and its reduction is a product of distinct monic irreducibles, then $A[X]/(P)$ is a product of complete discrete valuation rings with uniformizer p. If*

$$P = X^e + p(c_{e-1} X^{e-1} + \cdots + c_0), \qquad c_i \in A, \quad c_0 \in A^\times,$$

then $A[X]/(P)$ is a complete discrete valuation ring of uniformizer the image of X, and p is a unit times its e-th power. Such rings are integrally closed in their fraction fields.

Proof. Lift a factorization $\bar{P} = UV$ into coprime monic factors over F. Given monic lifts whose product is correct modulo p^l, add corrections $p^l U_1, p^l V_1$ to them, of degrees less than the respective factors, to correct the product modulo p^{l+1}. The map $(U_1, V_1) \mapsto U_1 V + U V_1$ is injective since U, V are coprime, and domain and codomain (polynomials of degree less than $\deg P$) have equal dimension. It gives the corrections. Completeness gives two exact monic factors. They generate the unit ideal in $A[X]$: the same map for the lifted factors on bounded coefficient lists has determinant a unit by its reduction and can therefore represent the polynomial 1. For comaximal factors the quotient by their product maps isomorphically to the product of the quotients, by their Bezout identity. Repeating splits all factors. For each resulting ring B the powers below the degree are an A-basis by monic division. Hence B is complete and $\bigcap_l p^l B = 0$, by coordinates. An element with nonzero reduction in its residue field has an inverse: lift a residue inverse, then invert $1 - pz$ by its convergent geometric series. Every nonzero element is therefore a power of p times a unit, and p is a nonzero divisor by freeness. This gives the first assertion.

In the second case put $\pi = X \bmod P$ and $B = A[\pi]$. The same power basis and argument about geometric series show that an element is a unit if its image in $F[X]/(X^e)$ is a unit, that is, if its constant coefficient is nonzero modulo p. Thus $c_{e-1}\pi^{e-1} + \cdots + c_0$ is a unit and π^e is a unit times p. In a nonzero sum $\sum_{i<e} a_i \pi^i$ the nonzero terms are units times the powers $e\, v_A(a_i) + i$ of π. These exponents are distinct. Factoring out the least leaves a unit, since the other terms are multiples of π. The element π is a nonzero divisor since π^e is one; the ring is therefore a domain with the stated valuation. The powers of p, π give the same completeness. In a monic equation over either valuation ring an element of negative valuation would give the leading term a unique least valuation. Dividing by that term contradicts that the maximal ideal is proper. This proves integral closure. □

Theorem 18.2. *Put* $n = N_1,\ \zeta = \zeta_n,\ \omega = (1+\sqrt{-d})/2$. *Then*

$$O_E = \mathbb{Z}[\omega,\zeta] = \bigoplus_{i=0}^{1}\bigoplus_{j=0}^{f-1}\mathbb{Z}\,\omega^i\zeta^j.$$

Proof. Lemma 12.1 gives the indicated order $\mathcal{B}$ with this basis. Fix a rational prime p; write $n = p^a h,\ p\nmid h$. The cyclotomic order localized in coefficients to $\mathbb{Z}_p$ is generated by ζ_h and, for $a>0$, ζ_{p^a}. These are powers of ζ, and conversely a Bezout identity for h, p^a expresses ζ as a product of powers of them. Their separate monic cyclotomic relations present this order: the resulting map is surjective and the two free modules have the same rank $\varphi(h)\varphi(p^a) = \varphi(n)$, by Lemma A.1 on units. A kernel in a free module that vanishes after passage to its fraction field is zero. The same argument permits adjoining ω by its separate quadratic relation in any order, since the total rank in Lemma 12.1 is $2f$.

The polynomial P_h of Lemma 10.2 divides X^h-1; its reduction is separable because $p\nmid h$. If $p\neq d$, the polynomial of ω is separable modulo p because its discriminant is $-d$, nonzero also in characteristic two. Apply Lemma 18.1 first to P_h and then in each resulting factor to the quadratic. If $a>0$, adjoin $\zeta_{p^a}-1$ in each factor. Its polynomial is

$$P_{p^a}(1+X) = \sum_{u=0}^{p-1}(1+X)^{up^{a-1}}.$$

The constant is p; the reduction is $X^{\varphi(p^a)}$, since multiplying by $(1+X)^{p^{a-1}}-1$ gives $(1+X)^{p^a}-1$. Thus the second case of Lemma 18.1 applies. If $p=d$, Proposition 10.3 gives $a=0$, so after adjoining ζ_h use $\sqrt{-d}$ and its polynomial X^2+d. This generates the same ring as ω, since d is odd. Again the lemma applies. In every case $\mathcal{B}\otimes\mathbb{Z}_p$ is a product of integrally closed rings, with total fraction algebra $E\otimes\mathbb{Q}_p$ by the power basis. A global algebraic integer satisfies the same monic equation in each local fraction field, hence lies in every $\mathcal{B}\otimes\mathbb{Z}_p$. Its rational coefficients in the displayed basis lie in $\mathbb{Z}_p$ for all p, so have no prime in their reduced denominators by Lemma A.1. They are integers. The opposite inclusion is Lemma 12.1. □

Lemma 18.3. *Let* α *have monic irreducible polynomial* $P\in\mathbb{Z}[X]$ *of degree* $h\geq 1$, *and put* $C=\mathbb{Z}[\alpha]$. *In* $\mathbb{Q}(\alpha)$, *the trace dual* $C^\vee = \{a : \mathrm{Tr}(aC)\subset\mathbb{Z}\}$ *is* $P'(\alpha)^{-1}C$.

Proof. For $\deg T<h$, interpolation on the distinct complex roots z of P writes $T(X)$ as the sum of $T(z)P(X)/((X-z)P'(z))$. This holds since the polynomials agree on all h roots and have degree less than h. Comparison of leading coefficients gives $\mathrm{Tr}(T(\alpha)/P'(\alpha)) = [X^{h-1}]T$. Here multiplication in the power basis diagonalizes upon evaluation at the h roots, exactly as in Lemma 12.1. For products of elements of C first take the remainder modulo P. On the power basis the matrix of $(x,y)\mapsto\mathrm{Tr}(xy/P'(\alpha))$ thus has integer entries, zeros for $i+j<h-1$ and ones for $i+j=h-1$. Reversing the columns makes it triangular with diagonal one. Its inverse is integral by its adjugate, proving the duality. □

Proposition 18.4. *Write* $\mathfrak{D}_E = \{u\in E : uO_E^\vee\subset O_E\}$, *the different of* $E/\mathbb{Q}$. *With* n,ζ *as in Theorem 18.2, set*

$$h_0' = 1-\varphi(n)/2-n/4, \qquad \varpi = \zeta^{h_0'}P_n'(\zeta)\sqrt{-d}.$$

Then $\mathfrak{D}_E = \varpi O_E,\ \bar\varpi = -\varpi$, *and*

$$\prod_{\tau\in\Phi}|\tau(\varpi)| = \mathrm{rd}_E^f, \qquad \mathrm{rd}_E = \sqrt{d}\,n\prod_{p\mid n}p^{-1/(p-1)}.$$

Proof. In the basis of Theorem 18.2, field trace of a product from the quadratic and cyclotomic factors is the product of the two traces: the embeddings vary independently by Proposition 10.3. Apply Lemma 18.3 to both factors. Products of their dual basis elements are dual to the product basis. It follows that $O_E^\vee = (P_n'(\zeta)\sqrt{-d})^{-1}O_E$. Here $4\mid n$

and $\varphi(n)$ is even, since the unit classes are paired by negation for $n > 2$. The reciprocal relation $P_n(X^{-1}) = X^{-\varphi(n)} P_n(X)$ follows from inversion of the primitive roots and the constant coefficient one (their inverse pairs have product one). Differentiate it at ζ to obtain $P_n'(\bar{\zeta}) = -\zeta^{2-\varphi(n)} P_n'(\zeta)$. As $\zeta^{2-\varphi(n)-2h_0'} = \zeta^{n/2} = -1$, the first two factors of ϖ have real product. Multiplication by this power of the unit ζ preserves the generated ideal.

The trace matrix identifies $O_E^\vee$ with the inverse trace matrix applied to $\mathbb{Z}^{2f}$, so its covolume relative to O_E is $1/|\Delta_E|$. Since $O_E^\vee = \varpi^{-1} O_E$, diagonalizing multiplication at the embeddings as in Lemma 12.1 gives $|\Delta_E| = \prod_{\sigma: E \to \mathbb{C}} |\sigma(\varpi)|$. Pairing conjugates proves the first product formula. For the second, the identity $X^n - 1 = \prod_{v|n} P_v(X)$ implies

$$P_n(X) = \prod_{j|n} (X^{n/j} - 1)^{\mu(j)}.$$

Indeed the exponent of each cyclotomic factor on the right is $\sum_{j|n/v} \mu(j) = \mathbf{1}_{v=n}$, by multiplication over prime divisors as in Proposition 9.6. Divide by $X - \zeta$ and evaluate the rational expression at ζ, isolating the factor $j = 1$. At a primitive j-th root ξ, the absolute product of $1 - \xi$ over its conjugates is $P_j(1) > 0$. We have $P_{p^a}(1) = p$ by the geometric sum, and $P_j(1) = 1$ for $j > 1$ with at least two prime divisors: evaluating $(X^j - 1)/(X - 1)$ at 1 gives $\prod_{h|j, h>1} P_h(1) = j$, and induction gives the claim. Each primitive j-th root occurs $f/\varphi(j)$ times as $\zeta^{n/j}$ varies over the conjugates of ζ, by the surjectivity of unit reduction in Lemma 3.2. The product over the f embeddings fixing K of $|P_n'(\zeta)|$ is consequently $n^f \prod_{p|n} p^{-f/(p-1)}$. Multiply by $d^{f/2}$ and take f-th roots. □

Proposition 18.5. *For $u \in \mathbb{Q}_p$ let $\{u\}_p$ be the rational number in $[0, 1)$ specified by its negative-position base-p digits. Thus $u - \{u\}_p \in \mathbb{Z}_p$ and its denominator is a power of p. Define $\mathbf{e}_p(u) = \exp(2\pi i \{u\}_p)$. For $E_p = E \otimes \mathbb{Q}_p$, $O_p = O_E \otimes \mathbb{Z}_p$, put $\psi_p(y) = \mathbf{e}_p(\operatorname{Tr}_{E_p/\mathbb{Q}_p} y)$. For a finite adele $y = (y_p)$, that is $y_p \in E_p$ and $y_p \in O_p$ for all but finitely many p, put*

$$\psi_f(y) = \prod_p \psi_p(y_p), \qquad \psi_f^0(y) = \psi_f(-y/\varpi).$$

These are additive characters, and $\psi_f(a) = \exp(2\pi i \operatorname{Tr}_{E/\mathbb{Q}} a)$ for diagonal $a \in E$. The annihilator of O_p under the pairing $(y, z) \mapsto \psi_p(-\bar{y} z/\varpi)$ is O_p at every p. The same holds for $\prod_p O_p$ under the product pairing.

Proof. Two fractions with power-of-p denominators differing by an element of $\mathbb{Z}_p$ differ by an integer. Therefore $\mathbf{e}_p$ is additive with kernel $\mathbb{Z}_p$. Integral elements of O_p have integral trace from the multiplication matrix in the basis of Theorem 18.2. The finite products make sense, also after division by ϖ, whose inverse has just finitely many primes in its basis denominators. For $r \in \mathbb{Q}$, $r - \sum_p \{r\}_p$ belongs to every $\mathbb{Z}_p$, since the fractions at the other primes are p-integral. Hence it is an integer, giving the formula at a diagonal element by taking $r = \operatorname{Tr} a$. By the same inverse trace matrix as in Proposition 18.4, the trace dual of O_p is $\varpi^{-1} O_p$. Thus $\psi_p(-\bar{y} z/\varpi) = 1$ for every $z \in O_p$ exactly when $\bar{y} \in O_p$, equivalently $y \in O_p$. Varying z at one prime at a time gives the global annihilator. □

19. Singular frame minors and the second moment

Definition 19.1. On a finite group J call $\theta : J \to \mathbb{R}_{\geq 0}$ a nonnegative weight of positive type if there are vectors u_a in a complex inner product space such that $\langle u_a, u_b \rangle = \theta(a^{-1} b)$. Write $\theta(S) = \sum_{a \in S} \theta(a)$ for $S \subset J$. If J acts on a finite set and v is a member, restricting θ to $\operatorname{Stab}(v)$, by setting it zero elsewhere, preserves positive type. Indeed use vectors having the entry u_a in coordinate av of the direct sum over the finite set and zero in the other coordinates.

Lemma 19.2 (A finite Fourier bound for a singular minor)**.** *Let p be prime, $F = \mathbf{F}_p$, $\pi : J \to \mathrm{GL}_r(F)$, and A, B be r-by-j matrices over F with $j \geq 1$. Write $\pi^\vee(s) = \pi(s)^{-\mathsf{T}}$.*

For a weight θ of Definition 19.1 set $M_\theta = \theta(J)$. Suppose for all $x \in F^j \setminus \{0\}$ that the θ-masses of the π-stabilizer of Ax and of the $\pi^\vee$-stabilizer of Bx are at most $Y \geq 0$. Then

$$(p-1)\,\theta\{s : \det(B^{\mathsf{T}}\pi(s)A) = 0\} \leq (1-p^{-j})M_\theta + p^{-j}(p^j-1)^2 p^{r/2} Y.$$

Proof. Put $\chi(t) = \exp(2\pi i t/p)$ for a residue represented by an integer. Geometric summation in one coordinate shows $\sum_{w \in F^r} \chi(z^{\mathsf{T}}w) = p^r \mathbf{1}_{z=0}$. In the direct sum of p^r copies of the space of the u_a, let $D_z u$ place u at coordinate z and zero elsewhere, and let $H_z u$ have entries $p^{-r/2}\chi(-z^{\mathsf{T}}w)u$ at $w \in F^r$. These two families each have orthogonal coordinate ranges for distinct z, by the displayed character sum. For nonzero $x, y \in F^j$ form the vectors

$$V = \sum_{a \in J} H_{\pi^\vee(a)By} u_a, \qquad W = \sum_{b \in J} D_{\pi(b)Ax} u_b.$$

On expanding $\|V\|^2$, a pair a, b contributes precisely when $a^{-1}b$ fixes By under $\pi^\vee$. Every such element occurs $|J|$ times and contributes its weight each time. Thus $\|V\|^2 \leq |J|Y$, and the same argument gives $\|W\|^2 \leq |J|Y$. In the mixed pairing each coefficient has the positive phase $\chi(y^{\mathsf{T}}B^{\mathsf{T}}\pi(a^{-1}b)Ax)$, since the first variable is conjugated. Cauchy–Schwarz therefore gives

$$\left| \sum_s \chi(y^{\mathsf{T}}B^{\mathsf{T}}\pi(s)Ax)\theta(s) \right| \leq p^{r/2} Y.$$

For fixed s, y, averaging the phase over $x \in F^j$ gives the indicator of y belonging to the left kernel of $B^{\mathsf{T}}\pi(s)A$. Summing over $y \neq 0$ counts its nonzero left kernel vectors. A singular matrix has at least $p-1$ such vectors. The $x = 0$ term in this restricted sum contributes $(1-p^{-j})M_\theta$; the remaining pairs number $(p^j-1)^2$ and each contributes at most $p^{r/2}Y$ before averaging. This proves the bound. □

Definition 19.3. At $p = g_0$ let $\mathcal{W}$ index the f split pairs in Proposition 15.2. In its chosen basis let J_w be the matrix of the integral hermitian form on the split pair, reduced on one factor. In that basis it is the symmetric invertible matrix in the proof of that proposition. Put $h = \nu + 1$. Let F_j be the matrix whose columns are the first j elements of the list $e_1, \ldots, e_\nu, z$, $1 \leq j \leq h$, in the integral basis of Definition 15.1. For $s \in \mathcal{G}$ write $s_w \in \mathrm{GL}_r(\mathbf{F}_p)$ for its component at w. The leading minors at w are

$$\Delta_{w,j}(s) = \det(F_j^{\mathsf{T}} J_w s_w F_j).$$

At w a prescribed vector condition means either that s_w fixes $F_h a$, or that $s_w^{-\mathsf{T}}$ fixes $J_w^{\mathsf{T}} F_h a$, for a specified $a \neq 0$ in $\mathbf{F}_p^h$. The latter condition expresses fixing the frame vector in the other component, since its matrix is $J_w^{-1} s_w^{-\mathsf{T}} J_w$. Given $S \subset \mathcal{W}$ and one condition D_w per $w \in S$, write $V(S, D)$ for their common solution set. Let $\Omega(S, D)$ be the subset of $V(S, D)$ where either $S = \mathcal{W}$, or at some $w \in \mathcal{W}$ all h leading minors are nonzero.

Proposition 19.4. *For the group and conditions of Definition 19.3, $[\mathcal{G} : V(S, D)] = (p^r - 1)^{|S|}$. Eventually $p^{h+r/2} \leq b^{6k}$ and $3h/p \leq b^{-4}$.*

Proof. Each of the vectors is nonzero because the frame vectors belong to a basis and J_w is invertible. In $\mathrm{GL}_r(\mathbf{F}_p)$ any nonzero vector can be sent to any other by extending both to bases. The same holds for the inverse-transpose action since inverse transpose is an automorphism. Cosets of the stabilizer biject to the $p^r - 1$ nonzero vectors. Different w occur in independent direct factors of Proposition 15.2. For the bounds, $r = k + 2\nu + 1$, $h = \nu + 1$, $k \geq x^5 - 2\sqrt{x}$, and $\nu \leq \sqrt{x}$ by Proposition 10.5. It follows that $11(h + r/2) = 11k/2 + 22\nu + 33/2 \leq 6k$ eventually. Theorem 11.4 gives $b^{10} \leq p \leq b^{11}$; also $h \leq b$ eventually by the exponential comparison in Proposition 10.5. These inequalities give both claims. □

Theorem 19.5. *Let $a : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $D_* > 0$, and put $C_S = (1 + 9b^{-4})D_* b^{-6k|S|}$. Work at sufficiently large x as in Proposition 19.4. Suppose for every S, D in Definition 19.3 that*

$$\sum_{s \in \Omega(S,D)} a(s) \leq (1 - b^{-4})C_S.$$

For any nonnegative weight θ of positive type bounded by a at every element and supported on $V(S, D)$, one has $\theta(\mathcal{G}) \leq C_S$. In particular if a itself has positive type then $\sum_s a(s) \leq (1 + 9b^{-4})D_*$.

Proof. Proceed by descending induction on $|S|$, proving the conclusion for all prescribed conditions and weights at once. For $S = \mathcal{W}$ the support lies in $\Omega(S, D)$, giving the claim. Otherwise take $w \notin S$. Suppose $M_\theta > C_S$. Restricting to any one additional condition at w preserves positive type by Definition 19.1. The new weight is still bounded by a. Induction bounds its mass by $C_{S\cup\{w\}} = b^{-6k}C_S < b^{-6k}M_\theta$.

For $1 \leq j \leq h$ take $A = F_j$, $B = J_w^{\mathsf{T}} F_j$ and $\pi(s) = s_w$ in Lemma 19.2. Each stabilizer there is one of the allowed extra conditions by padding vectors with zeros. Thus we can use $Y = b^{-6k}M_\theta$; by Proposition 19.4, $p^{j+r/2}Y \leq M_\theta$. Lemma 19.2 bounds the mass with $\Delta_{w,j} = 0$ by $2M_\theta/(p-1) \leq 3M_\theta/p$ since $p \geq 3$. On the support outside $\Omega(S, D)$, at least one of these minors vanishes at this particular w. Summing in j shows the mass outside is at most $3hM_\theta/p \leq b^{-4}M_\theta$. The assumed bound on a inside the set now yields $(1 - b^{-4})M_\theta \leq (1 - b^{-4})C_S$, a contradiction. The last assertion takes $S = \varnothing, \theta = a$. □

Corollary 19.6. *Use $\mathcal{G}, \rho, Q, c_0$ as in Proposition 15.7 and let R_i, $i = 0, 1$, be commuting orthogonal projections with $R = R_0R_1$ commuting with ρ. Set $a(s) = \|Q^*\rho(s)Q\|_{\mathrm{HS}}^2$. Suppose the mass hypothesis of Theorem 19.5 holds for this a and some $D_* > 0$. Suppose further that $d_0 = \dim(R\mathcal{H})$ satisfies $d_0 D_* \leq |\mathcal{G}|c_0^2$ and that*

$$\sum_j \|(Q^*Q - I)e_j\| \leq c_0 b^{-4}, \qquad \operatorname{tr}\left(Q^*(1 - R_i)Q\right) \leq 2c_0 b^{-4} \quad (i = 0, 1).$$

Then the conclusion on the selected family in Proposition 15.7 holds for all sufficiently large x.

Proof. The weight a is of positive type. Take, as its vectors, $U_t = \rho(t)QQ^*\rho(t)^*$ with inner product $\operatorname{tr}(U_t^* U_v)$; cyclic permutation as in Lemma 15.3 gives $a(t^{-1}v)$. By Theorem 19.5, $|\mathcal{G}|E_2 \leq (1 + 9b^{-4})D_*$. The operator $1 - R$ is bounded above as a quadratic form by $(1 - R_0) + (1 - R_1)$, because their difference after subtraction of $1 - R$ is $(1 - R_0)(1 - R_1)$, an orthogonal projection. The trace inequalities therefore give $E_1 \leq 5c_0 b^{-4} < c_0$. This implies $d_0 > 0$ by Lemma 15.3, and the dimension hypothesis gives $E_2 \leq (1 + 9b^{-4})c_0^2/d_0$. Finally $f \leq b$ eventually, since $f \leq x^{12}$ in the proof of Proposition 12.2, and $N - 1 \geq b^9$ by that proposition. Thus $(5 + 9/2)b^{-4} + f/(N - 1) \leq b^{-2}$ eventually. Proposition 15.7 now applies. □

20. The unipotent covolume at the principal level

Lemma 20.1. *Embed E diagonally in $\mathbb{C}^\Phi$ using the embeddings fixing K in Proposition 10.3. Real and imaginary coordinates carry ordinary Lebesgue measure. The covolume of O_E is $V_E = 2^{-f}\sqrt{|\Delta_E|}$. The image of $O_E^- = \{s \in O_E : \bar{s} = -s\}$ under $s \mapsto (\operatorname{Im} \tau s)_\tau$ has covolume $|\Delta_E|^{1/4}$ in $\mathbb{R}^\Phi$.*

Proof. The complex embedding matrix of an integral basis has absolute determinant $\sqrt{|\Delta_E|}$, by the trace calculation in Lemma 12.1. Each pair of conjugate rows changes from real and imaginary rows by a matrix of determinant of modulus two. The real coordinate embedding matrix thus gives the stated V_E.

Let $F = E^{\mathfrak{c}}$ be the fixed field of conjugation and $O_F = F \cap O_E$. Since $12 \mid N_1$, E contains both i and $\xi = \exp(2\pi i/12)$. We have $\xi - \bar{\xi} = i$, a unit. For every $x \in O_E$

the elements $b = (x - \bar{x})/i$ and $a = x - b\xi$ lie in O_F. The decomposition $x = a + b\xi$ is unique by conjugating and subtracting. Thus $O_E = O_F \oplus \xi O_F$, and O_F has rank f by the free-subgroup argument of Lemma 12.1. In a basis of O_F, its real embedding matrix T is nonsingular: the decomposition just obtained gives a full lattice in $\mathbb{C}^\Phi$. Its determinant in real coordinates for the two summands gives $V_E = |\det T|^2 \prod_\tau |\operatorname{Im} \tau\xi| = 2^{-f} |\det T|^2$, since $\tau(i) = \pm i$. Finally $O_E^- = iO_F$, whose imaginary embedding matrix differs from T by row signs. Its covolume is $|\Delta_E|^{1/4}$. □

Lemma 20.2. *Let $h \geq 1$. Let $B_p \in \mathrm{GL}_h(E_p)$ lie in $\mathrm{GL}_h(O_p)$ for all but finitely many p, and let*

$$\Lambda = \{x \in E^h : x \in B_p O_p^h \text{ for every } p\}.$$

In coordinates $\mathbb{C}^{h\Phi}$ its covolume is $V_E^h \prod_p |\det_{\mathbb{Q}_p} B_p|_p^{-1}$, where the determinant is taken on the underlying $\mathbb{Q}_p$-vector space. Its O_E-linear dual lattice of row vectors, locally specified by $xB_p \in O_p^h$, has covolume $V_E^h \prod_p |\det_{\mathbb{Q}_p} B_p|_p$.

Proof. There is a positive integer a with $aO_p^h \subset B_p O_p^h \subset a^{-1} O_p^h$ simultaneously for all p, by clearing denominators of the finitely many matrices and inverses in an integral basis. The quotient $a^{-1}O_E^h / aO_E^h$ is the product of the corresponding local quotients, by multiplying by a and using Lemma A.1 coordinatewise modulo a^2. Consequently these local lattices specify all the reductions of Λ, and it has full rank. Relative indices are the products of the local relative indices as seen by dividing the two counts above the common subgroup aO_E^h. For a matrix over $\mathbb{Q}_p$, integral row and column operations reduce it to diagonal form: move an entry of least valuation to the first position, use its integral entry ratios to clear its column and row, and repeat on the remaining block. Row and column operations with unit determinant preserve indices. A diagonal entry $p^e u$ contributes relative covolume p^e because $|\mathbb{Z}_p / p^j \mathbb{Z}_p| = p^j$ for $j \geq 0$; negative e gives the reciprocal by interchanging the lattices. Multiplying these factors gives $|\det_{\mathbb{Q}_p} B_p|_p^{-1}$. An index n of two ordinary full lattices multiplies the covolume by n: coset representatives give a union of n translates of a fundamental parallelepiped. This proves the first formula by Lemma 20.1. The dual row lattice is specified in column coordinates by the inverse transpose over E_p. Its $\mathbb{Q}_p$-determinant has inverse absolute value to the original: after extending scalars to K_p, diagonalize multiplication on the separable field factors by evaluation at their embeddings as in Lemma 12.1, and take the product of ordinary determinants. This gives the last formula. □

Definition 20.3. In Y_0 from Definition 15.1 number the hyperbolic pairs e_a, e_a^* from $a = 0$ to $\nu - 1$. Write M_c for the middle integral lattice locally obtained by applying $c_p \oplus 1$ to the middle lattice of that definition, where $c = (c_p)_p$ is any finite adelic isometry of the positive k-dimensional space, integral at almost every p. Adjoin the standard integral hyperbolic planes to obtain a lattice Λ_c, using Lemma 20.2. Let U be the rational unitary subgroup preserving the flag $\langle e_0, \ldots, e_a \rangle$, acting identically on each of its successive lines and on the middle quotient of the orthogonal flag. On each dual line it then acts identically by the pairing. Set

$$\Gamma_c = \{u \in U(E^+) : (u - I)\Lambda_c \subset g\Lambda_c\}, \qquad E^+ = E^{\mathfrak{c}}, \quad g = g_0 g_1.$$

Here $U(E^+)$ denotes the matrices on the E-space with entries in E satisfying these unitary and flag equations. Integrality of $(u - I)/g$ at every prime is precisely integrality everywhere and principal congruence at g_0, g_1. At a split pair the two congruences agree by inverse transpose and unimodularity. The group is closed under inverses since $u - I$ is strictly block upper triangular in the ordered flag and therefore nilpotent.

Lemma 20.4. *Remove the outermost remaining pair e_a, e_a^* of Definition 20.3. The intervening space, of dimension $n_a = 2(\nu - 1 - a) + k + 1$, has hermitian matrix J and integral*

lattice L consisting of M_c and the remaining hyperbolic planes. The projection of U onto the analogous inner flag group U' splits by $u' \mapsto \operatorname{diag}(1, u', 1)$. Its kernel consists of the matrices

$$H(l,t) = \begin{pmatrix} 1 & l & t \\ 0 & I & -J^{-1}l^* \\ 0 & 0 & 1 \end{pmatrix}, \qquad t + \bar{t} = -lJ^{-1}l^*.$$

On arithmetic subgroups the same projection splits. In the kernel the conditions are precisely

$$l = gx, \quad xL \subset O_E, \qquad t = -\omega\, lJ^{-1}l^* + gs, \qquad s \in O_E^-,$$

where ω is from Theorem 18.2.

Proof. In the outer decomposition the form has antidiagonal corner entries one and middle block J. A flag element fixes the first basis vector and preserves its perpendicular. Passing to its quotient by that vector gives u'. Multiplying by $\operatorname{diag}(1, u'^{-1}, 1)$ leaves an upper triangular block matrix with identity diagonal. Applying the hermitian identity to its last column and the middle columns forces its off-diagonal blocks and trace equation to be exactly the displayed ones. Substitution proves the converse and the splitting.

The inner lattice is hermitian self-dual at each finite prime: the initial Gram determinant is a unit by Proposition 15.2, the other hyperbolic planes are unimodular, and $c_p \oplus 1$ is an isometry. Thus $xL \subset O_E$ is equivalent locally to $J^{-1}x^* \in L_p$ for every p, and hence to the global membership by Lemma 20.2. In particular $xJ^{-1}x^* \in O_E$ is fixed by conjugation. Applied to the blocks of $H - I$, principal congruence now requires precisely $l = gx$ as stated and $t \in gO_E$. Since $\omega + \bar{\omega} = 1$, the number $-\omega lJ^{-1}l^*$ is one solution of the trace equation in gO_E (in fact it belongs to g^2O_E). The other solutions differ by gO_E^-. The inner subgroup preserves the corresponding congruence by the direct-sum lattice and its block diagonal splitting. □

Proposition 20.5. *For $\tau \in \Phi$ put*

$$\kappa_\tau = \frac{g\,\tau(\varpi)}{2\pi i} \in \mathbb{R}^\times, \qquad s_{a,\tau} = \kappa_\tau^{\nu-a}/\sqrt{N} \quad (0 \le a \le \nu).$$

At τ, scale e_a, e_a^* by $s_{a,\tau}, s_{a,\tau}^{-1}$, respectively, the positive middle basis by one, and z by $s_{\nu,\tau} = 1/\sqrt{N}$. These are scales on the original rational basis $x_1, \dots, x_k, z$ before replacing x_k in the integral lattice. Use the coordinates of the scaled basis for the real groups, so matrices transform by $u \mapsto D_\tau^{-1} u D_\tau$ with this diagonal D_τ. Use Lebesgue measure at each successive kernel of Lemma 20.4 on the complex row l and the real coordinate $\operatorname{Im} t$. Their product over places and layers is invariant under both left and right translation. Put $d_* = \nu(\nu + k)$. The covolume of Γ_c for this measure is independent of c and equals

$$\mathfrak{v} = g^{(2d_*+\nu)f} V_E^{d_*} |\Delta_E|^{\nu/4} \prod_{\tau \in \Phi} \prod_{a=0}^{\nu-1} (s_{a,\tau}^2)^{-(k+2(\nu-a))}.$$

Proof. Proposition 18.4 shows that κ_τ is real and nonzero. In a real kernel the real part of t is determined by l. Matrix multiplication sends the rows to their sum and adds the two entries t with the correction $-lJ^{-1}l'^*$. Translation thus preserves Lebesgue measure in $(l, \operatorname{Im} t)$ by first translating l and then translating the real coordinate by a quantity depending on l. The inner flag group acts on l on the right by the inverse middle matrix, of complex determinant one, and fixes t. This preserves the measure; the same statements in scaled coordinates follow by the diagonal change of basis. In the split extension the product measure is therefore left and right invariant, as is seen by integrating first over the kernel and then over U'. Iteration proves the assertion for the whole group.

For the layer a, the rational row lattice of x in Lemma 20.4 has covolume $N^f V_E^{n_a}$. Indeed the finite local matrix of its primal lattice is the identity on the inner hyperbolic planes and the middle matrix $(c_p \oplus 1)B$ on $x_1, \dots, x_k, z$, where B implements the basis change of Definition 15.1. We have $\det_E B = 1/\bar{\pi}_N$. The determinant of c_p times its conjugate

is one; the determinant over $\mathbb{Q}_p$ is the product over the embeddings of E_p, so its p-adic absolute value is one. The product of the finite absolute values of $\det_{\mathbb{Q}} B$ is N^f, since at infinity $|\det_{\mathbb{Q}} B| = N^{-f}$ by $\pi_N \bar{\pi}_N = N$, and the rational product formula is proved in Proposition 16.4. Apply the row part of Lemma 20.2.

The new row is gx with its entries at place τ multiplied by $s_{j,\tau}/s_{a,\tau}$, where $s_{j,\tau}$ here denotes the scale of the inner basis vector in column j. All the paired inner scales cancel in their product; the remaining factor is $s_{\nu,\tau} = 1/\sqrt{N}$. Thus the squared absolute Jacobian over the complex row variables contributes N^{-f}, canceling the N^f in the rational row covolume. The row lattice in the scaled coordinates has covolume

$$g^{2n_a f} V_E^{n_a} \prod_{\tau} (s_{a,\tau}^2)^{-n_a}.$$

After the row is fixed, the lattice of central translations is given by $g \operatorname{Im} \tau(s)/s_{a,\tau}^2$ for $s \in O_E^-$. By Lemma 20.1 its covolume is $g^f |\Delta_E|^{1/4} \prod_\tau (s_{a,\tau}^2)^{-1}$.

Choose half-open fundamental parallelepipeds for these two ordinary lattices. Every element of the kernel has exactly one left arithmetic translate with row and central coordinate in their respective parallelepipeds: first translate the row, using an arithmetic lift from Lemma 20.4, then use a unique central correction. Two possible lifts of the row differ by a central translation. For the split extension first reduce the projection to a fixed fundamental domain of the inner arithmetic group, then reduce the kernel variables. Uniqueness follows by projecting two putative representatives and then using uniqueness in the kernel. The measures of the domains multiply by the preceding product description. Finally $\sum_{a=0}^{\nu-1} n_a = d_*$ and $n_a + 1 = k + 2(\nu - a)$; substituting yields the formula. □

Corollary 20.6. *In the setting of Proposition 20.5, let the real unipotent group act on a measurable space and let $\beta \geq 0$ be measurable with $\sum_{\gamma \in \Gamma_c} \beta(\gamma y) = 1$ for every y. Assume $u \mapsto \beta(uy)$ is measurable for each y. Then $\int_U \beta(uy)\, \mathrm{d}u = \mathfrak{v}$.*

Proof. Partition the real group into the left translates of the fundamental domain constructed in Proposition 20.5. The lattice is countable, since its elements are matrices over a number field. Left invariance and monotone convergence from Lemma A.2 turn the integral into the integral of $\sum_{\gamma \in \Gamma_c} \beta(\gamma u y) = 1$ over that domain. □

21. The unipotent character and Poincaré series

Lemma 21.1. *The rational group $U(E^+)$ of Definition 20.3 is dense in its finite adelic group U_f. Here $E^+ = E^{\mathfrak{c}}$ and finite adelic groups have the restricted product topology, with the integral subgroups prescribed at all but finitely many primes. For each c, the real group U_∞ in the scaled coordinates of Proposition 20.5 admits a compact set C_c with $U_\infty = \Gamma_c C_c$, writing the rational group by its image at infinity.*

Proof. The diagonal E is dense in its finite adeles. In an integral basis the assertion reduces to $\mathbb{Q}$. Given finitely many targets in $\mathbb{Q}_p$, specified modulo $p^{a_p}\mathbb{Z}_p$, take an integer D supported there, with D times every target p-integral and $a_p + v_p(D) \geq 0$. By Lemma A.1 choose a single integer b in the resulting residue classes modulo $p^{a_p + v_p(D)}$. Then b/D approximates all targets and is integral away from these primes. For skew elements use the continuous projection $x \mapsto (x - \bar{x})/2$; it is an identity on skew adeles. The proof of Lemma 20.4 realizes the kernel over the finite adeles by the independent coordinates l and $s = t + \omega l J^{-1} l^*$, which is skew. These polynomial maps and their inverses are continuous on the restricted products, since their fixed coefficients are integral outside finitely many primes. Thus rational kernel elements are dense. The section for the inner group and multiplication identify the whole unipotent group with the successive kernels as a topological space. Applying the same argument to each proves density. For the second claim take closures of the finitely many bounded parallelepipeds in the proof of Proposition 20.5. Their compact product has

compact image in U_∞ under the coordinate reconstruction in Lemma 20.4, and contains the fundamental domain. □

Proposition 21.2. *In the rational basis put $v_0 = e_0, \dots, v_{\nu-1} = e_{\nu-1}, v_\nu = z$ and $\alpha(u) = \sum_{a<\nu} u(v_a, v_{a+1})$, with $u(i,j)$ denoting row i, column j. On U_f the map $\chi_f(u) = \psi_f(-\alpha(u)/(g\varpi))$ is a continuous character. At infinity let α_τ denote the same sum of entries in the scaled basis and put*

$$\chi_\infty(u) = \prod_{\tau\in\Phi} \exp\left(\alpha_\tau(u_\tau) - \overline{\alpha_\tau(u_\tau)}\right).$$

For rational u, $\chi_\infty(u_\infty)\chi_f(u_f) = 1$. For $u \in \Gamma_c$ both factors are 1.

Proof. In a product of two group elements, the entry (e_a, e_{a+1}) for $a+1 < \nu$ is the sum of the corresponding entries of the factors: column e_{a+1} has support in the flag up to $a+1$ with diagonal one, and row e_a vanishes in earlier flag positions. The same holds for $(e_{\nu-1}, z)$: outside the flag the column z equals the unit vector at z because the group acts as the identity on the middle quotient and preserves the perpendicular flag. This proves additivity. At every finite prime $\mathbf{e}_p$ is continuous, being constant on cosets of $\mathbb{Z}_p$, so Proposition 18.5 gives the finite character. For a rational element, consecutive scale ratios along the frame are $1/\kappa_\tau$. Thus with $a' = -\alpha(u)/(g\varpi)$, $\alpha_\tau(u_\tau) = -2\pi i\,\tau(a')$. The product at infinity is $\exp(-2\pi i \operatorname{Tr} a')$; Proposition 18.5 gives its inverse at the finite places. For Γ_c, the vectors e_{a+1}, z belong to Λ_c, and the e_a-coordinate of each $(u-I)v_{a+1}$ belongs to gO_E. Consequently $\alpha(u)/g \in O_E$, and Propositions 18.4 and 18.5 give $\chi_f(u_f) = 1$. □

Definition 21.3. Let G_∞ be the product of the real unitary groups of Y_0 in the scaled coordinates, and G_f its finite adelic unitary group. Let K_Y^0 stabilize the initial lattice in Definition 15.1, and K_Y be its principal level at g_0, g_1, so K_Y is normal in K_Y^0. Write Γ for the rational unitary group and put $X = G_\infty \times G_f/K_Y$, with diagonal left action of Γ. For the class representative c in Definition 20.3 let h_c act by c on the positive middle block, and identically on z and on the hyperbolic planes. Then $\Gamma_c = U(E^+) \cap h_c K_Y h_c^{-1}$ in finite coordinates, by the definition of Λ_c. Right translation R_k on functions of X acts by $(R_k t)(x, hK_Y) = t(x, hkK_Y)$ for $k \in K_Y^0$, so only the class of k in $\mathcal{G}$ matters.

Lemma 21.4. *Give each component of X the same left and right invariant smooth volume element on G_∞, whose positive normalization may be chosen separately on each archimedean group. The action of Γ is free, and for compact $D_1, D_2 \subset X$, only finitely many $\gamma \in \Gamma$ satisfy $\gamma D_1 \cap D_2 \neq \varnothing$. There is a measurable fundamental domain D for the left action. For measurable nonnegative b on X,*

$$\int_D \sum_{\gamma\in\Gamma} b(\gamma y)\,\mathrm{d}y = \int_X b(y)\,\mathrm{d}y.$$

For invariant functions integrable on the quotient, integration over D is independent of the choice of fundamental domain.

Proof. On a real unitary factor with Gram matrix J, the real tangent space at one is $\{A : A^\dagger = -A\}$, $A^\dagger = J^{-1}A^*J$; $A \mapsto (I+A)(I-A)^{-1}$ gives a chart near one with inverse $(g-I)(g+I)^{-1}$. Translate an ordinary volume density on this tangent space to obtain a smooth left invariant measure by integrating the density in charts. On overlaps the change of variables preserves the integral by the Jacobian transformation rule for densities. Right translations also preserve it: conjugation on the tangent space has determinant of modulus one. Indeed its complexification is conjugation on all complex matrices, since each matrix is a sum of a skew-adjoint matrix and i times a skew-adjoint matrix. On all matrices left and right multiplication by g, g^{-1} have determinants $(\det g)^r, (\det g)^{-r}$. The real determinant equals the complex determinant of the complexified operator. Use the product of these measures at the infinite places and counting measure on G_f/K_Y.

A compact subset meets only finitely many finite components. At each possible pair $hK_Y, h'K_Y$, the finite part of a translating γ belongs to $h'K_Y h^{-1}$, whose entries lie in $a^{-1}O_p$ for all p and some fixed integer a. Indeed the initial lattice has fixed rational basis denominators, the integral stabilizer has bounded entries relative to it, and h, h' are integral with integral inverses at almost every prime. The archimedean matrices of γ are bounded by the compact sets of the two real projections and their inverses. In the original rational basis this bounds every conjugate of every entry in $a^{-1}O_E$. This is a discrete lattice by Lemma 20.1, so only finitely many matrices occur. An element fixing a point has identity archimedean matrix by cancellation and so is identity over E.

The space is second countable. The archimedean group is a subspace of finitely many matrix spaces. There are countably many finite components: hK_Y^0 determines and is determined by its tuple of local lattices, all of which occur among the finitely many intermediate modules for some integer denominator bound a, by Lemma 20.2. The additional level index is finite by Proposition 15.2. Each point has an open neighbourhood disjoint from its translates by nonidentity elements: start inside a relatively compact neighbourhood and shrink to avoid the finitely many possible intersections, using freeness and the Hausdorff property. Take a countable cover O_j by such opens and, successively, keep in O_j only points outside all translates of the preceding opens. Their union is a measurable fundamental domain. Its translates partition X; the displayed identity follows by additivity and Lemma A.2. Intersecting two fundamental domains with each other's translates gives the asserted independence. □

Theorem 21.5 (Poincaré series and their inner products)**.** *For each of finitely many pairs (c, i) choose an archimedean continuous function $W_{c,i}$ satisfying $W_{c,i}(ux) = \chi_\infty(u)W_{c,i}(x)$, $u \in U_\infty$, and supported in $U_\infty C$ for some compact $C \subset G_\infty$. Such functions exist with $W_{c,i}(1) \neq 0$. One can choose measurable nonnegative β_c such that $\phi_{c,i} = \beta_c W_{c,i}$ is continuous with compact support, and its left Γ_c-sum equals $W_{c,i}$. Set*

$$P_{c,i}(x, hK_Y) = \sum_{\gamma\in\Gamma} \mathbf{1}_{\gamma_f hK_Y = h_c K_Y}\, \phi_{c,i}(\gamma_\infty x).$$

This is a locally finite sum defining a continuous Γ-invariant function with compact support modulo Γ. For each c there is a choice with $P_{c,i}(1, h_cK_Y) \neq 0$. With the inner product conjugate linear on the left, the formula for its translates is

$$\langle R_k P_{c',j}, P_{c,i}\rangle = \sum_{\substack{\gamma\in\Gamma \\ h_c^{-1}\gamma_f h_{c'}\in kK_Y}} \int_{G_\infty} \phi_{c,i}(x)\overline{\phi_{c',j}(\gamma_\infty^{-1}x)}\,\mathrm{d}x \qquad (k \in K_Y^0).$$

The same identity holds for measurable seeds on these finite cosets when both periodizations exist absolutely almost everywhere, belong to $L^2(\Gamma\backslash X)$, and the sum on the right with the absolute value of the integrand has finite integral.

Proof. Given a continuous compactly supported F on G_∞, set

$$W(x) = \int_{U_\infty} \overline{\chi_\infty(u)}F(ux)\,\mathrm{d}u.$$

The group U_∞ is closed by its flag equations. For x in a compact neighbourhood the integration is over a fixed compact subset of that subgroup, proving continuity by dominated convergence. Right invariance in Proposition 20.5 gives the required equivariance. The support is contained in $U_\infty \operatorname{supp} F$, a closed set since the second factor is compact. Choose $F \geq 0$, positive at 1, supported near enough to 1 that $\operatorname{Re}\chi_\infty(u) > 1/2$ on this support in U_∞; a continuous bump in a matrix chart gives such F. Then $\operatorname{Re} W(1) > 0$.

For each c choose a single compact C covering the finitely many supports modulo U_∞. By Lemma 21.1 they lie in $\Gamma_c C_c C$. Take a nonnegative compactly supported continuous ξ equal to one on $C_c C$; for instance use a cutoff in a matrix norm, since the real groups are closed

matrix subsets with compact bounded closed subsets. Set $\beta_c(x) = \xi(x)/\sum_{\delta\in\Gamma_c}\xi(\delta_\infty x)$ where the denominator is positive, and zero elsewhere. The sum is locally finite by Lemma 21.4 at the finite coset h_cK_Y, and is at least one on the supports of the W's. Thus each $\phi_{c,i}$ is continuous with compact support, also at points where the denominator is zero because those points lie outside the closed supports. Its periodization by Γ_c gives $W_{c,i}$, by Proposition 21.2 and the definition of β_c.

The seeds on X have compact support. The finiteness assertion of Lemma 21.4 yields locally finite sums; reindexing gives left invariance and the support descends to the image of the compact seed support. On each orbit the sum can also be taken over $\Gamma_c\backslash\Gamma$ with W in place of ϕ, by the preceding periodization. For the nonzero value, fix a relatively compact identity neighbourhood O at infinity and the compact $C_c \subset U_\infty$ above. There are only finitely many rational γ preserving h_cK_Y at finite places whose infinite part lies in $C_c\bar{O}$, again by Lemma 21.4. Shrink O so that C_cO avoids all those outside U_∞. This is possible by compactness of C_c and closedness of U_∞. Any such rational γ with $\gamma_\infty \in U_\infty O = \Gamma_c C_c O$ can be translated by Γ_c into this finite list, where it must now belong to U_∞. The rational flag equations then imply $\gamma \in \Gamma_c$. Taking the preceding F with support in O, only the identity coset in $\Gamma_c\backslash\Gamma$ contributes at $(1, h_cK_Y)$, giving the nonzero value $W(1)$.

For the inner product, unfold the series for $P_{c,i}$ on a fundamental domain by Lemma 21.4, leaving its seed paired on X against the invariant function $R_kP_{c',j}$. Only $h = h_c$ contributes. Expand the conjugated series and invert its rational index, obtaining precisely the finite condition and the integral displayed above. For compact seeds only finitely many overlaps occur. The quotient integrals exist: their absolute square integrals are finite by this same nonnegative unfolding applied to the absolute values of the seeds and their finitely many overlaps. For the measurable extension apply the unfolding identity first to the absolute values of the two expanded series. The specified finite absolute integral justifies interchanging all sums and integrals in the identity by Lemma A.2. □

22. Central projections at the split level

Lemma 22.1. *Let J be a finite group and write $[s]$ for the basis elements of $\mathbb{C}[J]$, with multiplication induced by the group law. For a finite-dimensional irreducible complex representation π, put $d_\pi = \dim \pi$, $\chi_\pi(s) = \operatorname{tr}\pi(s)$ and*

$$e_\pi = \frac{d_\pi}{|J|}\sum_{s\in J}\chi_\pi(s^{-1})[s].$$

In every unitary representation e_π acts as the orthogonal projection onto the sum of its irreducible subrepresentations equivalent to π, in finite dimension. On an arbitrary unitary representation it acts as a self-adjoint idempotent. Distinct irreducible types give mutually orthogonal central idempotents. The coefficient of $[1]$ in e_π is $d_\pi^2/|J|$.

Proof. Averaging any positive inner product over the group makes it invariant. In finite dimension, an invariant subspace has invariant orthogonal complement, so induction decomposes each representation into irreducibles. A commuting map between irreducibles is zero or an isomorphism by its kernel and image. An endomorphism commuting with an irreducible action is scalar by subtracting one of its complex eigenvalues, which exists by Lemma 16.2. On the complex space of matrices $T : \pi \to \pi'$ the average of $T \mapsto \pi'(s)T\pi(s)^{-1}$ over s is a projection onto the commuting maps, by reindexing. Its trace equals their dimension. On matrix units the trace of each summand operator is $\chi_{\pi'}(s)\chi_\pi(s^{-1})$. The average of these products is thus one for equivalent irreducibles and zero otherwise. The displayed e_π is central since traces are conjugacy invariant. It acts as a scalar on each irreducible, and the trace just calculated gives the scalar one for π and zero otherwise. These action identities imply the corresponding products and squares in the group algebra by applying them to the left regular action: an element acting as zero there sends $[1]$ to zero and is itself zero. Finally the involution $[s]^* = [s^{-1}]$ conjugating coefficients fixes e_π since the character is the

trace of a unitary matrix. This proves the general projection assertion and the claimed identity coefficient follows by evaluation at 1. □

Proposition 22.2. *For e the sum of any subset of the distinct projections in Lemma 22.1, the coefficients of $|J|e$ are algebraic integers. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, applying σ to the coefficients leaves a self-adjoint idempotent. Suppose ρ is a finite-dimensional unitary action for a positive definite rational Gram matrix S, and each entrywise transported action $s \mapsto \sigma(\rho(s))$ is also S-unitary. For any v and $s \in J$ one has*

$$\|\sigma(|J|\rho(e)\rho(s)v)\|_S^2 \leq |J|^2\|\sigma v\|_S^2.$$

For a pair (p, j) as in Definition 16.3, let $\mathcal{O}_j = \{z : |jz|_p \leq 1\}$. Any $\mathcal{O}_j$-module L stable under $\rho(J)$ is sent into itself by $|J|\rho(e)\rho(s)$.

Proof. Every s has order dividing $|J|$ by counting its cyclic cosets. Its matrix is diagonalizable, since it is annihilated by the polynomial $X^{|J|} - 1$ with distinct roots; the projections on the separate roots are given by polynomial interpolation. Thus its trace is a sum of $|J|$-th roots of unity. The coefficients in question are integral by Lemma 12.1 and lie in the cyclotomic field $\mathbb{Q}(\zeta_{|J|})$. An automorphism permutes these roots by a fixed unit exponent, by Lemma 10.2. On this field it commutes with complex conjugation, which raises the roots to their inverses. Hence the idempotent and self-adjoint identities from Lemma 22.1 persist under σ. In the transported unitary action the operator is an orthogonal projection, which is a contraction by orthogonal decomposition into image and kernel. This gives the norm bound. At p each root of unity has absolute value one by multiplicativity. Hence the images of $|J|e$'s coefficients have absolute value at most one by Lemma 16.2. Applying the resulting integral combination of group elements proves the last assertion. □

Definition 22.3. At $p = g_i$ write $A = M_\omega$ from Theorem 11.4, and let $\mathcal{M}_r$ be all monic degree r polynomials over $\mathbf{F}_p$. A label is a squarefree member with nonzero constant coefficient. It is allowed if none of its roots in $\mathbf{F}_p$ has A-th power one. It is primitive if it is the minimal polynomial of a generator of the multiplicative group of the field with p^r elements. A good tuple of length f is a tuple of allowed labels with at least one primitive member.

Lemma 22.4. *For $2 \leq r < p$, there are exactly $\varphi(p^r - 1)/r$ primitive labels, all allowed. Among all p^r members of $\mathcal{M}_r$, the proportion failing to be an allowed label is at most $(1 + A + r)/p$. The proportion which are primitive is at least*

$$\frac{1}{2r(2r\log p + 1)}.$$

Proof. In an algebraic closure of $\mathbf{F}_p$ (constructed in the first paragraph of the proof of Lemma 16.2) the roots of $X^{p^r} - X$ form a field: p-power Frobenius is additive and multiplicative by the binomial theorem, giving closure under addition, multiplication and inverses of nonzero elements. The derivative is -1, so the field has exactly p^r elements. Its multiplicative group is cyclic. Indeed for each prime power in the least common multiple L of the orders of its elements there is an element of that prime-power order, by taking a suitable power. Their product has order L, since the orders are coprime and the group abelian. All elements are roots of $X^L - 1$, so the group's order is at most L, proving cyclicity. Its generators, of which there are $\varphi(p^r - 1)$ by coprime exponents, each generate the entire field over $\mathbf{F}_p$. The field has vector-space dimension r by its cardinality. For such a generator ξ the elements $\xi, \xi^p, \ldots, \xi^{p^{r-1}}$ are distinct since $p^r - 1$ cannot divide $p^a - 1$ for $0 < a < r$. All are roots of the same minimal polynomial, and give precisely its r roots. Thus each polynomial is counted r times. Since $r > 1$ it has no root in $\mathbf{F}_p$, and is squarefree and allowed.

A monic polynomial with a forbidden root or zero as a root is divisible by $X - u$ for one of at most $1 + A$ possible u's. This excludes at most $(1 + A)p^{r-1}$ polynomials. A monic polynomial divisible by the square of a nonconstant monic polynomial of degree d is among

at most $p^d p^{r-2d}$ choices, for $1 \le d \le \lfloor r/2 \rfloor$. The sum is at most rp^{r-1}; division by p^r gives the claim. For any integer $n \ge 2$ with distinct prime factors $p_1 < \cdots < p_t$, reduction to prime powers and Lemma A.1 give $\varphi(n)/n = \prod_i (1 - p_i^{-1})$. Since $p_i \ge i+1$, this is at least $1/(t+1)$, and $t \le \log n / \log 2 \le 2 \log n$. Take $n = p^r - 1 \ge p^r/2$ to get the displayed lower bound. □

Proposition 22.5. *Let $J_0 = \mathrm{GL}_r(\mathbf{F}_p)$, and suppose that for all labels c of Definition 22.3 there are distinct irreducible representations π_c of J_0 whose dimension satisfies*

$$\frac{(\dim \pi_c)^2}{|J_0|} \ge (1 - r/p) p^{-r}.$$

In $\mathbb{C}[J_0^f]$ form the sum e_{good} of the tensor products $\bigotimes_w e_{\pi_{c_w}}$ over the good tuples. It is a self-adjoint central idempotent and its complementary share at the identity is bounded by

$$0 \le [1](1 - e_{\mathrm{good}}) \le \frac{f(1 + A + 2r)}{p} + \exp\left(- \frac{f}{2r(2r \log p + 1)} \right).$$

For both $p = g_0, g_1$ and sufficiently large x this bound is at most $b^{-8}/2$.

Proof. The group algebra of the product is the tensor product of the group algebras by their group-element bases. Distinct tuples give orthogonal projections by Lemma 22.1, and every identity coefficient is the product of the individual projection shares. The identity coefficient of any projection lies between zero and one: the normalized trace of its action in the regular representation equals its coefficient at one by the permutation basis, and the trace of an orthogonal projection is its rank. The share of e_{good} is therefore at least $(1 - r/p)^f$ times the counting probability of a uniform good tuple in $\mathcal{M}_r^f$. By Lemma 22.4 the chance of any forbidden coordinate is at most $f(1 + A + r)/p$, and the chance of no primitive coordinate is at most the indicated exponential by $1 - t \le \mathrm{e}^{-t}$. Finally $1 - (1 - r/p)^f \le fr/p$ by induction, yielding the bound.

For the scales, $A = (Md)^{s+6}$, $f \le x^{12}$, $r \le 3x^5$ and $s \le 32 \log x$ give $f(1 + A + 2r) \ll_{M,d} x^{C_{M,d}}$ for a constant independent of x. Theorem 11.4 gives $p \ge b^{10}$. The first summand is thus at most $b^{-8}/4$ eventually by the exponential comparison in Proposition 10.5. For the other, $f = \varphi(M) 3^{c_s} 2^s \ge 2^s = m^{22} \ge x^{11}$ by Proposition 10.3, and $\log p \le 11\ell$. Thus $f/(2r(2r \log p + 1)) \gg x/\ell$, which exceeds 9ℓ eventually because $\ell = x^\epsilon$, $0 < \epsilon < 1/100$. This gives $b^{-9} \le b^{-8}/4$ eventually. □

Lemma 22.6. *Let $L = p^r - 1$, and let c be primitive with root ξ. The product of its roots is $\xi^{L/(p-1)}$, a generator of $\mathbf{F}_p^\times$. If $u \in \mathbf{F}_p^\times$, $u^A = 1$, $A^2 \mid p - 1$, then $c^{[u]}(X) = u^r c(X/u)$ is primitive. If $r \ge 3$, $p \ge 5$ and $9 \mid p - 1$, the three polynomials obtained as u runs over the cube roots of unity are distinct. In particular one of them avoids any two prescribed labels.*

Proof. The root description of Lemma 22.4 gives the product, fixed by Frobenius and of order $p - 1$. Write $u = \xi^e$; since $u^A = 1$ and $A \mid L$, the exponent e is divisible by L/A. Every prime $b \mid L$ divides L/A: for $b \mid A$ this uses $A^2 \mid L$, and otherwise it follows immediately. Thus $1 + e$ is prime to L, proving $u\xi$ a generator. The new polynomial is monic with this root and degree r, hence is its minimal polynomial. Now take $A = 3$. The element ξ^3 has order $L/3$. This exceeds $p^{r-1} - 1$ for $p \ge 5$, so ξ^3 cannot be fixed by any Frobenius power of exponent $1, \dots, r - 1$. Equality of two scaled polynomials would give $u\xi = (v\xi)^{p^a} = v\xi^{p^a}$ for one such a, or for $a = 0$, by the list of roots. Cubing forces $a = 0$, so $u = v$. There are three such scalars by cyclicity and $3 \mid p - 1$. □

23. Counting definite classes in the ranks of the lifts

Lemma 23.1. *Let $O = O_E$ from Theorem 18.2, $h \ge 1$, and $C \in M_h(O)$ have nonzero determinant. Every O-submodule of $A = C^{-1}O^h/O^h$ is generated by at most h elements. In particular the number of such submodules is at most $|A|^h$, and $|A| = |\mathrm{N}_{E/\mathbb{Q}} \det C|$.*

Proof. Multiplication by C identifies A with O^h/CO^h. On an integral basis the resulting integer matrix has determinant $\mathrm{N}_{E/\mathbb{Q}} \det C$, as seen by evaluating at all embeddings, as in Lemma 12.1. Its absolute determinant is the index by the lattice volume calculation in Lemma 20.2. Denote this integer by a. It annihilates A, by the adjugate of that matrix. The ring O/aO is a product of O_p/aO_p over $p \mid a$ by Lemma A.1 in an integral basis. By Theorem 18.2 and its local proof each nonzero factor is itself a product of quotients $R/\pi^j R$ of discrete valuation rings. The action of the coordinate idempotents splits every O/aO-module and every submodule into these factors. Over each R the module A is a quotient of R^h. The inverse image M in R^h of any of its submodules is generated by at most h vectors: project M onto the first coordinate. If the image is nonzero, choose an element of least valuation there, which generates that ideal, and subtract a multiple of its lift to kill the first coordinate. Induct on h for the kernel, omitting the lift if the image is zero. Project the resulting generators to the quotient. Pad each factor's list by zeros to length h, and join corresponding elements by the product decomposition. These h elements generate since coefficients in the different factors can be chosen independently in O/aO. Counting their ordered lists gives $|A|^h$; when $a = 1$, the conclusion is immediate. □

Theorem 23.2. *For the standard definite unitary group on E^h, $h \geq 1$, let $\mathfrak{C}_h(J)$ denote its rational-by-level double cosets, with level J in its finite adelic group. Let K_h^0 be the stabilizer of the standard lattice. Set $D_* = dN_1$, $C_h = 54fhD_*$. Then*

$$|\mathfrak{C}_h(K_h^0)| \leq \left((1 + 2C_h^2)C_h^2\right)^{2fh^2}.$$

If J contains the principal level $K_h(G)$ of matrices in K_h^0 congruent to the identity modulo a positive integer G, then $|\mathfrak{C}_h(J)|$ is at most the right side times G^{2fh^2}.

Proof. The proofs of Lemmas 13.4 and 14.1 apply in any rank h: a coset is the isometry class of its lattice L, which is integral for the hermitian form everywhere by local isometry with O_p^h, and $D_* L^\# \subset L$ for its trace pairing. Thus the trace dual has shortest nonzero length at least $1/D_*$. Lemma 14.2 gives h vectors in L of trace length at most $27(2fh)D_* = C_h$, independent over E by the real spanning argument of Proposition 14.4. Let $Y \in \mathrm{GL}_h(E)$ have these columns and $C = Y^*Y$. Its entries belong to O, where $O = O_E$, and all their conjugates have modulus at most C_h^2 by Cauchy–Schwarz on the column vectors at each embedding. All embeddings commute with the involution as in the proof of Proposition 14.4. At each embedding $0 < \det(\sigma C) \leq C_h^{2h}$ by Lemma 5.1 and invertibility of Y. Consequently $|\mathrm{N}_{E/\mathbb{Q}} \det C| \leq C_h^{4fh}$.

Distinct algebraic integers differ by an element with nonzero integral norm, and so some archimedean absolute value of their difference is at least one. In the sup norm on $\mathbb{C}^\Phi$, open balls of radius $1/2$ about the elements with conjugates bounded by C_h^2 are therefore disjoint. They lie in the ball of radius $C_h^2 + 1/2$. Scaling real $2f$-dimensional volume bounds their number by $(1 + 2C_h^2)^{2f}$. There are at most the h^2-th power of this number of possible matrices C.

In the basis Y, the original lattice lies between O^h and $C^{-1}O^h$: the columns of Y lie in L, and pairing them against every member of L gives integral coordinates. For this fixed Gram matrix, Lemma 23.1 bounds the number of possible intermediate lattices by $C_h^{4fh^2}$. Knowing C and the intermediate lattice determines the rational isometry class. In fact if Y, Y' have the same Gram matrix, $Y'Y^{-1}$ is unitary and takes one lattice to the other whenever the intermediate lattices coincide. Multiplying the two counts yields the asserted bound.

Reduction of K_h^0 modulo G has kernel $K_h(G)$, and its image has at most $|O/GO|^{h^2} = G^{2fh^2}$ elements by its matrix entries and the integral basis of Theorem 18.2. Representatives of these cosets refine each double coset into at most this many classes. The natural map from classes at $K_h(G)$ onto those at J gives the final inequality. □

Corollary 23.3. *Put $t_1 = 5r,\ n_j = k + j(t_1+1)$ for $0 \le j \le m$. For all sufficiently large x, for all integers $1 \le h \le 2n_m,\ G \ge 1$ and all levels $J \supset K_h(G)$,*

$$\log|\mathfrak{C}_h(J)| \le x^{24}(1+\log G).$$

In particular a level containing $K_h(gN)$ has at most $\exp(x^{25})$ classes. For the column level K_c in rank k this bound holds as well.

Proof. Proposition 10.5 gives $h \ll x^{11/2}$, $f \le x^{12}$ and $D_* = dN_1 \le dx^{12}$ eventually by the estimates in Proposition 12.2. Hence $fh^2 \ll x^{23}$ and $\log C_h \ll_{d,M} 1+\log x$. Taking logarithms in Theorem 23.2 yields $O_{d,M}(x^{23}(1+\log x+\log G))$. The claimed inequality follows since $(1+\log x)/x$ tends to zero. For $G = gN$, Theorem 11.4 and Proposition 12.2 give $\log(gN) \le 22\ell + 30m\ell \ll x^{1/2+\epsilon}$, and eventually $1+\log(gN) \le x$. At g_0, g_1 principal congruence modulo gN implies the required identity reductions. At N it fixes the last row modulo N, so $K_k(gN) \subset K_c$ by Lemma 13.2. □

24. Collisions of top places at the nonsplit primes

Definition 24.1. Use A, D, G, M, N_1, U of Propositions 10.3 and 10.5, with A the subgroup of top places and D the distinguished set reducing to -1 mod M. For a prime $p < x_1$ with $\eta_d(p) \ne 1$, put

$$h = -p \bmod N_1, \quad o = \mathrm{ord}_M(-p), \quad L = \mathrm{ord}_{N_1}(-p), \quad j_0 = \min\{j \ge 0 : p^j \ge m\}.$$

These classes are units by Proposition 10.3. Call p short if $p \equiv \pm 1 \bmod m$. Call p served if $\eta_d(p) = -1$, it is not short, and o is odd with $o \le j_0$. For $V \subset G$ put

$$C_p(V) = \sum_{\tau \in A\setminus V} \sum_{\substack{e\ge 1,\ e \text{ odd or } p=d \\ (-p^e)\tau \in A\setminus V}} \frac{a}{p^e}.$$

The power in $-p^e$ is taken before multiplying by -1. This nonnegative sum converges by comparison with $Ua/(p-1)$.

Proposition 24.2. *For the primes of Definition 24.1 one has*

$$C_p(\varnothing) = \begin{cases} Ua\,p^{-o}/(1-p^{-2o}) & \eta_d(p) = -1 \text{ and } o \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

For every unserved prime in this range and all sufficiently large x, $mC_p(\varnothing) \le e_p/3$, with e_p as in Proposition 12.6.

Proof. For $\tau \in A$, membership of $(-p^e)\tau$ in A is equivalent to $-p^e \equiv 1 \bmod M$. If $p = d$, then $d \equiv M/2 - 1 \bmod M$ and $d^2 \equiv 1 \bmod M$. The two possible classes of $-d^e$ are -1 and $1 - M/2$; neither is 1 since $M \ge 18480$. In the inert case only odd e occur. For such e the class of $-p^e$ equals h^e and it lies in A exactly when $o \mid e$, by division by the order. If o is odd the exponents are $o, 3o, 5o, \dots$ for each of the U starting points. If o is even there are none. Summing the series proves the identity.

For p short, $C_p(\varnothing) \le (4/3)Ua/p$, since $p \ge 2,\ o \ge 1$. For an unserved prime which is not short, the only nonzero case has $o \ge j_0 + 1$. Then $p^o \ge pm$, giving $mC_p(\varnothing) \le (4/3)Ua/p$. In both cases $U \le f$ and $\ell^2 \ge 4/3$ eventually, giving the claimed bound. □

Lemma 24.3. *Let p have $\eta_d(p) = -1$. In an algebraic closure of $\mathbf{F}_p$ fix a primitive N_1-th root ξ and a root z of $X^2 - X + (1+d)/4$. Let $\mathfrak{p}_\tau$ be the kernel of the reduction map on O_E specified by $\omega \mapsto z,\ \zeta_{N_1} \mapsto \xi^\tau$, for $\tau \in G$. Write $\overline{\mathfrak{p}}$ for the image of an ideal under complex conjugation on E. For $\sigma, \tau \in G$,*

$$\mathfrak{p}_\sigma = \mathfrak{p}_\tau \iff \sigma/\tau \in \langle p^2 \rangle,$$
$$\mathfrak{p}_\sigma = \overline{\mathfrak{p}_\tau} \iff \sigma/\tau \in (-p)\langle p^2 \rangle.$$

All products and subgroups on the right are taken in G. The assertions hold for every choice of the two residue roots.

Proof. The prime p divides neither d nor N_1 by Proposition 10.3, hence is odd. Lemma 9.1 gives $(-d/p) = (p/d) = -1$, so the quadratic is irreducible over $\mathbf{F}_p$ and $z^p = 1 - z$. The roots of $X^{N_1} - 1$ are distinct by its derivative and form a cyclic group of order N_1 by the finite-subgroup argument of Lemma 22.4. In particular ξ exists. The factorization by cyclotomic polynomials in Lemma 10.2 shows $P_{N_1}(\xi) = 0$: a root of a factor P_b, $b \mid N_1$, has b-th power one, forcing $b = N_1$. The presentation and integral basis in Theorem 18.2 now give the reduction maps in the statement. Their images are finite integral domains, hence fields, since multiplication by a nonzero element is injective and then surjective on a finite set.

Two such maps into $\overline{\mathbf{F}}_p$ have the same kernel exactly when their values on the generators differ by one common power of Frobenius. Indeed their common residue field, if the kernels agree, has p^a elements for some a. Its image is the full set of roots of $X^{p^a} - X$, by the finite-field argument in Lemma 22.4. On a multiplicative generator the a Frobenius images exhaust its distinct conjugates, as in the proof there. This accounts for every embedding of that residue field. Conversely a power of Frobenius preserves the kernel. To preserve the specified image z the power has to be even. It sends ξ^τ to $\xi^{p^{2i}\tau}$, proving the first assertion. After conjugating in E the two images are $1 - z, \xi^{-\tau}$. Returning to z requires an odd power of Frobenius, which gives the second assertion. Nonnegative powers run through the entire cyclic subgroups since p is a unit modulo N_1. □

Theorem 24.4. *For a served prime the orbits of $\langle h \rangle$ that meet A correspond bijectively to unordered pairs of distinct primes $\{\mathfrak{p}, \bar{\mathfrak{p}}\}$ above p met by the top places. Their number is $Uo/L \le U/m$. Choose one prime in each such pair and call the resulting set Λ. Put $V = \{\sigma \in G : \mathfrak{p}_\sigma \in \Lambda\}$. Then V and hV partition these orbits, they avoid D, and*

$$|A \cap V| = |A \cap hV| = U/2, \qquad C_p(V) = 0.$$

More precisely the number of tops inducing each chosen prime is $L/(2o)$.

Proof. Write $m = 2^u,\ s = 22u$, so $2^s \mid N_1$ by Proposition 10.5 and Definition 10.1. Take x large enough that $u \ge 3$. The odd integer $b' = -p$ is neither 1 nor $-1 \bmod 2^u$. Of $b' - 1, b' + 1$ one has exact 2-adic order one and the other has order at most $u - 1$. Hence $v_2((b')^2 - 1) \le u$. Odd squares are 1 mod 8, so factoring successive differences of squares gives

$$v_2((b')^{2^j} - 1) = v_2((b')^2 - 1) + j - 1 \qquad (j \ge 1).$$

The order of $b' \bmod 2^s$ is a power 2^j since the group of odd residues has 2^{s-1} elements, and cosets of a cyclic subgroup all have its cardinality. The order exceeds one, and the identity above yields $j \ge s - u + 1 \ge u$. Reduction from N_1 to 2^s proves $m \mid L$.

By Lemma 24.3 the orbit of each τ consists of the two fibres over $\mathfrak{p}_\tau, \bar{\mathfrak{p}}_\tau$. They are distinct because h has even order L, so $h \notin \langle h^2 \rangle$. Consecutive powers of h alternate between the two fibres. An orbit through a top returns to A at exactly the multiples of o; there are L/o returns counting the starting point in one period. Since o is odd, $m \mid L/o$ and half the returns are in each fibre. Dividing the total number U of tops by L/o yields the asserted count and bound. The reductions modulo M of such an orbit are in the odd-order subgroup generated by $h \bmod M$. That subgroup cannot contain -1, an element of order two, so the orbit avoids D. Finally each admissible exponent e contributing to $C_p(\varnothing)$ is odd. Multiplication by $-p^e = h^e$ switches fibres. Two tops outside V in the same orbit cannot therefore give a term of $C_p(V)$. □

25. Elementary divisors and the size of transferred columns

Lemma 25.1. *Let O be a valuation ring, meaning a domain in whose fraction field either z or z^{-1} belongs to O for each $z \neq 0$. For $c_1, \dots, c_d \in O \setminus \{0\}$ put $P = \bigoplus_{i=1}^{d} c_i O \subset O^d$. Then*

$$\mathcal{I}(O^d, P; z) \quad \Longleftrightarrow \quad \prod_i c_i \mid z.$$

For a square matrix Y over O and $0 \neq u \in O$, consider $P' = YO^d + uO^d$. There exist $a_i \in O$, possibly zero, with $a_i \mid a_{i+1}$, such that in some basis P' is $\bigoplus_i c_i O$, where $c_i O = a_i O + uO$.

Proof. Apply Definition 15.5 to the d standard basis vectors. In any recombination landing in P, the i-th column of the matrix is divisible by c_i. This proves necessity. For sufficiency, start with any finite list of vectors. In coordinate i, if all entries are zero move to the next coordinate. Otherwise some nonzero entry divides every other entry: divisibility in O is a total preorder by the defining property of a valuation ring. Use the vector with that entry as pivot. Subtract suitable multiples of it from the other vectors to clear their i-th coordinates. Multiply the pivot vector by c_i. Every i-th coordinate is now in $c_i O$; all previously obtained coordinate conditions persist under these O-linear recombinations. The determinant at this stage is c_i, or one when the stage was skipped. Multiplying the matrices proves sufficiency, including for the empty list.

In a nonzero matrix first bring an entry dividing all entries to the upper left by row and column swaps. Clear its row and column by invertible elementary operations. All entries of the remaining block are still multiples of that entry. Induction gives a diagonal matrix with the indicated divisibilities, padding with zeros for a zero block. Column operations preserve the image module. Row operations are a change of basis preserving uO^d. In diagonal coordinates the assertion follows since of a_i and u one divides the other. □

Lemma 25.2. *For a domain O, an O-module M, $u \in O$ and endomorphisms Y_i of M, suppose $\mathcal{I}(M, Y_i M + uM; z_i)$ for $1 \leq i \leq h$. Then*

$$\mathcal{I}(M, Y_1 \cdots Y_h M + uM; \ \prod_{i=1}^{h} z_i).$$

No commutation between the operators is required.

Proof. The empty product gives M. To combine operators Y, Z, first recombine a list into $YM + uM$. Write each resulting vector as $Yv_i + uw_i$. Recombine the list of v_i's into $ZM + uM$. Applied to the first list, this same matrix lands in $YZM + uM$, since $Y(uM) \subset uM$. Determinants multiply, as in Lemma 15.6. Induction now gives the claim. □

Proposition 25.3. *Put $B = B_0'$ of Definition 12.3 and $L_{\mathrm{loc}} = \sum_p e_p \log p$ with e_p of Proposition 12.6. Uniformly on $t \leq v \leq t^2$, $L_{\mathrm{loc}} = o(B)$ as $x \to \infty$. Put $\beta_p = \lceil G_p - e_p \rceil$, where G_p is the gain of Proposition 12.5. The positive rational $c = \prod_p p^{-\beta_p}$ satisfies*

$$\log c \leq L_{\mathrm{loc}} - \sum_p G_p \log p, \qquad |cp^{\beta_p}|_p = 1 \quad \text{for all } p.$$

For $\iota = (p, j)$ of Definition 16.3, let $O_j = \{z \in \mathbb{C} : |jz|_p \leq 1\}$. Let M_j be an O_j-module of columns, $M_j' \subset M_j$ a submodule, and suppose some submodule $S_j \subset M_j$ satisfies $\mathcal{I}(M_j, S_j; p^{a_j})$ with an integer $0 \leq a_j \leq c_0 e_p$. Let $D_t : M_j' \to H_t$ be O_j-linear maps into complex vector spaces with specified O_j-submodules $P_{j,t}$. The allowed t's may depend on p. Assume $p^{-\beta_p} D_t z \in P_{j,t}$ for all $z \in S_j \cap M_j'$ and every allowed t.

Let W be a complex vector space, let $Q_s : M_j' \to W$ be O_j-linear and let $u_t : W \to H_t$ be complex linear. Suppose $u_t Q_s z = T_{s,t} D_t z$ for linear operators $T_{s,t}$ on H_t preserving $P_{j,t}$. For every family $y_i = cQ_{s_i} z_i$, $z_i \in M_j'$, in which at most b_T distinct indices s_i occur, there

is one square matrix A_j over O_j with

$$|j(\det A_j)|_p \ge \exp(-b_T c_0 e_p \log p), \qquad u_t\Big(\sum_l (A_j)_{il} y_l\Big) \in P_{j,t} \quad \text{for all } i,\ t.$$

Proof. The proof of Proposition 12.6 bounds L_{loc}/B by $15v(1+v)/\ell + 480/\ell^2$. By Theorem 11.4, $v^6 \le \ell$ eventually and $\ell = x^\epsilon \to \infty$. The bound tends uniformly to zero. Since $-\beta_p = \lfloor e_p - G_p \rfloor$, this c is the rational constructed there. Summing the floor inequality proves the logarithmic bound, and all factors at primes different from p are p-adic units. Thus cp^{β_p} is a unit of O_j, giving $cD_t z \in P_{j,t}$ for $z \in S_j \cap M_j'$.

Group the original list by the index s_i. In each group apply Definition 15.5 to the original z_i's. Every recombined vector belongs both to S_j and to M_j', the latter by linearity. Its normalized image under $T_{s,t}D_t$ lies in $P_{j,t}$ for every allowed t. The block diagonal matrix of these recombinations, conjugated by the permutation that grouped the list, has determinant dividing $p^{a_j b_T}$. Since elements of O_j have absolute value at most one after j, its determinant satisfies the asserted inequality. This also proves that smaller integers than β_p need no separate divisibility estimate: multiply $p^{-\beta_p} D_t z$ by the corresponding nonnegative power of p. □

Proposition 25.4. *Let D_0' send the ramified columns to vector-valued functions on a definite quotient of rank at most $2n_m$, at a level containing $K_h(gN)$ as in Corollary 23.3. Its squared norm in archimedean coordinates is the sum of squared norms at class representatives with rational weights in $(0,1]$. For every short column z of Corollary 14.5 assume that the values of $D_0'z$, after any $\sigma \in \mathrm{Aut}(\mathbb{C})$, have squared norm at most*

$$\exp\big(h_0 + 2B + 2\sum_p G_p \log p\big).$$

Let $e = \sum_{w\in\mathcal{G}} a_w[w]$ be a sum of the orthogonal central idempotents of Lemma 22.1. Assume that in these coordinates the translated values satisfy

$$(u_0 Q_s z)(y) = |\mathcal{G}| \sum_w a_w I_{w,s,y}(D_0'z)(y_{w,s}),$$

where the points $y_{w,s}$ may depend on y, and $I_{w,s,y}$ has operator norm at most one after every σ. Then for c of Proposition 25.3, eventually, uniformly in v, s, z, σ, the squared norm of $\sigma(u_0 c Q_s z)$ is at most $\exp(H - m\mathcal{R} + 6B)$.

Proof. By Proposition 22.2, $\sigma(e)$ is a self-adjoint idempotent in the left regular unitary representation. Applying it to the basis vector at the identity gives $\sum_w |\sigma a_w|^2 \le 1$. Hence $\sum_w |\sigma a_w| \le \sqrt{|\mathcal{G}|}$ by Cauchy–Schwarz. The sum of translates increases the squared pointwise bound by at most $|\mathcal{G}|^3$. Summation over class representatives costs at most $\exp(x^{25})$ by Corollary 23.3. From Proposition 15.2, each of the f split factors at g_i has at most $g_i^{r^2}$ matrices. By Theorem 11.4, $3\log|\mathcal{G}| \le 66 f r^2 \ell$. Together with the bound on r in Proposition 10.5, this gives $x^{25} + 3\log|\mathcal{G}| \le B$ for large x: $a \ge x\exp(x^\epsilon)$ dominates every fixed power of x by the exponential series. Every σ fixes c, so its squared scalar adds $2\log c \le 2B - 2\sum_p G_p \log p$, using Proposition 25.3 and $L_{\mathrm{loc}} \le B$. The gain sum cancels. Finally $h_0 \le H - m\mathcal{R} + B$ by Proposition 12.4. Adding the five contributions gives the stated bound. □

Corollary 25.5. *In the setting of Corollary 17.7, suppose the selected rows are $cQ_s z$ of Propositions 25.3 and 25.4, with their hypotheses for every required embedding and twist and with the same integral modules as in Corollary 17.4. Retain the other hypotheses of Propositions 15.7 and 17.6 on the row maps, completions and factorizations. Then*

$$m\mathcal{R} \le (2 + o(1)) f a \ell^4.$$

The $o(1)$ is uniform for $t \le v \le t^2$ for fixed M, d, ϵ. In particular these estimates, if established for the automorphic maps for every such choice of parameters, imply Dirichlet GRH.

Proof. Write $T_0 = m\mathcal{R},\ E_0 = B\ell,\ d_0 = \dim W$ and n for the number of independent rows selected. The selection bound and Proposition 25.3 give

$$\delta \le 3d_0 \log(1+b^2) L_{\mathrm{loc}} = o(d_0 E_0),$$

since $\log(1+b^2) \le 3\ell$ eventually. Put $V = n(H - T_0 + 6B)$, allowed by Proposition 25.4 and Lemma 5.1. In the proof of Proposition 17.6 the volume ratio has height at most $V + 2\delta + d_0 b$; its inverse square has height at most $2d_0(2E_0 - H)$. Proposition 16.4 applied to their product, with the first ratio squared, yields $d_0(H - 2E_0) \le V + 2\delta + d_0 b$. The lost dimensions satisfy $0 \le d_0 - n \le 4d_0/b^2$, and $-H + T_0 - 6B \le b^2 x^{99}$ by Proposition 12.4. Consequently

$$T_0 \le 2E_0 + 6B + b + 4x^{99} + 2\delta/d_0.$$

Here $B/E_0 = 1/\ell,\ b/E_0 \le 1/\ell^4$ since $a \ge b,\ f \ge 1$, and $x^{99}/E_0 \to 0$ by exponential growth of b. This proves the uniform estimate. Eventually its right side is at most $38fa\ell^4$; Theorem 10.8 gives the final implication. □

26. The diagrams in the chain of lifts

Definition 26.1. Use h_2, h, t_1, w_j of Definition 12.3 and $n_i = k + i(t_1+1)$ of Corollary 23.3. For $0 \le j < m$ and $\tau \in G$ define a diagram by its columns of height strictly less than $n = n_{j+1}$. There are $X_\tau = a\mathbf{1}_A(\tau)$ columns of height one, $d_\tau = w_1(\tau) - \lfloor n_1/2 \rfloor$ of height k, and $h - (t_1+1)$ of height n_i for each $1 \le i \le j$. All other multiplicities are zero. Denote the number of columns of height greater than one by r_τ, their number of boxes by b_τ, and put $N_j = \sum_\tau (X_\tau + b_\tau)$. For $V \subset G$ put $z_\tau = X_\tau \mathbf{1}_{\tau \notin V}$. At a prime p of Definition 24.1 write $I_p = \{e \ge 1 : e \text{ odd or } p = d\},\ \omega_p = (1 - \eta_d(p))/2,\ E_j = \lfloor \log N_j / \log p \rfloor$. Set $I_p^0 = I_p$ for $p \ne d$ and $I_d^0 = I_d \cup \{0\}$. For $e \in I_p^0,\ \sigma = (-p^e)\tau,\ s_e = p^{-e}$ put

$$K(\tau, e) = \min(z_\tau s_e, r_\sigma + z_\sigma) + r_\tau s_e + \min(b_\tau s_e, b_\sigma).$$

Define the number

$$J_p(j, V) = \omega_p \sum_{\tau \in G} \left(\sum_{e \in I_p} (X_\tau + b_\tau) s_e - \sum_{\substack{e \in I_p^0 \\ e \le E_j}} K(\tau, e) \right) - fn^3(E_j + 1).$$

Only positive exponents occur in the geometric sum. At $p = d$ the second sum also contains $K(\tau, 0)$, for the pairing with $-\tau$. Absolute convergence follows from geometric summation. Its leading coefficient per box is $\omega_p \sum_{e \in I_p} p^{-e}$, equal to $p/(p^2-1)$ at an inert prime and $1/(2(p-1))$ at $p = d$.

Lemma 26.2. *Eventually, uniformly in $j < m, \tau, p < x_1$ in the preceding ranges, all displayed multiplicities are nonnegative integers, $n^3 \le x^{49}$, and*

$$w(\tau) \le r_\tau \le w(\tau) + x^{49}, \qquad kw(\tau) \le b_\tau \le kw(\tau) + x^{49}, \qquad 1 \le E_j \le 3\ell/\log 2.$$

Writing $u_p = a/(mp)$, one has $x^{49} = o(u_p)$ uniformly and $r_\tau/p \le u_p$ eventually.

Proof. Here $k > 1,\ t_1 + 1 < h$ eventually by Proposition 10.5. By Definition 12.3, $w_1(\tau)$ is an integer at least $w(\tau) + h + h_2$, and $h > \lfloor n_1/2 \rfloor$ eventually. Thus $d_\tau \ge w(\tau)$, while $d_\tau \le w(\tau) + x^{42}$ by Proposition 12.4. Besides these columns there are exactly $j(h - t_1 - 1)$ higher ones, of height at most n_m. For the upper errors in boxes and columns it suffices to bound $kx^{42} + mn_m h$. Here $k = O(x^5)$, $m = O(\sqrt{x})$, $n_m = O(x^{11/2})$, $h = O(x^{40})$ by Proposition 10.5. This sum and n_m^3 are at most x^{49} for large x. Also $Ua \le N_j \le f(a + k \max_\tau w(\tau) + x^{49})$, and $\max w = a/x$ since $x_1 \ge x^{10^6}$. The bounds $f \le x^{12}$, $a \le 2xb$ in Proposition 12.4 and its proof show $\log N_j \le 3\ell$. Since $N_j \ge a \ge x_1 > p$, this proves the claims about E_j.

From $a \ge x\exp(\ell)$, $p < x_1 = \exp(v)$ and $v \le (10^6 \log x)^2 = o(\ell)$ it follows that u_p dominates every fixed power of x, uniformly. Finally $w(\tau) \le a/x$ and $m/x \to 0$, so $r_\tau \le a/x + x^{49} \le a/m$ eventually. □

Theorem 26.3. *For all sufficiently large x, every prime in Definition 24.1, every $j < m$, and every subset $V \subset G$,*

$$J_p(j,V) \;\ge\; \frac{G_p}{m} - C_p(V) + \omega_p\,|A \cap V|\,\frac{a}{x_1} - 30f\ell\,\frac{a}{mp}.$$

Proof. Set $\Delta = x^{49}$ and use u_p of Lemma 26.2. Fix τ, e with $e \in I_p$, $e \le E_j$, and abbreviate $s = s_e$, $w = w(\tau)$, $w' = w(\sigma)$. The contribution from the tall boxes satisfies

$$b_\tau s - \min(b_\tau s, b_\sigma) = (b_\tau s - b_\sigma)_+ \ge k(ws - w')_+ - \Delta,$$

since $b_\tau \ge kw$, $b_\sigma \le kw' + \Delta$, and $(t - \Delta)_+ \ge t_+ - \Delta$. Also $r_\tau s \le u_p$.

For the height-one columns, if $\tau \notin A$ both X_τ and z_τ vanish. If $\tau \in A \cap V$, their contribution after subtraction of the first minimum in K is as, equal to $(as - w')_+ + \min(as, w')$. At $e = 1$, $p < x_1$ and $w' \ge a/x_1$, so the minimum is at least a/x_1. If $\tau \in A \setminus V$ and $\sigma \notin A \setminus V$, the contribution is $(as - r_\sigma)_+ \ge (as - w')_+ - \Delta$. In the remaining case both places are in $A \setminus V$. The contribution is nonnegative, hence at least $(as - w')_+ - as$. Summing this last loss over the indicated pairs gives at most $C_p(V)$ after weighting by $0 < \omega_p \le 1$.

For $e > E_j$, the unsubtracted $X_\tau s$ and $b_\tau s$ already bound $\mathbf{1}_A(\tau)(as - w')_+$ and $k(ws - w')_+$; there is no extra $e = 1$ term since $E_j \ge 1$. Over all e, τ these two positive parts with coefficient ω_p give exactly G_p/m by Proposition 12.5. At most fE_j pairs have the error $2\Delta + u_p \le 3u_p$. At $p = d$ there is also the subtracted term at $e = 0$. Here d is fixed and $r_\tau \le a/x + \Delta \le u_p$ eventually, since $md/x \to 0$ and $\Delta = o(u_p)$. If $z_\tau > 0$ then $\tau \in A$, so $-\tau \notin A$ and $z_{-\tau} = 0$. Hence the first minimum in $K(\tau, 0)$ is at most u_p. The places $\tau, -\tau$ cannot both reduce to $-1 \bmod M$, so one of them is outside D and has weight a/x_1. Thus $\min(b_\tau, b_{-\tau}) \le ka/x_1 + \Delta \le u_p$, using $kmd/x_1 \to 0$. The total extra cost is at most $3fu_p$. The final subtraction in Definition 26.1 is at most $fu_p(E_j + 1)$ by Lemma 26.2. The total of these errors is at most $4f(E_j + 1)u_p \le 30f\ell u_p$, since $E_j \le 3\ell/\log 2$, $\log 2 > 1/2$, and $\ell \to \infty$. This proves the inequality. □

Proposition 26.4. *For a served prime use V, Λ of Theorem 24.4. Put $\delta_p = j_0 f(n_m + j_0)$. Eventually for every $j < m$,*

$$|\Lambda|\delta_p + j_0 \sum_{\tau \in V} r_\tau \;\le\; |A \cap V|\,a/x_1 + 3f\,a/(mp).$$

Consequently the sum over $0 \le j < m$ of $J_p(j,V) - |\Lambda|\delta_p - j_0 \sum_{\tau\in V} r_\tau$ is at least $G_p - (30\ell + 3)fa/p$. For an unserved prime the sum of $J_p(j,\varnothing)$ is at least $G_p - e_p/3 - 30f\ell a/p$.

Proof. Since $j_0 \le 1 + \log m/\log 2$, the quantity $(|\Lambda|/f)\delta_p$ is bounded by a fixed power of x, using $|\Lambda| \le f$, $f \le x^{12}$ and the bound on n_m in Lemma 26.2. The same is true of $j_0 x^{49}$. Both are at most $u_p = a/(mp)$ eventually by the proof of that lemma. All $\tau \in V$ avoid D by Theorem 24.4, so their common nominal weight is a/x_1. If $p < m$, then $mpj_0 \le m^2 j_0 \le x_1$ eventually by $m^2 \le 4x$, $x_1 \ge x^{10^6}$. Thus $j_0 a/x_1 \le u_p$. If $p \ge m$, then $j_0 = 1$ and servedness forces $o = 1$. The element $h = -p \bmod N_1$ then belongs to A, so every marked orbit is contained in A. In this case $j_0 \sum_{\tau\in V} w(\tau) = |A \cap V| a/x_1$. The bound on r_τ in Lemma 26.2 now gives the claimed cost in both cases. Subtract it from Theorem 26.3. At a served prime $\omega_p = 1$ and $C_p(V) = 0$, so the terms involving $|A \cap V|$ cancel. In the unserved case apply that theorem directly with $V = \varnothing$ and use Proposition 24.2. □

27. Changing frames to annihilate Frobenius pairings

Lemma 27.1. *Let O be a valuation ring as in Lemma 25.1. Let $L_1, \dots, L_w : O^h \to O$ be linear, and let $\alpha_1, \dots, \alpha_w \in O \setminus \{0\}$. There is a matrix $M \in M_h(O)$ with $\det M \mid \prod_i \alpha_i$ such that every row v satisfies $L_i(v) \in \alpha_i O$ for every i.*

Proof. Map the h standard basis vectors of O^h into O^w by the tuple of the L_i. Apply Lemma 25.1 to this list and $P = \bigoplus_i \alpha_i O$. Its recombination matrix has the stated determinant, and its rows give the desired combinations of standard basis vectors. When there are no constraints the identity matrix suffices. □

Definition 27.2. Fix j, p, V of Definition 26.1. Let $\kappa_{\tau,h}$ be the multiplicities there, $1 \le h < n = n_{j+1}$. Fix any positive integer d', let K_p be the valued field of Lemma 16.2 with valuation ring O, and take arbitrary vectors $y_{\tau,i} \in O^{d'}$ for $1 \le i < n$. A frame of height h at τ uses the first h such vectors. Let $\theta \in O \setminus \{0\}$ with $|\theta|_p < 1$ and set $\bar{O} = O/(p, \theta)$. For $e \ge 0$ define the pairing over $\bar{O}$ by

$$B_e(u, v) = \sum_{a=1}^{d'} u_a^{p^e} v_a.$$

It is additive in each variable and semilinear in the first, by raising sums to p^e in characteristic p.

Theorem 27.3. *With the preceding data there are matrices $M_{\tau,h} \in M_h(O)$ for each $\kappa_{\tau,h} > 0$. In the corresponding changed frame write $\tilde{y}_{\tau,h,i} = \sum_{l \le h} (M_{\tau,h})_{il} y_{\tau,l}$. Their determinants satisfy*

$$\prod_{\tau,h} |\det M_{\tau,h}|_p^{\kappa_{\tau,h}} \;\ge\; |\theta|_p^{\sum_{e \in I_p^0,\ e \le E_j} \sum_{\tau \in G} K(\tau,e)}.$$

For each exponent in the sum, $\sigma = (-p^e)\tau$, and all rows of contributing source and target frames of heights h, h', their reductions to $\bar{O}$ satisfy

$$B_e(\tilde{y}_{\tau,h,i}, \tilde{y}_{\sigma,h',i'}) = 0$$

except possibly when $h = 1$ and either $\tau \in V$, or $h' = 1$, $\sigma \in V$.

Proof. For each of the finitely many directed pairs (τ, e), write $Q = p^e$ and choose $\theta_e^Q = \theta$ in K_p. The root lies in O, since $|\theta_e|_p = |\theta|_p^{1/Q}$. For each component of $y_{\sigma,1}$ also choose a Q-th root in O. We impose the following linear congruences on the rows $v \in O^h$ of each change of basis. Their costs below are the sums of the exponents of $|\theta|_p$ in the divisors of Lemma 27.1, counted with the column multiplicities. Multiplicities of height zero do not occur.

First consider a source of height one with $\tau \notin V$. A target is relevant unless it has height one and $\sigma \in V$. One choice is to require the single coordinate of the change of source frame to lie in $\theta_e O$, at cost z_τ/Q. Its Q-th power vanishes modulo (p, θ). Another choice requires of each relevant target frame of height h' the single linear congruence

$$\sum_a y_{\tau,1,a}^Q \sum_{l \le h'} v_l y_{\sigma,l,a} \in \theta O.$$

This costs $r_\sigma + z_\sigma$. It kills the pairing also after any change of frame at the height-one source, by semilinearity. Use the cheaper choice. If there is no such source, impose nothing; the minimum then vanishes.

Next consider sources of height $h > 1$ and a target of height one, including a marked target. On each source frame impose the single linear congruence

$$\sum_a \Big(\sum_{l \le h} v_l y_{\tau,l,a} \Big) y_{\sigma,1,a}^{1/Q} \in \theta_e O.$$

Taking Q-th powers in $\bar{O}$ annihilates its pairing with $y_{\sigma,1}$, hence with any multiple of this target. The cost is at most r_τ/Q; when no target of this height occurs the congruence can be omitted.

Finally consider both heights greater than one. Requiring every coordinate of each tall source row to belong to $\theta_e O$ has cost b_τ/Q. Requiring every coordinate of each tall target row to belong to θO has cost b_σ. Either condition kills all these pairings; choose the less costly. Thus the total for this directed pair is at most $K(\tau, e)$ of Definition 26.1.

For each frame collect the congruences imposed by all directed pairs at either end, including both sets if the endpoints coincide. Apply Lemma 27.1 once to their full list. Every resulting determinant divides the product of its congruence scalars and is nonzero. Taking absolute values reverses the divisibility bound because $|x|_p \le 1$ on O. Multiplication with the stated multiplicities proves the inequality, and each imposed vanishing holds simultaneously. □

28. Integral coordinates for the differential minors

Lemma 28.1 (Straightening flag minors)**.** *For each τ take a matrix of independent variables $Y_{\tau,i,a}$, $1 \le i < n$, $1 \le a \le d'$. A minor for a column of height h is $[a_1, \dots, a_h]_\tau = \det(Y_{\tau,i,a_l})_{1\le i,l\le h}$. Fix the number of columns of every height at each τ and order the columns there by weakly decreasing height. Fill each box with an integer in $[1, d']$. A filling is standard if its entries strictly increase down columns and weakly increase from left to right along each row. Write P_T for the product of the column minors of a filling T. The P_T for standard T are a basis over $\mathbb{Z}$ of the integer span of all such products. After passage to any field they remain linearly independent. Over K_p, if a polynomial $P = \sum_T c_T P_T$ has coefficients of valuation at least $v_0 \in \mathbb{R}$, then $v_p(c_T) \ge v_0$ for each T. Here $v_p(z) = -\log|z|_p/\log p$, with $v_p(0) = +\infty$.*

Proof. A repeated entry in a column gives zero. Otherwise sort each column, multiplying by its permutation sign. If an adjacent pair $I = (i_1, \dots, i_h), J = (j_1, \dots, j_k)$, $h \ge k$, violates the row condition, choose $t \le k$ with $i_t > j_t$. Use the ordered list $D = (i_t, \dots, i_h, j_1, \dots, j_t)$ of length $h + 1$. The signed sum over its splittings into increasing sublists D_S, D_{S^c}, with $|S| = h - t + 1$, satisfies

$$\sum_S \operatorname{sgn}(S, S^c)[i_1, \dots, i_{t-1}, D_S]_\tau [D_{S^c}, j_{t+1}, \dots, j_k]_\tau = 0.$$

Here order within each sublist is inherited from D, and the sign is that of the permutation listing S before S^c. As a function of the $h + 1$ column vectors of the variables with labels in D, the sum is alternating and multilinear: interchanging two labels either changes one determinant's sign or pairs two shuffle terms with opposite signs. The second determinant uses the first k coordinates of its vectors, which have h coordinates. Expanding all $h + 1$ vectors in the h basis vectors gives zero by alternation. This polynomial identity remains valid when entries among the fixed columns and D agree.

The term taking $i_t, \dots, i_h$ on the left is the initial product, with coefficient one. Every other nonzero term, after sorting the columns, has a lexicographically smaller left column: it replaces one or more labels at least i_t by labels at most $j_t < i_t$. Rearrange the equality to express the initial product in these terms. All earlier columns stay fixed. Order the finite set of column words lexicographically, fixing also an order on the places. Induction on this order proves spanning over $\mathbb{Z}$.

Order monomials lexicographically by taking variables first by place, then by increasing row slot i, then by increasing label a, with an earlier variable having priority for higher exponents. For a strictly increasing column, the leading monomial is the diagonal product with coefficient one: in each successive row slot choose the smallest remaining entry. The leading monomial of a standard product records for every row of every place the multiset of entries. The weak increase along rows recovers their order uniquely. Therefore different standard products have distinct leading monomials. In any linear relation a largest leading

monomial with nonzero scalar coefficient could not be cancelled. This proves independence over any field and over $\mathbb{Z}$. For the valuation claim, recover the coefficient of the largest leading monomial of the expansion first, subtract its scalar times the corresponding integral minor product, and continue in decreasing order. The ultrametric inequality in Lemma 16.2 preserves the lower bound at every step. □

Lemma 28.2. *For a polynomial P in finitely many variables over the valued field K_p, there is a point in its valuation ring O in each coordinate at which $|P|_p$ equals the largest absolute value of its coefficients.*

Proof. For a nonzero polynomial divide by a coefficient of largest absolute value, obtaining an integral polynomial with nonzero reduction modulo the maximal ideal $\{x : |x|_p < 1\}$. The residue field is algebraically closed: lift a monic nonconstant polynomial to O, split it in K_p by Lemma 16.2, and note that every root has absolute value at most one, since a larger root would make the leading term strictly larger than all other terms. Reducing gives the factorization. Such a field is infinite, since in a field with q' elements the polynomial $1 + \prod_a (X - a)$, over its q' elements a, would have no root. A nonzero polynomial over an infinite field has a point where it does not vanish: in one variable this follows from the degree bound on the number of roots, and in several variables first make a nonzero coefficient in the last variable nonzero by induction. Lift this point coordinatewise to O. Evaluation of the normalized polynomial now has absolute value one. The zero polynomial causes no restriction. □

Definition 28.3. Keep the diagram $\kappa_{\tau,h}$, N_j, E_j, n of Definition 26.1. Fix $d' \geq 1$. Let Θ be a formal power series over K_p in variables $Z_{\tau,a,b}$, $1 \leq a \leq d'$, $1 \leq b \leq n$. A linear substitution L assigns to each variable a homogeneous linear form over O in a finite set of auxiliary variables. For $\varpi, \theta \in O$ of valuations ρ_p, ω_p respectively, put

$$\rho_p = \frac{1}{p-1} - \omega_p \sum_{e \in I_p} p^{-e}.$$

Say that Θ has integral coefficients of order N_j on the stated linear substitutions if every such substitution satisfying

$$\sum_{\tau,a,b} L_{\tau,a,b}^{p^e} L_{(-p^e)\tau,a,b} = 0 \pmod{(p, \theta)} \qquad (e \in I_p^0,\ e \leq E_j)$$

makes all coefficients of degree N_j in $\Theta(\varpi L)$ integral. Congruences are coefficientwise. Only terms of degree N_j in Θ enter this property.

For an output filling T' with entries in $[1, n]$ and the given shape, list its entries in a height h column c at τ as $b_{c,1}, \ldots, b_{c,h}$. Define the polynomial

$$F_{T'}(Y) = \left(\prod_{\tau,c} \det \left(\sum_a Y_{\tau,i,a} \frac{\partial}{\partial Z_{\tau,a,b_{c,l}}} \right)_{1 \leq i,l \leq h} \right) \Theta \Bigg|_{Z=0}.$$

Partial derivatives are formal derivatives of monomials; they commute. This polynomial has a unique expansion in the standard products $P_T(Y)$ of Lemma 28.1, of the indicated shape and input alphabet $[1, d']$. Indeed each determinant in the product expands as a sum over distinct increasing h-tuples of labels a, of their Y-minor times the corresponding determinant of derivatives. To see this, expand by multilinearity in the rows of the derivative matrix; repeated intermediate labels vanish by alternation, and collecting each remaining permutation gives the minor. Multiplication followed by Lemma 28.1 proves the assertion, also if $d' < h$ so that the expansion is zero.

Lemma 28.4. *For a prime p and $0 \leq t \leq N$ integers with $N \geq 1$, write $s_p(t)$ for the sum of the base-p digits. Then*

$$v_p(t!) = \frac{t - s_p(t)}{p-1}, \qquad \frac{t}{p-1} - (1 + \lfloor \log N / \log p \rfloor) \leq v_p(t!) \leq \frac{t}{p-1}.$$

Proof. Counting the positive multiples of each power p^l up to t gives $v_p(t!) = \sum_{l\ge1}\lfloor t/p^l\rfloor$. Insert the base-$p$ expansion of t and sum each digit times its finite geometric sum to obtain the identity. At most $1+\lfloor \log N/\log p\rfloor$ digits each contribute between zero and $p-1$, which proves both bounds. □

Theorem 28.5 (Divisibility of differential minors)**.** *Suppose Θ has the integrality property of Definition 28.3. Fix nonnegative integers $\gamma_{\tau,h,b}$. For $e\in I_p^0,\ e\le E_j,\ \sigma=(-p^e)\tau$, require*

$$\gamma_{\tau,1,b}+\gamma_{\sigma,h',b}\ge 1 \quad \text{whenever } \ \tau\in V \text{ or } (h'=1,\ \sigma\in V)$$

for all b and all contributing source and target heights in this formula. Put $m_{T'} = \sum_{\tau,c}\sum_{l=1}^{h}\gamma_{\tau,h,b_{c,l}}$. Each coefficient of $F_{T'}$ in its standard minor basis has valuation at least

$$J_p(j,V)-m_{T'}.$$

Proof. Evaluate at arbitrary $Y_{\tau,i,a}=y_{\tau,i,a}\in O$ and use the changed frames of Theorem 27.3. For every contributing τ,h, $1\le i\le h,\ 1\le b\le n$, introduce a variable $X_{\tau,h,i,b}$. There are at most $fn\sum_{h<n}h\le fn^3$ of them. In Definition 28.3 take the linear forms

$$L_{\tau,a,b}=\sum_{h,i}p^{\gamma_{\tau,h,b}}\tilde y_{\tau,h,i,a}X_{\tau,h,i,b}.$$

In characteristic p the p^e-th power distributes over this sum. For fixed τ,b, each term after summing over a in a coupling contains a pairing of the two changed frames. It vanishes modulo (p,θ) by Theorem 27.3 or, in an exempt pair, by the displayed charge condition because $p=0$ in that quotient. Hence every coefficient of degree N_j of $\Theta(\varpi L)$ lies in O.

Apply to this substituted series, at zero, the product of the column determinants of derivatives $\det(\partial/\partial X_{\tau,h,i,b_{c,l}})_{i,l\le h}$. An order N_j monomial derivative of multi-index α gives $\prod_i\alpha_i!$ times its degree N_j coefficient, by successive differentiation of powers. Lemma 28.4 gives $v_p(\alpha_i!)\ge\alpha_i/(p-1)-(E_j+1)$ for every i. Thus the differentiated value has valuation at least $N_j/(p-1)-fn^3(E_j+1)$, also after the signed additions in the determinants, by Lemma 16.2.

The same differentiated value equals

$$\varpi^{N_j}p^{m_{T'}}\left(\prod_{\tau,h}(\det M_{\tau,h})^{\kappa_{\tau,h}}\right)F_{T'}(y).$$

In fact the derivative in $X_{\tau,h,i,b}$ acts by $\varpi p^{\gamma_{\tau,h,b}}\sum_a\tilde y_{\tau,h,i,a}\partial/\partial Z_{\tau,a,b}$ before substitution. On each monomial this follows by the product rule, so also for repeated derivatives in the degree N_j polynomial. Factoring the scalars from each column and $M_{\tau,h}$ from the matrix of each determinant gives the expression. The determinant factor has valuation at most $\omega_p\sum_{e,\tau}K(\tau,e)$ by Theorem 27.3, with precisely the exponents $e\in I_p^0,\ e\le E_j$. Subtract its valuation and $N_j\rho_p+m_{T'}$ from the lower bound. By the definitions of J_p and ρ_p the result is $v_p(F_{T'}(y))\ge J_p(j,V)-m_{T'}$. Lemma 28.2 transfers this bound from integral evaluations to all monomial coefficients. Lemma 28.1 then gives it for all standard minor coordinates. □

29. Frobenius on the rows of a tight frame

Definition 29.1. Use the representatives $\tau:E\to E$ fixing K for the places in Proposition 10.3. A finite family $w_i\in N_1^{-1}O_E$ has the tight frame identity of length M' if M' is a positive integer and

$$\sum_i\tau(w_i)\overline{\sigma(w_i)}=M'\mathbf{1}_{\tau=\sigma}\qquad(\tau,\sigma\in G).$$

The family $w_i=\zeta_{N_1}^i,\ 0\le i<N_1$, has this identity with $M'=N_1$. Indeed the sum is the finite geometric sum $\sum_i\zeta_{N_1}^{(\tau-\sigma)i}$, which vanishes unless $\tau=\sigma \bmod N_1$. The length in this example is prime to each nonsplit p by Proposition 10.3.

Proposition 29.2. *Take any prime $p < x_1$ with $\eta_d(p) \neq 1$, an embedding $\iota : E \hookrightarrow K_p$, and an element θ of valuation ω_p. Let w_i satisfy Definition 29.1. For β in the valuation ring O of K_p, put $z_{\tau i} = \beta\iota(\overline{\tau(w_i)})$. All these entries are integral. With $\bar{z}$ denoting reduction modulo θO in the following formula, for every $e \in I_p^0$, $\tau, \sigma \in G$ one has*

$$\sum_i \bar{z}_{\tau i}^{p^e} \bar{z}_{\sigma i} = \bar{\beta}^{p^e+1} M' \mathbf{1}_{\sigma=(-p^e)\tau}.$$

Proof. Since the images under ι of algebraic integers have absolute value at most one by their monic integral equations, and $p \nmid N_1$, each image of w_i and its conjugates is integral at p. The quotient $O/\theta O$ has characteristic p, since $v_p(p) = 1 \geq \omega_p > 0$. We claim the two maps

$$w \longmapsto \overline{\iota(\overline{\tau(w)})}^{p^e}, \qquad w \longmapsto \overline{\iota(((-p^e)\tau)(w))}$$

from $O_E[1/N_1]$ to this quotient agree, with the outer bars denoting reduction. They agree on ζ_{N_1} by the exponents. If p is inert, $v_p(\theta) = 1$ and $\theta O = pO$. The image of ω is a root of its irreducible quadratic over $\mathbf{F}_p$, as in Lemma 24.3. In the subfield it generates, an odd power of Frobenius exchanges $1-\omega$ and ω. This applies to every $e \in I_p^0$ in the inert case. If $p = d$, then $v_p(\iota(\sqrt{-d})) = 1/2$ from its square, so ω and $1-\omega$ both reduce to $1/2$. Every Frobenius power fixes that element, including the identity at $e = 0$. Since ω, ζ_{N_1} generate O_E by Theorem 18.2 and N_1 is invertible in the quotient, the claim follows. Now multiply the claim for w_i by the reduction of $\iota(\overline{\sigma(w_i)})$ and sum. The tight frame identity with places $(-p^e)\tau, \sigma$ gives the formula, after inserting the common power of β. □

30. From a one-variable expansion to the theta product

Lemma 30.1. *Let R be a commutative ring of characteristic p, H a nilpotent ideal, and $a_0 \in \{1, 2\}$. For finitely many $l_i, t_i \in H$ suppose $t_i - t_i^{p^{a_0}} = l_i$. If $a_0 = 2$, suppose $\sum_i l_i^{p^e} l_i = 0$ for all odd $e \geq 1$. If $a_0 = 1$, suppose the same holds for all $e \geq 0$. Then*

$$\sum_i t_i^{p+1} = 0, \qquad \sum_i t_i^{p+p^{a_0}} = 0.$$

Proof. Substitution in $t_i = l_i + t_i^{p^{a_0}}$, using additivity of Frobenius, gives $t_i = \sum_{r \geq 0} l_i^{p^{a_0 r}}$; the iterated remainder vanishes and all sums are finite by nilpotence. For $A, B \geq 0$ one has

$$\sum_i l_i^{p^A} l_i^{p^B} = \left(\sum_i l_i^{p^{|A-B|}} l_i \right)^{p^{\min(A,B)}}.$$

Expand the two expressions in the assertion using the finite series for the t_i. The exponent pairs are respectively $(a_0 r + 1, a_0 s)$ and $(a_0 r + 1, a_0 s + a_0)$. For $a_0 = 2$ their differences are odd; for $a_0 = 1$ all possible differences occur among the hypotheses. Every term vanishes by the displayed identity. □

Theorem 30.2. *Use $p, \varpi, \theta, \rho_p, \omega_p$ of Definition 28.3 at a nonsplit prime, and $z_{\tau i}, M'$ of Proposition 29.2. Let c range over a finite index set. Write $Z_{\tau,c}$ for independent variables and $l_{ic}(Z) = \sum_\tau z_{\tau i} Z_{\tau,c}$. Suppose there are $S, \psi \in XO[[X]]$, $u_0, u_1, u_2 \in O$ and $a_0 \in \{1, 2\}$ with*

$$S - S^{p^{a_0}} \equiv X \pmod{\theta},$$
$$\psi \equiv u_0 X^2 + u_1 S^{p+1} + u_2 S^{p+p^{a_0}} \pmod{\theta}.$$

Require $p = d$ if $a_0 = 1$, and $u_0 = 0$ if $a_0 = 2$. For given $V_{ic} \in O[[X]]$ and $\kappa \in K_p^\times$ with $v_p(\kappa) = 1/(p-1) - \omega_p$, suppose the series Θ satisfies

$$\Theta(\varpi Z) = \exp\left(\kappa \sum_{i,c} \psi(l_{ic}(Z)) \right) \prod_{i,c} V_{ic}(l_{ic}(Z)).$$

Then for every $B \ge 1$ and every integral homogeneous linear substitution $Z_{\tau,c} = L_{\tau,c}$ satisfying

$$\sum_{\tau,c} L_{\tau,c}^{p^e} L_{(-p^e)\tau,c} \equiv 0 \pmod{\theta} \qquad (e \in I_p^0,\ e \le \lfloor \log B / \log p \rfloor),$$

the coefficients through degree B of $\Theta(\varpi L)$ belong to O. For $B = N_j$ and the indices $c = (a, b)$ in Definition 28.3, this supplies the integrality hypothesis of Theorem 28.5.

Proof. Take the polynomial ring over $O/\theta O$ in the substitution variables modulo all monomials of degree $> B$. Its positive-degree ideal is nilpotent and its characteristic is p, because $\omega_p \le 1$. Every series below can be evaluated in that ring on positive-degree arguments by finite truncation. Set $l_i' = l_{ic}(L),\ t_i' = S(l_i')$, with c included in the index i' in the following sums. By Proposition 29.2 and expansion of the p^e-th power,

$$\sum_{i,c} l_{ic}(L)^{p^e} l_{ic}(L) = \bar\beta^{p^e+1} M' \sum_{\tau,c} L_{\tau,c}^{p^e} L_{(-p^e)\tau,c}.$$

For $e \in I_p^0$ up to the stated cutoff this vanishes by hypothesis. Above the cutoff $p^e > B$, and each term vanishes by degree. The first congruence gives $t_i' - (t_i')^{p^{a_0}} = l_i'$. Lemma 30.1 therefore annihilates both sums of powers of t_i' in the second congruence. For $a_0 = 1$ the remaining quadratic sum of the l_i' vanishes by the $e = 0$ congruence at $p = d$. For $a_0 = 2$ its coefficient is zero. It follows that $\sum_{i,c} \psi(l_{ic}(L))$ vanishes in the truncated ring.

Thus through degree B every coefficient of the last sum has valuation at least ω_p and its constant term is zero. After multiplication by κ, the coefficients have valuation at least $1/(p-1)$. In the ℓ'-th power each coefficient of degree at most B then has valuation at least $\ell'/(p-1)$, by finite multiplication. The factorial estimate $v_p(\ell'!) \le \ell'/(p-1)$ follows from Lemma 28.4. Hence division by $\ell'!$ leaves these coefficients integral. Summation over ℓ' in the formal exponential is finite in each degree, and the other factors in the product have integral coefficients. This proves the first assertion. Finally $(p, \theta) = \theta O$ because $v_p(\theta) \le 1$. The congruences of Definition 28.3 are exactly the ones just used. □

Corollary 30.3. *In Theorem 30.2 suppose p is inert, β and u_1 are units, and $p \nmid M'$. Set $\Psi(Z) = \sum_{i,c} \psi(l_{ic}(Z))$, and now use bars for reduction to the residue field of O. For every σ, c,*

$$\frac{\partial \bar\Psi}{\partial Z_{\sigma,c}} = \bar u_1 M' \sum_{\substack{e \ge 1 \text{ odd},\ \tau \in G \\ (-p^e)\tau = \sigma}} \bar\beta^{p^e+1} Z_{\tau,c}^{p^e}.$$

Every displayed monomial has nonzero coefficient.

Proof. Here $a_0 = 2,\ u_0 = 0$. After reduction, the solution of $\bar S - \bar S^{p^2} = X$ with zero constant term is $\bar S(X) = \sum_{r \ge 0} X^{p^{2r}}$, by iteration as in Lemma 30.1 in every finite truncation. Its formal derivative is one. The derivative of $\bar S^{p+p^2}$ vanishes in characteristic p, while that of $\bar S^{p+1}$ is $\bar S^p$. Consequently differentiating $\bar\Psi$ gives $\bar u_1 \sum_{i,r} \bar z_{\sigma i} (\sum_\tau \bar z_{\tau i} Z_{\tau,c})^{p^{2r+1}}$. Apply Proposition 29.2 in the residue field to its coefficients. Each exponent determines e, and the map $\tau \mapsto (-p^e)\tau$ is a permutation of G. Thus the displayed monomials are distinct, and their coefficients are units by the three additional hypotheses. □

31. A normalized complex multiplication lattice and its theta function

Lemma 31.1. *Let $\Lambda = \lambda(\mathbb{Z} + \tau\mathbb{Z})$ with $\lambda \ne 0,\ \Im\tau > 0$, where $\Im$ denotes imaginary part. For a nonzero meromorphic function f, on a closed fundamental parallelogram whose boundary avoids zeros and poles, the integral of f'/f on its positively oriented boundary divided by $2\pi i$ counts its zeros minus its poles inside, with multiplicities. In particular a meromorphic Λ-periodic function that is not identically zero has equally many zeros and poles per fundamental parallelogram. An entire Λ-periodic function is constant.*

Proof. Zeros and poles are isolated by Lemma A.3, so there are finitely many in the compact parallelogram. Around one of order n, positive for a zero and negative for a pole, write $f = (z - z_0)^n h$ with h nonzero holomorphic locally. Then $f'/f = n/(z - z_0) + h'/h$. Remove small disjoint disks and use the triangle integral identity and polygonal approximation in Lemma A.3 on the region between their boundaries and the parallelogram. Integrating gives $2\pi i n$ from each disk. For a periodic logarithmic derivative the integrals on opposite sides cancel. One can translate a parallelogram slightly to avoid singularities on the boundary: in a bounded neighbourhood there are finitely many such points and the forbidden translations lie on finitely many line segments. An entire periodic function is bounded on a compact parallelogram and hence everywhere. Its derivative at any point has modulus at most this global bound divided by the radius of any circle centered there, by Lemma A.3. Letting the radius tend to infinity proves constancy. □

Proposition 31.2. *Put $y = \Im\tau > 0$, $A = |\lambda|^2 y$. On the lattice $\Lambda = \lambda(\mathbb{Z} + \tau\mathbb{Z})$ there is a unique odd entire function σ_Λ with derivative one at zero and transformation law*

$$\sigma_\Lambda(z + l) = s(l) \exp\left(\frac{\pi}{A}(\bar{l}z + |l|^2/2)\right) \sigma_\Lambda(z) \qquad (l \in \Lambda),$$

where $s(l) = 1$ if $l/2 \in \Lambda$ and $s(l) = -1$ otherwise. Its zeros are the lattice points, each simple.

Proof. Set

$$F(u) = \sum_{j \in \mathbb{Z}} \exp\left(\pi i (j + 1/2)^2 \tau + 2\pi i (j + 1/2)(u + 1/2)\right).$$

On every bounded set the Gaussian decay in $|j|^2$ dominates all powers of j and the exponential of a constant times $|j|$. Thus the series and its differentiated series converge uniformly there, giving an entire function by the disk formula of Lemma A.3. Integrating $F(u)\exp(-\pi i u)$ on $0 \le u \le 1$ isolates its nonzero $j = 0$ coefficient by elementary integration. Replacing j by $-j - 1$ gives $F(-u) = -F(u)$; shifting the index once gives

$$F(u + 1) = -F(u), \qquad F(u + \tau) = -\exp(-\pi i \tau - 2\pi i u) F(u).$$

In the integral of F'/F over a generic parallelogram for $1, \tau$ with zero in its interior, the two sides parallel to τ cancel. On translation by τ the logarithmic derivative decreases by $2\pi i$, so the other two sides contribute $2\pi i$. By Lemma 31.1 there is one zero with multiplicity inside. Oddness puts it at zero, so it is simple. The translation laws give the same result in all translates, whose boundaries contain no zeros.

For $u = z/\lambda$ put $\sigma_\Lambda(z) = \lambda \exp(\pi u^2/(2y)) F(u)/F'(0)$. It is odd and has derivative one at zero. At the generators $\lambda, \lambda\tau$ the required sign is negative. Their transformation factors follow from the two formulas for F, using $\tau - 2iy = \bar{\tau}$. For $l = m\lambda + n\lambda\tau$ the composed sign is $(-1)^{m+n+mn}$: adding λ to l introduces the cross phase $\exp(\pi i \Im(\bar{\lambda} l)/A) = (-1)^n$, and adding $\lambda\tau$ gives $(-1)^m$. These recurrences also run backwards. This sign is positive exactly when m, n are even. A second function with the stated properties vanishes at zero and hence at every lattice point. Its quotient by σ_Λ is entire since the latter zeros are simple. The quotient is periodic, therefore constant by Lemma 31.1, and the two derivatives at zero fix that constant as one. □

Proposition 31.3. *Let $O_K = \mathbb{Z} + \omega\mathbb{Z}$, $\omega = (1 + \sqrt{-d})/2$, using the embedding of positive imaginary part. For a lattice Λ as in Proposition 31.2 define*

$$\wp_\Lambda(z) = z^{-2} + \sum_{0 \ne l \in \Lambda} \left((z - l)^{-2} - l^{-2}\right) \qquad (z \notin \Lambda).$$

There are $\lambda \ne 0$, $t = \lambda/2$, $a_1 \in \mathbb{C}$ such that $\Lambda = \lambda O_K$ is stable under O_K, $2t \in \Lambda$, $t, \omega t, \bar{\omega} t \notin \Lambda$, and

$$\wp_\Lambda''(t) = 2, \qquad a_1^2 = 12\wp_\Lambda(t).$$

The functions $\xi(z) = \wp_\Lambda(z+t) - \wp_\Lambda(t)$ *and* $\phi(z) = (\wp'_\Lambda(z+t) - a_1\xi(z))/2$ *then satisfy*

$$\phi^2 + a_1\xi\phi = \xi^3 + \xi, \qquad \xi(0) = \xi'(0) = \phi(0) = 0, \quad \phi'(0) = 1.$$

Proof. The real linear map $(m,n) \mapsto \lambda(m+\tau n)$ is invertible, so its norm is bounded below by a positive constant times $\max(|m|,|n|)$, by compactness of the unit square's boundary. There are $8j$ integer pairs with this maximum $j \geq 1$. Consequently $\sum_{l\neq 0} |l|^{-3} < \infty$. The summands in the definition are $O_C(|l|^{-3})$ on a compact set C off the lattice for large $|l|$, by subtraction of the fractions. Termwise differentiation is justified locally by the disk formula in Lemma A.3. Thus $\wp_\Lambda$ is even and meromorphic, with principal part z^{-2} at zero and zero constant term there. Its derivative, $-2\sum_{l\in\Lambda}(z-l)^{-3}$, is periodic by absolute convergence. Each period difference of $\wp_\Lambda$ is constant; possible poles cancel by their identical principal parts. For either primitive generator $l = \lambda, \lambda\tau$ evaluate the difference at $-l/2$, which is not a pole. Evenness gives zero. This proves full periodicity.

Near zero write $\wp_\Lambda(z) = z^{-2} + c_2 z^2 + c_4 z^4 + O(z^6)$ and set $g_2 = 20c_2$, $g_3 = 28c_4$. Expansion at zero cancels all negative powers and the constant term in $\wp'^2_\Lambda - 4\wp^3_\Lambda + g_2\wp_\Lambda + g_3$. The resulting entire elliptic function is zero by Lemma 31.1. Differentiation yields $\wp''_\Lambda = 6\wp^2_\Lambda - g_2/2$, first away from the isolated zeros of $\wp'_\Lambda$, then throughout off the poles by continuity. Up to lattice translation, $\wp'_\Lambda$ has exactly one pole of order three. By oddness and periodicity it vanishes at the three nonzero half periods. They are distinct, so the counting formula of Lemma 31.1 makes each zero simple.

For the initial lattice O_K take $t_0 = 1/2$, so $\wp''_{O_K}(t_0) \neq 0$. Choose any fourth root $\lambda^4 = \wp''_{O_K}(t_0)/2$ and the required square root a_1 in $\mathbb{C}$, using Lemma 16.2. The defining series gives $\wp_{\lambda O_K}(\lambda z) = \lambda^{-2}\wp_{O_K}(z)$, hence $\wp''_\Lambda(t) = 2$. Stability follows since O_K is a ring by the monic quadratic relation for ω. In the basis $1, \omega$, the coordinates of $1/2, \omega/2$ and $(1-\omega)/2$ are not all integral in each pair, proving the half-period assertions. Finally put $e_t = \wp_\Lambda(t)$. The differential equation gives $12e_t^2 - g_2 = 4$. Subtract the cubic equation at t, where $\wp'_\Lambda(t) = 0$, from the one at $t+z$. The result is $(2\phi + a_1\xi)^2 = 4\xi^3 + 12e_t\xi^2 + 4\xi$, which yields the displayed equation. Differentiation at the half period gives the asserted initial values. □

Proposition 31.4. *Take the lattice of Proposition 31.3, with area* A. *For an entire function* F *and* $y \in \mathbb{C}$ *put*

$$(U_y F)(z) = \exp\Big(-\frac{\pi}{A}(z\bar{y} + |y|^2/2)\Big)\, F(z+y).$$

The normalized theta function

$$\theta_\Lambda(z) = \frac{(U_t\sigma_\Lambda)(z)}{(U_t\sigma_\Lambda)(0)} = \exp(-\pi z\bar{t}/A)\frac{\sigma_\Lambda(z+t)}{\sigma_\Lambda(t)}$$

is even and entire with $\theta_\Lambda(0) = 1$ *and simple zeros at* $t+\Lambda$. *Its multiplier at* $l \in \Lambda$ *is given by*

$$\theta_\Lambda(z+l) = s(l)\exp\Big(\frac{\pi}{A}(\bar{l}z + |l|^2/2 + \bar{l}t - l\bar{t})\Big)\,\theta_\Lambda(z).$$

For a prime p, *an isomorphism* $\iota : \mathbb{C} \to K_p$ *as in Definition 16.3, and torsion shifts* y_{ic} *with* $n'y_{ic} \in \Lambda$ *for some* $n' \geq 1$ *prime to* p, *the following formal theta product is well defined. Choose a frame* w_i *of Definition 29.1 of length prime to* p *and* $b' \in \mathbb{C}$ *with* $v_p(\iota b') \geq 0$, *and put*

$$F_y(X) = \sum_{j\geq 0} \iota\Big((U_y\theta_\Lambda)^{(j)}(0)/j!\Big)\, X^j,$$

$$\Theta(Z) = \prod_{i,c} F_{y_{ic}}\left(\sum_{\tau\in G} \iota(b'\overline{\tau(w_i)})\, Z_{\tau,c}\right).$$

For $b' = 1$, $y_{ic} = 0$ *its constant term equals one.*

Proof. The value $\sigma_\Lambda(t)$ is nonzero by Propositions 31.3 and 31.2. Their zero set and multiplier formulas give the zeros and the stated translation formula. Since $2t \in \Lambda$ but $t \notin \Lambda$, one has $s(2t) = -1$ and $\sigma_\Lambda(z-t) = -\exp(-2\pi z\bar{t}/A)\sigma_\Lambda(z+t)$. Using oddness of σ_Λ gives $\theta_\Lambda(-z) = \theta_\Lambda(z)$. Each translate is entire, so its indicated derivatives exist by Lemma A.3. Applying ι to coefficients gives formal series. The arguments have zero constant terms and are homogeneous linear, so in each fixed degree only that degree of each composed series can contribute before multiplication, and only finitely many terms enter the product coefficient. The last claim follows from $U_0\theta_\Lambda(0) = 1$. □

32. The discriminant along a cycle of index-two lattices

Lemma 32.1. *Put $O_K = \mathbb{Z}[\omega]$ as in Lemma 11.3. For a rank-two complex lattice Λ stable under O_K there is exactly one nonzero class t in $(\frac{1}{2}\Lambda)/\Lambda$ for which ωt and $\bar{\omega}t$ are both nonzero. Write $T = \omega t$, $S = \bar{\omega}t$ in this group, so $t = T+S$. The lattice $\Lambda' = \Lambda+(\omega/2)\Lambda = \Lambda+\mathbb{Z}T$, with any representative of T, contains Λ with index two. In $\mathbb{C}/\Lambda'$ the image of t is $S' = \bar{\omega}t'$ for the analogous distinguished point t'. Starting at O_K, after finitely many such enlargements one returns to a complex multiple of O_K.*

Proof. Multiplication by ω on the real vector space $\mathbb{C}$ has characteristic polynomial $X^2 - X + (d+1)/4$. In an integral basis of Λ its matrix is integral. Since $d \equiv 7 \bmod 8$, the reduced polynomial over $\mathbf{F}_2$ has the two distinct roots 0 and 1. The two-dimensional space $\Lambda/2\Lambda \simeq (\frac{1}{2}\Lambda)/\Lambda$ is the direct sum of two one-dimensional eigenspaces: the idempotents are $\omega, 1-\omega$ and neither is zero by the characteristic polynomial. The unique mixed nonzero vector has a nonzero component in both. The image of ω is generated by T, proving the index. The image of t in the next quotient is nonzero and killed by ω, so it is S', the unique nonzero such element of the new two-torsion group.

Write $I_n = (O_K + (\omega/2)O_K)^n$, the lattice after n steps starting from O_K, and $J_n = 2^n I_n = (2,\omega)^n$. Here products of ideals are finite sums of products of elements. Since $(2,\omega)(2,\bar{\omega}) = 2O_K$, we have $J_n\bar{J}_n = 2^n O_K$: the generators of the first product are divisible by 2 using $\omega\bar{\omega} = (1+d)/4$, and $2\omega + 2\bar{\omega} = 2$. Each enlargement has index two, so I_n has area 2^{-n} times the area of O_K. Multiplication by 2^n shows $[O_K : J_n] = 2^n$, by the lattice determinant calculation of Lemma 11.3. The same pigeonhole argument used there supplies nonzero $z_n \in J_n$ with $|z_n| \le 2(1+|\omega|)2^{n/2}$, by comparing the elements $a+b\omega$, $0 \le a,b \le \lceil 2^{n/2} \rceil$, modulo J_n. Then $z_n\bar{J}_n/2^n \subset O_K$ has index $|z_n|^2/2^n \le 4(1+|\omega|)^2$, again by areas. There are finitely many sublattices of these indices, since a sublattice of index h contains hO_K by the order of its finite additive quotient. Equality of two such sublattices at $n < n'$, followed by conjugation and rescaling, gives $J_{n'} = cJ_n$ for some $c \neq 0$. Multiply by $\bar{J}_n$ and use $J_{n'} = J_{n'-n}J_n$ to get $J_{n'-n} = cO_K$. It follows that $I_{n'-n}$ is homothetic to the initial lattice. □

Proposition 32.2. *For the lattice and the three classes of Lemma 32.1 put $e_P = \wp_\Lambda(P)$ for $P = t, T, S$. They are distinct. Define the nonzero values*

$$u = \frac{e_T - e_t}{e_S - e_t}, \qquad V = \frac{16u^2}{1-u}, \qquad D = \frac{16(u-1)^2}{u}.$$

These are unchanged when the lattice and the classes are scaled by the same nonzero complex number. For the next lattice denote the corresponding values by u', V', D'. Then

$$V' = -DV, \qquad (48 + V + D)V' = -4096.$$

Proof. For any nonzero half period P the elliptic function $\wp_\Lambda(z) - e_P$ has a double pole and a zero of order at least two at P, by Proposition 31.3 and its proof. Lemma 31.1 thus gives exactly that double zero per cell. In particular the three e_P are distinct and give the three roots of $4X^3 - g_2X - g_3$, by the same differential equation. Their sum is zero. Homothety invariance follows from the scaling formula for $\wp$ in Proposition 31.3.

Put $X(z) = \wp_\Lambda(z) - e_T$ and $B_T = (e_T - e_t)(e_T - e_S)$. The product $X(z)X(z+T)$ has its double zeros paired with its double poles. It is elliptic and entire, hence constant by Lemma 31.1. Evaluation at $z = t$, for which $t + T = S$, gives B_T. Also

$$\wp_{\Lambda'}(z) = \wp_\Lambda(z) + \wp_\Lambda(z+T) - e_T.$$

Indeed the right side has periods Λ' and the same double principal parts as the left side. Its constant term at zero in the Laurent expansion is zero. Their difference is constant and therefore zero. Since $e_t + e_S = -e_T$, subtraction of the value at t now gives

$$\wp_{\Lambda'}(z) - \wp_{\Lambda'}(t) = X(z) + 3e_T + B_T/X(z).$$

On the new torus the image of t is S' by Lemma 32.1. The two remaining nonzero half periods t', T' have representatives y with $2y = T \bmod \Lambda$: those with $2y = 0 \bmod \Lambda$ give only the identity and S' after quotienting by the order-two subgroup generated by T. For such y, $y + T = -y \bmod \Lambda$ and $X(y)^2 = B_T$. Let r_1, r_2 be the two values of $\wp_{\Lambda'}(y) - \wp_{\Lambda'}(t)$ with $y = t', T'$ in that order. They are distinct by the first part of the proof at Λ'. Hence they are the two numbers $3e_T \pm 2\sqrt{B_T}$. In particular

$$(r_1 - r_2)^2 = 16B_T, \qquad r_1 r_2 = (e_S - e_t)^2,$$

where the product uses $3e_T = (e_T - e_t) + (e_T - e_S)$. Since $u' = (r_1 - r_2)/r_1$, we obtain

$$V' = 16\,\frac{(r_1 - r_2)^2}{r_1 r_2} = 256\,u(u-1).$$

Substitution in the definitions gives $-DV = 256u(u-1)$ and $48 + V + D = -16/[u(u-1)]$. This proves both recurrences. □

Lemma 32.3. *In a nonarchimedean valued field extending $\mathbb{Q}_p$, let nonzero V_i, D_i have indices modulo a positive integer and satisfy $V_{i+1} = -D_i V_i,\ (48 + V_i + D_i)V_{i+1} = -4096$. Then $|D_i|_p = 1$ for every i.*

Proof. Put $r_i = |V_i|_p,\ c = |4096|_p$, with $0 < c \le 1$. When $r_i \le 1$ and $r_{i+1} > r_i$, the term D_i of modulus $r_{i+1}/r_i > 1$ strictly dominates the other two terms in $48 + V_i + D_i$. The ultrametric inequality in both directions gives $r_{i+1}^2 = c r_i$. Thus the interval $0 < r_i < c$ is forward invariant: either the modulus does not increase or the squared formula applies. If a maximum $r_i > 1$ occurred, then $|D_i|_p \le 1$ and V_i would strictly dominate in the same sum, giving $r_{i+1} = c/r_i < c$. Forward invariance would contradict return to the maximum. Thus all $r_i \le 1$. If any $r_i < c$, all would be $< c$ by invariance. For a step with $r_{i+1} \le r_i$ the three terms in the sum would all have modulus at most one, forcing $r_{i+1} \ge c$. Hence there could be no nonincreasing step, impossible on a cycle. We have shown $c \le r_i \le 1$. A strict increase would now give $r_{i+1}^2 = c r_i \le r_i^2$. Thus all moduli agree, so $|D_i|_p = r_{i+1}/r_i = 1$. □

Theorem 32.4. *For the normalized lattice and a_1 of Proposition 31.3, every p and every isomorphism $\iota : \mathbb{C} \to K_p$ satisfy*

$$|\iota(a_1^4 - 64)|_p = 1, \qquad |\iota(a_1)|_p \le 1.$$

Proof. Starting at $\Lambda = \lambda O_K$, the distinguished class of Lemma 32.1 is the previously chosen t by Proposition 31.3. Its cycle up to scaling is finite by that lemma, and V, D are scale invariant. Apply ι to Proposition 32.2 around the cycle, then apply Lemma 32.3. It remains to identify D. Set $\alpha = e_T - e_t,\ b_1 = e_S - e_t$. The derivative at e_t of $(X - e_t)(X - e_T)(X - e_S) = X^3 - (g_2/4)X - g_3/4$ gives $\alpha b_1 = 3e_t^2 - g_2/4 = 1$, since $\wp_\Lambda''(t) = 2$. Also $\alpha + b_1 = -3e_t = -a_1^2/4$. It follows that

$$a_1^4 - 64 = 16(\alpha - b_1)^2 = 16(u-1)^2/u = D.$$

The unit assertion follows. Adding 64, of p-adic absolute value at most one, gives $|\iota(a_1)|_p^4 \le 1$. □

33. Doubling, torsion values and the integral coordinate

Proposition 33.1. *Use $\Lambda, \sigma = \sigma_\Lambda, \theta = \theta_\Lambda, t, \xi, \phi, a_1$ of Propositions 31.3 and 31.4. Then*

$$\theta(z)^2 = (\wp_\Lambda(z) - \wp_\Lambda(t))\sigma(z)^2, \qquad \sigma(z)^2 = \xi(z)\theta(z)^2,$$
$$\theta(2z) = \theta(z)^4 - \sigma(z)^4,$$
$$\sigma(2z)^2 = (\theta(z)\sigma(z))^2\{4(\theta(z)^4 + \sigma(z)^4) + a_1^2(\theta(z)\sigma(z))^2\}.$$

Expressions with apparent poles are interpreted by continuation. The function $G = \phi\theta^3$ extends to an entire function and satisfies $G^2 + a_1\sigma^2\theta G = \sigma^6 + \sigma^2\theta^4$.

Proof. For $l \in \Lambda$ the extra phase in the multiplier of θ^2/σ^2 is $\exp(4\pi i \Im(\bar{l}t)/A)$, by Propositions 31.2 and 31.4. It equals one since $2t$ is a period and the oriented area of two periods is an integer multiple of A. The quotient is therefore periodic and is even with principal part z^{-2} at zero. Subtracting $\wp_\Lambda$ gives an entire elliptic function, whose value at t is $-\wp_\Lambda(t)$. Lemma 31.1 proves the first identity. As in the proof of Proposition 32.2, the product of $\wp_\Lambda(z) - e_t$ and $\wp_\Lambda(z+t) - e_t$ is the constant $(e_T - e_t)(e_S - e_t) = 1$ by Theorem 32.4 and its proof. This gives the second identity of the first line.

For fixed $w \notin \Lambda$ the quotient $\sigma(z+w)\sigma(z-w)/(\sigma(z)^2\sigma(w)^2)$ is periodic by the sigma multiplier, is even in z, and has principal part $-z^{-2}$ at zero. Add $\wp_\Lambda(z)$ and evaluate at $z = w$ to get the constant $\wp_\Lambda(w)$. Thus the quotient equals $\wp_\Lambda(w) - \wp_\Lambda(z)$. Letting $w \to z$ after division of the corresponding product identity by $z - w$ gives $\sigma(2z) = -\wp_\Lambda'(z)\sigma(z)^4$, since $\sigma'(0) = 1$. Substituting $z + t$ for z in the product identity and using Proposition 31.4 yields

$$\theta(z+w)\theta(z-w) = \theta(z)^2\{\theta(w)^2 - \xi(z)\sigma(w)^2\}.$$

At $w = z$ this gives the stated doubling formula for θ. The cubic equation in the proof of Proposition 31.3 at z, in the coordinate $X = \wp_\Lambda(z) - e_t$, reads $\wp_\Lambda'(z)^2 = 4X^3 + a_1^2X^2 + 4X$. Substitute $X = \theta(z)^2/\sigma(z)^2$ into the squared duplication formula for σ. This gives the other formula. All resulting identities extend over the excluded discrete sets by continuity. Finally ϕ has poles of order at most three at $t + \Lambda$ by its definition. The simple zeros of θ there cancel them. Multiplying the cubic for ϕ, ξ by θ^6 and using $\xi\theta^2 = \sigma^2$ gives the last identity. □

Theorem 33.2. *For a prime p and isomorphism $\iota : \mathbb{C} \to K_p$, define*

$$H_p(z) = \max\{|\iota(\sigma(z))|_p, |\iota(\theta(z))|_p\}.$$

Then for all $z \in \mathbb{C}$, $H_p(z) > 0$, $H_p(2z) = H_p(z)^4$, and $|\iota G(z)|_p \le H_p(z)^3$. At every torsion point y of $\mathbb{C}/\Lambda$ one has

$$\max(|\iota((U_y\sigma)(0))|_p, |\iota((U_y\theta)(0))|_p) = 1.$$

Proof. The zeros of σ, θ are disjoint by Propositions 31.2 and 31.4. Choose whichever of their two images has larger absolute value and call it c. Divide the images by c to get a, b with $\max(|a|_p, |b|_p) = 1$. Put $A_1 = \iota(a_1)$. By Theorem 32.4, A_1 is integral and $A_1^4 - 64$ is a unit. Applying Proposition 33.1 to the ratios $y_0 = \iota(\theta(2z))/c^4$, $x_0 = \iota(\sigma(2z))/c^4$ gives

$$y_0 = b^4 - a^4, \qquad x_0^2 = a^2b^2\{4(b^4 + a^4) + A_1^2a^2b^2\}.$$

Both are integral by these equations. If both vanished in the residue field, the first would give $\bar{a}^4 = \bar{b}^4 \neq 0$. Putting $r = (\bar{a}/\bar{b})^2$ yields $r^2 = 1$; the second gives $8 + \bar{A}_1^2 r = 0$. Squaring would give $\bar{A}_1^4 = 64$, contradicting the unit property, including in residue characteristic two. Hence the maximum is one after division by c^4. For G/c^3, meaning $\iota(G(z))/c^3$, its monic quadratic equation has integral coefficients by the last identity of Proposition 33.1. Its absolute value is therefore at most one, by the dominant-term argument of Lemma 28.2.

Write A for the area of Λ, and $h(y) = H_p(y)/|\iota(\exp(\pi|y|^2/(2A)))|_p$. At $y \in n^{-1}\Lambda$, replacing y by $y + l$, $l \in \Lambda$, in this expression preserves h. In fact the real part of the

exponent in both multipliers is cancelled by the change of $\pi|y|^2/(2A)$; the remaining phase is a root of unity because $\Im(\bar{l}y)/A \in n^{-1}\mathbb{Z}$. The extra phase for θ is a root of unity since $2t \in \Lambda$. Every such root has p-adic absolute value one by multiplicativity. Thus h is defined on the finite group $n^{-1}\Lambda/\Lambda$, and $h(2y) = h(y)^4$ by doubling. Two iterates $2^i y, 2^{i'} y$, $i < i'$, agree in that group, so $h(y)^{4^i} = h(y)^{4^{i'}}$. Positivity forces $h(y) = 1$. The definition of U_y at zero identifies this with the required maximum. □

Proposition 33.3. *Use the coefficientwise Taylor map at zero through ι of Proposition 31.4, and write $\Phi(Z), \Xi(Z)$ for the resulting Taylor series of ϕ, ξ. Put $A_1 = \iota(a_1)$ and let O be the valuation ring of K_p. There is a unique $x(X) \in XO[[X]]$ satisfying*

$$x = X^2 + A_1 X x - x^3.$$

It belongs to $X^2 O[[X]]$ and $\Xi = x(\Phi)$. The subring $O[[\Phi]] = \{F(\Phi(Z)) : F \in O[[X]]\}$ of $K_p[[Z]]$ is closed under formal differentiation in Z. For $F \in O[[\Phi]]$ and every $j \geq 0$, $j![Z^j]F \in O$. Here $[Z^j]$ extracts a coefficient at the indicated degree.

Proof. Theorem 32.4 gives $A_1 \in O$. The map $y \mapsto X^2 + A_1 X y - y^3$ increases by at least one the order of the difference of two elements of $XO[[X]]$, since $y^3 - z^3 = (y - z)(y^2 + yz + z^2)$. Starting at zero, the iterates stabilize coefficientwise to a fixed point in $X^2 O[[X]]$. The same increase shows uniqueness over K_p as well. By Proposition 31.3, $\Phi(Z) = Z + O(Z^2)$. It has a compositional inverse: in a proposed inverse starting with Z, at degree $n \geq 2$ exactly the linear term of Φ contributes the unknown coefficient with multiplier one. This gives a right inverse by recursion and the same uniqueness gives a two-sided inverse. Substitution in the cubic equation proves $\Xi = x(\Phi)$ by uniqueness.

Substituting the Taylor series in the identities of Proposition 31.3 gives $\Xi' = 2\Phi + A_1\Xi$. Differentiating the cubic then gives $(2\Phi + A_1\Xi)(\Phi' + A_1\Phi - 3\Xi^2 - 1) = 0$. The first factor starts with $2Z$, so the other factor vanishes. Hence $\Phi' = 1 + 3x(\Phi)^2 - A_1\Phi$ belongs to $O[[\Phi]]$. For $F_0 \in O[[X]]$ one has $(F_0(\Phi))' = F_0'(\Phi)\Phi'$, by differentiating monomials coefficientwise. This proves closure. All members have integral constant coefficient since $\Phi(0) = 0$. Taking the constant coefficient after j differentiations proves the last assertion. □

34. Removing torsion translates from the local expansion

Lemma 34.1. *For a nonnegative integer m, suppose F is entire and F/θ^m is Λ-periodic for the functions of Proposition 33.1. There are unique complex polynomials P, Q with*

$$F/\theta^m = P(\xi) + \phi Q(\xi).$$

Only terms of the form $\xi^a \phi^b$ with $b = 0, 1$, $2a + 3b \leq m$ occur. For each $\iota : \mathbb{C} \to K_p$ there is a finite constant C_F such that $|\iota(F(z))|_p \leq C_F H_p(z)^m$ for all z.

Proof. The quotient can have poles only at $t + \Lambda$, of order at most m. By Proposition 31.3, ξ and ϕ have leading Laurent terms $(z - t)^{-2}$ and $-(z - t)^{-3}$ there. The integers $2a + 3b$ with $a \geq 0$, $b \in \{0, 1\}$ attain zero and each integer at least two exactly once. Successively subtract the unique scalar multiple of the monomial that cancels a pole of any such order, starting at the largest. A possible simple pole also has to disappear: periodicity makes the boundary integral of the remaining function zero, and its sole possible residue is therefore zero by the disk excision in Lemma 31.1. The remaining entire elliptic function is constant. The distinct pole orders give uniqueness. Off the zeros of θ, multiplying a monomial by θ^m gives $\theta^{m-2a-3b}\sigma^{2a}G^b$ by Proposition 33.1. This identity extends everywhere. Theorem 33.2 bounds it by $H_p(z)^m$ at ι. The maximum of the absolute values of the finitely many coefficients, or one if all vanish, suffices for C_F. □

Proposition 34.2. *Let H_p be as in Theorem 33.2 and write $g_y(z) = \exp(-\pi(z\bar{y} + |y|^2/2)/A)$. For every $y \in n^{-1}\Lambda$ with $n \geq 1$, and every $z \in \mathbb{C}$,*

$$|\iota(g_y(z))|_p H_p(z + y) = H_p(z), \qquad H_p(nz) \leq H_p(z)^{n^2}.$$

Proof. Write $R_y(z)$ for the left side of the first identity divided by its right side. It is unchanged if y is replaced by $y+l$ with $l \in \Lambda$. Indeed the two transformation formulas of Propositions 31.2 and 31.4 give the same absolute factor on $H_p(z+y)$, since their multipliers differ by a sign. After multiplication by $g_{y+l}(z)/g_y(z)$, only $\exp(\pi i \Im(\bar{l}y)/A)$ remains, a root of unity at such a torsion point. For $f = \sigma, \theta$ the quotient $(U_y f)^{2n}/\theta^{2n}$ is periodic: under $z \mapsto z+l$ the quotient before raising to $2n$ picks up at most a sign times $\exp(2\pi i \Im(\bar{l}y)/A)$. By Lemma 34.1, R_y is bounded on all z, using the larger of the two bounds and taking a $2n$-th root. There are only n^2 classes of y, so a common finite bound $C \geq 1$ works for all of them.

Now $R_{2y}(2z) = R_y(z)^4$, by Theorem 33.2 and $g_{2y}(2z) = g_y(z)^4$. Hence $R_y(z)^{4^k} \leq C$ for all $k \geq 0$, giving $R_y(z) \leq 1$. Applying the same inequality at $-y, z+y$ yields the reverse bound, since $g_{-y}(z+y)g_y(z) = 1$. For the second assertion consider $S_n(z) = H_p(nz)/H_p(z)^{n^2}$. For $f = \sigma, \theta$ the function $f(nz)^2/\theta(z)^{2n^2}$ is periodic: a period l at z gives the period nl at nz with the same exponential raised to n^2, and squaring removes the signs. Thus S_n is bounded by the same lemma. Since $S_n(2z) = S_n(z)^4$, iteration gives $S_n(z) \leq 1$. □

Proposition 34.3. *In Lemma 34.1, suppose for a given ι that $|\iota(F(z))|_p \leq H_p(z)^m$ for every z. All coefficients of $\iota P, \iota Q$ then belong to the valuation ring O. In particular, writing $F_0(Z)$ for the transported Taylor series of θ at zero, the transported Taylor series of F belongs to $F_0(Z)^m O[[\Phi]]$.*

Proof. Suppose a coefficient of $\iota P, \iota Q$ of largest absolute value C satisfies $C > 1$. Divide both polynomials by that coefficient to obtain integral P_1, Q_1 whose reductions are not both zero. In the residue field put $D(X) = 4X^3 + \bar{A}_1^2 X^2 + 4X$, $A_1 = \iota(a_1)$. In odd characteristic this polynomial is nonzero from its cubic term. In characteristic two it is nonzero because $\bar{A}_1 \neq 0$ by Theorem 32.4. The residue field is infinite by Lemma 28.2. Choose x_0 avoiding the zeros of D and of one nonzero polynomial from $\bar{P}_1, \bar{Q}_1$. Lift x_0 to $x \in O$. Both roots of

$$Y^2 + A_1 x Y - x^3 - x = 0$$

exist in K_p and are integral by their monic equation. Their squared difference has reduction $D(x_0) \neq 0$, so their reductions are distinct. At least one root y gives $|P_1(x) + yQ_1(x)|_p = 1$, since an affine linear polynomial in the residue field nonzero at some point cannot vanish at two different points.

This pair is attained by $\iota\xi(z), \iota\phi(z)$ for some $z \notin t + \Lambda$. In fact $\xi - \iota^{-1}x$ has a double pole at t and therefore has a zero z by Lemma 31.1. The value of ξ at $-z$ is the same by periodicity and evenness of $\wp$, while its derivative changes sign. Thus the two values of $\iota\phi$ at $z, -z$ supply both roots by $\xi' = 2\phi + a_1\xi$ and their nonzero difference in characteristic zero. Since $x \in O$, $\sigma^2 = \xi\theta^2$ in Proposition 33.1 gives $H_p(z) = |\iota(\theta(z))|_p > 0$. The chosen pair would give $|\iota(F(z)/\theta(z)^m)|_p = C > 1$, contradicting the assumption. This proves integrality. The series conclusion follows by substituting $\Xi = x(\Phi)$ from Proposition 33.3 in the polynomial expression. □

Theorem 34.4. *Let $n \geq 1$, $p \nmid n$, $ny \in \Lambda$, and retain the series F_y, F_0, Φ of Propositions 31.4 and 33.3. There exists $V_y \in O[[X]]$ with*

$$F_y(nZ) = F_0(Z)^{n^2} V_y(\Phi(Z)).$$

Consequently, suppose at this p that $\Phi(\varpi X) \in O[[X]]$ and that $F_0(\varpi X) = \exp(\kappa\psi(X))V_0'(X)$, with $V_0' \in O[[X]]$ and the common data $S, \psi, \varpi, \theta, \kappa$ of Theorem 30.2. Every product of the actual torsion translates in Proposition 31.4 with denominator n then satisfies the conclusions of Theorem 30.2 for all B.

Proof. The quotient $(U_y\theta)(nz)/\theta(z)^{n^2}$ is periodic. The prime-to-p condition is immaterial for this first calculation. In the two multipliers a period l produces the same exponential of $n^2(\bar{l}z + |l|^2/2)\pi/A$. The remaining factor from the translation by y is $\exp(2\pi i \Im(\bar{l}ny)/A) = 1$.

For the sign, write $\chi(l) = s(l)\exp(2\pi i\Im(\bar{l}t)/A)$. One has $s(nl) = s(l)^{n^2}$ by parity, and the extra phase for t is a sign, whose n-th and n^2-th powers agree. Thus $\chi(nl) = \chi(l)^{n^2}$. By Proposition 34.2,

$$|\iota((U_y\theta)(nz))|_p \le H_p(nz) \le H_p(z)^{n^2}.$$

Proposition 34.3 with $m = n^2$ now proves the series equality.

Substitute $Z = \varpi X/n$. The integral factor remains an integral series in X/n since $\Phi(\varpi X)$ is integral with zero constant term. The common exponential for all the shifts in the product is $\exp(n^2\kappa\,\psi(X/n))$, and the remaining factors are integral series in X/n. In a tight frame replace $\beta = \iota b'$ by β/n when forming the linear arguments, and replace κ by $n^2\kappa$. These operations preserve the required integrality and the prescribed valuation since $p \nmid n$. Expansion of the product gives the common exponential of the sum: the formal identity $\exp(X)\exp(Y) = \exp(X+Y)$ follows by the binomial coefficients in each degree for commuting series of zero constant term. This gives exactly the hypotheses of Theorem 30.2. □

35. Multiplication on the integral local ring

Proposition 35.1. *For $0 \neq \alpha \in O_K$ put $q' = \alpha\bar{\alpha} \in \mathbb{Z}_{\ge 1}$ and $a = \iota(\alpha)$. Then $H_p(\alpha z) \le H_p(z)^{q'}$ for every z. In the notation of Theorem 34.4, put*

$$E_\alpha(Z) = F_0(aZ)/F_0(Z)^{q'}, \qquad f_\alpha(\Phi(Z)) = \Phi(aZ).$$

One has $E_\alpha \in O[[\Phi]]^\times$, $f_\alpha(X) \in XO[[X]]$. If q' is odd, then the reduction of f_α to the residue field is nonzero of order at most q'.

Proof. Let $\chi(l) = s(l)\exp(2\pi i\Im(\bar{l}t)/A)$ be the multiplier apart from the standard exponential in Proposition 31.4. For $l = \lambda(m+n\omega)$ one has $\chi(l) = (-1)^{m+mn}$ by the computation of $s(l)$ in Proposition 31.2 and $t = \lambda/2$. This is $(-1)^{|l/\lambda|^2}$, since $|m+n\omega|^2 = m^2+mn+((1+d)/4)n^2$ and $(1+d)/4$ is even. Consequently $\chi(\alpha l) = \chi(l)^{q'}$. Moreover the basic exponential multiplier at $\alpha z, \alpha l$ is the one at z, l raised to q'. As in Proposition 34.2, this makes $f(\alpha z)^2/\theta(z)^{2q'}$ periodic for $f = \sigma, \theta$. Lemma 34.1 bounds $H_p(\alpha z)/H_p(z)^{q'}$; the value at $2z$ equals its fourth power. Iterating gives the required inequality.

The functions $\theta(\alpha z)$ and $G(\alpha z)$ have ratios with $\theta(z)^{q'}$, respectively $\theta(z)^{3q'}$, that are periodic, since ϕ is elliptic and the signs for θ agree before squaring. They satisfy the bounds required by Proposition 34.3, by the first assertion and Theorem 33.2. The first ratio has constant term one at zero and hence an inverse in $O[[\Phi]]$, by recursively solving the product coefficients. Dividing the second ratio by the cube of the first gives f_α in $O[[X]]$; its constant term is zero.

For q' odd the element α acts as the identity on $\Lambda/2\Lambda$: its class in $O_K/2O_K \simeq \mathbf{F}_2 \times \mathbf{F}_2$ is a unit since $\alpha\bar{\alpha} = q'$, and the only unit is $(1,1)$. The ring decomposition follows directly from the two distinct roots of the quadratic for ω modulo 2. In particular $\alpha t = t \bmod \Lambda$. The ratio $\sigma(\alpha z)^2/\theta(z)^{2q'}$ is even. By Proposition 34.3 it is $P(\xi(z))$ with $\deg P \le q'$ and ιP integral. Indeed its possible term $\phi Q(\xi)$ must vanish: ξ is even, and $\phi(z) - \phi(-z) = \xi'(z)$ is not identically zero. Let $P_{q'}$ be the degree q' coefficient, taken as zero if necessary. Since $\xi\theta^2 = \sigma^2$, division by $\xi^{q'}$ and passage to $z = t$ give

$$P_{q'} = \sigma(\alpha t)^2/\sigma(t)^{2q'}.$$

At the two torsion points $t, \alpha t$ the theta value vanishes. The torsion formula in Theorem 33.2, together with $|\alpha t|^2 = q'|t|^2$, gives $H_p(\alpha t) = H_p(t)^{q'}$. Hence $|\iota P_{q'}|_p = 1$.

In the coordinate $X = \Phi(Z)$, the polynomial identity now reads

$$(\iota P)(x(X)) = x(f_\alpha(X))\,E_\alpha(Z)^2,$$

where $x(X) = X^2 + \cdots$ is the integral series of Proposition 33.3. The last factor is a unit in this coordinate. After reduction, the left side has order $2\,\mathrm{ord}_0(\overline{\iota P}) \le 2q'$, since its leading

polynomial coefficient is a unit. The right side has order $2\,\mathrm{ord}_0(\bar f_\alpha)$ if $\bar f_\alpha \neq 0$ and otherwise vanishes. Comparing gives the assertion. □

Proposition 35.2. *Define $\mathcal{L} = -\log F_0$ by the formal expansion of $-\log(1+u)$ at $u = F_0 - 1$. Then $\mathcal{L}(0) = \mathcal{L}'(0) = 0$ and for some $c_2 \in K_p$*

$$\mathcal{L}'' = \Xi + c_2, \qquad v_p(\iota(\sqrt{-d})c_2) \geq 0.$$

For every α of Proposition 35.1, the derivative of $q'\mathcal{L}(Z) - \mathcal{L}(aZ)$ belongs to $O[[\Phi]]$. The same integral-coordinate conclusions about $\Phi(aZ)$, $\exp(v\mathcal{L}(Z) - \mathcal{L}(aZ))$ and the derivative of its exponent hold for every $(a, v) \in O^2$ in the closure of $\{(\iota\alpha, \alpha\bar\alpha) : 0 \neq \alpha \in O_K\}$. Here closure is for the topology of the p-adic absolute value.

Proof. The series is well defined since $F_0(0) = 1$. Evenness of θ gives vanishing of the linear term. The function $-(\sigma'/\sigma)' - \wp_\Lambda$ is elliptic by the sigma multiplier and has no poles, by the local expansion at each simple zero of σ. It is constant by Lemma 31.1. In the formula for θ from Proposition 31.4 the additional exponential has linear logarithm. Taking two logarithmic derivatives therefore gives $\mathcal{L}''(Z) - \mathcal{L}''(0) = \Xi(Z)$. All these expressions use transported Taylor coefficients of complex germs. This commutes with addition, multiplication and differentiation by the power-series rules, and the derivative of the formal logarithm is the logarithmic derivative by its coefficient expansion.

By Proposition 35.1, E_α is integral and invertible in the local coordinate. Its logarithmic derivative is the derivative of $q'\mathcal{L} - \mathcal{L}(aZ)$, hence integral in the same coordinate by Proposition 33.3. Differentiate once more and evaluate at zero to see that $(\alpha\bar\alpha - a^2)c_2 \in O$, with the integer $\alpha\bar\alpha$ regarded in K_p. Use the two choices $\omega, \bar\omega$. The resulting multipliers are $-\iota(\omega\sqrt{-d})$ and $\iota(\bar\omega\sqrt{-d})$. Subtracting the first from the second gives the asserted bound since $\omega + \bar\omega = 1$.

To pass to the closure substitute the formal inverse of Φ for Z in $\Phi(aZ)$, the stated exponential and the derivative of its exponent. For fixed degree the coefficient is a polynomial in a, v over K_p: each composition uses finitely many powers in that degree and the exponent has constant term zero. It depends continuously on the pair. The valuation ring is closed by the ultrametric inequality, so integrality persists in the closure. The exponential has constant term one and is again invertible there. □

36. A formal coordinate at the nonsplit primes

Lemma 36.1. *Let p be a prime with $\eta_d(p) \neq 1$ in Proposition 10.3. Set $e = 2$ if p is inert and $e = 1$ if $p = d$, and put $q' = p^e$, $Q = q' - 1$. Then $p \geq 13$. There is a primitive Q-th root of unity $\zeta \in K_p$, and for every integer i the pair $(\zeta^i, \zeta^{(p+1)i})$ belongs to the closure in Proposition 35.2.*

Proof. The primes $2, 3, 5, 7, 11$ divide $18480 \mid M$, hence all split by Proposition 10.3. The polynomial $X^Q - 1$ has Q distinct roots in the algebraically closed field of Lemma 16.2 by its derivative. The cyclicity argument of Lemma 22.4 gives ζ. The reduction map on these roots is injective. Indeed if $w \equiv 1$ in the residue field then $1 + \cdots + w^{Q-1}$ reduces to $Q \neq 0$, so $w^Q - 1 = 0$ forces $w = 1$. Write $\xi = \zeta^i$, and use bars on residue classes in this proof.

The q' roots of $X^{q'} - X$ in the residue field form a field of that cardinality by the argument of Lemma 22.4. The reduced image of ιO_K is this field. For p inert, the quadratic for ω over $\mathbf{F}_p$ is irreducible and Frobenius takes its root to $1 - \bar\omega$, as in Lemma 24.3. For $p = d$, the identity $(2\omega - 1)^2 = -d$ reduces ω to $1/2$. It follows in both cases that a lift $\alpha \in O_K$ of $\bar\xi$ has complex conjugate reducing to $\bar\xi^p$. The lift is nonzero. Thus $\iota\alpha/\xi$ and $(\alpha\bar\alpha)/\xi^{p+1}$ reduce to one.

If $v_p(u-1) = v > 0$, the binomial theorem gives $v_p(u^p - 1) \geq \min(v+1, pv)$, with valuation $+\infty$ at zero. While finite, the successive valuations increase by at least $\min(1, (p-1)v) > 0$. Thus $u^{q'^j} \to 1$. Since $\xi^{q'} = \xi$, the norm pairs of $\alpha^{q'^j}$ tend to (ξ, ξ^{p+1}), as claimed. □

Theorem 36.2. *Use the notation of Lemma 36.1 and set $h = e/2$, so $h = \omega_p$. There are $\varpi, \pi \in O$ and a formal coordinate $T \in O[[\Phi]]$ with*

$$v_p(\varpi) = \frac{h}{Q} = \rho_p, \qquad \pi = \varpi^Q, \qquad O[[T]] = O[[\Phi]].$$

Write Z_T for the inverse of T under substitution. Then

$$T(Z) = Z + O(Z^{q'}), \quad Z_T(X) = X + \sum_{s\geq 1} c_s X^{1+Qs}, \quad Z_T' \in O[[X]], \qquad c_1 = -\pi^{-1}.$$

The series $S(X) = T(\varpi X)/\varpi$ belongs to $XO[[X]]$ and

$$S - S^{q'} \equiv X \pmod{\pi O[[X]]}, \qquad \Phi(\varpi X) \in XO[[X]].$$

Proof. Proposition 35.2 and Lemma 36.1 give $\Phi(\zeta^i Z) \in O[[\Phi]]$. Since $p \nmid Q$,

$$T(Z) = Q^{-1} \sum_{i=0}^{Q-1} \zeta^{-i} \Phi(\zeta^i Z)$$

belongs to this ring. The sum of the Q powers of a nontrivial Q-th root is zero by multiplication by the root minus one. Thus T selects exactly the terms of Φ of degrees $1 \bmod Q$. It starts with Z. Write $T = B(\Phi)$ with $B \in XO[[X]]$. The coefficient of X in B is one, so the recursive inversion of Proposition 33.3 gives $B^{-1} \in XO[[X]]$ and the asserted equality of rings. The equality $T(\zeta Z) = \zeta T(Z)$ gives $Z_T(\zeta X) = \zeta Z_T(X)$, yielding the stated support. Closure under differentiation in Proposition 33.3 gives $T' \in O[[T]]$ with constant term one. Differentiating $T(Z_T(X)) = X$ therefore gives $Z_T' = 1/T'(Z_T) \in O[[X]]$. In particular $v_p(c_s) \geq -v_p(1 + Qs)$.

Choose $\alpha = p$ at an inert prime and $\alpha = \sqrt{-d}$ at $p = d$. Then $\alpha\bar{\alpha} = q'$ is odd and $A = \iota\alpha$ has valuation h. By Proposition 35.1, $T(AZ_T(X)) = B(f_\alpha(B^{-1}(X)))$ is integral and its reduction has order at most q'. Indeed a substitution with unit linear coefficient preserves the first nonzero degree, on both sides of the composition. This series has only degrees $1 \bmod Q$ by the covariance under ζ. Its linear term reduces to zero. Thus its coefficient of degree q' is a unit. The first two terms of T are $Z - c_1 Z^{q'}$ through that degree by inversion. It follows that $c_1(A - A^{q'})$ is a unit, so $v_p(c_1) = -h$. Take a Q-th root ϖ of $-1/c_1$ by Lemma 16.2. The geometric series in Definition 28.3 gives $\rho_p = h/Q$. Its positive-exponent sum is $p/(p^2 - 1)$ at inert primes and $1/(p - 1)$ at the ramified prime.

For $s \geq 2$ we have $v_p(1 + Qs) \leq (s - 1)/2$. Indeed it vanishes unless $s \equiv 1 \bmod p$. In the latter case $s \geq p + 1$ and $1 + Qs \leq sp^2 \leq p^{(s-1)/2}$. For the second inequality, at $s = p + 1$ use $p + 1 \leq p^2$ and $p/2 \geq 4$. Increasing s multiplies its right side by $\sqrt{p}$ and its left side by $1 + 1/s \leq 2$. Hence for $s \geq 2$

$$v_p(c_s \varpi^{Qs}) \geq sh - (s - 1)/2 \geq h.$$

Consequently $Z_T(\varpi X)/\varpi$ is integral and is congruent to $X - X^{q'}$ modulo π. Its linear coefficient is one. Its integral compositional inverse is exactly S. Composing the congruence with S gives the first assertion about S. Finally $\Phi(\varpi X) = B^{-1}(\varpi S(X))$, which is integral. □

37. The logarithm of the normalized theta function

Lemma 37.1. *Retain $\zeta, Q, T, \mathcal{L}$ from Lemma 36.1, Theorem 36.2 and Proposition 35.2. Write P_j for the projection of a formal series onto degrees congruent to $j \bmod Q$, and $\gamma = p + 1$. Then $\exp(-\mathcal{L} + P_\gamma \mathcal{L})$ belongs to $O[[T]]^\times$ and has constant term one.*

Proof. Set $w = \zeta^\gamma$. In both cases $w^{p-1} = 1$, since $Q \mid p^2 - 1$. For $0 \leq i < Q$,

$$D_i = \exp(w^i \mathcal{L}(Z) - \mathcal{L}(\zeta^i Z))$$

belongs to $O[[T]]^\times$ by Proposition 35.2, Lemma 36.1 and the equality of formal rings in Theorem 36.2. It has constant term one.

Each $b_i = -w^{-i}/Q$ lies in the closure of the ordinary integers in K_p. Indeed the residue of w^{-i} is a root of $X^{p-1}-1$ in $\mathbf{F}_p$, since that polynomial already has all its roots among the nonzero elements of $\mathbf{F}_p$. Choose a positive integer a lifting that root. The binomial estimate in Lemma 36.1 gives $a^{p^j} \to w^{-i}$. Also, for each j Bezout's identity in Lemma A.1 gives an integer inverse to Q mod p^j. Its difference from $1/Q$ has valuation at least j, proving the claim. If b is a limit of integers, every $\binom{b}{k} = b(b-1)\cdots(b-k+1)/k!$ belongs to O, since this is a continuous polynomial for fixed k and its values at integers are integers. For a negative integer this follows from $\binom{-m}{k} = (-1)^k\binom{m+k-1}{k}$ for $m \geq 1$. Thus $D_i^{b_i} = \sum_{k\geq 0}\binom{b_i}{k}(D_i-1)^k$ belongs to $O[[T]]^\times$. This binomial series equals $\exp(b_i \log D_i)$. In an indeterminate u the binomial series and $\exp(b_i \log(1+u))$ both satisfy $(1+u)F' = b_i F,\ F(0)=1$; coefficient recursion gives uniqueness, and substitution of $D_i - 1$ is valid. Formal logarithm and exponential are inverse on series with the indicated constant terms, by differentiation and those constant terms. The geometric sum proof in Theorem 36.2 gives $P_\gamma \mathcal{L} = Q^{-1}\sum_i w^{-i}\mathcal{L}(\zeta^i Z)$. Adding the exponents in the product $\prod_i D_i^{b_i}$ now gives $-\mathcal{L} + P_\gamma\mathcal{L}$. □

Lemma 37.2. *Use $q', Q, h, \pi, \varpi, Z_T$ of Theorem 36.2. Put $\kappa_0 = \varpi^{p+1}/p$. Let $m, n \in K_p[[X]]$ satisfy $m \in X^3K_p[[X]]$, $n \in X^2K_p[[X]]$, $n' \in O[[X]]$ and $m' = nZ_T'$. Then for some $b_1, b_2 \in O$,*

$$\kappa_0^{-1}(P_{p+1}m)(\varpi X) \equiv b_1X^{p+1} + b_2X^{p+q'} \pmod{\pi O[[X]]}.$$

In particular the series on the left is integral. Here b_1, b_2 may be taken to be its actual coefficients at the displayed degrees.

Proof. Write $n_i = [X^i]n$ and $w_J = [X^{QJ}]Z_T'$. All w_J are integral, $w_0 = 1$ and $w_1 = q'c_1$ has valuation $e-h$ by Theorem 36.2. For $i \geq 1$ the coefficient in_i of n' is integral. The only contributing degrees $k \geq 3$ are $k = p+1+QI$ for integers $I \geq 0$. In the ramified case degree two also lies in the congruence class and has zero coefficient by hypothesis. At a contributing degree

$$k[X^k]m = \sum_{J=0}^{I} n_{p+Q(I-J)}w_J.$$

Indeed the derivative of Z_T has support in the multiples of Q. For $J > I$ the index of n is negative or else is one at $J = I+1$ in the ramified case, where $n_1 = 0$. Thus each term, after division by k and scaling, has valuation at least

$$1 + hI - v_p(k) - v_p(p+Q(I-J)) + v_p(w_J).$$

For $I = 0$ this is zero. For $I = 1$ one has $v_p(k) = 1$ because $k = p+q'$ and $p \geq 13$. For $J = 0$ the other integer is prime to p, giving lower bound h. For $J = 1$ it is p, and $v_p(w_1) = e-h$ gives lower bound $e-1 \geq 0$. This proves integrality at the two displayed degrees.

For $2 \leq I < p$, $k \equiv 1-I$ mod p is a unit. The other integer is prime to p except when $J = I$, in which case it is p. The lower bound is therefore at least $hI \geq h$. For $I \geq p$ we have $k^2 \leq p^{(I+1)/2}$. At $I = p$ one has $k \leq p^3+1 < 2p^3$ and $4p^6 \leq p^{(p+1)/2}$. Increasing I multiplies k^2 by at most $(1+1/I)^2 < \sqrt{p}$, proving the bound by induction. The two positive integers $k, p+Q(I-J)$ have product at most k^2, so the sum of their valuations is at most $(I+1)/2$. The displayed lower bound is at least $1+hI-(I+1)/2 \geq h$, since $h \geq 1/2$. This proves divisibility by π at all other degrees. □

Theorem 37.3. *Let p, ϖ, π, S, q' be as in Theorem 36.2 and F_0 the transported Taylor series of the normalized theta function of Proposition 31.4. There are $\psi \in XO[[X]]$, $V \in O[[X]]^\times$, $u_0, u_1, u_2 \in O$ and $\kappa \in K_p^\times$ such that*

$$v_p(\kappa) = 1/(p-1) - \omega_p, \qquad F_0(\varpi X) = \exp(\kappa\psi(X))V(X),$$
$$\psi(X) \equiv u_0X^2 + u_1S(X)^{p+1} + u_2S(X)^{p+q'} \pmod{\pi O[[X]]}.$$

One can take $u_0 = 0$ when p is inert. At every nonsplit p, the theta products of Proposition 31.4 with any common torsion denominator prime to p satisfy the integrality conclusion of Theorem 30.2 for every B.

Proof. Take $\kappa = -\kappa_0$ as in Lemma 37.2. Then $v_p(\kappa) = (p+1)h/Q - 1 = 1/(p-1) - h$ in both cases. Write $d_2 = [Z^2]\mathcal{L}$ and $\mathcal{L}_1 = \mathcal{L} - d_2 Z^2$. By Proposition 35.2, $\mathcal{L}_1'' = \Xi$; it starts in degree four because Ξ starts in degree two and the constant and linear terms of $\mathcal{L}$ vanish. Set $m = \mathcal{L}_1(Z_T)$ and $n = \mathcal{L}_1'(Z_T)$. The chain rule gives $m' = nZ_T'$ and $n' = \Xi(Z_T)Z_T'$. Proposition 33.3 and Theorem 36.2 make the latter integral. The orders of m, n are at least four and three. Thus $M(X) = \kappa_0^{-1}(P_{p+1}m)(\varpi X)$ has the form in Lemma 37.2.

The relation $Z_T(\zeta X) = \zeta Z_T(X)$ gives $P_{p+1}m = (P_{p+1}\mathcal{L}_1)(Z_T)$ by the root-of-unity average in Lemma 37.1. Since $Z_T(\varpi S(X)) = \varpi X$, it follows that for $\psi(X) = \kappa_0^{-1}(P_{p+1}\mathcal{L})(\varpi X)$ we have

$$\psi(X) = u_0 X^2 + M(S(X)).$$

Indeed, at inert p, $2 \not\equiv p+1 \bmod (p^2-1)$, giving $u_0 = 0$. At $p = d$, degree two is retained and gives $u_0 = d_2\varpi^2/\kappa_0$. Proposition 35.2 bounds $v_p(d_2) \geq -1/2$, since 2 is a unit and $v_p(\iota\sqrt{-d}) = 1/2$. Here $v_p(\varpi^2/\kappa_0) = 1-(p-1)h/Q = 1/2$, so $u_0 \in O$. Take $u_1 = b_1$, $u_2 = b_2$ from Lemma 37.2. Composition with $S \in XO[[X]]$ preserves the congruence. Lemma 37.1, evaluated at ϖX, gives an integral unit V, since $T(\varpi X) = \varpi S(X)$. The definition of ψ now gives the asserted factorization of $F_0 = \exp(-\mathcal{L})$. Use Theorem 36.2 for $\Phi(\varpi X)$ and apply Theorem 34.4 with $\theta = \pi$ to obtain the conclusion for all the translates. □

38. The first exceptional coefficient at an inert prime

Proposition 38.1. *If p is inert, the coefficient u_1 constructed in Theorem 37.3 is a unit. More precisely*

$$u_1 = \frac{[Z^{p-1}]\Xi(Z)}{p+1} \in O^\times.$$

Proof. Here $q' = p^2$, so Theorem 36.2 gives $T(Z) = Z + O(Z^{p^2})$. Thus for any $f \in O[[T]]$ all coefficients through degree p in Z are integral. In particular $[Z^{p-1}]f' = p[Z^p]f \in pO$. For such f let $\lambda(f)$ denote the residue of $[Z^{p-1}]f$ in the field $O/\{v_p > 0\}$. The ring $O[[T]]$ is closed under differentiation by Proposition 33.3 and Theorem 36.2, and $\lambda(f') = 0$.

Propositions 31.3 and 33.3 give

$$(\Xi')^2 = 4\Xi^3 + A_1^2\Xi^2 + 4\Xi, \qquad \Xi'' = 6\Xi^2 + A_1^2\Xi + 2.$$

The second identity follows from the first by differentiation and cancellation of the nonzero series $2\Xi'$. Thus for $j \geq 0$

$$(\Xi^j\Xi')' = (4j+6)\Xi^{j+2} + (j+1)A_1^2\Xi^{j+1} + (4j+2)\Xi^j.$$

The term with the derivative of Ξ^0 is zero for $j = 0$. Apply λ and set $r = (p-1)/2$. The integer $4j+6$ is a unit for $0 \leq j \leq r-2$, since it is even and strictly between zero and $2p$, with p odd. Now $\lambda(1) = 0$. If $\lambda(\Xi) = 0$, the recurrence gives $\lambda(\Xi^j) = 0$ for every $j \leq r$. Yet $\Xi(Z) = Z^2 + O(Z^3)$ by Proposition 33.3 and $\Phi'(0) = 1$; hence $\lambda(\Xi^r) = 1$. This contradiction proves $[Z^{p-1}]\Xi \in O^\times$.

In Lemma 37.2 the coefficient at $p+1$ is $p[X^{p+1}]m$. The inverse Z_T agrees with X through degree $p+1$, so this is $p[Z^{p+1}]\mathcal{L}_1$ in Theorem 37.3. Taking the coefficient of degree $p-1$ in $\mathcal{L}_1'' = \Xi$ gives the stated formula. □

Corollary 38.2. *Use the tight frames of Proposition 29.2. For the actual theta products of Proposition 31.4 with one common torsion denominator n_0 prime to p, the coefficients of the differential minors in Theorem 28.5 have valuation at least $J_p(j,V) - m_{T'}$ for every set V and every charge system obeying its displayed condition. At an inert prime with $p \nmid M'$ and $\beta \in O^\times$, the derivative formula of Corollary 30.3 holds for the common exponent of the*

translates in the normalization of Theorem 34.4, with β/n_0 in place of β. Each indicated monomial has a unit coefficient.

Proof. Theorems 37.3 and 30.2 establish the linear substitution hypothesis of Theorem 28.5 for these products, after the simultaneous change of argument in Theorem 34.4. Thus that theorem gives the stated lower bound for each coefficient of a differential minor. For the last assertion the argument scale β/n_0 is a unit. The scalar n_0^2 is absorbed into κ there. Proposition 38.1 supplies the unit u_1, so Corollary 30.3 applies. □

39. Integral derivatives at every finite place

Proposition 39.1. *Fix any prime p, any coefficient embedding $\iota : \mathbb{C} \to K_p$, and $0 \neq \delta \in O_K$. Put $A = \iota(\bar{\delta})$, $N = \delta\bar{\delta}$. For every $y \in \mathbb{C}$ with $\delta y \in \Lambda$, one has*

$$F_y(AZ) = F_0(Z)^N V_y(\Phi(Z)), \qquad V_y \in O[[X]].$$

Proof. Use the multiplier $\chi(l)$ of Proposition 35.1. The quotient $(U_y\theta)(\bar{\delta}z)/\theta(z)^N$ is periodic. In fact for a lattice period l, the two quadratic exponential multipliers match because $|\delta|^2 = N$. The signs match by $\chi(\bar{\delta}l) = \chi(l)^N$. The remaining phase of the translate is $\exp(2\pi i\Im(\bar{l}\delta y)/\mathrm{area}(\Lambda)) = 1$. Indeed for two lattice vectors $\lambda(m + n\omega), \lambda(m' + n'\omega)$ the quotient of the imaginary part of their hermitian product by the area is the integer $mn' - nm'$. The factor U_y and the signs are those of Proposition 31.4. Since $\delta y \in \Lambda$ implies $Ny \in \Lambda$, Propositions 34.2 and 35.1 give

$$|\iota((U_y\theta)(\bar{\delta}z))|_p \leq H_p(\bar{\delta}z) \leq H_p(z)^N.$$

The numerator is entire. Proposition 34.3 therefore gives the desired equality after taking Taylor series. □

Theorem 39.2. *Use $c_2 = \mathcal{L}''(0)$ of Proposition 35.2 and choose $0 \neq t \in O$ with $t^2c_2 \in O$. Fix a finite set of shifts y_i with $\delta y_i \in \Lambda$ as in Proposition 39.1. For finitely many variables $Z_1, \ldots, Z_s$, take linear forms $L_i = \sum_j c_{ij}Z_j$ with $c_{ij}/(At) \in O$. Then every mixed partial derivative at zero of*

$$\mathcal{T}(Z) = \prod_i F_{y_i}(L_i)$$

belongs to O, including the derivative of order zero. The condition $t^2c_2 \in O$ is necessary for this conclusion uniformly in the indicated products and forms, for each fixed $t \in O \setminus \{0\}$, when $\delta = 1$. It always suffices to take $v_p(t) = v_p(d)/4$. In particular every coefficient system satisfying $c_{ij}/(Ad) \in O$ obeys the assertion.

Proof. Put $\mathcal{A} = O[[\Phi]][t\mathcal{L}', F_0]$ in $K_p[[Z]]$. All its elements have integral constant term, by Propositions 33.3 and 35.2. The derivation $t\,d/dZ$ preserves this algebra. On its base ring this follows from Proposition 33.3 and $t \in O$. On the two generators the derivatives are

$$t^2(\Xi + c_2) \in O[[\Phi]], \qquad -(t\mathcal{L}')F_0 \in \mathcal{A}.$$

The ring rules prove preservation on the whole algebra. Thus $(t\,d/dZ)^k H(0) \in O$ for every $H \in \mathcal{A}$, $k \geq 0$, where evaluation is after differentiation.

Write $L_i/A = t\sum_j u_{ij}Z_j = tW_i$ with $u_{ij} \in O$. By Proposition 39.1 each $F_{y_i}(L_i)$ is $H_i(tW_i)$ with $H_i \in \mathcal{A}$. Formal differentiation with respect to Z_j changes this to $u_{ij}((t\,d/dZ)H_i)(tW_i)$. The product rule therefore keeps every mixed derivative a finite O-linear combination of products of the same kind. Evaluating at zero proves the assertion.

For necessity take one zero shift, one variable and $L = tZ$. The second derivative of $F_0(tZ)$ at zero is $-t^2c_2$ by $F_0 = \exp(-\mathcal{L})$ and $\mathcal{L}'(0) = 0$. Finally Proposition 35.2 gives $v_p(c_2) \geq -v_p(d)/2$. A fourth root of d exists by Lemma 16.2 and has valuation $v_p(d)/4$. With that choice, $t^2c_2 \in O$ and $d/t \in O$, proving the last two assertions. □

40. A mean criterion for the prime remainder

Lemma 40.1. *Fix d, M as in Definition 8.1 and $0 < \eta < 1$. There is a constant $A > 0$, depending only on d, M, η, with the following property. For $t \geq 1$ put*

$$k = \lceil At \rceil, \quad \mu = (k+1)/3, \quad J_t = [(1-\eta)\mu, (1+\eta)\mu], \quad w_t(v) = \frac{v^k \mathrm{e}^{-3v}}{\int_{J_t} u^k \mathrm{e}^{-3u}\,\mathrm{d}u} \quad (v \in J_t).$$

Eventually $J_t \subset [t, t^2]$. Uniformly for $0 \leq X \leq \mathrm{e}^t$,

$$\left| \int_{J_t} (R_{d,M}(\mathrm{e}^v) - X R_{d,1}(\mathrm{e}^v)) w_t(v)\,\mathrm{d}v \right| = O_{d,M,\eta}(\mathrm{e}^{-2t}).$$

Proof. Put $F(v) = R_{d,M}(\mathrm{e}^v)$, $G(v) = R_{d,1}(\mathrm{e}^v)$. By Proposition 8.2 and Lemma 6.1 there are common $R > 3, K < \infty$ bounding their moments as in that lemma. Let Y have the full probability density $3^{k+1} v^k \mathrm{e}^{-3v}/k!$ on $v \geq 0$. Its normalization and moment generating integral $\mathbb{E}\mathrm{e}^{sY} = (1 - s/3)^{-(k+1)}$ for $s < 3$ were proved in Theorem 6.2. It follows that

$$|\mathbb{E}(F(Y) - XG(Y))| \leq 3K(1 + \mathrm{e}^t)(3/R)^k.$$

Set $h = \min\{\eta - \log(1+\eta), -\eta - \log(1-\eta)\} > 0$. Positivity follows by integration of $1/u$ on the intervals from 1 to $1+\eta$ and from $1-\eta$ to 1. On the upper tail, integrate the inequality $\mathbf{1}_{Y \geq (1+\eta)\mu} \leq \mathrm{e}^{s(Y - (1+\eta)\mu)}$ with $s = 3\eta/(1+\eta)$. The moment formula gives $\mathbb{P}(Y \geq (1+\eta)\mu) \leq ((1+\eta)\mathrm{e}^{-\eta})^{k+1}$. On the lower tail take $s = -3\eta/(1-\eta)$ in the same exponential bound with threshold $(1-\eta)\mu$ and reversed inequality on Y. This gives $\mathbb{P}(Y \leq (1-\eta)\mu) \leq ((1-\eta)\mathrm{e}^{\eta})^{k+1}$. Thus $\mathbb{P}(Y \notin J_t) \leq 2\mathrm{e}^{-h(k+1)}$. Integrating v^j against the density gives $\mathbb{E}Y^j = (k+j)!/(3^j k!)$ by the same elementary integral, so $\mathbb{E}(1+Y)^4 = O((k+4)^4)$. For real square-integrable functions the squared integral of their product is bounded by the product of their squared norms: expand the nonnegative integral of $(af - g)^2$ as a quadratic in a. Using this inequality with the indicator of the complement, and using the quadratic bound of Proposition 8.2, yields

$$\mathbb{E}[|F(Y) - XG(Y)|\mathbf{1}_{Y \notin J_t}] \ll_{d,M} (1 + \mathrm{e}^t)(k+4)^2 \mathrm{e}^{-h(k+1)/2}.$$

Choose A so that $A\log(R/3) > 4$, $Ah/2 > 4$ and $(1-\eta)A/3 > 1$. Both bounds for the integrals are $O_{d,M,\eta}(\mathrm{e}^{-2t})$, since every fixed power of t is $o(\mathrm{e}^t)$ by its exponential series. Conditioning on J_t divides their difference by $\mathbb{P}(Y \in J_t) \geq 1/2$ eventually and gives the stated density. The endpoints of J_t now give both inclusions in $[t, t^2]$ for all sufficiently large t. □

Theorem 40.2. *Use the data of Theorem 10.8. Fix $0 < \eta < 1$, set $t = 10^6 \log x$, and choose J_t, w_t by Lemma 40.1. Write $\mathcal{R}_x(v)$ for $\mathcal{R}$ at $x_1 = \mathrm{e}^v$, keeping m, a, S, U fixed. Then as $x \to \infty$*

$$R_{d,M}(x) = \frac{1}{maS} \int_{J_t} m\mathcal{R}_x(v) w_t(v)\,\mathrm{d}v + O_{d,M,\eta}(x^{-2\cdot 10^6}).$$

In particular the averaged inequality

$$\int_{J_t} m\mathcal{R}_x(v) w_t(v)\,\mathrm{d}v \leq Cfa\ell^4$$

for fixed $C = C(d, M, \epsilon, \eta)$ and all sufficiently large x gives $R_{d,M}(x) \leq C\varphi(M)x^{-1/2+5\epsilon} + O_{d,M,\eta}(x^{-2\cdot 10^6})$. If this inequality for the average holds for each M, d of Definition 10.1 and for arbitrarily small fixed $\epsilon > 0$ in the range of Proposition 10.5, then $L(s, \chi) \neq 0$ for every Dirichlet character and $\operatorname{Re} s > 1/2$ at every regular point. The principal character modulo q has a simple pole at $s = 1$ of residue $\varphi(q)/q$. Under the same hypothesis Conjecture 1 follows.

Proof. For every $v \in J_t$, Proposition 10.7 gives

$$R_{d,M}(x) = \frac{\mathcal{R}_x(v)}{aS} + R_{d,M}(\mathrm{e}^v) - (U/S)R_{d,1}(\mathrm{e}^v).$$

All functions in this formula are continuous in v by Proposition 8.2 and the identity itself. The range of x_1 is admissible by Lemma 40.1 and Theorem 10.8. The ratio $U/S \le \ell = x^{\epsilon} \le \mathrm{e}^t$ by Proposition 10.5. Integrate with w_t and use Lemma 40.1; this proves the first formula. Since $m \ge \sqrt{x}$ and $f = \varphi(M)U$ by Propositions 10.5 and 10.3, the averaged inequality gives the displayed upper bound. Given $1/2 < \beta < 1$, choose one of the available ϵ with $5\epsilon < \beta - 1/2$. Both terms in the bound are then $o(x^{\beta-1})$, with d, M fixed. Corollary 9.8 and Proposition 3.3 prove the remaining assertions. □

41. The size of the main height and the individual volume ratios

Proposition 41.1. *Use $t = 10^6 \log x$, $t \le v \le t^2$, and the data of Definition 12.3. Uniformly in this range,*

$$H \ge 16mk\,\ell S\,a/x > 0, \qquad \frac{m|\mathcal{R}_x(v)|}{H} \ll_{d,M} \frac{x(1+v)^2}{k} = o(1)$$

for all sufficiently large x, with ϵ fixed. In particular $n(H - m\mathcal{R}_x(v) + 6B) \le d_0(H - m\mathcal{R}_x(v) + 6B)$ whenever $0 \le n \le d_0$, where $B = fa\ell^3$.

Proof. In $P(y,\tau)$ of Definition 12.3, with $0 < y \le a$, a positive summand requires $n < yx_1/a \le x_1 = \mathrm{e}^v$, since the least possible weight is a/x_1. By Definition 8.1, $\lambda_{d,1}(n) \le 2\log n$ for $n \ge 1$. Integral comparison of the harmonic sum gives $0 \le P(y,\tau) \le 2yv(1+v)$. The formula for J in Proposition 12.4 and the bound $G_0 \ge 19\ell$ in Proposition 12.2 give

$$J(w_0(\tau)) \ge (G_0 + 1 + \kappa_{d,1})w_0(\tau) \ge 18\ell w(\tau)$$

eventually. Here $w_0(\tau) \ge w(\tau)$, both are positive, and $\log(w_0(\tau)!) \ge 0$. Since $v \le t^2$, we have $v(1+v) = o(\ell)$ by Proposition 10.5. Consequently $D_0 \ge 16\ell \sum_\tau w(\tau) \ge 16\ell S a/x$, using the S places of weight a/x alone. Multiplying by mk gives the first bound.

By Definition 10.6, $\log a - \bar{w}$ is a weighted mean of $\log x$ and v and lies between zero and v. Thus

$$|\mathcal{R}_x(v)| \le Ua\{2v(1+v) + v + |\kappa_{d,1}|\}.$$

Divide by the lower bound for H/m and use $U/S \le \ell$ from Proposition 10.5. Finally $k = r + 1 - m \ge x^5/2$ eventually by that proposition. It follows that $x(1+v)^2/k \to 0$ uniformly. Thus $H - m\mathcal{R}_x(v) > 0$ eventually, giving the last inequality. □

Proposition 41.2. *Let U, T be finite-dimensional complex Hilbert spaces with $D = \dim U > 0$, and $A : T \to U$ a surjective linear map. For a parameter t' in a set with a distinguished member 0, let $u_{t'} : U \to V_{t'}$, $G_{t'} : T \to V_{t'}$ be linear maps of finite-dimensional complex Hilbert spaces. Suppose $u_{t'}^* G_{t'} = \gamma_{t'} A$ with $\gamma_{t'} \ne 0$. Write $L_{t'} = u_{t'}(U)$ and $P_{t'}$ for the orthogonal projection onto $L_{t'}$. For a fixed positive real h, let*

$$\lambda(t') = Z(u_{t'}), \qquad \Delta(t') = \det\left(hP_{t'}G_{t'}G_{t'}^*|_{L_{t'}}\right).$$

Both numbers are positive, and

$$\frac{\lambda(0)}{\lambda(t')} = \frac{\Delta(t')}{\Delta(0)}\left|\frac{\gamma_0}{\gamma_{t'}}\right|^{2D}.$$

Suppose that parameters t_ι have been selected for all $\iota = (p,j)$ of Definition 16.3. If real a_∞, a_p, C_0, C_1 satisfy

$$|\sigma\Delta(0)| \le \mathrm{e}^{Da_\infty} \quad (\sigma \in \mathrm{Aut}(\mathbb{C})), \qquad |j\Delta(t_\iota)|_p \le \mathrm{e}^{Da_p},$$
$$a_\infty + \sum_{p<M'} a_p \le C_0 \quad \textit{eventually as } M' \to \infty,$$
$$\mathrm{ht}\left(\left|\frac{\gamma_0}{\gamma_{t_\iota}}\right|^2\right) \le C_1,$$

then $\mathrm{ht}(\lambda(0)/\lambda(t_\iota)) \le D(C_0 + C_1)$.

Proof. Surjectivity of A implies injectivity of A^*: if $A^*z = 0$ then z is orthogonal to $A(T) = U$. Taking adjoints gives $G_{t'}^* u_{t'} = \bar{\gamma}_{t'} A^*$, an injection. Hence $u_{t'}$ is injective, and $G_{t'}^*$ is injective on $L_{t'}$. For $z \in L_{t'}$ the quadratic form of the compressed operator at z is $h\|G_{t'}^* z\|^2$. It is positive definite, so its determinant is positive by the Gram determinant formula of Lemma 5.1. Lemma 17.5 gives

$$\lambda(t')\Delta(t') = h^D |\gamma_{t'}|^{2D} \det(AA^*).$$

Indeed $\det(AA^*) = Z(A^*) > 0$ by Definition 17.2. Divide the identity for 0 by the identity for t'. For the height of $\Delta(t_\iota)/\Delta(0)$, take $\Delta(0)$ as the nonzero witness of Definition 16.3. The stated archimedean and finite bounds give cost at most DC_0. Multiply by the D-th power of the scalar ratio using Proposition 16.4. □

42. The comparison before enlarging its error terms

Theorem 42.1. *Fix x, v in the range of Proposition 41.1 and use the parameters of Definition 12.3. Put $B = fa\ell^3$, $b_0 = B/(mk)$, and $L_{\mathrm{loc}} = \sum_p e_p \log p$ as in Proposition 25.3. Let W have dimension $d_0 > 0$ and let $U_{t'} : W \to H_{t'}$ be the maps of Corollary 17.4, all injective for $t' = 0, t_\iota$. Suppose the completions and recombinations in that corollary exist, with parameters satisfying*

$$n \ge (1 - 4/b^2)d_0, \qquad \beta + g \le b^3/4, \qquad \delta \le 3d_0 \log(1 + b^2) L_{\mathrm{loc}},$$
$$C_\sigma\big((U_0 x_i)_{i \le n}\big) \le \exp\{n(H - m\mathcal{R}_x(v) + 6B)\} \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

For $1 \le j \le m$ suppose the comparison maps of Proposition 41.2 exist with $D = d_0$; denote their functions by $\lambda_j, \Delta_j, \gamma_j$. Assume the determinant bounds of that proposition with $a_\infty \le 2rb_0$ and $\sum_{p<M'} a_p \le rb_0$ eventually for each j. Require

$$\mathrm{ht}\left(\left|\frac{\gamma_j(0)}{\gamma_j(t_\iota)}\right|^2\right) \le 10mx^{999} + k(6b_0 - D_0).$$

Finally suppose $Z(U_{t'}) = |c(t')|^{d_0} \prod_{j=1}^m \lambda_j(t')$ with $c(t') \ne 0$ and $\mathrm{ht}(|c(t_\iota)/c(0)|^{-2}) \le 2m^2 t_1 x^{999}$. Then, for all sufficiently large x,

$$m\mathcal{R}_x(v) \le (12 + 3r/k)B + (10 + t_1)m^2 x^{999} + b + 6\log(1 + b^2)L_{\mathrm{loc}}$$
$$\ll_{d,M,\epsilon} fa\ell^3(1 + v)^2.$$

If these hypotheses hold on the intervals J_t of Theorem 40.2 for fixed d, M, ϵ, η and every sufficiently large x, then

$$R_{d,M}(x) \le C_{d,M,\epsilon,\eta} x^{-1/2+4\epsilon}(1 + \log x)^2$$

for some positive constant $C_{d,M,\epsilon,\eta}$. If they hold for all M, d of Definition 10.1 and arbitrarily small fixed $\epsilon > 0$ as in Proposition 10.5, the nonvanishing and pole assertions of Theorem 40.2 and Conjecture 1 follow.

Proof. For each stage Proposition 41.2 gives

$$\mathrm{ht}(\lambda_j(0)/\lambda_j(t_\iota)) \le d_0\{3rb_0 + 10mx^{999} + k(6b_0 - D_0)\}.$$

By Proposition 16.4 their product has height at most

$$d_0\{(6 + 3r/k)B + 10m^2 x^{999} - H\},$$

using $mkb_0 = B$, $mkD_0 = H$. Set $Z_\iota = Z(U_{t_\iota})/Z(U_0)$. Squaring the product of ratios and using the scalar height and the factorization gives

$$\mathrm{ht}(Z_\iota^{-2}) \le 2d_0\{(6 + 3r/k)B + (10 + t_1)m^2 x^{999} - H\}.$$

For the opposite direction apply Corollary 17.4. The cost for the missing $d_0 - n$ vectors is at most $d_0 b$. By Proposition 41.1, the exponent for the given n vectors is at most $d_0(H - m\mathcal{R}_x(v) + 6B)$. Thus

$$\mathrm{ht}(Z_\iota) \le d_0(H - m\mathcal{R}_x(v) + 6B + b) + 2\delta.$$

Multiply the square of this family by its inverse square. Proposition 16.4 forces the sum of the corresponding height bounds to be nonnegative. After division by $2d_0$, the terms H cancel and the claimed explicit inequality follows.

By Proposition 12.6 and its proof,

$$L_{\mathrm{loc}} \le B\{15v(1+v)/\ell + 480/\ell^2\}.$$

The factor $\log(1+b^2) \le 3\ell$ therefore contributes $O(B(1+v)^2)$. Also $r/k \to 1$, $t_1 = 5r$ and $m \ll \sqrt{x}$, $r \ll x^5$ by Proposition 10.5. Since $B \ge xb\ell^3$, every fixed power of x is $o(B)$ by the exponential series for $b = \exp(x^\epsilon)$; also $b \le B$. This proves the second bound, uniformly in v. On J_t we have $v = O_{d,M,\eta}(\log x)$ by Lemma 40.1. Integrate the second bound there and use the identity in Theorem 40.2. Division by maS costs at most $f/(mS) \le \varphi(M)\ell/\sqrt{x}$ in the coefficient of $\ell^3(1+\log x)^2$. The error in that identity is $O_{d,M,\eta}(x^{-2\cdot 10^6})$, proving the one-sided bound. For any $1/2 < \beta' < 1$, choose an available ϵ with $4\epsilon < \beta' - 1/2$; the squared logarithm is $o(x^{\beta' - 1/2 - 4\epsilon})$ by the exponential series. Corollary 9.8 now proves the last assertion. □

43. Finite and archimedean bounds for the compressed determinant

Lemma 43.1. *Let L be a D-dimensional complex vector space, $D \ge 1$, and ℓ_a a finite family of linear forms on L whose common kernel is zero. Fix $\iota : \mathbb{C} \to K_p$ as in Definition 16.3. There exist indices $a_1, \dots, a_D$ and a basis $z_1, \dots, z_D$ of L with*

$$\ell_{a_i}(z_j) = \mathbf{1}_{i=j}, \qquad |\iota(\ell_a(z_j))|_p \le 1 \quad (a,\ 1 \le j \le D).$$

Proof. In any basis of L, put the coordinate rows of the ℓ_a in a matrix B. Its columns are independent by the common kernel condition. Elementary elimination, using a nonzero entry as a pivot and clearing its column, produces D independent rows: each step reduces the column rank of the remaining block by one. Thus some D-rowed square minor is nonzero. Among the finitely many ordered row choices choose a minor with maximum p-adic absolute determinant through ι. Its forms are a basis of the dual, so there is a basis z_j with the first asserted equality, obtained by inverting this square matrix. In the expansion $\ell_a = \sum_j \ell_a(z_j)\ell_{a_j}$, replacing row j of the chosen determinant by ℓ_a multiplies its determinant by $\ell_a(z_j)$, by multilinearity and alternation. Maximality yields the second assertion. □

Proposition 43.2. *Let $G : T \to V$ be a linear map of finite-dimensional complex Hilbert spaces, let $L \subset V$ have dimension $D > 0$, and let P be orthogonal projection onto L. Assume $G^*|_L$ is injective. Fix $h > 0$, $\iota : \mathbb{C} \to K_p$ and finitely many f_a spanning T. Apply Lemma 43.1 to $\ell_a(z) = \langle Gf_a, z\rangle$ on L, obtaining indices a_i and basis z_j. Then*

$$\det(hPGG^*|_L) = \det(\langle Gf_{a_i}, P(hGG^* z_j)\rangle)_{i,j=1}^{D}.$$

In particular suppose $\mathcal{I}, \mathcal{J} \subset V$ satisfy $Gf_a \in \mathcal{I}$ for every a, and $hGG^* z \in \mathcal{J}$ whenever $z \in L$ and $|\iota\ell_a(z)|_p \le 1$ for every a. If

$$\left|\iota \det(\langle y_i, Py'_j\rangle)_{i,j=1}^{D}\right|_p \le \mathrm{e}^{Db_p} \qquad (y_i \in \mathcal{I},\ y'_j \in \mathcal{J}),$$

then $|\iota \det(hPGG^*|_L)|_p \le \mathrm{e}^{Db_p}$. It suffices that the displayed bound hold for the indicated single pair of families attached to a maximizing minor.

Proof. If all $\ell_a(z)$ vanish then $\langle f_a, G^* z\rangle = 0$ for every a, so $G^* z = 0$ by spanning, and $z = 0$. Thus the lemma applies. For an endomorphism M of L, the i,j entry of its matrix in the basis z_j is $\ell_{a_i}(Mz_j)$, by duality. Apply this to $M = hPGG^*|_L$. By the bounds in

Lemma 43.1, every hGG^*z_j belongs to $\mathcal{J}$. The identity of determinants proves both versions of the inequality. □

Proposition 43.3. *In Proposition 43.2 realize $T = \mathbb{C}^I$ with norm given by a positive definite rational Gram matrix, and V inside a complex coordinate space whose positive definite rational Gram matrix induces its norm. Use the same matrix for the norm of the coordinatewise image σv under any $\sigma \in \mathrm{Aut}(\mathbb{C})$, for T and the ambient space of V respectively. Suppose $h \in \mathbb{Q}_{>0}$, and for all σ, $x \in T$, $z \in L$ one has*

$$h\|\sigma(Gx)\|^2 \le \mathrm{e}^{b_\infty}\|\sigma x\|^2, \qquad \|\sigma(PGG^*z)\| \le \|\sigma(GG^*z)\|.$$

Then $|\sigma \det(hPGG^*|_L)| \le \mathrm{e}^{Db_\infty}$ for every σ.

Proof. In these coordinates rationality of the two Gram matrices gives

$$\sigma\langle y, z\rangle = \langle \sigma' y, \sigma z\rangle, \qquad \sigma' = \mathfrak{c}\sigma\mathfrak{c},$$

where $\mathfrak{c}$ denotes complex conjugation, as in Lemma 17.1. Each eigenvalue α of $hPGG^*|_L$ has a nonzero eigenvector z. Indeed its characteristic polynomial splits by Lemma 16.2, and vanishing of the determinant of $M - \alpha I$ gives a nontrivial kernel by elimination. Set $a = G^*z$, which is nonzero by injectivity. For a fixed σ choose $x = (\sigma')^{-1}(\sigma a)$ coordinatewise in T. Then

$$\sigma\langle Gx, z\rangle = \sigma\langle x, a\rangle = \|\sigma a\|^2.$$

Multiply the left side by $\sigma\alpha$ and use the eigenvector equation in the second argument of the pairing. Cauchy–Schwarz gives

$$|\sigma\alpha|\,\|\sigma a\|^2 \le h\,\|\sigma'(Gx)\|\,\|\sigma(PGa)\| \le h\,\|\sigma'(Gx)\|\,\|\sigma(Ga)\|.$$

Here the last inequality is the projection hypothesis at this particular z. The pairing inequality follows by subtracting the orthogonal projection onto the line of one vector as in Lemma 5.1. Apply the norm hypothesis with (σ', x) and (σ, a). Since $\sigma' x = \sigma a$, their product bounds the right side by $\mathrm{e}^{b_\infty}\|\sigma a\|^2$. Cancellation gives $|\sigma\alpha| \le \mathrm{e}^{b_\infty}$. The determinant is the product of the D eigenvalues counted with multiplicity by the constant term of the characteristic polynomial. Multiplication yields the assertion. □

Corollary 43.4. *In Theorem 42.1 the determinant estimates are implied by the following tests for each stage. In the notation of Proposition 41.2, use the tests of Proposition 43.3 at $t' = 0$ with $b_\infty = 2rb_0$. For every $\iota = (p, j')$ use the single minor of Proposition 43.2 at $t' = t_\iota$ and embedding j', with $b_p = r\beta_p$, where $\sum_{p<M'} \beta_p \le b_0$ eventually. All other hypotheses of Theorem 42.1 have their stated force.*

Proof. Injectivity of $G^*_{t'}$ on $L_{t'}$ was proved in Proposition 41.2. The two determinant bounds follow directly from Propositions 43.3 and 43.2. Their exponents have precisely the sums required in Theorem 42.1. □

44. The arithmetic lattice of the theta frames

Proposition 44.1. *Put $N_c = N_1$, $\zeta = \zeta_{N_c}$ and $D_c = P'_{N_c}(\zeta)$ as in Proposition 18.4. For any prime l and any embedding $\varphi : E \to K_l$,*

$$v_l(\varphi D_c) = \begin{cases} a_l - 1/(l-1), & a_l = v_l(N_c) > 0, \\ 0, & l \nmid N_c. \end{cases}$$

If $l \mid N_c$, the root $\xi_l = \zeta^{N_c/l}$ satisfies $v_l(\varphi(1-\xi_l)) = 1/(l-1)$. For i, k distinct modulo l the same is true of $\xi_l^i - \xi_l^k$.

Proof. For a positive integer h prime to l, reduction to the residue field is injective on the h-th roots of unity. Indeed a root u reducing to one has $1 + u + \cdots + u^{h-1}$ reducing to $h \neq 0$, forcing $u = 1$. When $l \nmid N_c$ this makes every difference between $\varphi\zeta$ and a distinct conjugate a unit. The product formula for the derivative at a simple root gives $v_l(\varphi D_c) = 0$.

For $a = a_l > 0$ let $h = N_c/l^a$. In characteristic l the only l-th root of unity is one since $X^l - 1 = (X-1)^l$. If ξ is primitive of order l, each $(1-\xi^i)/(1-\xi)$, $1 \le i < l$, therefore reduces to $i \neq 0$. Evaluating $(X^l-1)/(X-1) = \prod_{i=1}^{l-1}(X-\xi^i)$ at $X = 1$ shows that $(l-1)v_l(1-\xi) = 1$. This proves the claims about ξ_l and its powers.

The identity

$$P_{N_c}(X)P_h(X^{l^{a-1}}) = P_h(X^{l^a})$$

holds because a root of the polynomial on the right has order hl^j, $0 \le j \le a$, and the second factor on the left accounts for exactly $j < a$; the roots are simple in characteristic zero. At $X = \varphi\zeta$ the derivative of the right side has valuation a. In fact P_h' evaluated at a primitive h-th root is a unit by the first argument of this proof. The factor $P_h((\varphi\zeta)^{l^{a-1}})$ has valuation $1/(l-1)$. To see this, the argument of P_h is of order hl, so by coprime exponents it is uniquely $u\xi$ with u a primitive h-th root and ξ a primitive l-th root. In the product over primitive h-th roots only the factor $u\xi - u$ reduces to zero. It has the asserted valuation. Differentiating the polynomial identity at $\varphi\zeta$ and subtracting this valuation gives the formula. □

Definition 44.2. Write $n_K = (d+1)/4 = \omega\bar{\omega}$. For $l \mid N_c$, $\varphi : E \to K_l$, define $d(l,\varphi) = v_l(\varphi D_c)$ if $\varphi\omega$ has zero residue, and $d(l,\varphi) = 0$ otherwise. Set it to zero for $l \nmid N_c$. Define the additive subgroup

$$\mathcal{A} = \{x \in E : v_l(\varphi x) \ge -d(l,\varphi) \text{ for every } l, \varphi\}.$$

For a finite-place embedding $\iota : \mathbb{C} \to K_p$, put $\varphi_\tau(x) = \iota(\overline{\tau(x)})$ for $\tau \in G$. All these maps agree on K. Thus $d_\iota = d(p, \varphi_\tau)$ is independent of τ by Proposition 44.1. Choose any $s_\iota \in K_p^\times$ of valuation d_ι. Such a choice is either 1 or $\varphi_1 D_c$.

Proposition 44.3. *One has $O_E \subset \mathcal{A} \subset N_c^{-1}O_E$. For $x, y \in \mathcal{A}$,*

$$\sum_{\tau \in G} \tau(x)\overline{\tau(y)} = \mathrm{Tr}_{E/K}(x\bar{y}) \in O_K.$$

For each ι, τ as in Definition 44.2 and $x \in \mathcal{A}$, $s_\iota\varphi_\tau(x)$ is integral.

Proof. An element of E is in O_E precisely when it is integral at every (l,φ) of Definition 44.2. A monic integer equation forces all valuations of a root to be nonnegative: a negative valuation would make the leading term strictly larger in absolute value than all other terms. Conversely the elementary symmetric functions of the conjugates of an element are the rational coefficients, up to signs, of its monic minimal polynomial. All the conjugates in K_l extend to embeddings of E there by adjoining roots, as in Lemma 16.2. Hence all coefficients are integral at each l. A rational number with that property has integer denominator one in lowest terms by Lemma A.1. The characterization follows. Since $0 \le d(l,\varphi) \le v_l(N_c)$, it implies both inclusions.

Every prime $l \mid N_c$ divides n_K. Indeed $d + 1 = (M/2)(2u+1)$ for an integer u, every odd prime of N_c divides M, and $16 \mid M$. Thus the two reductions of $\omega, \bar{\omega} = 1 - \omega$, in some order, are $0, 1$ at every such embedding by their sum and product. Exactly one of φ and $\varphi \circ \mathfrak{c}$ selects the positive allowance in Definition 44.2 whenever that allowance is positive. Proposition 44.1 makes the valuation of D_c the same at both. It follows that $D_c x\bar{y} \in O_E$ by the characterization just proved. Write $D_c x\bar{y} = F(\zeta)$ with $F \in O_K[X]$ and $\deg F < f$ by Theorem 18.2. The interpolation identity in the proof of Lemma 18.3, applied to the f roots of P_{N_c} over K, gives

$$\mathrm{Tr}_{E/K}(x\bar{y}) = [X^{f-1}]F \in O_K.$$

Those roots give exactly the embeddings in G by Proposition 10.3. Each commutes with conjugation on E, as is seen on ζ, ω; hence the sum is that trace. The last assertion is the defining inequality for $\mathcal{A}$ at φ_τ. □

45. Tight frames of every sufficiently large integral length

Lemma 45.1. *For N_c, n_K, ω in Definition 44.2 there is $t_K \in O_K$ with*

$$0 \le \mathrm{Tr}_{K/\mathbb{Q}}(t_K) < 3N_c, \qquad 0 < \mathrm{N}_{K/\mathbb{Q}}(t_K) \le 4n_K N_c^2,$$
$$\mathrm{Tr}_{K/\mathbb{Q}}(t_K) \equiv 1 \bmod N_c, \qquad \mathrm{N}_{K/\mathbb{Q}}(t_K) \equiv 0 \bmod N_c.$$

For each $l^a \| N_c$ and each $\varphi : E \to K_l$, $\varphi t_K \equiv 1 \bmod l^a O$ on the branch where $\varphi\omega$ has zero residue, and $\varphi t_K \equiv 0 \bmod l^a O$ on the other branch.

Proof. By Proposition 44.3, $l \mid n_K$. The two roots of $F(X) = X^2 - X + n_K \bmod l$ are $0, 1$. Their derivatives $2X - 1$ are units, including at $l = 2$. Given a root $u \bmod l^j$, the lifts $u + l^j b$ have $F(u + l^j b) \equiv F(u) + l^j b F'(u) \bmod l^{j+1}$. Exactly one residue $b \bmod l$ prolongs the root. Thus there are two roots $u_0, u_1 \bmod l^a$ reducing to $0, 1$, interchanged by $u \mapsto 1 - u$. Evaluation at this pair identifies $O_K / l^a O_K$ with $(\mathbb{Z}/l^a\mathbb{Z})^2$. Indeed $O_K = \mathbb{Z}[\omega]$ with basis $1, \omega$, by the quadratic ring calculation in Theorem 18.2. The evaluation matrix on these two elements has determinant $u_1 - u_0$, a unit.

Choose the element evaluating to $(1, 0)$ for each l. The Chinese remainder calculation of Lemma A.1 combines the two coefficients to give $t_K = A + B\omega$ with $0 \le A, B < N_c$. At an embedding φ, one of $\varphi\omega - u_0, \varphi\omega - u_1$ is a unit. Their product lies in $l^a O$, since $F(X) \equiv (X - u_0)(X - u_1) \bmod l^a$. This proves the claimed congruences. Conjugation interchanges the two components, so the trace and norm reduce to $1, 0 \bmod N_c$. Finally $\mathrm{Tr}(t_K) = 2A + B$ and $\mathrm{N}(t_K) = |t_K|^2 \le N_c^2 (1 + \sqrt{n_K})^2 \le 4 n_K N_c^2$. The first component shows $t_K \ne 0$. □

Theorem 45.2. *Put $N_c = N_1$, $\log_2 u = \log(u)/\log(2)$. For any integers $R \ge 0$ and*

$$L \ge 24 d N_c^4 (1 + \log_2 N_c)(R + 1)$$

there is a finite family $w_i \in \mathcal{A}$ with integer multiplicities $b_i \ge \max(1, R)$ such that

$$\sum_i b_i \, \tau(w_i) \overline{\sigma(w_i)} = L \, \mathbf{1}_{\tau = \sigma} \qquad (\tau, \sigma \in G).$$

For every finite-place embedding ι there is an index i with $s_\iota \varphi_\tau(w_i)$ a unit for every $\tau \in G$, with the notation of Definition 44.2. The family and the multiplicities can be taken independent of ι. The length condition holds in particular for $L \ge (dN_c)^6 (R + 1)$.

Proof. Let t_K be as in Lemma 45.1. Start with the N_c vectors ζ^j, $0 \le j < N_c$. They lie in $O_E \subset \mathcal{A}$ and have length N_c by Definition 29.1, with each multiplicity one. For each $l^a \| N_c$ put $r_l = N_c / l$, $h_l = N_c / l^a$, $\xi = \zeta^{r_l}$ and $s_l = t_K / l^a$. Form a second family by taking the following vectors for all such l:

$$v_{l,j,k,i} = \zeta^j \left(\mathbf{1}_{k=0} \xi^i + s_l (\xi^i - \xi^k) \right), \qquad 0 \le j < r_l, \quad 0 \le k, i < l.$$

These are integral at places of residue characteristic other than l. At the l-adic embeddings where ω has nonzero residue, s_l is integral. On the other branch, a nonzero $s_l(\xi^i - \xi^k)$ has valuation $-a + 1/(l-1)$ by Proposition 44.1. This is the precise allowance of Definition 44.2. Hence all the vectors belong to $\mathcal{A}$.

Fix $\tau, \sigma \in G$, regarded as exponents on ζ. Set $u = \tau(\xi)$, $w = \sigma(\xi)$. The sums of the l powers of u and w both vanish, and $\sum_{i=0}^{l-1} (u\bar{w})^i = l \mathbf{1}_{u=w}$. Upon multiplying the two factors in parentheses in the vector formula, conjugating the second, and summing over i, k, the constant term is $l \mathbf{1}_{u=w}$. The terms linear in $s_l, \bar{s}_l$ sum to $l(s_l + \bar{s}_l)\mathbf{1}_{u=w}$, since $k = 0$ in their other factors. The mixed quadratic sum is $2l^2 |s_l|^2 \mathbf{1}_{u=w}$; the two cross sums in the product $(u^i - u^k)(\bar{w}^{\,i} - \bar{w}^{\,k})$ vanish separately. When $u = w$, $l \mid \tau - \sigma$. The sum of the remaining $\zeta^{j(\tau - \sigma)}$ over $j < r_l$ is then r_l for $\tau = \sigma$ and zero otherwise. Thus this block has length

$$C_l = N_c (1 + s_l + \bar{s}_l + 2l|s_l|^2) = N_c + h_l \left(\mathrm{Tr}(t_K) + 2l \frac{\mathrm{N}(t_K)}{l^a} \right).$$

By Lemma 45.1 this is a positive integer, congruent to $h_l \neq 0 \bmod l$. Every other prime dividing N_c divides h_l and hence C_l. The sum $C = \sum_{l|N_c} C_l$ over prime divisors is therefore coprime to N_c. The bounds of the lemma give $C_l \leq N_c + 3N_c^2 + 8n_K N_c^3 \leq 12dN_c^3$. There are at most $\log_2 N_c$ prime divisors, since the product of a' distinct primes is at least $2^{a'}$. Consequently $C + N_c \leq 12dN_c^3(1 + \log_2 N_c)$.

Let $R_0 = \max(1, R)$. Our length hypothesis implies $L \geq (R_0 + N_c)(C + N_c)$, since $R_0 + N_c \leq 2N_c(R + 1)$. Reduce $L - R_0(C + N_c)$ modulo N_c. Since C is invertible there, it has the same residue as BC for some $0 \leq B < N_c$. The difference is nonnegative by the length bound and equals AN_c for $A \in \mathbb{Z}_{\geq 0}$. Give the first family multiplicity $A + R_0$ everywhere and the second multiplicity $B + R_0$. The preceding identities give exactly the asserted length.

If $d_\iota = 0$, the vector 1 in the first family supplies the unit simultaneously at all τ. Otherwise $p = l \mid N_c$ and every φ_τ sees the branch where t_K is a unit. Take the vector with $j = 0, k = 1, i = 0$ in that block. It is $t_K(1 - \xi)/l^a$, of valuation $-d_\iota$ for every such embedding. Its product with s_ι has valuation zero. Finally $1 + \log_2 N_c \leq N_c$ for integral $N_c \geq 2$, since $2^{N_c - 1} \geq N_c$ by induction, and $24 \leq d^5 N_c$ by Proposition 10.3 and the size of N_c. These inequalities prove the last length comparison. □

46. One algebraic normalization for all the theta frames

Theorem 46.1. *In Definition 44.2 there is a single $s_0 \in \overline{\mathbb{Q}}^\times \subset \mathbb{C}^\times$ with*

$$v_p(\iota s_0) = d_\iota \quad \textit{for all } (p, \iota), \qquad |\sigma s_0|^2 = \mathrm{rd}_E / \sqrt{d} \quad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Consequently in Theorem 45.2 one may use $s_\iota = \iota s_0$ at all embeddings simultaneously.

Proof. Take a common positive multiple D of the integers $l - 1$ for $l \mid N_c$ prime, and put $e_l = D(v_l(N_c) - 1/(l - 1)) \in \mathbb{Z}_{\geq 0}$. Let $P_l = (l, \bar{\omega}) \subset O_K$ and

$$I = \prod_{l|N_c} P_l^{e_l}, \qquad m_0 = \prod_{l|N_c} l^{e_l}$$

with both products over primes. For each l, $P_l \bar{P}_l = lO_K$: its generators $l^2, l\omega, l\bar{\omega}, n_K$ belong to lO_K, and $l\omega + l\bar{\omega} = l$. Also $P_l + \bar{P}_l = O_K$ since $\omega + \bar{\omega} = 1$. Their positive powers remain comaximal. Indeed, if $a + b = 1$ belongs termwise to two such ideals then every term of $(a + b)^{2n-1}$ belongs to one of their n-th powers. At distinct primes the corresponding powers contain relatively prime rational integers. Products of ideals each comaximal with a fixed ideal remain comaximal with it, by multiplying elements congruent to one modulo the fixed ideal. Hence $I^k, \bar{I}^k$ are comaximal, with product $m_0^k O_K$, for $k \geq 1$. For comaximal ideals the product equals the intersection, and the quotient by their product maps onto the product of quotients: if $a + b = 1$ as above then $xb + ya$ realizes the given pair of residues x, y. As O_K has rank two over $\mathbb{Z}$, and conjugation exchanges the quotients, this gives $[O_K : I^k] = [O_K : \bar{I}^k] = m_0^k$.

For every $k \geq 1$ choose a nonzero $z_k \in I^k$ with $|z_k| \leq 2(1 + |\omega|)m_0^{k/2}$ by the square grid pigeonhole argument of Lemma 11.3. Then $J_k = z_k \bar{I}^k / m_0^k \subset O_K$, by $I^k \bar{I}^k = m_0^k O_K$. By the lattice area calculation of that lemma its index is $|z_k|^2 / m_0^k \leq 4(1 + |\omega|)^2$. Only finitely many such sublattices occur, again by Lemma 11.3. Equality of two J_k's for $k < k'$ yields $I^{k'} = bI^k$ for some $b \in K^\times$ by conjugation and rescaling. Multiplication by $\bar{I}^k$ and cancellation of the rational factor m_0^k give $I^h = bO_K,\ h = k' - k \geq 1$. Thus $b \in O_K$ and $|b|^2 = m_0^h$ by the index just computed.

Choose $s_0^{Dh} = b$ by Lemma 16.2. At any embedding ι, the least valuation of the generators of P_l is zero except when $p = l$ and $\iota\bar{\omega}$ has zero residue. In that case $\iota\omega$ is a unit and $\bar{\omega}\omega = n_K$ shows $v_l(\iota\bar{\omega}) \geq 1$, so the least valuation is one. The same minimum holds over all nonzero members of the ideal, since all integral linear combinations have valuation at least the minimum. Minima add on products, by taking products realizing the two minima.

It follows from $I^h = bO_K$ that $v_p(\iota b) = he_p$ on the selected branch at $p \mid N_c$ and is zero otherwise. The selected branch is exactly where $\varphi_1\omega = \iota\bar{\omega}$ has zero residue. Division by Dh and Proposition 44.1 give the required valuation.

Every complex automorphism sends $b \in K$ to b or $\bar{b}$. Thus

$$|\sigma s_0|^2 = m_0^{1/D} = N_c \prod_{l \mid N_c} l^{-1/(l-1)} = \mathrm{rd}_E / \sqrt{d}$$

by Proposition 18.4. The last assertion follows from the valuation formulation of Theorem 45.2. □

Corollary 46.2. *Any two nonzero complex numbers satisfying the finite-place valuations of Theorem 46.1 and its modulus condition for automorphisms fixing K differ by a root of unity. For any formal series in finitely many variables, substitution of either scalar times every variable gives at each homogeneous degree coefficients differing by one common root of unity. In particular their absolute values agree after every archimedean or finite-place embedding.*

Proof. Write w for the quotient. Every complex automorphism σ takes K to itself. If its restriction there is conjugation, then $\mathfrak{c}\sigma$ fixes K and $|\sigma w| = |\mathfrak{c}\sigma w|$. Thus $|\sigma w| = 1$ for every σ. The transcendence basis argument in Proposition 16.4 shows w is algebraic. The finite-place hypotheses give $v_p(\iota w) = 0$ for every embedding, so w is integral by the minimal polynomial argument in Proposition 44.3, applied in $\mathbb{Q}(w)$. If its degree is D', each positive power has degree at most D' and integer minimal polynomial. Every root has modulus one, since each embedding extends to an automorphism of $\mathbb{C}$ by Lemma 16.2. The elementary symmetric coefficients are at most $2^{D'}$ in absolute value by the triangle inequality. Finitely many such monic integer polynomials exist, and each has finitely many roots. Two distinct positive powers of w therefore agree. Since $w \neq 0$, it is a root of unity. A homogeneous coefficient of degree j acquires the factor w^j, which has absolute value one after any embedding since it has finite multiplicative order. □

47. The alternating form of the arithmetic frame lattice

Lemma 47.1. *Let $\mathcal{A}, D_c, t_K$ be as in Definition 44.2, Proposition 44.1 and Lemma 45.1. Inside E one has*

$$\mathcal{A} = O_E + (t_K/D_c)O_E, \qquad \mathcal{A}\overline{\mathcal{A}} = D_c^{-1}O_E.$$

Here the product of two O_E-modules consists of all finite sums of products of their elements.

Proof. The valuation inequalities in Definition 44.2 are stable under addition and multiplication by O_E. Both 1 and t_K/D_c satisfy them. Indeed at a prime $l^a \| N_c$ the branch with no allowance has $v_l(t_K) \geq a$ by Lemma 45.1, while $v_l(D_c) \leq a$ on both branches by Proposition 44.1. At primes outside N_c the denominator is a unit. Conversely, for $z \in \mathcal{A}$, both $D_c z$ and $(1 - t_K)z$ are integral at every place. For the latter assertion the congruence $t_K \equiv 1 \bmod l^a$ at each branch with allowance absorbs the possible denominator of z. The integrality characterization in Proposition 44.3 thus puts both elements in O_E. The identity $z = (1 - t_K)z + (t_K/D_c)(D_c z)$ proves the first formula.

The same proposition gives $D_c x\bar{y} \in O_E$ for all $x, y \in \mathcal{A}$. For the reverse inclusion the product contains $1, t_K/D_c, \bar{t}_K/\bar{D}_c$. The derivative reciprocity formula in Proposition 18.4 gives $\bar{D}_c/D_c = -\zeta^{2-f} \in O_E^\times$, so the product also contains $\bar{t}_K/D_c$. Lemma 45.1 gives $1 - t_K - \bar{t}_K \in N_c\mathbb{Z}$, and $N_c/D_c \in O_E$ by Proposition 44.1 and the same integrality characterization. Therefore

$$D_c^{-1} = t_K/D_c + \bar{t}_K/D_c + (1 - t_K - \bar{t}_K)/D_c \in \mathcal{A}\overline{\mathcal{A}}.$$

Closure under O_E proves the second equality. □

Proposition 47.2. *Fix $n \geq 1$, put $g = fn$, and on E^n set*

$$\mathcal{E}(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}\left(\sum_{k=1}^{n} \bar{x}_k y_k/\sqrt{-d}\right) = \frac{2}{\sqrt{d}}\Im\left(\sum_{\tau\in G,k} \overline{\tau(x_k)}\,\tau(y_k)\right).$$

Set $\sigma_0 = (2\pi/\sqrt{d})^{1/2} > 0$, and $r(x) = (\sigma_0\tau(x_k))_{\tau,k} \in \mathbb{C}^g$. On $\mathbb{C}^g$ use $H(u,z) = \sum_{\tau,k} \bar{u}_{\tau k} z_{\tau k}$ and $E_H(u,z) = \Im H(u,z)/\pi$. Then $E_H(rx,ry) = \mathcal{E}(x,y)$. The group $r(\mathcal{A}^n)$ is a full lattice, equal to its dual for E_H. For $0 \neq \delta \in O_K$ set $N = \delta\bar{\delta}$ and $\Gamma = r(\bar{\delta}\mathcal{A}^n)$. Its dual $\Gamma^\# = \{v : E_H(v,\Gamma) \subset \mathbb{Z}\}$ satisfies

$$\Gamma^\# = r(\delta^{-1}\mathcal{A}^n) = N^{-1}\Gamma, \qquad [\Gamma^\# : \Gamma] = N^{2g}.$$

Proof. For $z \in E$, taking conjugate pairs in the field trace gives

$$\mathrm{Tr}_{E/\mathbb{Q}}(z/\sqrt{-d}) = 2\Im\mathrm{Tr}_{E/K}(z)/\sqrt{d},$$

since the selected embeddings fix K by Proposition 10.3. Applying this identity to $\sum_k \bar{x}_k y_k$ proves both formulas for the alternating form. The real extension of $r : E^n \to \mathbb{C}^g$ is an isomorphism. Indeed, in each copy of E the embeddings on the powers of ζ have the invertible evaluation matrix at the f distinct primitive roots. On the basis $1, \omega$ the two embeddings of K have distinct values at ω. The product basis of Theorem 18.2 therefore has invertible evaluation matrix on all $2f$ embeddings. A real linear relation on the selected f rows implies the conjugate relation on the other f, hence is zero. Equality of real dimensions proves the claim. Proposition 44.3 sandwiches $\mathcal{A}$ between O_E and $N_c^{-1}O_E$. It is a full free lattice: in integer coordinates any subgroup of $\mathbb{Z}^h$ has an integer basis, by projecting to its first coordinate, choosing a lift of a generator of that image if nonzero, and adjoining an inductively chosen basis of the kernel. Its rank here is $2f$ by the two inclusions.

First test a rational point in one coordinate. For $x \in E$, the condition $\mathcal{E}(x,\mathcal{A}) \subset \mathbb{Z}$ is equivalent to $\bar{x}\mathcal{A} \subset D_c^{-1}O_E$. In fact one can multiply any test element of $\mathcal{A}$ by every element of O_E and stay inside $\mathcal{A}$. The trace test therefore says precisely that $\bar{x}\mathcal{A}/\sqrt{-d} \subset O_E^\vee$, and Proposition 18.4 gives the stated equivalence. If $x \in \mathcal{A}$, Lemma 47.1 gives the inclusion. Conversely, multiply the inclusion by $D_c\overline{\mathcal{A}}$ and use the same lemma to obtain $\bar{x}O_E \subset \overline{\mathcal{A}}$, proving $x \in \mathcal{A}$. Coordinate tests and addition give self-duality among points of $r(E^n)$. Every point of the real dual is in this rational space. Relative to a basis of $r(\mathcal{A}^n)$ its coordinates are obtained by inverting the matrix of E_H on that basis and applying it, up to transpose, to an integer column. The matrix is integral by the rational test and is nonsingular, since $E_H(u,iu) = H(u,u)/\pi > 0$ for $u \neq 0$. Its inverse has rational entries by the adjugate formula. Finally $E_H(z,\bar{\delta}\,y) = E_H(\delta z, y)$, so the dual of Γ is $\delta^{-1}r(\mathcal{A}^n)$. The two descriptions follow from $\delta\bar{\delta} = N$; division by N in each of $2g$ independent integral basis directions gives the stated index. □

48. Theta products on the polarized torus

Lemma 48.1. *For the lattice $\Lambda = \lambda O_K$ and functions $\theta, \sigma, G_\Lambda = \phi\theta^3$ of Propositions 31.4 and 33.1, put $A = |\lambda|^2\sqrt{d}/2$. For $a,b,c \in \mathbb{Z}_{\geq 0}$ let $P = \theta^a(\sigma^2)^b G_\Lambda^c$ and $k = a + 2b + 3c$. For $y \in \mathbb{C}$ write*

$$(U_y^{[k]}P)(z) = \exp\Big(-k\pi(\bar{y}z + |y|^2/2)/A\Big)\,P(z+y).$$

This function is entire. For $l = \lambda\alpha \in \Lambda$ it satisfies $U_l^{[k]}P = (-1)^{k|\alpha|^2}P$. If $y \in \lambda K$, the number $(U_y^{[k]}P)(0)$ is an algebraic integer.

Proof. Write $\alpha = u + v\omega$, $u,v \in \mathbb{Z}$. The sign of σ is $(-1)^{u+v+uv}$ by Proposition 31.2. For θ, Proposition 31.4 gives an additional factor $\exp(\pi(\bar{l}t - l\bar{t})/A) = (-1)^v$, since $t = \lambda/2$. Its sign is therefore $(-1)^{u+uv}$. Now $|\alpha|^2 = u^2 + uv + n_K v^2$ and $n_K = (d+1)/4$ is even by Proposition 10.3. This proves the sign formula in degree one. The sign in degree two of σ^2 is one. The meromorphic function ϕ is periodic by its defining formula in

Proposition 31.3, so G_Λ has the sign of θ^3, first off its apparent singularities and then everywhere by Proposition 33.1. Multiplication proves the law and entireness for P.

For $y \in \lambda K$, clearing the rational coefficients on the basis $1, \omega$ gives $my \in \Lambda$ for some $m \geq 1$. Theorem 33.2 gives

$$|\iota P(y)|_p \leq H_p(y)^k = |\iota \exp(\pi |y|^2/(2A))|_p^k$$

at every finite isomorphism in Definition 16.3. Here the bound on the factor G_Λ is the cubic bound of that theorem. Thus $w = (U_y^{[k]} P)(0)$ has $|\iota w|_p \leq 1$ at every such isomorphism. To verify the algebraic conclusion, suppose that w were transcendental. Extend it to a transcendence basis of $\mathbb{C}/\mathbb{Q}$. For each $j \geq 1$ the assignment $w \mapsto 2^{-j} w$, fixing the remaining basis elements, gives an automorphism of the rational function field and extends to $\mathbb{C}$ by the argument of Lemma 16.2. Compose a fixed isomorphism at 2 with these automorphisms. The resulting absolute values of the images of w are $2^j |\iota w|_2$, unbounded. Hence w is algebraic. Every conjugate of w in each K_p occurs under some such isomorphism by Lemma 16.2. The rational coefficients of its monic minimal polynomial are consequently integral at every p, being signed elementary symmetric functions of these conjugates. Prime factorization in Lemma A.1 puts them in $\mathbb{Z}$. □

Theorem 48.2. *Use $g, r, H, E_H, \Gamma, \delta, N$ of Proposition 47.2 and let (w_i, b_i) be a family in $\mathcal{A}$ satisfying the tight-frame identity of length L in Theorem 45.2. Choose $0 \neq \alpha \in O_K$ with $L|\alpha|^2 = N$ and put $a' = \lambda \alpha$. For $1 \leq k \leq n$ set*

$$B_{ik}(z) = \frac{a'}{\bar{\delta}\, \sigma_0} \sum_{\tau \in G} \overline{\tau(w_i)} z_{\tau k}, \qquad \mathcal{S}(z) = \prod_{i,k} P_{ik}(B_{ik}(z)),$$

where P_{ik} is any monomial of degree b_i in Lemma 48.1. Define for $v, z \in \mathbb{C}^g$

$$(W_v S)(z) = \exp(-H(v,z) - H(v,v)/2) S(z+v).$$

Then $\mathcal{S}$ is entire. For $u = r(h) \in \Gamma$,

$$W_u \mathcal{S} = \chi_\Gamma(u) \mathcal{S}, \qquad \chi_\Gamma(u) = (-1)^{\mathrm{Tr}_{E/K} \sum_k \bar{h}_k h_k} = \exp\bigl(\pi i E_H(\bar{\omega} u, u)\bigr).$$

For $\gamma \in E^n$ set $y_{ik} = -B_{ik}(r\gamma)$. Then

$$(W_{-r\gamma} \mathcal{S})(z) = \prod_{i,k} (U_{y_{ik}}^{[b_i]} P_{ik})(B_{ik}(z)).$$

In particular its value at zero is an algebraic integer. For every $t' \in \mathbb{C}$ the Taylor series at zero after the substitution $z \mapsto t'z$, viewed as a formal series in the coordinates of z, is the product of the one-variable Taylor series of the indicated translates with $B_{ik}(t'z)$ substituted. Taking $P_{ik} = \theta^{b_i}$ gives $\mathcal{S}(0) = 1$.

Proof. The tight-frame identity implies

$$\sum_{i,k} b_i \frac{\pi}{A} \overline{B_{ik}(v)} B_{ik}(z) = H(v,z).$$

Indeed in each block k the coefficient of $\bar{v}_{\tau k} z_{\nu k}$ in this sum equals $\pi |a'|^2 L \mathbf{1}_{\tau=\nu}/(AN\sigma_0^2)$. This equals $\mathbf{1}_{\tau=\nu}$, by the formulas for A, σ_0 and $L|\alpha|^2 = N$. It follows by multiplication of exponentials that the formula for the translate holds for every v in place of $-r\gamma$ and shifts $B_{ik}(v)$. The product of entire functions on linear forms is entire. At $v = -r\gamma$ we have

$$y_{ik} = -(a'/\bar{\delta}) \,\mathrm{Tr}_{E/K}(\gamma_k \bar{w}_i) \in \lambda K$$

by Proposition 10.3. Lemma 48.1 now gives integrality of the product at zero. To see explicitly that products of algebraic integers are integral, the integer span of products of powers of the factors, with each exponent less than its monic degree, is finitely generated, contains one and is preserved by multiplication by their product. On these generators multiplication is given by an integral square matrix. Its characteristic polynomial annihilates

each generator by the adjugate identity for $XI - C$, and hence annihilates the product acting on one. Taylor expansion and finite multiplication give the asserted series identity by the disk formula of Lemma A.3, or by absolute convergence of the one-variable expansions on every bounded polydisk.

For $h = \bar{\delta}\, x,\ x \in \mathcal{A}^n$, $B_{ik}(rh) = a' \mathrm{Tr}_{E/K}(x_k \bar{w}_i) \in \Lambda$ by Proposition 44.3. The translation formula and Lemma 48.1 give the sign with integer exponent $\sum_{i,k} b_i |B_{ik}(rh)/\lambda|^2$. Using the hermitian identity above at $v = z = rh$ this exponent is

$$\frac{A}{\pi|\lambda|^2} H(rh, rh) = \sum_{\tau,k} |\tau(h_k)|^2 = \mathrm{Tr}_{E/K} \sum_k \bar{h}_k h_k.$$

Since $\Im\omega = \sqrt{d}/2$, the first expression is also $E_H(\bar{\omega} rh, rh)$. This proves the claimed multiplier. Finally $\theta(0) = 1$ by Proposition 31.4. □

Corollary 48.3. *Let $R \geq 0$ be an integer. For every positive integer b such that $b^2 \geq 24 d N_c^4 (1 + \log_2 N_c)(R+1)$, the lattice $r(b\mathcal{A}^n)$ admits the section of Theorem 48.2 with value one at the origin and every multiplicity at least $\max(1, R)$.*

Proof. Take $\delta = b,\ L = N = b^2,\ \alpha = 1$. Theorem 45.2 supplies the frame. Choose every factor to be the indicated power of θ in Theorem 48.2. □

49. Fourier coordinates for the arithmetic theta space

Lemma 49.1. *Use the data g, Γ, N, H, E_H of Proposition 47.2. There is an integral basis $e_1, \ldots, e_g, b_1, \ldots, b_g$ for Γ satisfying*

$$E_H(e_j, e_k) = E_H(b_j, b_k) = 0, \qquad E_H(e_j, b_k) = N \mathbf{1}_{j=k}.$$

The e_j form a complex basis of $\mathbb{C}^g$. Write $ez = \sum_j z_j e_j$ and $b_j = eB_j$, where B_j is column j of a complex matrix $B = X + iY$ with real X, Y. The matrices X, Y are symmetric and Y is positive definite. The real Gram matrix $M = (H(e_i, e_j))$ satisfies $MY = \pi N I$.

Proof. First work on the lattice $\Gamma_0 = r(\mathcal{A}^n)$ of Proposition 47.2. The map $x \mapsto E_H(x, \cdot)$ identifies it with its integral dual: invert the nonsingular real pairing to represent any integer linear functional and apply self-duality. Take a basis element u. Its image is primitive since the map and its inverse are integral. Its values on an integral basis have gcd one, so Lemma A.1 supplies v with $E_H(u, v) = 1$. For each x its difference from $E_H(x, v)u + E_H(u, x)v$ is perpendicular to both. This splits off $\mathbb{Z}u + \mathbb{Z}v$ orthogonally. On the remaining direct summand the pairing is again unimodular, since any integral linear functional on it can be extended by zero to the first summand, and the representing vector of such an extension is perpendicular to u, v. Induction, with the free subgroup argument of Proposition 47.2, gives g pairs. Multiplying the basis by $\bar{\delta}$ multiplies H, hence E_H, by N, producing the required basis for Γ.

On the real span U of the e_j, E_H vanishes. If $u, iu \in U$, then $0 = E_H(u, iu) = H(u, u)/\pi$, so $u = 0$. Thus $U \cap iU = 0$, and real independence proves complex independence of the g vectors. The Gram matrix M is real symmetric positive definite. In these coordinates $H(eu, ez) = \bar{u}^t M z$. The equations $E_H(e_j, b_k) = N\mathbf{1}_{j=k}$ give $MY = \pi N I$, so $Y = \pi N M^{-1}$ is symmetric positive definite. Finally the imaginary part of $\bar{B}^t M B$ is $X^t MY - Y^t MX = \pi N (X^t - X)$. The vanishing of $E_H(b_j, b_k)$ proves symmetry of X. □

Theorem 49.2. *Let $\mathcal{T}(\Gamma)$ denote the complex vector space of entire functions on $\mathbb{C}^g$ such that $W_u S = \chi_\Gamma(u) S$ for all $u \in \Gamma$, with W, χ_Γ of Theorem 48.2. Then $\dim_{\mathbb{C}} \mathcal{T}(\Gamma) = N^g$. In the basis of Lemma 49.1 choose a real column t with $\exp(2\pi i t_j) = \chi_\Gamma(e_j)$. For $S \in \mathcal{T}(\Gamma)$, the function*

$$h(z) = \exp(-z^t M z/2 - 2\pi i t^t z)\, S(ez)$$

has a normally convergent expansion $\sum_{m\in\mathbb{Z}^g} a_m \exp(2\pi i m^t z)$*, meaning absolute uniform convergence on compact sets. The coefficients* $\{a_m : 0 \le m_j < N\}$ *are arbitrary independent coordinates. Thus* $\mathcal{T}(\Gamma_0)$ *has dimension one for* $\Gamma_0 = r(\mathcal{A}^n)$*.*

Proof. For $u, v \in \mathbb{C}^g$, multiplying the exponentials defining W gives

$$W_u W_v = \exp(\pi i E_H(u,v))\, W_{u+v}.$$

For $u = r(h), v = r(h')$ in Γ, $T = \mathrm{Tr}_{E/K} \sum_k \bar{h}_k h'_k \in O_K$ by Proposition 44.3, since $\bar{\delta}\mathcal{A} \subset \mathcal{A}$. Write $T = a + c\omega$ with $a, c \in \mathbb{Z}$. The cross term in the integer exponent of $\chi_\Gamma(u+v)$ is $2\Re T = 2a + c$; by Proposition 47.2, $E_H(u,v) = c$. Hence

$$\chi_\Gamma(u+v) = \exp(-\pi i E_H(u,v))\chi_\Gamma(u)\chi_\Gamma(v).$$

These two formulas show that it suffices to impose the equations for the generators of the integral basis. They also give the equations for negatives because $W_{-u} = W_u^{-1}$.

The equations at the e_j say precisely that h is periodic under $\mathbb{Z}^g$. The equations at the b_j become

$$h(z + B_j) = d_j \exp(-2\pi i N z_j)\, h(z), \qquad d_j = \chi_\Gamma(b_j) \exp(-2\pi i t^t B_j - \pi i N B_{jj}).$$

Indeed the linear exponent contributed by the Gaussian equals $(\bar{B}_j - B_j)^t M z = -2\pi i N z_j$, and its constant contribution is $(\bar{B}_j - B_j)^t M B_j/2 = -\pi i N B_{jj}$, by Lemma 49.1.

For a periodic entire h define a_m by integrating $h(z)\exp(-2\pi i m^t z)$ over the real unit cube. Moving this cube to the parallel cube of imaginary part $y \in \mathbb{R}^g$ leaves the integral unchanged: in each coordinate integrate on the boundary of the intervening rectangle, whose vertical sides cancel by periodicity, and use the disk and contour identities of Lemma A.3. Iteration over coordinates is valid by continuity on the compact rectangles and Lemma A.2. Taking $y_j = -A\,\mathrm{sign}(m_j)$ shows for every $A > 0$ that

$$|a_m| \le C_A \exp\left(-2\pi A \sum_j |m_j|\right)$$

where C_A is a maximum of $|h|$ over a compact box. The Fourier series therefore converges normally by products of geometric sums. Its restriction to the real cube has the same coefficients as h, so it agrees there by the product Fejér-kernel argument of Lemma 14.2. Successive application of the one-variable identity principle of Lemma A.3 gives agreement on $\mathbb{C}^g$.

Let $\mathbf{e}_j$ be the j-th integer coordinate column. The equations at B_j are now equivalent, by coefficient comparison, to

$$a_{m+N\mathbf{e}_j} = d_j^{-1} \exp(2\pi i m^t B_j)\, a_m.$$

For m_0 a residue representative $0 \le (m_0)_j < N$, their unique solution throughout its residue class is

$$a_{m_0+N\ell} = a_{m_0} \left(\prod_j d_j^{-\ell_j}\right) \exp\left(2\pi i m_0^t B\ell + \pi i N \left(\ell^t B \ell - \sum_j \ell_j B_{jj}\right)\right) \qquad (\ell \in \mathbb{Z}^g).$$

Substitution at $\ell + \mathbf{e}_j$ verifies the recursion because $B^t = B$. Each multiplier is nonzero, so the same recursions determine all negative steps. Conversely give the finitely many a_{m_0} arbitrary values and use this formula. Since Y is positive definite, its quadratic form has a positive minimum on the real unit sphere. Thus the coefficients decay as $O(\exp(-c\|\ell\|^2 + C\|\ell\|))$ with $c > 0$. On compact subsets the additional Fourier exponent and each derivative introduce at most an exponential linear in $\|\ell\|$ and a polynomial. The series and its differentiated series converge there by products of one-dimensional Gaussian sums, as in Proposition 31.2. They give an entire function with the required translation equations. Reversing the defining transformation of h constructs S. The coordinate count and its specialization $\delta = 1$ follow. □

50. Orthogonality of the theta translates

Proposition 50.1. *On $\mathcal{T}(\Gamma)$ of Theorem 49.2, the operators W_v for $v \in \Gamma^{\#}$ preserve the space. Every nonzero vector spans $\mathcal{T}(\Gamma)$ under these translates. Every complex-linear operator commuting with all of them is scalar. It suffices in each assertion to use one representative of each class in $\Gamma^{\#}/\Gamma$.*

Proof. The addition law proved in Theorem 49.2 gives $W_u W_v = \exp(2\pi i E_H(u,v)) W_v W_u$. For $u \in \Gamma$ and $v \in \Gamma^{\#}$ these operators commute, proving preservation. Changing v by an element of Γ multiplies its operator on $\mathcal{T}(\Gamma)$ by a nonzero scalar by the same law.

Let e_j, b_j, N, t be as in Lemma 49.1 and Theorem 49.2. Write $U_j = \exp(-2\pi i t_j/N) W_{e_j/N}$ and $V_j = W_{b_j/N}$. These shifts are in the dual lattice by Proposition 47.2. In the Fourier coordinates of the dimension theorem, U_j replaces $h(z)$ by $h(z + \mathbf{e}_j/N)$. Indeed the real translation by $\mathbf{e}_j/N$ cancels the linear and constant terms from the quadratic exponential, leaving exactly $\exp(2\pi i t_j/N)$ before normalization. It follows that the line of coefficients supported on the residue class $m \bmod N$ is an eigenspace of each U_j with eigenvalue $\exp(2\pi i m_j/N)$. The lines exhaust the space by that theorem. Its projection on the line of m is

$$\prod_{j=1}^{g} \left(\frac{1}{N} \sum_{h=0}^{N-1} \exp(-2\pi i h m_j/N)\, U_j^h \right)$$

by the finite geometric sum on each eigenspace. Moreover

$$U_i V_j = \exp(2\pi i \mathbf{1}_{i=j}/N) V_j U_i$$

by the commutation relation. Each V_j therefore carries the line of m onto the line of $m + \mathbf{e}_j$, since it has inverse $W_{-b_j/N}$. These moves are transitive on all residue classes, also when $N = 1$. The span of the translates of a nonzero vector is stable under all shifts by the addition law, so the displayed projections recover a nonzero line in that span, and the moves recover all lines. An operator commuting with the U_j preserves each line, and commuting with the V_j makes its scalars equal on all of them. This proves both conclusions. □

Theorem 50.2. *Fix a nonzero $V_0 \in \mathcal{T}(\Gamma_0)$ with $\Gamma_0 = r(\mathcal{A}^n)$, and choose any set D of representatives for $\Gamma^{\#}/\Gamma$. Then $|D| = N^{2g}$ and each $a \in D$ has the form $r(x)$ with $\delta x \in \mathcal{A}^n$. For every $F \in \mathcal{T}(\Gamma)$ there are complex numbers c_a such that*

$$F = \sum_{a \in D} c_a W_a V_0.$$

For every complex-linear functional ℓ on $\mathcal{T}(\Gamma)$ one has the identity in that space

$$N^g \sum_{a \in D} \ell(W_a V_0) W_{-a} F = N^{2g} \ell(F) V_0.$$

In particular this formula holds for all of the sections in Theorem 48.2, with the same set D.

Proof. Proposition 47.2 gives the cardinality and the rational representatives. Theorem 49.2 gives the nonzero one-dimensional space at Γ_0. Since $\Gamma \subset \Gamma_0$ and the sign in Theorem 48.2 has the same expression on both lattices, $V_0 \in \mathcal{T}(\Gamma)$. The spanning statement follows from Proposition 50.1.

For a linear operator T on $\mathcal{T}(\Gamma)$, put $A(T) = \sum_{a \in D} W_{-a} T W_a$. The conjugation in a summand depends only on the class of a, because $W_{-a} = W_a^{-1}$ and a scalar in W_a cancels. Conjugating $A(T)$ by any W_b, $b \in \Gamma^{\#}$, permutes these classes by translation by b or $-b$, by the addition law in Theorem 49.2. Thus $A(T)$ is a scalar by Proposition 50.1. In coordinates $\operatorname{tr}(BC) = \sum_{i,j} B_{ij} C_{ji} = \operatorname{tr}(CB)$. Consequently its trace is $|D| \operatorname{tr}(T)$, whereas the trace of a scalar is N^g times that scalar by Theorem 49.2. Choose $T(z) = \ell(z) F$. Its trace is $\ell(F)$, by the coordinates of the column F and the row ℓ. Applying the formula for $A(T)$ to V_0 and multiplying by N^g gives the claimed identity. The final application uses the translation equation of Theorem 48.2. □

51. The finite Heisenberg action on theta sections

Definition 51.1. Put $L_0 = \mathcal{A}^n$, $\Gamma_0 = r(L_0)$, and let

$$\mathcal{F} = \bigcup_{m \geq 1} \mathcal{T}(m\Gamma_0).$$

Here $\mathcal{T}(m\Gamma_0)$ uses $\delta = m$ in Theorem 49.2. For $m \mid m'$ the larger periods $m'\Gamma_0$ impose a subset of the equations at $m\Gamma_0$, with the same sign. The union is directed by divisibility and thus is a complex vector space of entire functions. In the rational space E^n put

$$\widehat{L}_0 = \prod_p (L_0 \otimes \mathbb{Z}_p), \qquad V_f = \bigcup_{D \geq 1} D^{-1}\widehat{L}_0.$$

Its elements are finite adeles, and the subgroups $h\widehat{L}_0$, $h \geq 1$ form a basis of open neighborhoods at zero. Extend the rational alternating form $\mathcal{E}$ of Proposition 47.2 by scalars to $\mathcal{E}_p$ at each p, and set

$$c(X, Y) = \prod_p \mathbf{e}_p(\mathcal{E}_p(X_p, Y_p)/2) \qquad (X, Y \in V_f).$$

Theorem 51.2. *There is a unique family of complex-linear operators T_X on $\mathcal{F}$, indexed by $X \in V_f$, for which $T_X S$ is locally constant in X for every S, and $T_\gamma S = W_{-r(\gamma)} S$ for $\gamma \in E^n$. They satisfy*

$$T_X T_Y = c(X, Y) T_{X+Y}, \qquad T_0 = 1, \qquad T_X^{-1} = T_{-X}.$$

With V_0 as in Theorem 50.2, $\mathcal{F}$ is spanned by the $T_X V_0$. The linear map

$$S \longmapsto F_S, \qquad F_S(X) = (T_X S)(0)$$

is injective and has locally constant values in the variable X. It satisfies $F_{T_Y S}(X) = c(X, Y) F_S(X + Y)$.

Proof. Relative to a $\mathbb{Z}$-basis of L_0 provided by Proposition 47.2, $\widehat{L}_0 = \prod_p \mathbb{Z}_p^{2g}$. For $D, h \geq 1$, every $X \in D^{-1}\widehat{L}_0$ has a representative $\gamma \in D^{-1}L_0$ modulo $h\widehat{L}_0$. Indeed reduce each coordinate of DX modulo Dh at the primes dividing Dh and combine the residues to an integer coordinate by Lemma A.1. At all other primes Dh is a unit. At rational points $E^n \cap h\widehat{L}_0 = hL_0$, since a rational coordinate integral at every prime is integral. These facts prove density and identify the intersections used below. The form $\mathcal{E}_p$ is integral on $L_0 \otimes \mathbb{Z}_p$ by its integral matrix in Proposition 47.2. Thus almost all local factors of c equal one by Proposition 18.5. The product is a bicharacter, and on diagonal $\gamma, \eta \in E^n$ the same proposition gives

$$c(\gamma, \eta) = \exp(\pi i \mathcal{E}(\gamma, \eta)).$$

It is locally constant on V_f^2: for both arguments in $D^{-1}\widehat{L}_0$, perturbations in $2D\widehat{L}_0$ change $\mathcal{E}_p/2$ by p-integral quantities everywhere. On rational arguments the displayed law for $T_\gamma T_\eta$ therefore follows from Theorem 49.2.

Suppose $S \in \mathcal{T}(m\Gamma_0)$ and $\gamma \in D^{-1}L_0$. If $h \in 2mDL_0$, then $W_{-rh}S = S$. In fact this is a period with sign one by the quadratic formula of Theorem 48.2, since h is twice an element of L_0. Also $\mathcal{E}(\gamma, h) \in 2\mathbb{Z}$. The rational addition law thus gives $T_{\gamma+h}S = T_\gamma S$. The function $T_\gamma S$ lies in $\mathcal{T}(2mD\Gamma_0)$: each period in the latter lattice already acts on S by its required sign, and commutes with $W_{-r\gamma}$ since its pairing with $r\gamma$ is integral. This proves that rational operators carry $\mathcal{F}$ into itself.

For arbitrary $X \in D^{-1}\widehat{L}_0$, take a rational representative in $D^{-1}L_0$ modulo $2mD\widehat{L}_0$ and use its action on S. Two such representatives differ in $2mDL_0$, so the definition is independent of the choice. If D or the level m is enlarged to a multiple, the finer choice has the same action: it differs modulo the older neighborhood and lies in the old $D^{-1}L_0$ when X lies in $D^{-1}\widehat{L}_0$. For the latter assertion, it is a rational point of $D^{-1}\widehat{L}_0$, so the coordinate test above applies. Passing to common multiples proves independence for any

two choices of m, D. The action is locally constant on each open $D^{-1}\widehat{L}_0$; density gives uniqueness. Linearity follows by choosing a common level for finitely many sections and a common representative. The rational addition law extends to all X, Y by local constancy and density. Indeed $T_Y S$ is constant near each fixed Y, and applying T_X to that one section is locally constant near any fixed X. The identity and inverse statements use alternation of $\mathcal{E}_p$.

Theorem 50.2 at $\delta = m$ spans the whole m-th level by rational translates of V_0, proving the assertion for the union. For rational γ, the definition of W gives

$$F_S(\gamma) = \exp(-H(r\gamma, r\gamma)/2) S(-r\gamma).$$

The vectors $r(E^n)$ contain the rational span of a full real lattice by Proposition 47.2, so they are dense in $\mathbb{C}^g$. Thus $F_S = 0$ implies $S = 0$ by continuity. Evaluation of the operator law at zero gives the translation formula for F_S. □

Corollary 51.3. *For each product S constructed in Theorem 48.2, $F_S(X)$ is an algebraic integer for every $X \in V_f$. For $y \in E^n$ the radial Taylor series of $T_y S$ at the origin has the product formula of that theorem. The finite sum of Theorem 50.2 also holds after applying $S \mapsto F_S$ to its two sides. In this identity every shift $r(x)$ can be represented by T_{-x} with $\delta x \in L_0$.*

Proof. Since $N = \delta\bar{\delta}$, $N\Gamma_0 \subset \Gamma$. Hence $S \in \mathcal{T}(N\Gamma_0)$. By Theorem 51.2, for a given X its translate of S equals the translate at one rational representative $\gamma \in E^n$. Theorem 48.2 proves integrality of its value at zero and gives the entire product formula for every rational translate. The vacuum identity follows by linearity and the description of its shifts in Theorem 50.2. □

52. Counting the nonzero frequencies in the orbital sieve

Lemma 52.1. *Let $O = O_E$. For $0 \neq J \subset O$ an integral ideal put $\mathrm{N}J = [O : J]$. For each rational prime p, write $O \otimes \mathbb{Z}_p = \prod_{w|p} O_w$ as in the proof of Theorem 18.2. Denote the valuation with value one at a uniformizer by ord_w and the cardinality of the residue field by q_w. At each such w the completion of J is the ideal with some integral exponent $b_w \geq 0$. The exponents determine J, are zero except at finitely many places, and $\mathrm{N}J = \prod_w q_w^{b_w}$. For real $s > 1$,*

$$\sum_{0 \neq J \subset O} (\mathrm{N}J)^{-s} \leq \left(\sum_{h \geq 1} h^{-s} \right)^{2f}.$$

For $0 \neq y \in E$, put $J(y) = \{z \in O : yz \in O\}$. Its exponents are $b_w = \max(0, -\mathrm{ord}_w y)$, and

$$\prod_{\tau \in G} |\tau y|^2 = \prod_w q_w^{\mathrm{ord}_w y}.$$

Proof. Choose $0 \neq v \in J$. Multiplication by v on the integral basis in Theorem 18.2 has nonzero integer determinant. The positive absolute value a of that determinant satisfies $aO \subset vO \subset J$ by the adjugate. Thus O/J is finite. The ring O/aO is the product of O_w/aO_w over $w \mid p$, $p \mid a$, by the Chinese remainder calculation in Lemma 23.1. Coordinate idempotents split ideals of that product. In each factor every nonzero preimage ideal is a uniformizer power, by taking a member of smallest valuation. Each successive quotient of two adjacent powers has q_w elements by multiplication by a uniformizer power. This proves the claimed description modulo aO, and the index formula. In comparing two ideals take a common multiple of their two integers a; the description proves uniqueness from all the exponents. At most $2f$ factors occur over p, since each contributes at least one dimension to O/pO over $\mathbf{F}_p$. Also $q_w \geq p$. For any finite set of ideals, summing over all possible exponents at the primes in its support bounds its sum by the finite product of $(1 - p^{-s})^{-2f}$ there. Expanding the geometric series and using prime factorization in Lemma A.1 bounds

this by the displayed power of $\sum h^{-s}$. The latter sum converges by integration of t^{-s} from one to infinity. The inequality for the nonnegative infinite sum follows.

Choose an integer $a \ge 1$ with $ay \in O$. Membership of $z \in O$ in $J(y)$ says precisely that $(ay)z \in aO$. Testing this finite congruence in each of the factors just described gives the asserted exponents of $J(y)$. For integral $v \ne 0$ the index of vO is also $\prod_\tau |\tau v|^2$: on the real span, multiplication by v is diagonal on the complex embeddings, of real determinant this product, by Proposition 47.2; index equals the ratio of fundamental volumes as in Lemma 20.2. The formula for $J = vO$ gives $\prod_w q_w^{\mathrm{ord}_w v}$ for the same index. Taking the quotient of these identities at ay and a proves the formula for y. □

Lemma 52.2. *For J as in Lemma 52.1 put $J^{-1} = \{y \in E : yJ \subset O\}$. For real numbers $R_\tau \ge 1$, the number of $y \in J^{-1}$ with $|\tau y| \le R_\tau$ for all τ is at most*

$$9^f \,\mathrm{N}J \prod_\tau R_\tau^2.$$

Proof. For distinct y, y' in this set put $z = y - y'$. Then $zJ \subset O$ is a full sublattice and

$$1 \le [O : zJ] = \mathrm{N}J \prod_\tau |\tau z|^2$$

by the same real determinant and fundamental-volume calculation as in Lemma 52.1. After dividing coordinate τ by R_τ, the largest coordinate distance between the two points is at least $d_* = (\mathrm{N}J \prod_\tau R_\tau^2)^{-1/(2f)}$. Open balls of radius $d_*/2$ in this maximum norm are disjoint and lie in the ball of radius $1 + d_*/2$. Their real volumes are proportional to the radius to power $2f$, because the ball is a product of f complex disks. Thus every finite subset has cardinality at most $(1 + 2/d_*)^{2f} \le (3/d_*)^{2f}$, since $0 < d_* \le 1$. This gives the stated finite count for the whole set. □

Theorem 52.3. *Let T be any finite set of finite places of E. For a nonzero T-integer y, meaning $\mathrm{ord}_w y \ge 0$ at $w \notin T$, put*

$$G_T(y) = \prod_{w \in T} q_w^{\mathrm{ord}_w y}, \qquad D_T(y) = \prod_{w \in T} q_w^{\max(0, -\mathrm{ord}_w y)}.$$

For $s, t > 2$, $j \ge 0$ real, $c_\tau \in \mathbb{C}^\times$, and $X_\tau \ge |c_\tau|^{-1}$, put $B_\tau = \max(1, |c_\tau|^{-1})$ and $d_{j,t}(u) = u^j \min(1, u)^t$ for $u > 0$. The sum over all nonzero T-integers satisfies

$$\sum_y \frac{G_T(y)^{j/2}}{D_T(y)^s} \prod_\tau d_{j,t}(|c_\tau \tau y|^{-1}) \le \left(\frac{9t}{t-2} \left(\sum_{h \ge 1} h^{-(s-1)} \right)^2 \right)^f \prod_\tau B_\tau^2 X_\tau^j.$$

For $s = t = 6$ the factor in parentheses is less than 15. In particular, if $B \ge 1$ and $X_\tau \le B$ for all τ, the sum for these exponents is at most $(15B^2)^f \prod_\tau X_\tau^j$.

Proof. The generalized index of multiplication by y on the local integer rings at T is $G_T(y)$: multiplication by a uniformizer scales additive volume by q_w^{-1}, because its image in O_w has that index by Lemma 52.1. Multiplication by a unit preserves the ring and its index. The denominator index, the index in $\prod_{w \in T} O_w$ of the elements whose product by y stays integral, is $D_T(y)$ by the same lemma. The T-integer condition gives

$$D_T(y) = \mathrm{N}J(y), \qquad G_T(y)^{1/2} \le \prod_\tau |\tau y|.$$

The second inequality follows by inserting the factors with nonnegative exponents outside T in its norm formula. All factors containing power j in a summand can therefore be bounded together by $\prod_\tau X_\tau^j$.

It remains to sum $D_T(y)^{-s} \prod_\tau \min(1, B_\tau/|\tau y|)^t$. Fix $J(y) = J$; all those y belong to J^{-1}. For each $u > 0$, $B' > 0$ we have

$$\min(1, B'/u)^t = \int_1^\infty t v^{-t-1} \mathbf{1}_{u \le B'v} \, \mathrm{d}v$$

by integration of a power. On a finite set of points with denominator J, apply this identity in each coordinate and interchange the finite sum and the integrals. The resulting count in the product of disks of radii $B_\tau v_\tau$ is at most $9^f \mathrm{N}J \prod_\tau B_\tau^2 v_\tau^2$ by Lemma 52.2. The integrals now separate, each giving $t/(t-2)$. Multiplication by $(\mathrm{N}J)^{-s}$ and summation over J use the ideal-series estimate of Lemma 52.1 at $s-1>1$. This proves the bound on every finite sum, hence on the full nonnegative sum. With $s=t=6$, $\sum_{h\ge1} h^{-5} \le 1+2^{-5}+\int_2^\infty u^{-5}\,\mathrm{d}u = 67/64$. The numerical inequality $(27/2)(67/64)^2 < 15$ proves the last two assertions. □

53. Positive Fourier kernels on the adelic quotient

Definition 53.1. Write $E_\infty = \mathbb{C}^G$, $\sigma a = (\tau a)_\tau$, $\widehat{O} = \prod_w O_w$, $E_f = \bigcup_{D\ge1} D^{-1}\widehat{O}$, and $\mathbb{A}_E = E_\infty \times E_f$. Embed $a \in E$ diagonally as $[a] = (\sigma a, a)$. Use Lebesgue measure on E_∞ and additive measure on E_f normalized by $\mu(\widehat{O}) = 1$. For $y \in E$ put

$$\mathcal{P}_y(z,x) = \exp\left(4\pi i\, \Re \sum_\tau (\tau y) z_\tau\right) \psi_f(-yx).$$

For integrable G_1 on $\mathbb{A}_E$ write $\widehat{G}_1(y) = \int_{\mathbb{A}_E} \mathcal{P}_y G_1$. The mass of a fundamental parallelepiped for σO_E is $V_E = 2^{-f}\sqrt{|\Delta_E|}$ by Lemma 20.1.

Lemma 53.2. *The measure of each coset of $m\widehat{O}$ is m^{-2f} for integral $m \ge 1$. Every $x \in E_f$ is congruent to some $a \in E$ modulo $m\widehat{O}$, uniquely modulo mO_E. Thus*

$$p_m(z,x) = z - \sigma a \bmod \sigma(mO_E)$$

defines an additive map to a real torus which is trivial on the diagonal E. The integral Fourier indices of this torus are parametrized by $y \in (m\varpi)^{-1}O_E$, with ϖ from Proposition 18.4. The corresponding exponentials pull back along p_m to the functions $\mathcal{P}_y$ of Definition 53.1.

Proof. In an integral basis of O_E the quotient by $m\widehat{O}$ inside $\widehat{O}$ has m^{2f} elements, by the coordinate Chinese remainder argument of Theorem 51.2. The measure used here can be constructed by successively partitioning $[0,1)$ into equally long half-open intervals labelled by compatible residue classes modulo $j!\widehat{O}$, $j = 1, 2, \ldots$. At each step divide an interval equally among the lifts of its class. Compatible classes specify a unique point of $\widehat{O}$ by the p-adic digit description in Lemma 16.2. Pushing forward length gives probability m^{-2f} for every coset modulo $m\widehat{O}$, by choosing a divisible factorial. This measure is translation invariant. Indeed translation permutes cosets at each level. Equality for their finite unions extends to all Borel sets: the class of equality for two probability measures is closed under complementation and countable disjoint union. For a family of sets closed under intersections, its smallest class with these two closure properties is also closed under intersections. To see this, fix one set of the family and apply the two closure properties to the sets whose intersection with it is in the class, using differences of nested sets. Then fix an arbitrary set of the class and repeat. Such a class is a sigma algebra, since countable unions can now be made disjoint. The cosets generate the Borel sets because they form a countable base. Extend to E_f by summing translates of this measure on its countably many $\widehat{O}$-cosets. Their countability, the congruence assertion for x, and its uniqueness follow by the proof of Theorem 51.2 with any integral basis of O_E. These facts give the asserted properties of p_m.

The real pairing $(y,z) \mapsto 2\Re\sum_\tau y_\tau z_\tau$ is nondegenerate, by testing a single coordinate against its complex conjugate. On diagonal vectors it is the rational trace pairing. An integral Fourier index means a real linear form with integer values on $\sigma(mO_E)$. In the basis of mO_E its dual for this pairing is rational: the rational trace matrix is invertible by Proposition 18.4. The same proposition gives that lattice of indices as $(m\varpi)^{-1}O_E$. For such y, $\psi_f(-y(x-a)) = 1$ since $x - a \in m\widehat{O}$ and my belongs to the trace dual at each place, by Proposition 18.5. The same proposition gives $\psi_f(-ya) = \exp(-2\pi i \mathrm{Tr}_{E/\mathbb{Q}}(ya))$. This is precisely the phase in the pullback of the exponential at $z - \sigma a$. □

Theorem 53.3. *Let $G_1 : \mathbb{A}_E \to \mathbb{R}_{\ge 0}$ be integrable, and $A_0 \subset E$ finite. Suppose G_1 is lower semicontinuous at each $[a]$, $a \in A_0$, meaning that for every $\eta > 0$ its value nearby is at least $G_1([a]) - \eta$. Then*

$$\sum_{a \in A_0} G_1([a]) \le V_E^{-1} \left(\widehat{G}_1(0) + \sum_{0 \ne y \in E} |\widehat{G}_1(y)| \right).$$

The last sum is the supremum of its finite subsums. With the measure normalization of Definition 53.1 the leading constant is exactly V_E^{-1}.

Proof. Fix $\eta > 0$. Choose m so that each neighborhood in the semicontinuity hypothesis contains $[a] + B \times m\widehat{O}$ for a common open ball B about zero in E_∞. This is possible by the finite-adelic neighborhood basis of Definition 51.1 in integral coordinates. Shrink B so that these neighborhoods are disjoint and its image is injective on the torus $E_\infty/\sigma(mO_E)$. Distinct a's have distinct archimedean images. A full lattice has a shortest nonzero length bounded away from zero by invertibility of its real basis matrix.

In real lattice coordinates on that torus form the product kernel

$$K_h(u) = \prod_{j=1}^{2f} h^{-1} \left| \sum_{l=0}^{h-1} \exp(2\pi i l u_j) \right|^2 \qquad (h \ge 1).$$

It is nonnegative. Its constant Fourier coefficient is one and all coefficients lie in $[0,1]$: the coefficient at k in one coordinate is $\max(0, 1 - |k|/h)$ by multiplying the two finite sums. Its integral for torus probability is one, and the integral outside any neighborhood of zero tends to zero by the product concentration argument of Lemma 14.2. In Lebesgue measure on B we therefore have $\int_B K_h \to m^{2f} V_E$, the real fundamental volume. On $[a] + B \times m\widehat{O}$ its pullback under p_m is $K_h(z - \sigma a)$ for archimedean coordinate z, by Lemma 53.2. The finite-coordinate mass is m^{-2f}. Thus nonnegativity and the disjoint neighborhoods give

$$V_E \sum_{a \in A_0} \max(0, G_1([a]) - \eta) \le \liminf_{h \to \infty} \int_{\mathbb{A}_E} G_1 (K_h \circ p_m).$$

Each integral on the right is a finite linear combination of $\widehat{G}_1(y)$ with constant coefficient one and the other coefficients in $[0,1]$, by Lemma 53.2. It is at most the parenthesis in the assertion by the triangle inequality. Let $\eta \downarrow 0$.

For sharpness take a half-open fundamental parallelepiped P of σO_E centered at zero, whose interior contains zero, and $G_1 = \mathbf{1}_{P \times \widehat{O}}$, $A_0 = \{0\}$. The value at zero is one and the integral is V_E. Integration of a nontrivial additive character on $\widehat{O}$ vanishes: translating by a point of nontrivial value multiplies the integral by that value and preserves the measure. Thus only indices in $\varpi^{-1} O_E$ can contribute by Proposition 18.5. For nonzero such indices, integration of the corresponding nonconstant integer exponential over P gives zero by the elementary one-coordinate integral. Equality holds with these choices. □

Corollary 53.4. *Let T be a finite set of finite places, $C = E_\infty \times \prod_{w \in T} E_w$, and give each O_w additive mass one. Let $l_\tau > 0$, $m_l(z, x) = ((l_\tau z_\tau)_\tau, (x_w)_{w \in T})$, and $U_T = \{(z, x) : x_w \in O_w \text{ for } w \notin T\}$. For $F : C \to \mathbb{R}_{\ge 0}$ integrable, lower semicontinuous at the images under m_l of a finite set of diagonal T-integers A_0, define $G_1 = \mathbf{1}_{U_T}(F \circ m_l)$. Put*

$$d_l = (V_E \prod_\tau l_\tau^2)^{-1}, \qquad \psi_C(u) = \prod_\tau \exp(u_\tau - \bar{u}_\tau)\, \psi_f(-u_T^0/\varpi),$$
$$\xi(y) = \big((2\pi i\, \tau y/(l_\tau\, \tau\varpi))_\tau, (y_w)_{w \in T}\big).$$

Here u_T^0 means extension by zero to the other finite places. Then

$$\sum_{a\in A_0} F(m_l([a])) \le d_l \left(\int_C F + \sum_{\substack{y\neq 0\\ y \text{ a } T\text{-integer}}} \left| \int_C \psi_C(\xi(y)u) F(u)\,\mathrm{d}u \right| \right).$$

Proof. The set U_T is open and closed in the restricted product topology of Definition 53.1. Its finite-coordinate measure factors into mass one on $\prod_{w\notin T} O_w$ and the specified measure on $\prod_{w\in T} E_w$. This follows first on cosets of a product of open subgroups by taking their finite quotient indices, and then on Borel sets by the measure argument in Lemma 53.2. Integration in these coordinates and in the real coordinates follows first for indicators of measurable rectangles, then for nonnegative measurable functions by the same intersection argument on bounded boxes and increasing simple-function approximation in Lemma A.2. Real coordinate scaling in each complex factor divides the integral by l_τ^2. Consequently G_1 is integrable and meets the hypotheses of Theorem 53.3.

Index its transforms by y/ϖ, $y \in E$. If y is nonintegral at some $w \notin T$, translation by an element in O_w changes the character nontrivially while preserving G_1. Such an element exists by the local trace dual in Proposition 18.5, applied with the other components zero; the product trace pairs the field factors separately. Thus the transform vanishes. For a T-integer the factors outside T are trivial by the same trace dual. In the remaining variables put $u_\tau = l_\tau z_\tau$. The archimedean exponent at y/ϖ is $\sum_\tau((\xi(y)u)_\tau - \overline{(\xi(y)u)_\tau})$, since $iv - \overline{iv} = 2i\Re v$. The finite exponent has exactly the factor in ψ_C by Definition 53.1. The change of variables therefore gives the asserted integrals and the factor d_l after applying Theorem 53.3. □

54. Iteration of the one-sided orbital sieve

Lemma 54.1. *For a finite set A in a torsion-free additive group, nonnegative weights $b(s)$ and a nonzero group element h,*

$$\sum_{\substack{s,t\in A\\ s-t=h}} b(s)b(t) \le \frac{1}{4}\left(\sum_{s\in A} b(s)\right)^2.$$

The factor $1/4$ is attained for a suitable two-element set.

Proof. In each coset of $\mathbb{Z}h$ meeting A, choose one representative. Each element of the coset is that representative plus a unique integer multiple of h, by torsion freeness. Color the elements by the parity of that integer. A pair with difference h joins opposite colors, and a given unordered pair occurs in at most one orientation, since $2h \neq 0$. Its total weighted sum is bounded by the product of the total weights B_0, B_1 of the two colors. The inequality $4B_0B_1 \le (B_0+B_1)^2$ follows from $(B_0-B_1)^2 \ge 0$. Give both points of $\{0,h\}$ weight one to obtain equality. □

Definition 54.2. Let C, ψ_C be as in Corollary 53.4, $U = C^\times$, and $u : C \to U$ replace each zero component by one, retaining all nonzero components. Fix an integer $g_0 \ge 1$ with g_0^{-1} a T-integer. Define $I(v) = ((v_\tau^{-1})_\tau, (g_0^{-1}v_w^{-1})_{w\in T})$ for $v \in U$. Fix $n \ge 0$. A family of kernels $J_i : U^i \times U^i \to \mathbb{C}$, $0 \le i \le n+1$, has bounds E_0, β, Λ_i if $E_0 \ge 0$, $\beta : U \to \mathbb{R}_{\ge 0}$, $\Lambda_i : U^i \to \mathbb{R}_{\ge 0}$ and the following properties hold. Write (v,z) for a list with z appended to v.

(1) $|J_0| \le E_0$. For $0 \le i \le n$, $v, w \in U^i$, the function $K(x) = J_{i+1}((v,u(x)),(w,u(x)))$ is integrable and

$$\int_C K = J_i(v,w), \qquad \left| \int_C \psi_C(\xi x) K(x)\,\mathrm{d}x \right| \le \beta(\xi)\Lambda_i(v)\Lambda_i(w) \quad (\xi \in U).$$

(2) J_{n+1} is continuous. For every finite list $v_1, \dots, v_b \in U^{n+1}$ with common last entry, the matrix $(J_{n+1}(v_j, v_k))_{j,k}$ is positive semidefinite, meaning hermitian with nonnegative quadratic form.

(3) For $1 \le i \le n$, $v, w \in U^i$ with last entries z, z' and $\xi = I(z) - I(z') \in U$,

$$|J_i(v, w)| \le \beta(\xi)\Lambda_i(v)\Lambda_i(w).$$

Theorem 54.3. *Use Definition 54.2. Let $A \subset E^\times$ be a finite set of T-units, meaning s, s^{-1} are T-integers for all $s \in A$. For $0 \le i \le n$ and $\tau \in G$ let $l_{\tau i} > 0$, and write*

$$P_i(s) = ((l_{\tau i}\tau s)_\tau, (s_w)_{w\in T}), \quad c_i = (V_E \prod_\tau l_{\tau i}^2)^{-1}, \quad C_i = \prod_{a<i} c_a.$$

For $p = (s_0, \dots, s_{i-1}) \in A^i$ put $p^* = (P_0(s_0), \dots, P_{i-1}(s_{i-1}))$ and set

$$M_i = \sum_{p\in A^i} \Lambda_i(p^*), \quad Q_i = M_i^2/C_i, \qquad H_A = \{g_0^{-1}(s^{-1} - t^{-1}) : s, t \in A,\ s \ne t\}.$$

Define the two frequency sums, using the supremum of finite subsums for the first,

$$e_i = \sum_{\substack{y\ne 0 \\ y \text{ a } T\text{-integer}}} \beta\big(((2\pi i_{\mathbb{C}}\, \tau y/(l_{\tau i}\,\tau\varpi))_\tau, (y_w)_{w\in T})\big)\,,$$

$$d_i = \sum_{y\in H_A} \beta\big(((g_0\tau y/l_{\tau i})_\tau, (y_w)_{w\in T})\big)\,.$$

Here $i_{\mathbb{C}} = \sqrt{-1}$ in the complex exponential convention of Definition 3.1. Suppose the e_i are finite. For each $A' \subset A$ and $0 \le i \le n$,

$$\frac{1}{C_{i+1}} \Re \sum_{s\in A'} \sum_{p,q\in A^i} J_{i+1}((p^*, P_i(s)), (q^*, P_i(s))) \ \le\ E_0 + \sum_{a=0}^{i} Q_a e_a + \frac{1}{4}\sum_{a=1}^{i} Q_a d_{a-1}.$$

In particular, if $Q_a \le H$ and all the displayed frequency sums are at most ϵ, the last upper bound is at most $E_0 + (1 + 5i/4)H\epsilon$.

Proof. For $x \in C$ let $F_i(x)$ be the sum on $p, q \in A^i$ of $J_{i+1}((p^*, u(x)), (q^*, u(x)))$. The first requirement of Definition 54.2 gives integrability, $\int F_i = \sum_{p,q} J_i(p^*, q^*)$, and the Fourier bound $|\int \psi_C(\xi x)F_i(x)\,\mathrm{d}x| \le \beta(\xi)M_i^2$. The function is real and nonnegative, and lower semicontinuous at $P_i(s)$ for $s \in A$. To prove these properties, consider any finite list of tuples of length j, summing J_j over every ordered pair of tuples. At $j = n + 1$, for tuples sharing the final component the sum is real and nonnegative by the quadratic form at the vector of ones. It is continuous in the tuples by the same definition. For $j \le n$ the sum is the integral of the sum at $j + 1$ with one common entry $u(z)$ appended. Induct downwards. Nonnegativity passes through the integral. For a convergent sequence of lists the limiting sum is at most the lower limit of the successive sums: at each fixed z apply the preceding stage, then apply $\int \liminf H_j \le \liminf \int H_j$ for nonnegative functions. This inequality follows by applying Lemma A.2 to the increasing sequence $\inf_{j\ge a} H_j$. This proves lower semicontinuity on the product of the unit groups, whose topology has a countable neighborhood basis at each point from the real and finite valuation balls. Finally $x \mapsto u(x)$ is continuous at $P_i(s)$, since all components there are nonzero. This proves the assertions about F_i.

Write $D_i(A')$ for the unnormalized left side, including its real part. Corollary 53.4 gives $D_i(A') \le c_i(\int F_i + e_i M_i^2)$, because $P_i(s) = m_l([s])$. At $i = 0$ the integral is $J_0 \le E_0$, proving the initial bound. At $i \ge 1$, group the terms of $\int F_i$ according to the two final entries $s, t \in A$ of p, q. The contribution with $s = t$ is $D_{i-1}(A)$. For $s \ne t$, set $h = g_0^{-1}(s^{-1} - t^{-1})$. It is a nonzero T-integer, and

$$I(P_{i-1}(s)) - I(P_{i-1}(t)) = ((g_0\tau h/l_{\tau,i-1})_\tau, (h_w)_{w\in T}) \in U.$$

Use the last bound in Definition 54.2. For each s, let $a(s)$ be the sum of $\Lambda_i(p^*)$ over paths p ending at s. The map $s \mapsto g_0^{-1}s^{-1}$ injects A into the torsion-free additive group of E. For a fixed $h \neq 0$, Lemma 54.1 bounds the sum of $a(s)a(t)$ at that difference by $M_i^2/4$. Consequently the real off-diagonal contribution to $\int F_i$ is at most $d_{i-1}M_i^2/4$. We have shown

$$\frac{D_i(A')}{C_{i+1}} \le \frac{D_{i-1}(A)}{C_i} + Q_i(e_i + d_{i-1}/4).$$

Induction from $i = 0$ proves the displayed estimate, and summing the two finite constant bounds proves its corollary. □

55. The orbital sieve at the arithmetic scales

Lemma 55.1. *Use the arithmetic data of Propositions 12.2 and 20.5. Fix M, d, ϵ as in Proposition 10.5. Put $h = \nu + 1 = m/2$, $l_{\tau i} = |\kappa_\tau|^{2(\nu-i)}/N$ for $0 \le i \le \nu$, and $\rho_\tau = |\kappa_\tau|^{2\nu+1}/(gN)$. Uniformly on $t \le v \le t^2$ for sufficiently large x,*

$$b^{19} \le |\kappa_\tau| \le b^{23}, \quad N \le b^{60h}, \quad V_E \le b^f, \quad k/h^9 \longrightarrow \infty.$$

The frequency scales satisfy

$$f^{-1}\sum_\tau \log \rho_\tau = (m-1)G_0 - \log(gN) \le -m\bar{w} - \tfrac{25}{2}\ell.$$

Proof. By the differentiated product for P_{N_1} in Proposition 18.4, for $z = \tau\zeta_{N_1}$ one has

$$|P'_{N_1}(z)| = N_1 \prod_{\substack{j \mid N_1 \\ j>1}} |1 - z^{N_1/j}|^{\mu(j)}.$$

Every $z^{N_1/j}$ is primitive of order j. Its distance to one is between $1/N_1$ and 2: reducing its angle to the shorter arc gives the lower bound $2\sin(\pi/j) \ge 4/j$ for $j \ge 2$, by the chord inequality $\sin u \ge 2u/\pi$ on $[0, \pi/2]$, which follows from $\sin'' u = -\sin u \le 0$. Only squarefree j contribute, and their number is bounded in terms of M since N_1 has the same prime divisors as M by Proposition 10.3. It follows that $\big|\log|\tau\varpi|\big| = O_{M,d}(1 + \log N_1) = O_{M,d}(1 + \log x)$ by the definition of ϖ in Proposition 18.4 and Proposition 10.5. The range $20\ell \le \log g \le 22\ell$ in the proof of Proposition 12.2 and the formula for κ_τ in Proposition 20.5 prove the first bound. The estimate on N is the upper bound on $\log N$ of Proposition 12.2. By Lemma 20.1, $V_E = (\mathrm{rd}_E/2)^f$, and Lemma 12.1 gives the stated estimate. The bounds $h \le \sqrt{x}$, $k = r + 1 - m \ge x^5/2$ eventually by Proposition 10.5 give the asserted divergence. The product formula for $\prod_\tau |\tau\varpi|$ in Proposition 18.4 gives $f^{-1}\sum_\tau \log|\kappa_\tau| = G_0$. Substitute this and $\log N \ge m\bar{w} + (m-1)G_0 - 15\ell/2$ from Proposition 12.2; then $\log g \ge 20\ell$ proves the last inequality. □

Theorem 55.2. *Use the scales of Lemma 55.1. In Theorem 54.3 set $n = \nu$, $g_0 = g$ and take any finite T containing every finite place dividing g. Put $j_* = k - h^7$, $K_* = h^6$. Let $B_\tau > 0$ be constants. For $\xi = (\xi_\infty, \xi_T) \in U$, define $G_T(\xi_T), D_T(\xi_T)$ by the local products in Theorem 52.3 with y_w replaced by ξ_w. Suppose the bounds in Definition 54.2 satisfy*

$$\beta(\xi) \le b^{2001h^9 f}\frac{G_T(\xi_T)^{j_*/2}}{D_T(\xi_T)^6}\prod_\tau B_\tau\, d_{j_*,6}(|\xi_\tau|^{-1}).$$

For $p = (s_0, \dots, s_{i-1}) \in A^i$ put $z_j = \prod_{a<j} s_a$, $L_{\tau j} = \prod_{a<j} l_{\tau a}$. Assume

$$\Lambda_i(p^*) \le \prod_{j=1}^{i}\left(\frac{G_T(z_j)^{K_*/2}}{D_T(z_j)^6}\prod_\tau d_{K_*,6}(|L_{\tau j}\tau z_j|^{-1})\right).$$

For $A' \subset A$, let $D_i(A')$ be the real sum of the kernel entries in Theorem 54.3 and C_i the product there. Uniformly over these finite sets and $0 \le i \le \nu$, at all sufficiently large x one

has

$$\frac{D_i(A')}{C_{i+1}} \le E_0 + b^{2063h^9 f} \prod_\tau B_\tau \rho_\tau^{j_*}.$$

In particular, if a positive number $\mathfrak{d}$ *satisfies*

$$\mathfrak{d} \prod_\tau B_\tau \rho_\tau^{j_*} \le b^{-7kf},$$

the error is at most $\mathfrak{d}^{-1} b^{-6kf-10}$.

Proof. The lower bound on k in Lemma 55.1 ensures $j_* \ge 0$. For the transforms defining e_i in Theorem 54.3, the reciprocal of the absolute value of the coefficient multiplying τy is $l_{\tau i}|\kappa_\tau|/g$ by Proposition 20.5. For d_i it is $l_{\tau i}/g$. Both are at most ρ_τ, since $|\kappa_\tau| \ge 1$, and $\max(1, \rho_\tau) \le b^{46h}$ by Lemma 55.1. Every member of H_A is a nonzero T-integer because g^{-1} and the inverses of the entries are T-integers. Theorem 52.3 bounds both sums by

$$15^f b^{(2001h^9+92h)f} \prod_\tau B_\tau \rho_\tau^{j_*}.$$

Here the stronger version with individual B_τ in the frequency theorem is used only to bound the damping radii $\max(1, \rho_\tau)$; the constants B_τ in the present statement simply multiply the result.

For the prefix bound, $L_{\tau j}^{-1} \le N^j$ by Lemma 55.1. A tuple is determined by its successive z_j, $1 \le j \le i$, because $s_{j-1} = z_j/z_{j-1}$, with $z_0 = 1$. Thus the sum of the products bounding Λ_i is bounded by the product of independent sums over all possible nonzero T-integers at each prefix. The j-th sum is at most $15^f N^{j(K_*+2)f}$ by Theorem 52.3. In particular

$$M_i \le 15^{if} N^{(K_*+2)i(i+1)f/2}.$$

This also holds if the prefix weight vanishes when a z_j is nonintegral at T, and carries $G_T(z_j)^{K_*/2}$ without a denominator when integral: there $D_T(z_j) = 1$.

By definition $C_i^{-1} = V_E^i \prod_{a<i,\tau} l_{\tau a}^2$. Each $l_{\tau a}^2 \le b^{92(h-1)}$. Using $N \le b^{60h}$, $i(i+1) \le h^2$, $15 \le b$, and $V_E \le b^f$, we obtain

$$Q_i \le b^{(60h^9+120h^3+92h^2+3h)f} \le b^{61h^9 f}$$

eventually, since $h \to \infty$. Apply Theorem 54.3; its prefactor $1 + 5\nu/4$ is at most b eventually by Proposition 10.5. Including $15 \le b$, the error is at most $b^{(2062h^9+92h+2)f} \prod_\tau B_\tau \rho_\tau^{j_*}$, which proves the first estimate. Finally $k/h^9 \to \infty$ and $f \ge 1$, so eventually $2063h^9 f + 10 \le kf$. This proves the last error bound under its stated numerical hypothesis. □

56. The row parameters and the factorial cancellation

Definition 56.1. Use r, k, m, h_2, w_j of Definition 12.3. Put $\Delta_w = 2r^8$, $W_\tau = N_1 \lceil w(\tau)/N_1 \rceil$. For $0 \le i < r$ the row parameters at τ are the integers

$$t_{i,\tau} = \begin{cases} k/2 - 1 - i, & 0 \le i < k-1, \\ h_2 - w_{i+2-k}(\tau) = -W_\tau - (i+2-k)\Delta_w, & k-1 \le i < r. \end{cases}$$

Here r is a power of three, m is even for the sufficiently large x under consideration, and $k = r + 1 - m$ is even. Define the normalized Vandermonde product and the numerical constants

$$\mathcal{D}_\tau = \prod_{0 \le i < l < r} \frac{t_{i,\tau} - t_{l,\tau}}{l - i}, \qquad \mathcal{D} = \prod_{\tau \in G} \mathcal{D}_\tau,$$

$$U_j = \prod_{a=0}^{j-1} \frac{2\pi^{a+1}}{a!}, \qquad C_{\mathrm{B}} = \frac{U_k}{U_r} \pi^{(m-1)k}.$$

Proposition 56.2. *The integers $t_{i,\tau}$ in Definition 56.1 are strictly decreasing in i, and $\mathcal{D}_\tau$ is a positive integer. Its product with C_{B} equals*

$$\frac{2^{1-m}}{(k-1)!}\left(\frac{\Delta_w}{\pi}\right)^{m(m-1)/2}\left(\prod_{j=1}^{m-1} j!\right)\prod_{\substack{0\le i<k-1\\ 1\le j\le m}}(W_\tau+j\Delta_w+k/2-1-i).$$

For all sufficiently large x, uniformly for $\tau\in G,\ t\le v\le t^2$, we have

$$m(k-1)\log w(\tau)\le \log\mathcal{D}_\tau+\sum_{l=k-1}^{r-1}\log(l!),$$

$$\log(C_{\mathrm{B}}\mathcal{D}_\tau)\le m(k-1)\log w(\tau)+O\Big(mk/x+m^2\log x\Big).$$

Proof. The first $k-1$ entries have successive differences one and the final m entries have successive differences Δ_w. Each gap between the two groups is one of the factors in the displayed product. The smallest is $W_\tau+\Delta_w+1-k/2\ge w(\tau)\ge 1$ by Proposition 12.2 and $\Delta_w=2r^8$. Strict decrease follows.

Put $n_i=t_{i,\tau}-t_{r-1,\tau}\ge 0$. The matrix with entries $\binom{n_i}{j}$, $0\le i,j<r$, is integral by the binomial recursion, with $\binom{u}{j}=0$ for $0\le u<j$. The polynomial $\binom{X}{j}$ has leading coefficient $1/j!$. Change its columns to powers of X by triangular elimination. The absolute determinant becomes $\prod_{i<l}(n_i-n_l)/\prod_{j<r}j!$. Indeed the determinant of monomial evaluations in the last variable vanishes at all previous nodes and has their determinant as leading coefficient; induction yields the product of pairwise differences. Since $\prod_{i<l}(l-i)=\prod_{l<r}l!$, this absolute determinant equals $\mathcal{D}_\tau$, proving the integer assertion.

In the numerator Vandermonde of $t_{i,\tau}$, the upper group contributes $\prod_{l=0}^{k-2}l!$, canceling exactly this part of its denominator. The lower group contributes $\Delta_w^{m(m-1)/2}\prod_{j=1}^{m-1}j!$. Consequently all the remaining numerator factors are at least one, and the $m(k-1)$ mixed factors are at least $w(\tau)$, proving the first logarithmic inequality. By $r=k+m-1$ and summation of the arithmetic progression $\sum_{l=k}^{r-1}(l+1)$, the constant in the definition equals

$$C_{\mathrm{B}}=2^{1-m}\pi^{-m(m-1)/2}\prod_{l=k}^{r-1}l!.$$

This cancels all the remaining factorial denominators except $(k-1)!$. It proves the exact formula.

Each mixed gap is at most $w(\tau)+N_1+m\Delta_w+k$. By Proposition 10.5 the additive error is $O_M(x^{42})$. Here $w(\tau)\ge a\exp(-t^2)\ge xb\exp(-t^2)$ in the stated range by Definitions 10.6 and 12.3 and Proposition 12.2. Since $t^2=o(\ell)$ and $b=\exp(\ell)$, the relative error is at most $1/x$ eventually with d,M,ϵ fixed. The inequality $\log(1+u)\le u$ for $u\ge 0$, by integration of $(1+u)^{-1}$, bounds the logarithm of the mixed product by $m(k-1)(\log w(\tau)+1/x)$. Every difference within the lower group is at most $m\Delta_w$ and $\log(m\Delta_w)=O(\log x)$ by Proposition 10.5. The factors $2^{1-m},\pi^{-m(m-1)/2},1/(k-1)!$ are at most one. This proves the second logarithmic bound. □

Theorem 56.3. *Use the data of Theorem 55.2 and Definition 56.1. For every fixed $0<\eta<25/2$, uniformly for $t\le v\le t^2$,*

$$b^{2063h^9f}\mathcal{D}\prod_{\tau\in G}C_{\mathrm{B}}\rho_\tau^{k-h^7}\le b^{-(25/2-\eta)kf}$$

for all sufficiently large x, where $h=m/2$ as in Lemma 55.1. In particular, whenever the two bounds on β,Λ_i in Theorem 55.2 hold with $B_\tau=C_{\mathrm{B}}$, its conclusion takes the form

$$\frac{D_i(A')}{C_{i+1}}\le E_0+\mathcal{D}^{-1}b^{-(25/2-\eta)kf}\quad(A'\subset A,\ 0\le i\le\nu).$$

Proof. Sum the last inequality of Proposition 56.2 and the estimate on the frequency scales of Lemma 55.1, the latter multiplied by $f(k-h^7)$. After adding $2063h^9f\ell$ and dividing by f, the logarithm of the left side is at most

$$-\tfrac{25}{2}k\ell + 2063h^9\ell + \tfrac{25}{2}h^7\ell + m(h^7-1)\bar{w} + O(mk/x + m^2\log x).$$

Here $\bar{w} = f^{-1}\sum_\tau \log w(\tau)$ by Definition 10.6. We have $0 \le \bar{w} \le 2\ell$ by the bounds on the weights in Proposition 12.2. Since $m = 2h$, the terms involving h^9, h^7, mh^7 divided by k tend to zero by Lemma 55.1. Also $m/(x\ell) \to 0$, $m^2\log x/(k\ell) \to 0$ by Proposition 10.5. These comparisons are uniform in the window. The terms after the leading term are therefore at most $\eta k\ell$ eventually. Exponentiating proves the first assertion. The second follows by inserting this estimate in Theorem 55.2. □

57. Finite norm congruences for the column lattice

Lemma 57.1. *Let $F = E^{\mathfrak{c}}$ and $O_F = O_E \cap F$ as in Lemma 20.1. For each prime p, $O_F \otimes \mathbb{Z}_p$ is a product of at most f complete discrete valuation rings A. For each such factor, write $Q = |A/\pi A|$ for the residue cardinality, where π is a uniformizer. Then $Q \ge p$. The corresponding algebra $B = O_E \otimes_{O_F} A$ is either $A \times A$ with its factors interchanged, or a complete discrete valuation ring with uniformizer π and residue field of cardinality Q^2. In the latter case the involution induces $z \mapsto z^Q$ on residues.*

Proof. By Lemma 20.1, $O_E = O_F \oplus \xi O_F$ with $\xi\bar{\xi} = 1$, $(\xi - \bar{\xi})^2 = -1$. This decomposition holds over $\mathbb{Z}_p$, where subtracting the conjugate of $a + b\xi$ still detects b. Thus the fixed ring of $O_E \otimes \mathbb{Z}_p$ is $O_F \otimes \mathbb{Z}_p$. Theorem 18.2 and Lemma 18.1 decompose $O_E \otimes \mathbb{Z}_p$ into complete discrete valuation rings with finite residue fields. Conjugation permutes their identity idempotents, since the only idempotents in a domain are zero and one. On an exchanged pair the invariant ring projects isomorphically onto either factor. On a fixed factor it is the valuation ring of the fixed subfield: the condition of lying in the original valuation ring is nonnegative valuation, and the restricted value group is a nonzero subgroup of $\mathbb{Z}$, since it contains the value of p. A fixed element of least positive valuation is a uniformizer there. The fixed ring is closed under compatible limits, and its residue field embeds in the finite residue field of the original ring. This proves the decomposition asserted for $O_F \otimes \mathbb{Z}_p$. Its rank over $\mathbb{Z}_p$ is f by Lemma 20.1. Modulo p each factor contributes at least one dimension, and each residue field contains $\mathbf{F}_p$.

Over any such A, the algebra B is given by the monic polynomial $X^2 - (\xi + \bar{\xi})X + 1$, by the same basis. Its discriminant is -1, so it has distinct roots over the algebraic closure of the residue field, including in characteristic two. Apply Lemma 18.1. In the split case conjugation exchanges the roots. In the field case the two residue roots are interchanged by $z \mapsto z^Q$: this map fixes every element of the field of size Q by cyclicity in the proof of Lemma 22.4, and its fixed points satisfy the degree-Q equation $z^Q = z$. It cannot fix the quadratic generator. □

Definition 57.2. At a factor A of Lemma 57.1, set $H(z) = \sum_{i=1}^k \bar{z}_i z_i$ on B^k. For $t \in A$ and $n \ge 1$ put

$$A_n(t) = Q^{-(2k-1)n}\#\{z \in (B/\pi^n B)^k : H(z) \equiv t \bmod \pi^n A\}, \qquad A_0(t) = 1.$$

In this section $k \ge 1$. Write $\sigma = 1$ in the split case and $\sigma = (-1)^k$ in the field case. On A, B^k take the translation-invariant probability measures determined by uniform counting on finite quotients. Thus $Q^{-n}A_n(t)$ is the probability of the indicated congruence.

Proposition 57.3. *In Definition 57.2 one has*

$$A_1(t) = \begin{cases} 1 - \sigma Q^{-k}, & t \notin \pi A, \\ 1 + \sigma(Q-1)Q^{-k}, & t \in \pi A. \end{cases}$$

The reduction of H has a surjective differential at every residue vector other than the zero vector.

Proof. Write $C_k(u)$ for the unnormalized count over the residue field at u. Begin with $C_0(u) = \mathbf{1}_{u=0}$. In the field case the one-coordinate norm is z^{Q+1} by Lemma 57.1. The cyclic group of order Q^2-1 has kernel of size $Q+1$ under this power. Its image is the full group of nonzero fixed elements by its size $Q-1$. Only zero has zero norm. Summing the one-coordinate counts at $u - H(z)$ gives $C_{k+1}(u) = (Q+1)Q^{2k} - QC_k(u)$. In the split case the norm is xy. There are $Q-1$ pairs at $u \neq 0$ and $2Q-1$ at zero, giving $C_{k+1}(u) = (Q-1)Q^{2k} + QC_k(u)$. Substitution in these recurrences, starting at $k=1$ with the counts just given, proves the formulas after division by Q^{2k-1}.

The differential is $v \mapsto \sum_i (\bar{z}_i v_i + \bar{v}_i z_i)$. For an inert nonzero coordinate its image is the image of the residue trace. This trace has nonzero image: $s + s^Q$ cannot vanish on all Q^2 elements, being a nonzero polynomial of degree Q. The trace is linear over the fixed field and hence onto it. In the split case the derivative of a product xy is $x\,dy + y\,dx$, which is onto whenever either entry is nonzero. □

Theorem 57.4. *Set $u = 1 - \sigma Q^{-k}$ and $v = 1 + \sigma(Q-1)Q^{-k} - Q^{1-2k}$. For $n \geq 2$ one has*

$$A_n(t) = \begin{cases} u, & t \notin \pi A, \\ v, & t \in \pi A \setminus \pi^2 A, \\ v + Q^{2-2k} A_{n-2}(t/\pi^2), & t \in \pi^2 A. \end{cases}$$

For $k \geq 2$, $|A_n(t) - 1| \leq 2Q^{1-k}$ for every n, t, and $\alpha_A(t) = \lim_{n\to\infty} A_n(t)$ exists for every t; at $t = 0$ it equals $v/(1-Q^{2-2k})$. In all ranks and for $t \neq 0$ the sequence is constant as soon as $n > v_A(t)$, where $v_A(\pi) = 1$. In rank one at $t = 0$ the split values are $A_n(0) = 1 + n(1-Q^{-1})$; in the field case the values are 1 at even depths and Q^{-1} at odd depths.

Proof. Above any nonzero residue solution, to lift a solution modulo π^j to one modulo π^{j+1} we add $\pi^j z'$ to a representative. The linear term modulo π^{j+1} is $\pi^j\, dH(z')$, with the differential at that residue solution. The quadratic term is divisible by π^{2j}. Proposition 57.3 gives exactly Q^{2k-1} choices at each step. This part of the normalized count is thus u at a unit value and v at a nonunit, since at zero one residue solution has been removed. Above the zero vector write $z = \pi z'$. Its norm is $\pi^2 H(z')$. If $\pi^2 \mid t$, the vector z' is specified modulo π^{n-1} and its congruence only uses it modulo π^{n-2}. This extra digit has Q^{2k} choices, giving the normalized contribution $Q^{2-2k} A_{n-2}(t/\pi^2)$; for $n = 2$ this interpretation gives $A_0 = 1$.

For $k \geq 2$ put $c = Q^{1-k} \leq 1/2$. One has $|u - 1| \leq c$ and $|v - 1| \leq (1 - 1/Q)c + Q^{-k}c \leq c$. Proposition 57.3 handles $n = 1$. Inductively $c + c^2(1 + 2c) \leq 2c$, proving the bound in the last case as well. Iterating the recurrence at a nonzero t ends at a unit or at a value of valuation one, at a level at least one or two, respectively, when $n > v_A(t)$. At $t = 0$ both parities converge to the sum of the same geometric series when $k \geq 2$, since $Q^{2-2k} < 1$. In rank one the coefficient Q^{2-2k} is one. At zero the increment in two steps is $2(1 - Q^{-1})$ in the split case and zero in the field case. The initial values at depths zero and one now give the two formulas asserted. □

58. The finite Fourier integral for one represented element

Lemma 58.1. *Use Definition 57.2, and let χ be an additive character of A of conductor $n \geq 1$, meaning it is trivial on $\pi^n A$ and nontrivial on $\pi^{n-1} A$. For uniform probability on B^k,*

$$G(\chi) = \int_{B^k} \chi(H(z))\, \mathrm{d}z, \qquad |G(\chi)| = Q^{-kn}.$$

For the trivial character $G(1) = 1$. There are $Q^n - Q^{n-1}$ characters of conductor n. For any $t \in A$ the sum of $G(\chi)\chi(-t)$ over characters of conductor at most n, including the trivial character, is $A_n(t)$.

Proof. For a finite abelian group, characters on a subgroup extend across a cyclic quotient generated by a of order d modulo the subgroup: choose any d-th root of the old value at da as the value at a. This gives exactly d extensions. Beginning with the trivial subgroup and adjoining one element at a time shows the dual of a finite abelian group has the group's order. A character nontrivial at a given nonzero b is obtained by extending one from $\langle b\rangle$. Multiplication by this character shows the sum of all characters at b is zero. Applied to $A/\pi^n A$, this counts the characters and writes the indicator of a residue class as $Q^{-n}\sum_\chi \chi(H(z)-t)$. Definition 57.2 gives the last assertion.

In the squared absolute value of $G(\chi)$ write the two variables as $z+y, z$. The inner average in z is the average of the character with argument $\sum_i \operatorname{Tr}_{B/A}(\bar y_i z_i)$, times $\chi(H(y))$. The trace here is $b+\bar b$. The bilinear pairing $(y,z)\mapsto \operatorname{Tr}(\bar y z)$ on B has determinant a unit. Indeed in $1,\xi$ the matrix has entries $2, \xi+\bar\xi, \xi+\bar\xi, 2$ and determinant $-(\xi-\bar\xi)^2=1$. If any coefficient of y in this basis has valuation less than n, the image ideal of the linear form on B^k therefore contains $\pi^{n-1}A$. The average of its character vanishes by translation by an argument with character different from one. For $y\in\pi^n B^k$, both characters are one. The measure of the latter set is Q^{-2kn}, giving the claimed absolute value. □

Theorem 58.2. *Let $k\ge 3$. Index the factors of Lemma 57.1 by v, with residues Q_v, and write A_v, B_v for their rings. For any set T of such factors and $t_v\in A_v$ at each $v\notin T$, the product $P_T(t)=\prod_{v\notin T}\alpha_{A_v}(t_v)$ converges to a positive number. Its logarithm, and the logarithm of any product with the factors replaced by $A_{n_v}(t_v)$ at arbitrary levels $n_v\ge 0$, have absolute value at most $12f\,2^{1-k}$. At $t_v=1$ for all $v\notin T$ one has the exact product*

$$P_T(1)=\prod_{v\notin T}(1-\sigma_v Q_v^{-k}),\qquad |\log P_T(1)|\le 4f\,2^{-k}.$$

For $T=\varnothing$ and $t=(t_v)\in\prod_v A_v$, put $F_f=\bigcup_{m\ge1} m^{-1}\prod_v A_v$. Give $\prod_v B_v^k$ probability measure. Normalize additive measure on F_f to give the trace dual of $\prod_v A_v$ mass one. With the character of Proposition 18.5, the absolutely convergent integral satisfies

$$\int_{F_f}\left(\int_{\prod_v B_v^k}\psi_f\big(z(H(y)-t)/2\big)\,\mathrm{d}y\right)\mathrm{d}z=P_\varnothing(t).$$

Proof. Theorem 57.4 bounds the distance of each factor from one by $2Q_v^{1-k}\le 1/2$. Integrating $1/u$ between it and one bounds its logarithm by $4Q_v^{1-k}$. There are at most f factors above each p and $Q_v\ge p$ by Lemma 57.1. For $s>1$ monotonicity gives $\sum_{j\ge2} j^{-s}\le 2^{-s}(1+2/(s-1))$. With $s=k-1$ this proves the uniform bound and convergence. At one the exact value is Theorem 57.4, and the same argument uses $2Q_v^{-k}$ and $s=k$.

For $z,w\in F_f$ the character $\psi_f(zw/2)$ equals the product of the rational finite characters at $\operatorname{Tr}_{F\otimes\mathbb{Q}_p/\mathbb{Q}_p}(z_p w_p)$: the trace from E of a scalar in F is twice its trace from F. Thus no invertibility of two in any A_v is required. Write $D^\vee$ for the annihilator of $\widehat A=\prod_v A_v$. It is also the intersection with F_f of the trace dual of $\widehat O_E=\prod_v B_v$ under the full E-trace, because $\operatorname{Tr}_{E/F}(\xi/i)=1$ and $\operatorname{Tr}_{E/F}(\widehat O_E)=\widehat A$. In an integral basis of O_F the trace matrix is nonsingular: its real form on nonzero vectors is $\sum_\tau x_\tau^2>0$ by the embedding calculation in Lemma 20.1. It identifies $F_f/D^\vee$ with the additive characters of $\widehat A$ factoring through some $\widehat A/m\widehat A$. Indeed in the resulting dual basis the classes killed by m are $m^{-1}\widehat{\mathbb{Z}}^f/\widehat{\mathbb{Z}}^f$, giving m^f distinct characters by the rational character calculation of Proposition 18.5; this exhausts the characters by Lemma 58.1. Here $\widehat{\mathbb{Z}}=\prod_p\mathbb{Z}_p$. The finite quotients decompose over the factors v, so each character tuple is trivial except at finitely many v, and every such tuple occurs by choosing m divisible by its finitely many conductors in the local rings.

The inner integral is constant on each coset of $D^\vee$ and equals $\prod_v G(\chi_v)\chi_v(-t_v)$ there by the finite-quotient calculation of Lemma 58.1. Its absolute integral is finite. The sum of the

absolute local terms at positive conductor is at most $\sum_{n\ge1} Q_v^{(1-k)n} \le (4/3)Q_v^{1-k}$. The sum of these bounds over v is finite by the preceding series estimate. Expanding finite products and bounding $\prod(1+c_v) \le \exp(\sum c_v)$ justifies the absolutely convergent expansion over all tuples. Each local character sum has value $\alpha_{A_v}(t_v)$ by Lemma 58.1 and Theorem 57.4. This proves the integral formula. □

Corollary 58.3. *Use the arithmetic data of Proposition 12.2 and a finite set T of factors as above. Give every B_v^k and A_v mass one. Off T impose integrality on vectors. At finitely many $v \notin T$ impose also $H(z_v) \in t_v + \pi_v^{n_v} A_v$, with $n_v \ge 0$ and $t_v \in \mathrm{Frac}(A_v)$. Multiply the resulting measure by $\prod_v Q_v^{n_v}$, taking $n_v = 0$ at unused places. For every fixed $D > 0$ and all sufficiently large x the result is at most $1 + b^{-D}$, uniformly in T, the values and the exponents. Integrating an additional nonnegative test depending only on T multiplies its integral by that same factor after normalization off T.*

Proof. A nonintegral target at a constrained place makes that congruence empty. Otherwise independence of finite coordinate sets and Definition 57.2 give exactly $\prod_{v\notin T} A_{n_v}(t_v)$ after normalization. The identity with a test follows first for indicators by multiplication of measures and then by increasing nonnegative step functions, as in Lemma A.2. By Proposition 12.2 and Lemma 55.1, $f \le x^{12}$ eventually, $k/h^9 \to \infty$, $h = m/2 \ge \sqrt{x}/2$. Since $\ell = x^\epsilon$ with $0 < \epsilon < 1/10$ in Proposition 10.5, $12f2^{1-k} = o(\exp(-D\ell))$. Theorem 58.2 and $\exp u \le 1 + 2u$ for sufficiently small $u \ge 0$, by the exponential series, prove the estimate. □

59. Unit vectors and the masses of the definite lattices

Lemma 59.1. *On E^j write $h(u,v) = \sum_i \bar{u}_i v_i$, $H(u) = h(u,u)$. If $x, y \in E^j$ have the same nonzero norm, a rational isometry sends x to y. At every factor B of Lemma 57.1, the integral isometry group of B^j is transitive on vectors with $H(x) = 1$.*

Proof. For $x \ne y$ put $u = x - y$, $a = h(u,x)$. Then $a + \bar{a} = H(u) \ne 0$, since the form is positive definite in every complex embedding. In particular $a \ne 0$. The map $z \mapsto z - uh(u,z)/a$ sends x to y. Its two cross terms and its quadratic term in h cancel because $1/a + 1/\bar{a} = H(u)/(a\bar{a})$. Hence it is an isometry, since a linear map preserving a nondegenerate form in finite dimension is injective and onto. When $x = y$ use the identity.

For split $B = A \times A$, a unit vector is a column and covector (v, λ) satisfying $\lambda(v) = 1$ over A. The kernel of λ has a free basis: use an invertible coefficient of λ to solve for one coordinate. Adjoining v to that basis gives an invertible matrix, and the action by a matrix on columns and its inverse transpose on covectors sends a standard pair to this pair.

In the field case every unit of A is a norm of a unit of B. The residue assertion was proved in Proposition 57.3. Given a solution modulo π^n, correcting it by a factor $1 + \pi^n w$ corrects its norm at the next digit by $\mathrm{Tr}(w)$ times the old unit residue. This trace is onto by the same proposition. Compatible corrections converge by Lemma 57.1. A norm-one x involves a unit coordinate, so adjoining suitable coordinate vectors gives a basis. Subtract $xh(x,z)$ from those other basis elements. They form a basis of $x^\perp$, with unimodular hermitian Gram matrix since the full basis determinant is a unit and the first block is 1. Any positive-rank free lattice with unimodular hermitian Gram matrix over B has a vector of unit norm. Otherwise all residue norms would vanish and expansion at $u + cw$ would give $\mathrm{Tr}(c\,h(u,w)) = 0$ for all residues c, u, w. Trace surjectivity would force the entire residue matrix to vanish. Scale a lift of a vector of unit norm to have norm one by the unit norm surjectivity just proved. Continuing on its complement produces an integral orthonormal basis starting with the given vector. □

Definition 59.2. For $j \ge 1$ let $\mathcal{L}_j$ consist of the rational isometry classes of full O_E-lattices in E^j locally isometric at every finite place to the standard B_v^j. Write $U(L)$ for the rational isometries preserving L. Set $\mathfrak{m}_j = \sum_{[L]\in\mathcal{L}_j} |U(L)|^{-1}$, and give the zero lattice mass $\mathfrak{m}_0 = 1$. For $t \in F$ totally positive put $r_L(t) = \#\{x \in L : H(x) = t\}$.

Theorem 59.3. *The classes and stabilizers in Definition 59.2 are finite, each $r_L(t)$ is finite, and for all $j \geq 1$*

$$\sum_{[L]\in\mathcal{L}_j} \frac{r_L(1)}{|U(L)|} = \mathfrak{m}_{j-1}.$$

Passing from the full integral level to a subgroup of finite index I multiplies both the class mass and the weighted sum of $r_L(t)$ by I.

Proof. Use Lemma 13.4 in rank j. Its lattice description applies to every class here: a lattice and the standard lattice lie between two common scalar multiples of the latter. Away from the divisors of that scalar the identity can be used as local isometry. A vector of norm t satisfies $\sum_{\tau,i} |\tau(x_i)|^2 = \sum_\tau \tau(t)$, so it lies in a bounded subset of a real lattice and there are only finitely many possibilities.

Mark a vector x of norm one in L. Its pairing with any $y \in L$ is integral at every finite place by the local isometries. Thus $h(x, y) \in O_E$ by Theorem 18.2, and $L = O_E x \perp (L \cap x^\perp)$ by subtracting $xh(x, y)$. Use Lemma 59.1 to take x to the last standard vector rationally. Its orthogonal complement lattice is locally standard by the integral assertion of that lemma. Its rational class is independent of the chosen isometry since two choices differ by a rational isometry fixing the last vector. Conversely adjoining the standard last vector to any lattice of rank $j - 1$ constructs a marked lattice, inverse to this assignment up to isometry. The stabilizer of a marked pair is precisely the isometry group of its complement, by restriction and extension by the identity on the last line. In a finite group an orbit of size a has stabilizer of order $|U(L)|/a$, by the bijection with its cosets. Summing orbit contributions $a/|U(L)|$ proves the formula.

For a fixed full-level class the additional level consists of a finite coset set of size I, on which $U(L)$ acts. The stabilizer of a point is exactly the isometry group preserving that choice of level. Applying the same orbit identity to this set with the constant weights 1 and $r_L(t)$ proves the last assertion. □

Proposition 59.4. *For Lebesgue measure on $\mathbb{C}^j$ the image under $z \mapsto \sum |z_i|^2$ has density $\pi^j u^{j-1}/(j-1)!$ on $u > 0$. For $j \geq 3$ define*

$$B_j(t) = \frac{|\Delta_E|^{1/4}}{V_E^j} \prod_\tau \frac{\pi^j\, \tau(t)^{j-1}}{(j-1)!}\, P_\varnothing(t)$$

at totally positive $t \in O_F$, using Theorem 58.2. If the normalized class average of $r_L(1)$ equals $B_j(1)$, then the consecutive masses satisfy the exact formula

$$\frac{\mathfrak{m}_j}{\mathfrak{m}_{j-1}} = \left(\frac{\mathrm{rd}_E^{j-1/2}(j-1)!}{(2\pi)^j}\right)^f \prod_v (1 - \sigma_v Q_v^{-j})^{-1}.$$

The absolute error in its logarithm after omitting the product is at most $4f\, 2^{-j}$.

Proof. For one complex coordinate the disk of squared radius u has area πu, by integrating in polar coordinates. Summing independent nonnegative squared radii convolves their densities: integration from 0 to u sends $\pi(\pi^i v^{i-1}/(i-1)!)$ to $\pi^{i+1}u^i/i!$. Repeated integration proves the density. In the stated implication the normalized class average is $\mathfrak{m}_{j-1}/\mathfrak{m}_j$ by Theorem 59.3. Substitute $V_E = 2^{-f}|\Delta_E|^{1/2}$ from Lemma 20.1 and $\mathrm{rd}_E = |\Delta_E|^{1/(2f)}$ from Lemma 12.1. Taking the reciprocal yields the displayed formula, and Theorem 58.2 bounds the logarithm. □

60. The rank-one Hecke identity at one split place

Definition 60.1. Let A be a complete discrete valuation ring of the kind in Lemma 57.1, $L = \mathrm{Frac}(A)$, $Q = |A/\pi A|$, and $|\pi| = Q^{-1}$. Fix $j \geq 2$. On compactly supported locally

constant complex functions on $M_{j,2}(L)$ put

$$U_b\phi(X) = |\det b|^{-1}\phi(b^{-1}X), \qquad V_c\phi(X) = |\det c|^{j/2}\phi(Xc)$$

for $b \in \mathrm{GL}_j(L)$, $c \in \mathrm{GL}_2(L)$, and put $\phi_0 = \mathbf{1}_{M_{j,2}(A)}$. These commuting actions preserve the integral of $|\phi|^2$ for matrix measure giving $M_{j,2}(A)$ mass one. Indeed a diagonal multiplication has modulus given by the product of its entry moduli by the coset counts, and an invertible matrix is a product of integral basis changes and a diagonal matrix by the elimination in Lemma 20.2. There are two columns and j rows.

Lemma 60.2. *Let $\mathcal{D}$ be the set of index-Q inverse images in A^2 of hyperplanes of its residue space, using row coordinates. Choose c_D with $A^2c_D = D$, $\det c_D = \pi$. For every linear functional λ invariant under all U_b and every $g \in \mathrm{GL}_2(L)$, define $F(g) = \lambda(V_g\phi_0)$. Then*

$$\sum_{D\in\mathcal{D}} F(gc_D^{-1}) = (Q^{j/2} + Q^{1-j/2})F(g).$$

Proof. For rank s let $[s]_Q = (Q^s - 1)/(Q - 1)$, with $[0]_Q = 0$. The hyperplanes in residue dimension s are the kernels of the nonzero linear forms modulo scalar, so there are $[s]_Q$. Their inverse images have bases of determinant π up to a unit. Lift a residue basis putting the hyperplane in standard position (the lifted determinant is a unit), and multiply a complementary basis vector by π. Adjust one basis element by a unit to prescribe determinant π. This constructs the c_D, and constructs b_H for index-Q column lattices b_HA^j indexed by hyperplanes H in residue dimension j.

For an integral matrix X of residue rank r, the number of these column lattices containing its two columns is $[j - r]_Q$. The number of the row lattices D containing all its rows is $[2 - r]_Q$. Indeed row rank equals column rank by elimination over the residue field. From $[j - r]_Q = [j - 2]_Q + Q^{j-2}[2 - r]_Q$ we get the identity of functions

$$Q^{-1}\sum_H U_{b_H}\phi_0 = [j - 2]_Q\phi_0 + Q^{j-2-j/2}\sum_D V_{c_D^{-1}}\phi_0.$$

Outside the integral matrices every indicator occurring here is zero. Apply λV_g, which is left invariant by Definition 60.1. The left side gives $Q^{-1}[j]_QF(g)$. Now $Q^{-1}[j]_Q - [j - 2]_Q = Q^{j-2} + Q^{-1}$, giving the eigenvalue. □

Theorem 60.3. *If $j > 2$ and F from Lemma 60.2 is bounded on $\mathrm{SL}_2(L)$, then $F(g) = 0$ on $\mathrm{SL}_2(L)$. In particular $\lambda(\phi_0) = 0$.*

Proof. For $a \in L^\times$ one has $V_{aI_2} = U_{a^{-1}I_j}$ by Definition 60.1, so $F(ag) = F(g)$. Apply Lemma 60.2 twice and set $\Lambda = Q^{j/2} + Q^{1-j/2}$. For $g \in \mathrm{SL}_2(L)$,

$$\Lambda^2F(g) = \sum_{D,D'} F(g\,\pi c_D^{-1}c_{D'}^{-1}).$$

Each argument on the right has determinant one. There are $(Q + 1)^2$ terms. With $u = Q^{j/2-1} > 1$ one has $\Lambda - (Q+1) = (u-1)(Q-u^{-1}) > 0$. Taking absolute values and the supremum over g yields $\Lambda^2 \sup|F| \le (Q+1)^2 \sup|F|$. Its finiteness forces it to be zero. □

61. Reduction for the real subfield in rank one

Definition 61.1. For $F = E^{\mathfrak{c}}$ set $\widehat{A} = \prod_v A_v$, $F_\infty = \mathbb{R}^\Phi$ and $\mathbb{A}_F = F_\infty \times F_f$, using the rings of Lemma 57.1 and the finite adeles of Theorem 58.2. Use the usual real absolute values and $|z|_v = Q_v^{-v_A(z)}$ at finite places. An idele is a tuple of nonzero components which are units at almost all finite places. Set $s(a) = \prod_\tau |a_\tau| \prod_v |a_v|_v$ on ideles. For $r > 0$ write $d(r)$ for the idele equal to r at all infinite places and 1 elsewhere. For a an idele and $u \in \mathbb{A}_F$, put $m(a) = \mathrm{diag}(a, a^{-1})$ and $n(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$. The group $\mathrm{SL}_2(\mathbb{A}_F)$ consists of the tuples of determinant-one matrices integral at almost all finite places. Neighborhoods have

unrestricted integral components away from finitely many places. The same convention uses unit components for the group of ideles.

Lemma 61.2. *The real image of O_F has covolume $V_F = |\Delta_E|^{1/4}$. For $n \geq 1$ and an invertible adelic n-by-n matrix g with integral inverse and entries at almost all finite places, the lattice*

$$\{xg_\infty : x \in F^n,\ xg_f \in \widehat{A}^n\} \ \subset\ (F_\infty)^n$$

has covolume $V_F^n s(\det g)$. In particular $s(a) = 1$ for every $a \in F^\times$. The quotient of $\mathbb{A}_F$ by the additive group F has a compact set of representatives in the sense that every class meets the set. There exists a compact set of ideles meeting every $F^\times$-orbit of ideles with $s(a) = 1$.

Proof. Lemma 20.1 gives the determinant V_F in an integral basis of O_F. For the finite congruences repeat the integer-basis count in Lemma 20.2: choose one integer clearing all finite denominators of g, g^{-1}. The local quotient indices multiply by the Chinese remainder argument in that lemma. An integral elementary row or column operation over A_v preserves the index, and a diagonal entry π_v^e has index Q_v^e when $e \geq 0$, with the inverse index for $e < 0$. The preimage by g_v thus contributes $|\det g_v|_v$ to the covolume. The real multiplier contributes $\prod_\tau |\det g_\tau|$ by change of basis. For rational scalar $g = a$, the lattice xa in the assertion is O_F itself; here integrality at all places is seen in its integral basis. The scalar product formula follows.

Given a finite adele, the same basis and the Chinese remainder argument in Lemma 53.2 supply a rational element of F congruent to it modulo $\widehat{A}$. Translation by O_F then puts the infinite part in a fixed closed fundamental parallelepiped. That parallelepiped times $\widehat{A}$ is compact. Indeed the integral rings are inverse limits of finite quotients, and a sequence in their countable product has a convergent subsequence by successive extraction on the countably many residue coordinates. The digit metrics give the product topology, for example by summing bounded coordinate distances with coefficients 2^{-i}.

If $s(a) = 1$, the first assertion with $n = 1$ gives a lattice of covolume V_F, independently of a. By the cube argument in Lemma 13.4, there is $0 \neq b \in F$ with $ba_f \in \widehat{A}$ and all $|\tau(b)a_\tau| \leq C$ for a fixed $C > 1$. The product formula now gives

$$1 \leq \prod_v |ba_v|_v^{-1} = \prod_\tau |\tau(b)a_\tau| \leq C^f.$$

In particular every real modulus is at least $C^{-(f-1)}$. At finite places the nonnegative valuation exponents have product of their Q_v-powers at most C^f. There are only finitely many possibilities by $Q_v \geq p$ and the bound on the number of places over p in Lemma 57.1. Each choice of exponents gives a translate of $\prod_v A_v^\times$, compact by the same digit argument. The indicated archimedean bounds complete the construction of a compact set. □

Theorem 61.3. *There are $r_0 > 0$ and a compact $\mathcal{K} \subset \mathrm{SL}_2(\mathbb{A}_F)$, depending only on F, such that each $g \in \mathrm{SL}_2(\mathbb{A}_F)$ has the form*

$$g = \gamma\, m(d(r))\, \kappa, \qquad \gamma \in \mathrm{SL}_2(F), \quad r \geq r_0, \quad \kappa \in \mathcal{K}.$$

Proof. By Lemma 61.2 the row lattice defined by g has covolume V_F^2. The cube argument in Lemma 13.4 gives a nonzero $w \in F^2$ with wg_f integral everywhere and $\|(wg)_\tau\|_2 \leq C_F$ at all infinite places for a constant $C_F \geq 1$. Complete w to the bottom row of a matrix γ_1 of determinant one over F, by solving the single nontrivial linear equation for its first row. At a real place a rotation sends that bottom row of $\gamma_1 g$ to $(0, b_\tau)$ with $b_\tau = \|(wg)_\tau\|_2 > 0$. At a finite place take a component b_v of the row of maximal absolute value. Dividing the row by b_v gives an integral row containing a unit. Complete it to an integral determinant-one matrix by solving the determinant equation using that unit. Its inverse takes the original row to $(0, b_v)$. These changes of basis give a factorization $\gamma_1 g = n(u)m(a)k$, where k lies in the compact product of the rotation groups and $\mathrm{SL}_2(A_v)$, and $a = b^{-1}$. At almost every

finite place b_v is a unit because w has a unit entry there and g_v is integrally invertible. Therefore $s(a) \geq C_F^{-f}$.

Put $r = s(a)^{1/f} \geq C_F^{-1}$. The idele $a/d(r)$ has size one. By Lemma 61.2, left multiplication of $\gamma_1 g$ by a rational $m(c)$ puts $ca/d(r)$ in a fixed compact set of ideles. The unipotent parameter becomes c^2u. Left multiplication by a rational upper unipotent now puts that parameter in the fixed compact set of additive representatives from the same lemma. Write the result $n(u')m(d(r))m(a')k$. Moving $m(d(r))$ to the left replaces u' by $d(r)^{-2}u'$. The finite coordinates are unchanged, while r^{-2} is bounded. These parameters all lie in a compact set by taking the continuous image of the compact set of u' times the interval $[0, C_F^2]$ of real multipliers on the infinite coordinates. The remaining factors lie in compact sets as well. Inverting the rational multipliers gives the asserted representation. □

62. Quadratic Fourier transformations in rank one

Definition 62.1. Write $H(y) = \sum_{i=1}^{j} \bar{y}_i y_i$ on $\mathbb{A}_E^j$, and $B(x, y) = H(x + y) - H(x) - H(y)$. On $\mathbb{A}_F$ use

$$\Psi(u) = \exp(2\pi i \textstyle\sum_\tau u_\tau)\psi_f(-u_f/2).$$

It equals one on F by Proposition 18.5. A test is a finite linear combination of products of real Schwartz functions (smooth with all derivatives decreasing faster than any inverse power) and locally constant compactly supported finite functions. Each finite function is a finite sum of indicators of cosets of $m\widehat{O}_E^j$ for a common integer $m \geq 1$: cover its compact support by finitely many cosets of constancy and refine their levels. For the proof below the real functions needed are $\prod_\tau \exp(-\pi z_\tau H(y_\tau))$ with $\Re z_\tau > 0$. Put $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. On the generators of Definition 61.1 use

$$W(n(u))\phi(x) = \Psi(-uH(x))\phi(x),$$
$$W(m(a))\phi(x) = s(a)^j \phi(ax),$$
$$W(w)\phi(x) = \int_{\mathbb{A}_E^j} \phi(y)\Psi(-B(x, y))\,\mathrm{d}y.$$

In this Fourier integral take Lebesgue measure at infinity and give $\widehat{O}_E^j$ finite mass V_E^{-j}. In particular the diagonal copy of E^j has covolume one.

Lemma 62.2. *Each of the three operators in Definition 62.1 is a product of local unitary operators. The Fourier operator squared sends $\phi(x)$ to $\phi(-x)$. For rational generators the functional $\Theta(\phi) = \sum_{x \in E^j} \phi(x)$ is invariant.*

Proof. Use local Fourier measure at a real place equal to 2^j times Lebesgue measure on $\mathbb{C}^j$, since the real bilinear form is $2\Re \sum_i \bar{x}_i y_i$. At a finite place give B_v mass $[B_v^\vee : B_v]^{-1/2}$ in each coordinate, where $B_v^\vee$ is its annihilator under the finite character of $B(x, y)$. The trace matrix in Proposition 18.5 gives $\prod_v [B_v^\vee : B_v] = |\Delta_E|$. For any full additive lattice D at that place its annihilator has reciprocal mass: for cosets of two nested lattices their dual quotient has the same size by the invertible trace matrix, or by using its inverse transpose in a $\mathbb{Z}_p$-basis. Thus the local transform of a coset indicator is its mass times a character times $\mathbf{1}_{D^\vee}$, by translation orthogonality. Applying this twice gives reflection in zero, and multiplying two such formulas and integrating gives preservation of the inner product, by the orthogonality in Lemma 58.1. The product of these measures is the measure prescribed in Definition 62.1, since $2^f/\sqrt{|\Delta_E|} = 1/V_E$ by Lemma 20.1.

For ordinary real Fourier transformation with phase $\exp(-2\pi i x \cdot y)$, integration by parts and differentiation under the integral give rapid decay of the transform and all its derivatives on Schwartz functions. Inserting $\exp(-\pi\delta|y|^2)$ into the inverse integral yields convolution with $\delta^{-n/2}\exp(-\pi|x|^2/\delta)$ in real dimension n. This follows by multiplying the one-dimensional calculation of Lemma 4.2 and interchanging the absolutely integrable integrands. These Gaussian kernels have mass one and their mass outside each fixed

neighborhood of zero tends to zero by scaling. Their convolution converges to the original Schwartz function, by boundedness and uniform continuity. Dominated convergence also removes the inserted Gaussian on the Fourier side. This proves inversion, and substitution into the integral of the product of two transforms proves preservation of the inner product by absolute integration. The coordinate change for B gives precisely the stated real measure. The other generators are unitary by their unit-modulus multiplier and the Jacobian $s(a)^{2j}$ respectively.

For rational u, a the character is trivial on rational arguments, and a permutes the rational vectors with $s(a) = 1$ by Lemma 61.2. To prove invariance under w take a finite coset indicator at $c + m\widehat{O}_E^j$, choosing $c \in E^j$ by Lemma 53.2. The rational vectors lie at $c + mO_E^j$, with archimedean covolume $m^{2fj} V_E^j$. Periodize the real test on this lattice as in Lemma 4.2. Rapid decay gives uniform convergence of the periodization and absolute summability of its Fourier coefficients, since the number of lattice points in a real cube of side $R \geq 1$ is $O((1 + R)^{2fj})$ by a basis change. The product Fejér kernels from Lemma 53.2 (or the one-dimensional kernels in Lemma 4.2 in each coordinate) recover this series. Its dual indices are exactly the rational elements of the finite annihilator, by the trace-dual calculation of Lemma 53.2 with conjugation in the first argument. The real phase at c is the inverse of the finite phase there, since Ψ is trivial on F. The reciprocal covolume is the finite mass of $m\widehat{O}_E^j$ in Definition 62.1. This gives $\Theta(W(w)\phi) = \Theta(\phi)$, with absolute convergence. □

Theorem 62.3. *The local versions of these operators define $W(g)$ up to a scalar of modulus one for $g \in \mathrm{SL}_2(\mathbb{A}_F)$, with $W(g)W(g')$ such a scalar times $W(gg')$. Products of rational generators representing the same rational matrix give exactly the same operator. Thus their choice defines an ordinary action of $\mathrm{SL}_2(F)$ preserving Θ. All $W(g)$ commute with local unitary isometries acting by $\phi(y) \mapsto \phi(h^{-1}y)$. On any compact set of g, the finite transforms of a fixed finite test range over finitely many functions up to unit scalars. At a real place a fixed radial Gaussian transforms to a scalar times $\exp(-\pi z H)$. On a compact set the scalars are bounded in modulus and z ranges over a compact subset of $\Re z > 0$.*

Proof. At a single place use the operators

$$D(u, v)\phi(x) = \Psi\big(B(v, x) + B(u, v)/2\big)\phi(x + u)$$

with the local character. Direct substitution, and change of variables in the Fourier integral, show that conjugation by $W(n(b)), W(m(a)), W(w)$ sends the column $(v, u)^{\mathsf{T}}$ indexing D to $(v + bu, u)^{\mathsf{T}}$, $(av, a^{-1}u)^{\mathsf{T}}$, and $(u, -v)^{\mathsf{T}}$, respectively. These matrices generate the local group: a nonzero bottom pivot permits elementary triangular reduction and w exchanges pivots. Consequently two words for the same matrix differ by a unitary operator commuting with all translations and character multipliers.

At a finite place such a word sends test functions to test functions. The image of $\mathbf{1}_D$, for an additive lattice D, is supported in D by invariance under all multipliers from its annihilator, using the double annihilator computation in Lemma 62.2. Translation invariance by D makes the image constant there. The scalar is the same for every coset by translation, and for all lattices by subdividing across a common smaller one. Thus it is scalar on tests. At a real place the word maps Schwartz functions continuously to Schwartz functions: the integrals and integration by parts in Lemma 62.2 bound each derivative times a polynomial by finitely many such bounds for the original function. Differentiate translation and modulation commutation to get commutation with every ∂_i and multiplication by x_i in real coordinates. The Gaussian $h = \exp(-\pi|x|^2)$ is killed by each $\partial_i + 2\pi x_i$. Its image is therefore ch, by differentiating its quotient by h. The span of the modulations of h is dense in L^2: if u is orthogonal to all of them then the Fourier transform of $u\bar{h} \in L^1$ vanishes. Convolving with the Gaussian kernels of Lemma 62.2 then gives zero, by Fourier integration of the Gaussian, while these convolutions converge to $u\bar{h}$ in L^1. For the latter assertion translations are continuous in L^1: first use finite sums of rectangular indicators, whose translated

symmetric differences have volumes tending to zero, then approximate an integrable function by those sums. Such approximation follows by truncating the function and its domain and approximating its level sets in measure by finite unions of rectangles, from the definition of Lebesgue measure by countable box covers. The orthogonal-complement criterion used here follows by minimizing distance to a closed subspace: the parallelogram identity makes a minimizing sequence Cauchy, and varying the limit in each subspace direction gives orthogonality. Thus the word equals the same scalar on every Schwartz function by unitarity. In both cases the scalar has modulus one.

At every finite place away from the finitely many divisors of the different, the local vacuum $\mathbf{1}_{B_v^j}$ is fixed by the chosen w, integral unipotents and diagonal units. On integral arguments Ψ_v is trivial, by the trace calculation for F in Theorem 58.2. An integral determinant-one matrix has a unit bottom pivot after a possible exchange, so it is a product of these integral generators. Use words fixing this vacuum at all such places. This defines the product on every test: almost all its factors are vacua. Changes of words can now give nontrivial scalars at only finitely many places. On rational generators use the products in Definition 62.1. By Lemma 62.2, two rational words giving the same matrix preserve Θ; their scalar ratio is one since Θ is nonzero on a positive Gaussian times the integral finite indicator. Commutation with isometries holds on the three generators by preservation of B, H and the additive measure. For the last point a unitary determinant has norm one, giving absolute Jacobian one also in the split case.

At each finite place a fixed test and its Fourier transform are bounded in support and constant on cosets of some open lattice. The test is therefore fixed by sufficiently small upper unipotents and by diagonal units sufficiently close to one. The same assertion for its Fourier transform gives small lower unipotents by conjugation. Near the identity the upper and lower unipotent and diagonal factorization uses a unit pivot and continuous parameters, so the test is fixed projectively by an open subgroup. A compact set meets finitely many cosets of that subgroup. For compact adelic sets, all but a fixed finite set of components lie in the integral groups: cover by finitely many basic restricted-product neighborhoods. At the real places the three changes of Gaussian parameter are $z \mapsto z + 2iu,\ z \mapsto a^2 z,\ z \mapsto 4/z$, with scalar multipliers $1, |a|^j, (2/z)^j$. The last formula follows from Lemma 4.2 for real $z > 0$ and analytic continuation under the integral for $\Re z > 0$ by Lemma A.3. In a neighborhood of any given matrix the pivot and word can be chosen continuously. A finite cover of a compact set gives the bounds on modulus and parameter. □

63. A bound in the rank-one cusp

Lemma 63.1. *For $t \in F$ set $D(t) = \prod_v Q_v^{\max(0,-v_A(t))}$ if $t \neq 0$, and $D(0) = 1$. For real $j > 2,\ R \geq 1$,*

$$\sum_{t \in F} D(t)^{-j} \prod_{\tau} (1 + |\tau(t)|/R)^{-j} \ \leq \ \left(\frac{3j}{j-1} \sum_{n \geq 1} n^{1-j} \right)^f R^f.$$

Proof. The ideal $J(t) = \{a \in O_F : at \in O_F\}$ has index $D(t)$. Indeed the proof of Lemma 52.1 applies to O_F, using its integral basis and the factors in Lemma 57.1. In particular the exponents determine the ideal and the number of places above p is at most f, so that the sum of $(\mathrm{N}J)^{1-j}$ over nonzero integral ideals J is at most $(\sum_{n \geq 1} n^{1-j})^f$. For $t = 0$ take $J = O_F$. If t, t' are distinct members of J^{-1}, the inclusion $(t - t')J \subset O_F$ gives $1 \leq \mathrm{N}J \prod_\tau |\tau(t - t')|$ by scaling real covolumes as in Lemma 61.2. In the box $|\tau(t)| \leq R_\tau$, $R_\tau \geq 1$, rescale each coordinate by R_τ^{-1}. Distinct points have maximum distance at least $\delta = (\mathrm{N}J \prod R_\tau)^{-1/f}$. Cubes of side δ about them are disjoint up to boundaries and lie in a cube of side $2 + \delta$. Hence there are at most $3^f \mathrm{N}J \prod R_\tau$ points. Bound each weight $(1 + |\tau(t)|/R)^{-j}$ by $\int_1^\infty j b^{-j-1} \mathbf{1}_{|\tau(t)| \leq Rb} \, \mathrm{d}b$. Apply the box estimate and integrate. Summing over the possible $J(t)$ yields the result by the ideal sum above. □

Definition 63.2. Fix $j \geq 3$. Take a test ϕ of Definition 62.1 with radial Gaussians at the infinite places, allowing finite linear combinations. Let K be an open subgroup of the full finite integral unitary level K^0 in rank j that fixes its finite test. Such a subgroup exists by the coset description of tests: matrices sufficiently close to the identity preserve each coset at the finitely many exceptional places. Write $I = [K^0 : K]$ and let h range over the finite class representatives at this level in Definition 13.3. Put $\Gamma_h = \Gamma \cap hKh^{-1}$ and $(h\phi)(y) = \phi(y_\infty, h^{-1}y_f)$. Set

$$\mathcal{T}(\phi) = \frac{1}{I\mathfrak{m}_j} \sum_{[h]} \frac{\Theta(h\phi)}{|\Gamma_h|}, \qquad I_t(\phi) = \int_{\mathbb{A}_E^j} \phi(y)\Psi(-tH(y))\,\mathrm{d}y, \qquad \mathcal{E}(\phi) = \phi(0) + \sum_{t \in F} I_t(\phi).$$

For I_t one can also take $t \in \mathbb{A}_F$.

Proposition 63.3. *The series defining $\mathcal{E}$ converges absolutely. Both $\mathcal{T}, \mathcal{E}$ are independent of the representatives and level in Definition 63.2 and invariant under $W(\gamma)$ for $\gamma \in \mathrm{SL}_2(F)$. For $r \geq 1$ and any compact set of $\kappa \in \mathrm{SL}_2(\mathbb{A}_F)$, write $\phi' = W(\kappa)\phi$, with any of the unit-scalar choices of Theorem 62.3. Then uniformly on that set,*

$$\sum_{t \in F} |I_t(W(m(d(r)))\phi')| = O_\phi(r^{(2-j)f}).$$

The constant may also depend on F, j and the compact set.

Proof. For a finite test and finite adele s the absolute value of its finite quadratic integral is at most $CD(s)^{-j}$, with D defined by the same finite product as in Lemma 63.1 (take factor one at zero). To check this at almost all places take the integral vacuum, where Lemma 58.1 gives the exact modulus with conductor $\max(0, -v_A(s))$. At a remaining place write the test as a sum of indicators of $c + \pi^e B_v^j$, $e \geq 0$. Substituting $y = c + \pi^e z$ adds a linear character and a constant to the quadratic phase with parameter $s\pi^{2e}$. The local character Ψ_v has annihilator $\pi^{-d_v}A_v$ against A_v, for some $d_v \geq 0$: this is the F-trace dual calculation in Theorem 58.2. The induced conductor on A_v is $n = \max(0, -v_A(s) - 2e - d_v)$. The squared integral calculation in Lemma 58.1 still bounds the modulus of the probability average by Q_v^{-jn}: the extra linear character evaluated at the difference has modulus one and does not change the orthogonality in the other variable. The case $n = 0$ uses the bound one. Restoring the fixed volume of the coset proves the bound with a constant depending on the test. Multiplication proves the finite assertion, uniformly for finitely many tests.

At infinity substitute $z = ry$ into the integral. The measure and the multiplier in Definition 62.1 give a factor r^{-jf}. The remaining Gaussian integral at τ is $(z_\tau + 2i\tau(t)/r^2)^{-j}$ using ordinary complex Lebesgue measure and the Gaussian normalization of Lemma 4.2. If z_τ ranges over a compact subset of the right half-plane this has modulus at most $C(1 + |\tau(t)|/r^2)^{-j}$; use a positive lower bound for $\Re z_\tau$ on bounded arguments and the triangle inequality on large ones. The finite tests and Gaussian parameters after κ satisfy exactly these uniform conditions by Theorem 62.3. Apply Lemma 63.1 with $R = r^2$ to prove the bound and convergence.

For $\mathcal{T}$, a change of left representative permutes rational vectors and preserves the real radial test by the rational unitary matrix at infinity. Right multiplication by K preserves the finite test. Refining K multiplies the class sum and its total weight by the same index by the orbit counting in Theorem 59.3. Use an intersection for any two choices. Theorem 62.3 and Lemma 62.2 give invariance under rational matrices. For $\mathcal{E}$, let P be the upper triangular subgroup of $\mathrm{SL}_2(F)$. The cosets $P\backslash\mathrm{SL}_2(F)$ are determined by the projective bottom row, since matrices with proportional bottom rows differ by a left upper triangular matrix. Representatives are 1 and $wn(t)$, $t \in F$. Every $W(p)$ for $p \in P$ fixes evaluation at zero by the scalar product formula in Lemma 61.2. The sum of $(W(\gamma)\phi)(0)$ over these representatives equals $\mathcal{E}(\phi)$. The ordinary rational action in Theorem 62.3 and absolute convergence permit reindexing by right translation. $\square$

Theorem 63.4. *For every test of Definition 63.2,*

$$\sup_{g\in \mathrm{SL}_2(\mathbb{A}_F)} |(\mathcal{T}-\mathcal{E})(W(g)\phi)| < \infty.$$

Proof. Use Theorem 61.3 and rational invariance in Proposition 63.3. Scalar changes in Theorem 62.3 have modulus one. It suffices to bound the difference at $W(m(d(r)))\phi'$ with $r \ge r_0$ and $\phi' = W(\kappa)\phi$ for compact κ. There are finitely many finite components up to scalar, so one level suffices and there are finitely many h to consider by Theorem 59.3. For each of these, the finite support restricts the rational vectors to a subset of $M^{-1}O_E^j$ for a fixed integer M, by clearing finite denominators. The real factors for nonzero vectors have modulus at most $Cr^{jf}\exp(-cr^2\sum_\tau H(x_\tau))$, $c>0$. The lattice is discrete, has positive least length, and has $O((1+H_0)^{2fj})$ points of length at most H_0 by a basis change. Summing over unit intervals in length therefore bounds its nonzero contribution by $O(r^{-1})$ for $r \ge 1$: factor out half the Gaussian at the least positive length and sum the remaining fixed Gaussian. The zero term of the normalized class sum is $r^{jf}\phi'(0)$ by Theorem 59.3. This is exactly the first term of $\mathcal{E}(W(m(d(r)))\phi')$. Proposition 63.3 bounds the other terms by $O(r^{(2-j)f})$. The range $r_0 \le r \le 1$, when present, only enlarges the compact set of transforms, and the same bounds at $r=1$ suffice. □

64. The rank-one Siegel–Weil identity

Lemma 64.1. *Let v lie above a completely split rational prime outside the different. Write $L = \mathrm{Frac}(A_v)$, $A = A_v$, and write a vector in $(E\otimes_F L)^j$ as two columns $x, y \in L^j$. Its norm is $x^{\mathsf{T}}y$. With A^j of mass one define*

$$\widetilde{\phi}(x,z) = \int_{L^j} \phi(x,y)\Psi_v(-z^{\mathsf{T}}y)\,\mathrm{d}y, \qquad X = [\,x\ z\,].$$

This is an invertible map on compact locally constant tests and sends the integral vacuum to ϕ_0 of Definition 60.1. Under this map the unitary isometry acting by $\phi(b^{-1}x, b^{\mathsf{T}}y)$ induces U_b, and the local generators $n(u), m(a), w$ induce the operators $V_{n(u)}, V_{m(a)}, V_w$ of that same definition.

Proof. The finite character Ψ_v has annihilator A against A since the place is unramified of rational residue degree one, by Proposition 18.5 and the identification of the character in Theorem 58.2. Hence the finite Fourier inversion and the transform of the indicator of A^j follow from Lemma 62.2. Substitution $y' = b^{\mathsf{T}}y$ gives $|\det b|^{-1}\widetilde{\phi}(b^{-1}x, b^{-1}z) = U_b\widetilde{\phi}(X)$. The norm phase for $n(u)$ replaces z by $z+ux$. For $m(a)$, substitution $y' = ay$ cancels its Jacobian against the factor $|a|^j$ and replaces (x,z) by $(ax, a^{-1}z)$. For w the full Fourier kernel is $\Psi_v(-x^{\mathsf{T}}y' - y^{\mathsf{T}}x')$; the second Fourier transform in the variable paired with x' evaluates the x'-argument at $-z$ by reflection. The result is $\widetilde{\phi}(-z,x)$. These three changes of columns are right multiplication of X by their stated matrices. □

Theorem 64.2. *For every $j \ge 3$ and every test of Definition 63.2 one has*

$$\mathcal{T}(\phi) = \phi(0) + \sum_{t\in F}\int_{\mathbb{A}_E^j} \phi(y)\Psi(-tH(y))\,\mathrm{d}y.$$

Proof. A finite test is a finite sum of pure products with integral vacuum at almost all places, by its coset description in Definition 62.1. Work with one such product. Choose a prime $p \equiv 1 \bmod dN_1$ outside its exceptional places and the different by Corollary 7.5. By Lemma 13.2 we can choose $v \mid p$ with the split model of Lemma 64.1. Freeze the test away from v. Let λ on local matrix tests be $\mathcal{T}-\mathcal{E}$ applied to this product after inverting the partial Fourier transform at v.

Both functionals are invariant under each local isometry $(x,y) \mapsto (bx, b^{-\mathsf{T}}y)$. For $\mathcal{E}$, this preserves norm, zero and the product measure, since the two absolute determinants multiply to one. For $\mathcal{T}$, write h for the isometry supported at v. Choose a product level K fixing

the test, small enough that K and hKh^{-1} are both contained in K^0. Right translation $g \mapsto gh$ carries the classes at hKh^{-1} bijectively to those at K. The stabilizers have equal order since $g(hKh^{-1})g^{-1} = (gh)K(gh)^{-1}$. The two indices in K^0 are also equal. At the only relevant place the group is $\mathrm{GL}_j(L)$; on invertible matrices the measure $|\det X|^{-j}\,\mathrm{d}X$, with $\mathrm{d}X$ additive matrix measure, is invariant under left and right multiplication. This follows by the row or column Jacobian as in Definition 60.1. An open integral subgroup has positive finite mass by a compact cover with finite-index cosets. Conjugation preserves its measure, so the index equality follows by counting these cosets. Apply the independence of the level in Proposition 63.3. By Lemma 64.1 the resulting λ is invariant under every U_b.

For $g \in \mathrm{SL}_2(L)$ embed g at v and put the identity elsewhere. Then $|\lambda(V_g\phi_0)|$ is precisely the absolute value of $(\mathcal{T}-\mathcal{E})(W(g)\phi)$ by Lemma 64.1 and Theorem 62.3. It is bounded on this group by Theorem 63.4. Theorem 60.3 thus gives $\lambda(\phi_0) = 0$. Since the original test at v is the integral vacuum, this is the claimed equality. Linear combination treats the general test. □

65. Siegel's formula for one represented element

Theorem 65.1. *Let $j \geq 3$, $\alpha \in F$ be totally positive. Fix a compactly supported locally constant complex test Φ_f on E_f^j, and take any open $K \subset K^0$ fixing it. Use the classes $[h]$, stabilizers Γ_h and index I of Definition 63.2. Give $\widehat{O}_E^j$ mass one for $\mathrm{d}x_f$, and the trace dual of $\widehat{A}$ mass one for $\mathrm{d}z_f$. Then*

$$\frac{1}{I\mathfrak{m}_j}\sum_{[h]}\frac{1}{|\Gamma_h|}\sum_{\substack{y\in E^j\\ H(y)=\alpha}}\Phi_f(h^{-1}y) = \frac{V_F}{V_E^j}\prod_{\tau}\frac{\pi^j\,\tau(\alpha)^{j-1}}{(j-1)!}$$
$$\cdot\int_{F_f}\left(\int_{E_f^j}\Phi_f(x)\psi_f\big(z(H(x)-\alpha)/2\big)\,\mathrm{d}x_f\right)\mathrm{d}z_f.$$

Every inner rational sum is finite on the support of the test and the outer finite-adelic Fourier integral converges absolutely.

Proof. Use $\phi(y) = \exp(-\pi\sum_\tau H(y_\tau))\Phi_f(y_f)$. On $\mathbb{A}_F$ take infinite Lebesgue measure and finite measure giving $\widehat{A}$ mass $1/V_F$. A half-open real fundamental parallelepiped of O_F times $\widehat{A}$ is a fundamental domain Ω of volume one. Indeed Lemma 61.2 gives representatives modulo $\widehat{A}$ in the finite coordinates, unique modulo O_F, and the half-open convention chooses the remaining translate. The measure on the quotient is invariant by translation: partition a translate of Ω by its intersections with the disjoint sets $a+\Omega$, $a \in F$ and translate the pieces back. A character $u \mapsto \Psi(\beta u)$, $\beta \in F$, integrates to zero there unless $\beta = 0$. In fact a nonzero β admits a translation where its character is different from one, by varying a single real coordinate; the character descends by Definition 62.1.

Apply Theorem 64.2 to $W(n(u))\phi$, multiply by $\Psi(\alpha u)$, and integrate over Ω. The level on the theta side can be used for every u because the phase depends only on the norm. Its summands are absolutely dominated by a fixed decaying Gaussian on finitely many fractional lattices, as in Theorem 63.4. Orthogonality retains the rational vectors of norm α. Their real Gaussian factor is $\exp(-\pi\sum_\tau \tau(\alpha))$; they lie in bounded balls in these lattices, so their counts on the support are finite. The term $\phi(0)$ on the other side integrates to zero as $\alpha \neq 0$. The terms of the series are $I_{t+u}(\phi)\Psi(\alpha u)$, $t \in F$. Since $\Psi(\alpha t) = 1$, unfolding the translates of Ω gives $\int_{\mathbb{A}_F} I_u(\phi)\Psi(\alpha u)\,\mathrm{d}u$.

For absolute integration in this step, the finite bound in the proof of Proposition 63.3 is $CD(u_f)^{-j}$. Over A_v of mass one the shell $v_A(u) = -n$, $n > 0$, has mass at most Q_v^n. The integral over finite adeles with all A_v of mass one is therefore bounded by $\prod_v(1+\sum_{n\geq 1}Q_v^{(1-j)n})$. This is finite by $j > 2$ and the prime sum in Theorem 58.2. At a real coordinate the bound is $C(1+|u_\tau|)^{-j}$, also integrable. Thus the unfolding and separation

into the real coordinate factors and the finite factor are justified by absolute integration after the inner quadratic integral has been taken.

At a real coordinate set $a = \tau(\alpha) > 0$. By Proposition 59.4 its Fourier integral at a is the inverse transform of the density $\mathbf{1}_{t>0}\pi^j t^{j-1}\exp(-\pi t)/(j-1)!$. It has value $\pi^j a^{j-1}\exp(-\pi a)/(j-1)!$: insert the Gaussian cutoff in the outer integral as in Lemma 62.2; the convolution converges at a by boundedness of the density and continuity there. Removing the cutoff is justified by the integrable transform bound just proved. The Gaussian factors now cancel on both sides. The finite vector measure in Definition 62.1 gives $\widehat{O}_E^j$ mass V_E^{-j}. In an integral basis of O_F its trace matrix is $T^{\mathsf{T}}T$ with real embedding matrix T of determinant of absolute value V_F by Lemma 20.1. Thus $[\widehat{A}^\vee : \widehat{A}] = V_F^2$ by the same basis calculation as in Proposition 18.5; the finite measure on the outer variable gives $\widehat{A}^\vee$ mass V_F. The remaining finite phase is precisely the one displayed, by the sign in Definition 62.1. The two changes to the stated measures give V_F/V_E^j. □

Corollary 65.2. *For every $j \geq 3$ and every totally positive integral $\alpha \in O_F$,*

$$\frac{1}{\mathfrak{m}_j}\sum_{[L]\in\mathcal{L}_j}\frac{r_L(\alpha)}{|U(L)|} = B_j(\alpha)$$

with B_j of Proposition 59.4. In particular

$$\frac{\mathfrak{m}_j}{\mathfrak{m}_{j-1}} = \left(\frac{\mathrm{rd}_E^{j-1/2}(j-1)!}{(2\pi)^j}\right)^f \prod_v (1-\sigma_v Q_v^{-j})^{-1}.$$

The logarithm of the product has absolute value at most $4f\,2^{-j}$.

Proof. Take $K = K^0$, $\Phi_f = \mathbf{1}_{\widehat{O}_E^j}$ in Theorem 65.1. By Lemma 13.4 the inner sum is the representation count of its lattice. The finite integral equals $P_\varnothing(\alpha)$ by Theorem 58.2, and $V_F = |\Delta_E|^{1/4}$ by Lemma 61.2. This is exactly the definition of $B_j(\alpha)$. The special case $\alpha = 1$ satisfies the hypothesis proved sufficient in Proposition 59.4, giving the ratio and its logarithmic bound. □

66. The column level and local norm measures

Proposition 66.1. *Extend a finite isometry of the definite space by the identity on z, e_a, e_a^* in Definition 15.1. The inverse image of the full integral stabilizer in this extension is a subgroup of K^0 of index $(N^k-1)^f$. The inverse image of the row level imposing identity reduction also at g_0, g_1 is exactly K_c of Definition 13.3. In $\mathcal{G}$ let $\bar{H}$ be the image of the definite integral isometries preserving the row lattice, and $\bar{S}$ the image of the finite integral subgroup generated by these isometries and the unipotent group of Definition 20.3. Then*

$$|\bar{H}| = \prod_{i=0}^{1}|\mathrm{GL}_k(\mathbf{F}_{g_i})|^f, \qquad |\bar{S}| = |\bar{H}|\,g^{f\nu(2k+2\nu+1)}.$$

Proof. Away from N the change of middle lattice basis is integral with integral inverse because $\pi_N\bar{\pi}_N = N$. At N use the components of Lemma 13.2, first taking one with π_N divisible by N. Projection of the middle lattice to $X_0 = \langle x_1, \ldots, x_k\rangle$ along z is the full integral lattice in this component because $\bar{\pi}_N$ is a unit. An isometry fixing z and preserving the middle lattice therefore acts integrally invertibly on X_0 here. Its partner is the inverse transpose and is also integrally invertible. Write A for the matrix on X_0. For $i < k$ the image of x_i in the new basis has z-coordinate $-A_{ki}/\pi_N$ and v-coordinate $\bar{\pi}_N A_{ki}$. The image of $v = x_k/\bar{\pi}_N + z/N$ has coordinates $A_{ik}/\bar{\pi}_N$ for $i < k$, A_{kk} at v, and $(1-A_{kk})/N$ at z. The chosen component thus requires precisely that the last row of A reduce to the last standard row. In the partner it requires the last column of the inverse transpose to reduce to the last standard column, the very same condition. Inverses preserve the condition as

well. Proposition 13.5 counts the index, and at g_0, g_1 integral change of basis carries identity congruence to itself. This proves the level equality.

At a chosen component above either g_i, all the matrices on X_0 occur by integral lifting and the split description in Lemma 13.2, independently across components. In the ordered flag use the line blocks of the e_a, the middle block of size $k+1$, and the dual lines in reverse order. In one split component the unipotent matrices are precisely the upper block triangular matrices with identity diagonal blocks. Indeed the flag and dual flag give the two sequences of invariant subspaces, and identity on their quotients gives the diagonal blocks. There are no further equations in that component: on the partner the inverse adjoint is forced, and satisfies the flag conditions by orthogonality. All residues lift by lifting arbitrary off-block entries. There are $\binom{r}{2} - \binom{k+1}{2} = \nu(2k + 2\nu + 1)$ such entries. The definite matrices normalize this unipotent group and intersect it trivially by the middle diagonal block. Multiplication of the two finite subgroups is therefore a bijection from their Cartesian product onto their generated subgroup. Multiplying the orders over the two primes and the f components gives the formula. □

Definition 66.2. Let T be a nonempty finite set of places of F and put $F_T = \prod_{v\in T} \mathrm{Frac}(A_v)$, $E_T = \prod_{v\in T}(B_v \otimes_{A_v} \mathrm{Frac}(A_v))$. Give the integral rings and their finite Cartesian powers mass one. Write $L_n = \prod_{v\in T} \pi_v^n A_v$. For $\gamma \in F_T$, compact locally constant $\varphi : E_T^k \to \mathbb{C}$, and $k \geq 3$, put

$$d_T(\gamma, \varphi) = \lim_{n\to\infty} \frac{1}{|L_n|} \int_{H(x)-\gamma\in L_n} \varphi(x)\, \mathrm{d}x.$$

Here and below $|L|$ for an additive compact open set denotes its measure in its specified ambient space. Let $K_{c,T}$ be the projection of the column level to the factors over T.

Lemma 66.3. *The limit in Definition 66.2 exists uniformly in γ and is a continuous function of γ. It is locally constant wherever every component of γ is nonzero. It is a positive linear functional on nonnegative tests.*

Proof. Use the additive character $\psi_f(z \cdot /2)$ restricted to these places and give the dual of $\prod_{v\in T} A_v$ mass one in the z-variable. Integrating a character over $L_n^\vee$ gives $|L_n|^{-1}\mathbf{1}_{L_n}$ by the trace pairing and the finite orthogonality in Lemma 58.1. The approximants therefore have Fourier expressions with z restricted to $L_n^\vee$. By the proof of Proposition 63.3, the absolute value of the quadratic Fourier integral of the test is bounded by $CD(z)^{-k}$ independently of γ. It is integrable by the shell calculation in Theorem 65.1. The tail outside $L_n^\vee$ tends to zero and is independent of γ in this bound. Every finite-level approximant is locally constant, so uniform convergence implies continuity. Positivity follows from the original integrals.

For a scalar $u \in F_T$ sufficiently close to one at all places we can write $u = a\bar{a}$ with a arbitrarily close to one in E_T. In the field case apply the successive trace correction of Lemma 59.1 beginning with 1 at any prescribed positive digit depth. In a split pair take $(u, 1)$. Multiplication by all such a sufficiently close to one fixes the test, by compact support and its finite coset decomposition. It preserves the vector measure and multiplication by u preserves each L_n. The substitution $x = ay$ therefore gives $d_T(u\gamma, \varphi) = d_T(\gamma, \varphi)$. When every component of γ is nonzero these $u\gamma$ form a neighborhood of it. □

Theorem 66.4. *Let $k \geq 3$ be the column rank, let h_c represent the classes at K_c, and let $\beta \in F$ be totally positive and integral off T. For a $K_{c,T}$-invariant compact locally constant test φ on E_T^k one has*

$$\frac{1}{|\mathcal{C}(K_c)|}\sum_c \sum_{\substack{y\in E^k,\ H(y)=\beta \\ (h_c^{-1}y)_v\in B_v^k\ \ (v\notin T)}} \varphi((h_c^{-1}y)_T) = \frac{V_F}{V_E^k}\prod_\tau \frac{\pi^k \tau(\beta)^{k-1}}{(k-1)!} \cdot P_T(\beta) d_T(\beta_T, \varphi).$$

Here P_T is the product outside T in Theorem 58.2. For every fixed $D > 0$ it lies between $(1 + b^{-D})^{-1}$ and $1 + b^{-D}$ at all sufficiently large arithmetic scales, uniformly in β, T.

Proof. Use φ at T and the integral vacuum outside T in Theorem 65.1. The product test is fixed by K_c. All its class stabilizers have order one and $|\mathcal{C}(K_c)| = I_c \mathfrak{m}_k$ by Proposition 13.5. The Fourier integral over T equals $d_T(\beta_T, \varphi)$ by the proof of Lemma 66.3. The complementary integral factors absolutely and is $P_T(\beta)$ by the proof of Theorem 58.2. The logarithmic bound there is $o(b^{-D})$ by the scale estimate in Corollary 58.3. Exponentiating gives both inequalities. □

67. The one-sided Siegel bound on the middle quadric

Definition 67.1. Take T as in Definition 66.2 containing every place over N. On the middle space of Definition 15.1 use its integral basis $x_1, \dots, x_{k-1}, v, z$ at T and give the integral coordinate lattice mass one for $\mathrm{d}e$. Write $q(e) = h(e, e)$, $s(e) = h(z, e)$. For $u \in E_T$, $y \in E_T^k$ put $P(u, y) = \sum_i y_i x_i - uz/N$ in the rational middle basis. For $b \in F_T$ and compact locally constant G on the middle space define

$$\delta_b(G) = \lim_{n \to \infty} |L_n|^{-1} \int_{q(e) - b \in L_n} G(e) \, \mathrm{d}e,$$

using L_n in the scalar space F_T from Definition 66.2.

Proposition 67.2. *The above limit defines a Borel measure supported on $q(e) = b$, finite on compact sets. Put $\beta_b(u) = b + u\bar{u}/N$. For compact locally constant G one has*

$$\delta_b(G) = N^f \int_{E_T} d_T(\beta_b(u), G(P(u, \cdot))) \, \mathrm{d}u.$$

If every component of $\beta_b(t)$ is nonzero then for all sufficiently small compact open $L \subset E_T$ about zero, and also for measurable subsets of those L,

$$\int_{s(e) \in t + L} G(e) \, \mathrm{d}\delta_b(e) = N^f |L| \, d_T(\beta_b(t), G(P(t, \cdot))).$$

Proof. In the integral basis the coordinates of $P(u, y)$ are $(y_1, \dots, y_{k-1}, \bar{\pi}_N y_k, -y_k/\pi_N - u/N)$. Consequently $s(P(u, y)) = u$, $q(P(u, y)) = H(y) - u\bar{u}/N$; the determinant from (y, u) to e is $-1/\pi_N$. In the notation of Lemma 52.1, the product of the finite absolute values of π_N is $\prod_\tau |\tau \pi_N|^{-2} = N^{-f}$. All factors off T are one by integrality of $\pi_N, \bar{\pi}_N$ and their product N. The change of variables thus gives $\mathrm{d}e = N^f \, \mathrm{d}y \, \mathrm{d}u$, by the determinant and index computation in Lemma 20.2. The composed test $G(P(u, y))$ is a finite sum of products $\chi_i(u)\varphi_i(y)$ of compact open coset indicators with scalar coefficients, since P is a linear isomorphism. Lemma 66.3 gives uniform convergence of the tube averages for each φ_i. Integration over the compact supports in u yields the first formula. For nonnegative tests the limit is nonnegative, and it is additive on disjoint compact open sets.

To obtain an actual measure, partition the space into a countable union of compact open shells using increasing scalar dilates of the integral lattice. Each shell has finite mass by the formula just proved. Refine its finitely many coordinate boxes successively into smaller residue boxes with diameters tending to zero. Partition a real interval of length equal to the shell mass into intervals of the respective child masses at each stage. Compatible boxes specify a unique point by completeness; endpoints of the intervals form a null set. The resulting image of length realizes the prescribed masses of all boxes, including when the shell mass is zero. The uniqueness argument of Lemma 53.2 extends equality on the boxes to all Borel sets. Outside $q = b$, continuity of q gives a neighborhood where a sufficiently small tube is empty, and a countable base covers this complement by null sets. This proves support on the stated fiber.

In the product decomposition of $G(P(u, y))$, each χ_i is constant near t, and $\beta_b(u)$ is continuous there. Lemma 66.3 now makes the integrand constant near t when $\beta_b(t)$ has all

its components nonzero. The first formula also applies with an extra compact open cutoff in $s(e) = u$. It gives the indicated equality of measures near t on compact opens and hence on Borel subsets by the preceding uniqueness argument, and on their completions. □

Theorem 67.3. *Let $t \in E$ satisfy $|\tau t| > N$ at every infinite place, and put $\beta = (t\bar{t} - N^2)/N$. Use the column class representatives in Theorem 66.4. Let $\mathcal{S}$ be a finite set of distinct pairs (c, y) with $y \in E^k$, $H(y) = \beta$, $(h_c^{-1} y)_v \in B_v^k$ for $v \notin T$. Give these pairs weights $a_{c,y} \geq 0$. Assume β integral off T, which holds if $\mathcal{S}$ is nonempty. Let $G \geq 0$ be a Borel function on the middle space. Suppose for each pair there is a neighborhood of $P(t, (h_c^{-1} y)_T)$ throughout which $G(k'e) \geq a_{c,y}$ for all k' in the image of $K_{c,T}$ acting on the middle space. Then for all sufficiently small measurable neighborhoods $L \subset E_T$ of zero,*

$$|L| \sum_{(c,y) \in \mathcal{S}} a_{c,y} \;\leq |\mathcal{C}(K_c)|\, c_*\, P_T(\beta) \prod_{\tau} \left(|\tau t|^2 - N^2\right)^{k-1} \int_{s(e) \in t + L} G(e)\, \mathrm{d}\delta_{-N}(e),$$

$$c_* = \frac{V_F}{V_E^k} \left(\frac{\pi^k}{(k-1)!\, N^k} \right)^f .$$

In particular one may replace $P_T(\beta)$ by $1 + b^{-6}$ at all sufficiently large arithmetic scales, uniformly in T, t subject to these conditions.

Proof. For each pair choose a compact open neighborhood within the neighborhood in the hypothesis and saturate it by the compact group $K_{c,T}$. Its saturation is open as a union of translates, and compact as the image of the group times the original compact set. All its points have G-value at least the designated weight. Take $\Phi(e)$ to be the maximum of the weights of these saturations containing e, taking zero elsewhere. Then $\Phi \leq G$ is compact, locally constant and invariant under $K_{c,T}$, with $\Phi(P(t, (h_c^{-1} y)_T)) \geq a_{c,y}$. Compactness of the group here follows from integrality and the closed unitary and congruence equations by Proposition 66.1 and the compactness in Lemma 61.2.

The test $\varphi(y') = \Phi(P(t, y'))$ is invariant under the definite group $K_{c,T}$, since its action extends by the identity on z. It is compactly supported and locally constant by Proposition 67.2. Apply Theorem 66.4 and retain the pairs in $\mathcal{S}$ by nonnegativity. All components of β are nonzero at the finite places, since it is a totally positive field element. Proposition 67.2 at $b = -N$ identifies $|L|\, d_T(\beta_T, \varphi)$ with $N^{-f} \int_{s(e) \in t+L} \Phi(e)\, \mathrm{d}\delta_{-N}(e)$. The remaining scalar $\prod_\tau \tau(\beta)^{k-1}$ contributes $N^{-(k-1)f} \prod_\tau (|\tau t|^2 - N^2)^{k-1}$. Combining both powers of N gives exactly c_*. Use $\Phi \leq G$ and the exterior product bound in Theorem 66.4. □

68. The normalization of the orbital packets

Definition 68.1. Retain k, r, ν, put $h = \nu + 1$, $c = r - k = 2\nu + 1$, $d_* = \nu(k + \nu)$, and use U_j of Definition 56.1. Write

$$C = |\mathcal{C}(K_c)|, \quad J_c = \binom{a + k - 1}{k - 1}^U, \quad u = g^{(2d_* + \nu)f}, \quad S_0 = |\bar{S}|, \quad C_{\mathrm{op}} = \frac{U_k}{U_r} \frac{2\pi^k}{(k-1)!},$$

$$P_{\mathrm{pkt}} = S_0\, u\, C\, c_*, \qquad Z_{\mathrm{sc}} = \prod_{\tau} \prod_{a'=0}^{\nu} (s_{a',\tau}^2)^{k + 2(\nu - a')}, \qquad Z_{\mathrm{pkt}} = \frac{C^2}{\mathfrak{m}_r} C_{\mathrm{op}}^f V_E^h (\mathfrak{v} Z_{\mathrm{sc}})^2.$$

Here c_* is the constant of Theorem 67.3, $\mathfrak{v}, s_{a',\tau}$ are from Proposition 20.5, and S_0 is computed in Proposition 66.1. For each integer $j \geq 3$ put $P_j = \prod_v (1 - \sigma_{v,j} Q_v^{-j})$ with $\sigma_{v,j} = 1$ at split places and $(-1)^j$ at inert ones.

Theorem 68.2. *The two packet constants of Definition 68.1 satisfy the exact identity*

$$\frac{P_{\mathrm{pkt}}}{Z_{\mathrm{pkt}}} = (1 - N^{-k})^{-f} \prod_{j=k+1}^{r} P_j^{-1}.$$

For $k \ge 3$ the absolute value of its logarithm is at most $2f N^{-k} + 4f\,2^{-k}$. In particular, for any fixed $D > 0$ and all sufficiently large arithmetic scales, $P_{\mathrm{pkt}} \le (1 + b^{-D}) Z_{\mathrm{pkt}}$, uniformly throughout the linear window of Theorem 40.2.

Proof. Write $H_0 = |\bar{H}|$. Propositions 13.5 and 66.1 give $C = H_0 (N^k - 1)^f \mathfrak{m}_k$, $S_0 = H_0 u$. The omitted last scale in the product of Proposition 20.5 contributes $(s_{\nu,\tau}^2)^k = N^{-k}$ at every place. Hence $\mathfrak{v} Z_{\mathrm{sc}} = u N^{-kf} V_E^{d_*} V_F^{\nu}$, using Lemma 61.2. Substitution gives

$$\frac{P_{\mathrm{pkt}}}{Z_{\mathrm{pkt}}} = (1 - N^{-k})^{-f} \frac{\mathfrak{m}_r}{\mathfrak{m}_k} \left(\frac{U_r}{2U_k} \right)^f V_F^{1-2\nu} V_E^{-(k+h+2d_*)}.$$

The factor 2 here comes from the value $2\pi^k/(k-1)!$ in C_{op}: differentiating the real ball volume with respect to radius at radius one gives twice the density with respect to squared radius in Proposition 59.4.

Set $J = \sum_{l=k}^{r-1} l = c(k+\nu) = 2d_* + k + \nu$, by summing the arithmetic progression. By Corollary 65.2,

$$\frac{\mathfrak{m}_r}{\mathfrak{m}_k} \left(\frac{U_r}{U_k} \right)^f = (\mathrm{rd}_E^{J + c/2} 2^{-J})^f \prod_{j=k+1}^{r} P_j^{-1}.$$

Indeed the factor from U_{l+1}/U_l is $2\pi^{l+1}/l!$; it cancels the factorial in the mass quotient at each l. By Lemmas 20.1 and 61.2, $V_E = (\mathrm{rd}_E/2)^f$, $V_F = \mathrm{rd}_E^{f/2}$. Together with $k + h + 2d_* = J + 1$, $c = 2\nu + 1$, these equalities cancel the powers of 2 and of rd_E exactly.

Corollary 65.2 gives $\sum_{j=k+1}^{r} |\log P_j| \le 4f \sum_{j=k+1}^{\infty} 2^{-j} = 4f\,2^{-k}$. Also $-\log(1 - N^{-k}) \le 2N^{-k}$ by integration of $1/(1-v)$ on $0 \le v \le N^{-k} \le 1/2$. These estimates prove the bound. Proposition 10.5 gives $k = r + 1 - m \ge x^5 - 2\sqrt{x}$; $f \le x^{12}$ eventually by Proposition 12.2. Thus the bound is $o(b^{-D})$, as in Corollary 58.3. This proves the last assertion. □

69. Comparison of the two numerical energies

Definition 69.1. For the quantities of Definitions 68.1 and 56.1 put

$$E_1 = S_0 C J_c, \qquad E_2 = \frac{(C J_c)^2}{\mathcal{D} \mathfrak{m}_r}, \qquad T_0 = b^{kf/4}, \qquad Z_0 = 2(b/g)^{kf} T_0^2.$$

The ordinal subscripts in this section distinguish two real numbers. The column dimension here is CJ_c by Proposition 13.5. The positive integer $\mathcal{D}$ is defined from the row parameters.

Lemma 69.2. *Uniformly for $t \le v \le t^2$ in the arithmetic construction,*

$$f^{-1} \log \mathcal{D} = m(k-1)\bar{w} - \sum_{l=k-1}^{r-1} \log(l!) + O(mk/x + m^2 \log x),$$

$$\frac{\log J_c}{fk\ell} = \frac{1}{\varphi(M)} + o(1).$$

For every fixed $D > 0$ and at all sufficiently large x, the constructed primes satisfy $J_c \le b^{2kf}$, $(N^k - 1)^f \ge (N^k/b)^f$, and

$$\prod_{\tau} C_{\mathrm{B}}\, b^{7k} \rho_\tau^{k - h^7} \le \mathcal{D}^{-1} b^{-D}.$$

Proof. In the exact Vandermonde of Proposition 56.2 the first $k-1$ rows cancel the factorials with indices $0, \dots, k-2$. Each of the $m(k-1)$ mixed differences lies between $w(\tau)$ and $w(\tau)(1+1/x)$ eventually by the mixed-factor estimate in its proof. The $m(m-1)/2$ remaining differences have logarithms between zero and $\log(m\Delta_w) = O(\log x)$. Taking logarithms and averaging proves the first formula with both error bounds.

For J_c write

$$\log \binom{a+k-1}{k-1} = (k-1)\log a - \log((k-1)!) + \sum_{i=1}^{k-1} \log(1 + i/a).$$

The sum is between zero and k^2/a. Also $\log((k-1)!) \le k\log k$; we have $xb \le a \le 2xb$, $k = O(x^5)$, $\log x = o(\ell)$ by Propositions 12.2 and 10.5. Now use $U/f = 1/\varphi(M)$ from Proposition 10.3. This proves the second formula and $J_c \le b^{2kf}$ eventually. Since $N \ge 2$ and $1/b \to 0$, also $1 - N^{-k} \ge 1/b$. Finally Theorem 56.3 with $\eta = 1$ absorbs b^{7kf+D}, since $(25/2 - 1 - 7)kf > D$ eventually. □

Theorem 69.3. *For any auxiliary primes satisfying the ranges of Theorem 11.4 and Proposition 12.2, uniformly for $t \le v \le t^2$,*

$$\frac{\log(E_2/E_1)}{kf\ell} = \frac{\log g}{\ell} - \frac{15}{2} + \frac{1}{\varphi(M)} + o(1).$$

For the primes produced in the proof of Theorem 11.4, this expression tends to $25/2+1/\varphi(M)$. For every allowed choice in those ranges, for all sufficiently large x one has

$$T_0^{44} E_1 \le E_2, \qquad T_0^{18} Z_0 E_2 \le E_1, \qquad Z_0 \le 1.$$

Proof. Use $c = m - 1 = 2\nu + 1$, $J = \sum_{l=k}^{r-1} l = c(k+\nu)$. The two class and subgroup identities in the proof of Theorem 68.2 give

$$E_2/E_1 = (N^k - 1)^f \, \mathfrak{m}_k J_c/(u\mathcal{D}\mathfrak{m}_r).$$

Taking the product of the mass ratios of Corollary 65.2 over $k+1, \dots, r$ gives

$$\frac{1}{f}\log\frac{\mathfrak{m}_r}{\mathfrak{m}_k} = (J + c/2)\log \mathrm{rd}_E + \sum_{l=k}^{r-1} \log(l!) - (J+c)\log(2\pi) + O(2^{-k}).$$

The sum of the absolute logarithmic errors after division by f is at most $4\,2^{-k}$ by Theorem 68.2. Use $\log u/f = (J-k)\log g$. From Lemma 69.2, writing $G_0 = \log(g\,\mathrm{rd}_E/(2\pi))$ as in Proposition 12.2, the logarithm of E_2/E_1 divided by f is

$$\begin{aligned} & k\log N - m(k-1)\bar{w} - JG_0 + k\log g \\ & \quad + c\big(\log(2\pi) - \tfrac{1}{2}\log \mathrm{rd}_E\big) + \log((k-1)!) + f^{-1}\log J_c \\ & \quad + O(mk/x + m^2\log x + 2^{-k}). \end{aligned}$$

Here $|\log(1 - N^{-k})| \le 2N^{-k}$. The same level-prime proposition gives $\log N = m\bar{w} + cG_0 - 15\ell/2 + O(\sqrt{\ell})$, $G_0 = O(\ell)$, and $\bar{w} = O(\ell)$; also $\log \mathrm{rd}_E = O(\log x)$ by Lemma 12.1 and Proposition 10.5. Since $J = ck + c\nu$, the leftover terms $m\bar{w}$, $c\nu G_0$, $c\log \mathrm{rd}_E$ are $o(k\ell)$ by $m = O(\sqrt{x})$, $k \ge x^5 - 2\sqrt{x}$. The factorial contributes at most $k\log k = o(k\ell)$. The displayed errors and $k\sqrt{\ell}$ are also $o(k\ell)$. The limit for J_c in Lemma 69.2 now gives the asserted expansion.

In the proof of Theorem 11.4 each $\log g_i$ is $10\ell + O_d(\ell^{1/6})$, since only the fixed finite number $H_d + 1$ of intervals of width $\ell^{1/6}$ are used. Hence $\log g/\ell \to 20$ for that choice. For arbitrary primes in the stated intervals $\log g/\ell \ge 20$. Consequently the first ratio has normalized logarithm at least $25/2 + o(1) > 11$, proving $E_2 \ge T_0^{44} E_1$. The normalized logarithm of $T_0^{18} Z_0 E_2/E_1$ equals $-3/2 + 1/\varphi(M) + o(1)$, since $\log Z_0/(kf\ell) = 3/2 - \log g/\ell + o(1)$. It is negative eventually as $\varphi(M) \ge 1$. The expression for Z_0 also gives $Z_0 \le 1$. □

70. Hyperbolic coordinates on the finite norm fiber

Lemma 70.1. *Use the finite set T, the scalar and vector measures and the middle space of Definitions 66.2 and 67.1. Write $A_T = \prod_{v\in T} A_v$, $B_T = \prod_{v\in T} B_v$ for the integral rings and $E_T^- = iF_T$ for the skew elements. Give iA_T mass one in E_T^-. Put $\xi = \exp(2\pi i/12)$ and $\eta = \xi/i$. The integral middle space has an orthogonal sum decomposition into the hyperbolic plane spanned by v, v', with $v' = -z + N\eta v$ and $h(v,v') = 1$, and the span of $x_1, \dots, x_{k-1}$ with its standard form. For $e = uv + tv' + \sum_j y_j x_j$ put $H(y) = \sum_j \bar{y}_j y_j$. For $t \in E_T^\times$, write*

$$U(\gamma, t, y, \alpha) = \bar{t}^{-1}\eta(\gamma - H(y)) + t\alpha, \qquad \gamma \in F_T, \quad \alpha \in E_T^-.$$

For each fixed t, y this gives an affine bijection $(\gamma, \alpha) \mapsto u$ carrying $\mathrm{d}\gamma\, \mathrm{d}\alpha$ to $\mathrm{d}u$; its first inverse coordinate is $q(e)$.

Proof. Lemma 20.1 gives $\xi - \bar{\xi} = i$ and $O_E^- = iO_F$. In the basis $1, \xi$ the same formulas hold after completion. The elements with zero trace are precisely iA_T in B_T and iF_T in E_T, since division by the skew unit i makes them fixed. The element $\eta \in B_T$ has $\eta + \bar{\eta} = 1$, so every $b \in B_T$ has the unique form $b = \eta(b + \bar{b}) + b_-$ with $b_- \in iA_T$. The matrix on v, z in Proposition 15.2 gives $h(v', v') = -N + N(\eta + \bar{\eta}) = 0$. The change from v, z to v, v' has unit determinant. Thus in the new coordinates

$$q(e) = \bar{t}u + t\bar{u} + H(y), \qquad \mathrm{d}e = \mathrm{d}u\, \mathrm{d}t\, \mathrm{d}y.$$

For fixed invertible t, let $n(t) = t\bar{t} \in F_T^\times$. Multiplication of E_T by $\bar{t}$ has additive volume factor $\prod_{v \in T} |n(t)_v|_v$. Indeed its determinant on the two-dimensional algebra over each fixed field is $n(t)_v$, by the basis $1, \xi$ of Lemma 57.1; the measure factor follows by the diagonal index calculation in Lemma 61.2. For the proposed chart $\bar{t}u = \eta(\gamma - H(y)) + n(t)\alpha$. The trace splitting just proved identifies the two integral lattices with B_T, hence takes the product of their normalized Haar measures to the normalized measure on E_T, by counting residue boxes. Scaling α by $n(t)$ contributes the same factor as multiplication of u by $\bar{t}$. The factors cancel, and tracing this formula proves both bijectivity and the assertion on the first coordinate. □

Theorem 70.2. *Let $k \geq 3$, and let δ_γ be the measure of Proposition 67.2 in the ranks of that proposition. For every $\gamma \in F_T$ and every nonnegative Borel test G one has*

$$\int G(e)\, \mathrm{d}\delta_\gamma(e) = \int_{E_T^\times} \int_{E_T^{k-1}} \int_{E_T^-} G\left(U(\gamma, t, y, \alpha)v + tv' + \sum_j y_j x_j \right) \mathrm{d}\alpha\, \mathrm{d}y\, \mathrm{d}t.$$

In particular the set where t is a nonunit of the algebra E_T has δ_γ-measure zero. Every E_T-linear isometry of the middle space preserves δ_γ.

Proof. First take G compact, locally constant and nonnegative. Choose integers $a_v \geq 0$ such that all coordinates u, t, y_j on its support lie in $D = \prod_{v \in T} \pi_v^{-a_v} B_v$. There is a constant C_D such that for every $c \in F_T$ and integer $n \geq 1$,

$$\int_{D^{k-1}} \mathbf{1}_{H(y) - c \in L_n}\, \mathrm{d}y \leq C_D |L_n|.$$

Indeed, after $y_v = \pi_v^{-a_v} y_v'$, the target at v becomes $\pi_v^{2a_v} c_v$ and the depth is $n + 2a_v$. A nonintegral target gives zero. For an integral target Theorem 57.4 in rank $k - 1 \geq 2$ bounds the local quotient by $Q_v^{(2k-4)a_v}(1 + 2Q_v^{2-k})$ after division by Q_v^{-n}. The product of these finitely many bounds suffices.

For a positive integer B, let $D_B \subset D$ consist of those t whose valuation in every field component of E_T is at most B, with valuation of zero taken as infinity. This is compact open and consists of invertible elements. The volume of $D \setminus D_B$ tends to zero, since in each field component a ball of depth $B + 1$ has volume its residue cardinality to the power $-B - 1$, and there are finitely many components. At fixed u, t, apply the preceding bound with $c = \gamma - \bar{t}u - t\bar{u}$. The normalized mass of the part of the tube with $t \in D \setminus D_B$, weighted by G, is at most $\sup G\, |D| C_D |D \setminus D_B|$ for every n.

On D_B apply Lemma 70.1 at fixed t, y. The restricted tube integral divided by $|L_n|$ is the mean, for $s \in \gamma + L_n$, of the chart integral in the statement with $t \in D_B$ and s in place of γ. For $s \in \gamma + L_1$ its integrand is supported in a fixed compact set of t, y, α: $y \in D^{k-1}$, $t \in D_B$, and $\alpha = t^{-1}(u - \bar{t}^{-1}\eta(s - H(y)))$ with $u \in D$. The inverses of D_B form a compact set by continuity. On this compact set the integrand is locally constant in all variables. A finite cover by product neighborhoods at $s = \gamma$ shows that every sufficiently small such mean is its value at γ. Hence the restricted measure is the displayed chart integral over D_B. Let B tend to infinity. The uniform tube bound removes the omitted region on the

left. Monotone convergence gives the integral over invertible t on the right. This proves the formula for compact open indicators and proves local finiteness of the image measure.

Compact open boxes form a countable generating family. Uniqueness on increasing compact shells as in Proposition 67.2 now proves equality of the two Borel measures and the formula for all the asserted tests. For an isometry g the identity $\overline{\det g}\det g = 1$ follows by taking determinants of the isometry relation in the nonsingular Gram matrix. The additive Jacobian over F_T is consequently one: extend the determinant calculation of Lemma 70.1 to matrices by elementary elimination, whose elementary shears have unit Jacobian. In split factors one performs the elimination on the two components separately. Each tube around $q = \gamma$ is g-stable. Taking limits on compact open tests proves invariance on boxes and hence on their generated Borel sets. □

Corollary 70.3. *In Theorem 67.3 the integral over the slab $s(e) \in t + L$ can be evaluated in the hyperbolic chart of Theorem 70.2 on the fiber with $\gamma = -N$, with exactly the factor $c_* P_T(\beta)$ of that theorem. The same chart formula holds with any integral trace-one element in place of η.*

Proof. Apply Theorem 70.2 to the product of G and the indicator of the slab for Borel L. Sets in its completion have Borel representatives with null difference, using the local equality of measures in Proposition 67.2 for the compact open tests and exhaustion by such tests. The same conclusion also follows by taking the completion of the equal measures in the chart formula. For a second integral trace-one element the trace splitting in Lemma 70.1 still holds, as does the change to the corresponding integral hyperbolic basis. The tube measure uses only q and the integral volume on the ambient space. The preceding proof therefore gives the same measure. □

71. Integral characters from finite-field eigenvalues

Definition 71.1. For a finite group J, a class function is a complex function constant on conjugacy classes. Put $\langle F, H\rangle_J = |J|^{-1}\sum_{g\in J}\overline{F(g)}H(g)$. Let $R(J)$ be the integer span of the characters of finite dimensional complex representations, called the virtual characters. It is a ring for pointwise addition and multiplication, by direct sums and tensor products. An elementary subgroup is a direct product of a cyclic group of order prime to a prime ℓ and a group of order a power of ℓ, allowing trivial factors. For $H \le J$ and a class function λ on H define

$$(\mathrm{Ind}_H^J \lambda)(g) = |H|^{-1} \sum_{\substack{x\in J\\ x^{-1}gx\in H}} \lambda(x^{-1}gx).$$

Lemma 71.2. *A class function F on a finite group J belongs to $R(J)$ if and only if its restriction to every elementary subgroup H belongs to $R(H)$.*

Proof. Induction in Definition 71.1 sends a character to a character. Given an H-space V, take a copy of V for each left coset xH with a chosen representative. If $gx = x'h$, let g carry the x-copy to the x'-copy by h. The group law follows by composing the two coset decompositions. Its trace is the displayed sum, since precisely the fixed cosets contribute. The same formula gives

$$F\,\mathrm{Ind}_H^J \lambda = \mathrm{Ind}_H^J((F|_H)\lambda).$$

The restriction of a virtual character is virtual by restricting the representations. It will therefore suffice to prove that the ideal $I \subset R(J)$ generated by $\mathrm{Ind}_H^J\lambda$ for elementary H and one-dimensional characters λ contains 1.

Write $m = |J|$, $O = \mathbb{Z}[\exp(2\pi i/m)]$, and put $S = O^b$ with one coordinate for each of the b conjugacy classes. The eigenvalues of every representing matrix of g are m-th roots of unity by the order argument in Proposition 22.2. Thus $R(J) \subset S$. The ring S is a finite module over $\mathbb{Z}$, generated by the first m powers of the chosen root in each coordinate. These

elements generate S over $R(J)$ as well. If $IS = S$, express each of these generators as a combination of them with coefficients in I. The adjugate of the identity minus the coefficient matrix shows that its determinant kills every generator and hence 1. This determinant is 1 modulo I, so $1 \in I$.

If IS is proper, its projection to O is proper in at least one coordinate g, since multiplication by the coordinate idempotents separates the components. Choose a maximal ideal of O containing that proper ideal. For existence enumerate the countable ring O, adjoining successive elements to the ideal whenever the resulting ideal is proper. The union is proper and maximal. The quotient K' is a field generated over $\mathbb{Z}$ as a module by finitely many elements. Every element of this field satisfies a monic polynomial over $\mathbb{Z}$, by applying the same adjugate argument to its multiplication on a list of module generators. It has positive characteristic ℓ: in characteristic zero, substituting $1/2$ in a monic integer polynomial and clearing denominators would give an odd integer equal to zero. Since K' is a field, ℓ is prime. In this coordinate every generator of I must vanish modulo the maximal ideal.

Write $g = cw = wc$ with c of order prime to ℓ and w of ℓ-power order. Such factors are powers of g by the coprime congruences in Lemma A.1. For every character its values at g, c have the same reduction in K': diagonalize g, so c, w are diagonal in the same basis, and every ℓ-power root reduces to one by $X^{\ell^s} - 1 = (X-1)^{\ell^s}$ in characteristic ℓ. The centralizer $Z_J(c)$ has a subgroup P whose order is the largest ℓ-power ℓ^a dividing $|Z_J(c)|$. Indeed let any group of order $\ell^a u$, with $\ell \nmid u$, act by left translations on subsets of size ℓ^a. Their number is congruent to u modulo ℓ, by the coefficient of X^{ℓ^a} in $(1 + X^{\ell^a})^u$. Some orbit has size prime to ℓ. Its stabilizer acts freely on the subset and has order a multiple of ℓ^a by counting cosets, so has order exactly ℓ^a.

Take $C = \langle c \rangle$, $H = CP = C \times P$. For any character λ of C extend it trivially on P. Counting the preimages of each conjugate in the induction formula gives

$$(\mathrm{Ind}_H^J \lambda)(c) = \frac{|Z_J(c)|}{|H|} \sum_{z \in C \cap \{xcx^{-1} : x \in J\}} \lambda(z).$$

Here an element in H conjugate to c has trivial P-component by its order. The prefactor is an integer prime to ℓ. The set summed over contains c. Multiply these sums by $\lambda(c)^{-1}$ and sum over the $|C|$ characters of C. By the geometric sum for powers of a cyclic generator the result is $|C|$, which is nonzero in K'. Consequently some induced character has nonzero reduction at c and at g. This contradicts properness of the projected ideal. Thus $1 \in I$. Multiplying any expression for 1 in these generators by F, the displayed induction identity proves the assertion whenever the restrictions are virtual. □

Lemma 71.3. *Let P be a finite group of prime-power order ℓ^a. Over an algebraically closed field of characteristic different from ℓ, every irreducible representation of P is induced from a one-dimensional character of a subgroup.*

Proof. Induct on $|P|$. For a nontrivial kernel apply induction to the quotient, and pull the inducing character back to the preimage of its subgroup. Cosets and their actions in the two induction constructions agree. For abelian P the commuting matrices are simultaneously diagonalizable since each is annihilated by $X^{|P|} - 1$ with distinct roots. To diagonalize a commuting family, split into eigenspaces successively, all of them invariant for the remaining matrices. An irreducible space then has dimension one.

Now take a faithful irreducible representation of a nonabelian P. Every nontrivial ℓ-group has nontrivial center, since counting the conjugacy orbits modulo ℓ shows that its center has size divisible by ℓ. Apply this to $P/Z(P)$. Some noncentral a has all its conjugates in $aZ(P)$, so $A = \langle a, Z(P) \rangle$ is abelian and normal. It has at least two weight spaces, since in a faithful action an element acting by a scalar is central. The group P permutes these spaces transitively by irreducibility. A weight space is irreducible for its stabilizer H. Indeed, translating any of its proper nonzero H-invariant subspaces over the weight-space orbit

would give a proper nonzero P-invariant direct sum. The whole space, expressed as this orbit of weight spaces, is the induced representation from H, which is proper. Induct inside H. Induction from a subgroup of H, followed by induction from H, is induction from the original subgroup, since products of the two sets of coset representatives represent its cosets in P and give the same action. □

Theorem 71.4. *Let J be a finite group, p a prime, and V a finite-dimensional representation of J over an algebraic closure of $\mathbf{F}_p$. Let m be any positive integer prime to p divisible by the prime-to-p part of $|J|$. Fix an isomorphism ι of the group of m-th roots of unity in this closure with the complex m-th roots of unity. For each integer $b \geq 0$,*

$$\beta_b(g) = \sum_{\lambda} \iota(\lambda)^b \quad (g \in J)$$

is a virtual character. The sum ranges over the eigenvalues of g on V counted with their algebraic multiplicities.

Proof. All eigenvalues have order dividing m, by the order of g and $X^{p^n} - 1 = (X-1)^{p^n}$ in characteristic p. Both groups of m-th roots have m elements by separability and are cyclic by the elementary exponent proof in Lemma 22.4. For $b = 1$ take an elementary subgroup $H = C \times P$ as in Definition 71.1, with $|P|$ an ℓ-power. In H the elements of p-power order form a direct factor H_p commuting with a complement D of order prime to p. If $\ell = p$, take $D = C$; otherwise split off the p-primary part of the cyclic factor C. For $h = dy$ with $d \in D,\ y \in H_p$, the eigenvalues of h are those of d with the same multiplicities. In fact d diagonalizes, and on each eigenspace y is unipotent and commutes with d.

Over the given field the restriction to D is a direct sum of irreducibles. Average any linear projection onto an invariant subspace over D to get an invariant complement, since $|D|$ is invertible. If D is cyclic all summands have dimension one. Otherwise its central cyclic prime-to-p factor acts by a single weight on each irreducible, and the remaining ℓ-group acts irreducibly there. By Lemma 71.3, each such summand has a coset basis induced from a one-dimensional character, with the cyclic factor acting by scalars. Lift the nonzero entries in this basis by ι, preserving products and permutations. These entries have order dividing m, being values of characters of subgroups of D. This gives complex representations whose eigenvalues at d are precisely the lifts of the old eigenvalues with multiplicity. Indeed a permutation cycle of length e with product of its entries equal to h_0 has characteristic polynomial $X^e - h_0$. Here $p \nmid e$ as the cycle length divides the order of d. The distinct roots lift bijectively to those of $X^e - \iota(h_0)$, since all the roots are eigenvalues of d and ι is injective. Inflating the direct sum along $H \to D$ realizes $\beta_1|_H$ as a character.

For general b perform the power operation on the eigenvalues of this complex representation of H. If its elementary symmetric functions are e_j, they are the characters of the exterior powers by diagonalizing each group element. For $b \geq 1$ its b-th power sum is an integer polynomial in these e_j: equate coefficients of X^b in

$$\left(\sum_{j \geq 0} (-1)^j e_j X^j\right)\left(\sum_{d \geq 1} p_d X^d\right) = -X \frac{\mathrm{d}}{\mathrm{d}X} \prod_i (1 - \lambda_i X), \qquad p_d = \sum_i \lambda_i^d.$$

The identity follows by expanding each $\lambda_i X/(1-\lambda_i X)$; the constant coefficient of the first factor is 1, so the recursion uses only integer operations. For $b = 0$ the function is the dimension times the trivial character. In each case $\beta_b|_H \in R(H)$. Lemma 71.2 proves virtuality on J. □

72. Parabolic induction of finite class functions

Definition 72.1. Fix a finite field K of size Q and characteristic p, and a rank bound $R \geq 1$. Set $G_j = \mathrm{GL}_j(K)$ for $0 \leq j \leq R$, with G_0 the trivial group. Consider sequences F

of class functions on the G_j. Write $F(g)$ for the value on an automorphism of any such K-space, using a basis. For g on V and a, b on A, B respectively, define

$$(F * H)(g) = \sum_{\substack{W \subset V \\ gW = W}} F(g|W) H(g|V/W),$$

$$(\Delta F)(a, b) = \frac{1}{|\operatorname{Hom}_K(B, A)|} \sum_{u : B \to A} F\begin{pmatrix} a & u \\ 0 & b \end{pmatrix}.$$

In the first formula $g|V/W$ denotes the induced quotient map. The second formula is used when $\dim A + \dim B \le R$. For functions of two matrix arguments, the product $*$ sums independently over invariant subspaces of both spaces, restricting in both arguments in the first factor and taking quotients in both in the second. Put $\mathbf{e}(g) = \mathbf{1}_{\dim V = 0}$ and $\mathbf{u}(g) = 1$. A class function supported in positive rank is cuspidal if $(\Delta F)(a, b) = 0$ whenever both ranks are positive. For functions of ranks i, j, the pairing on two matrix arguments is the normalized product-group pairing in Definition 71.1.

Lemma 72.2. *The product $*$ is associative with unit $\mathbf{e}$. If F, H are supported in ranks i, j, their product at rank $i + j$ is induction from the subgroup preserving an i-plane, of the product class function on the plane and its quotient. Consequently $*$ preserves virtual characters in each rank and*

$$\langle F * H, B' \rangle_{G_{i+j}} = \langle F \otimes H, \Delta B' \rangle_{G_i \times G_j}.$$

Proof. Counting flags $U \subset W \subset V$ stable under g gives the same three factors on $U, W/U, V/W$ in either parenthesization. The unique zero subspace and the full space give the unit assertions. All i-planes form one orbit of G_{i+j} by choosing bases. Their stabilizer consists of the block upper triangular matrices in Definition 72.1. The fixed-plane sum for g is the trace formula in Definition 71.1, since its fixed cosets are precisely the invariant planes. Ordinary induction preserves virtual characters by Lemma 72.2 and its proof. For any subgroup $P \le J$ that trace formula also gives $\langle \operatorname{Ind}_P^J f, h \rangle_J = \langle f, h|_P \rangle_P$: interchange the sums and substitute $g = xyx^{-1}$, $y \in P$; each x then gives the same sum. Averaging over the upper right block in P first gives exactly $\Delta B'$, as required. □

Proposition 72.3. *For the operations of Definition 72.1,*

$$\Delta(F * H) = (\Delta F) * (\Delta H).$$

Proof. Fix a, b. Choose the upper right block $u : B \to A$ uniformly. For an invariant subspace $W \subset A \oplus B$ put $X = W \cap A$, $Y = \operatorname{pr}_B W$. Then $aX = X$, $bY = Y$. At fixed X, Y, possible subspaces before the stability condition are parametrized by all maps $l : Y \to A/X$, whose graphs describe W/X in $(A/X) \oplus B$. Extend l to a map $B \to A$ by lifting in bases and taking zero on a complement of Y. Conjugating by the shear with that off-diagonal block changes $X \oplus Y$ to W and translates u by a fixed map. Hence the average at W, including its stability indicator, equals the one for $W = X \oplus Y$.

Choose complements $A = X \oplus A'$, $B = Y \oplus B'$. The condition is now $u_{A',Y} = 0$, which has probability $Q^{-\dim A' \dim Y}$. Conditional on it, the two blocks $u_{X,Y}$ and $u_{A',B'}$ are independent uniform maps. The diagonal actions on W are $a|X, b|Y$, and those on the quotient are $a|A/X, b|B/Y$. The remaining block of u does not enter the two induced actions. Thus their expected product is the product of the corresponding two block averages. The number $Q^{\dim A' \dim Y}$ of possible graphs cancels the stability probability. Summing over X, Y gives the assertion. □

Definition 72.4. Take m to be the prime-to-p part of $|G_R|$. It is divisible by the prime-to-p part of $|G_j|$ for $j \le R$, by embedding G_j on a direct summand and counting cosets. Fix an embedding ι of the m-th roots of unity as in Theorem 71.4. For any multiplicative character

$\theta = \iota^b$ of these roots with $b \geq 0$, put $\beta_\theta(g) = \sum_\lambda \theta(\lambda)$ over the eigenvalues with multiplicity. Define

$$\mu = \sum_{n=0}^{R} (-1)^n (\mathbf{u} - \mathbf{e})^{*n}, \qquad D_\theta F = \beta_\theta F, \qquad C_\theta = (D_\theta \mu) * \mathbf{u}.$$

Multiplication in D_θ is pointwise. A rank part means the sequence restricted to the specified rank and zero in all other ranks.

Theorem 72.5. *Each positive rank part of C_θ is a virtual character and is cuspidal. On the ranks in Definition 72.1 one has*

$$\mu * \mathbf{u} = \mathbf{u} * \mu = \mathbf{e}, \qquad \Delta\mu = \mu \otimes \mu, \qquad \Delta C_\theta = C_\theta \otimes \mathbf{e} + \mathbf{e} \otimes C_\theta.$$

Proof. A product of $R+1$ functions vanishing at rank zero vanishes at every rank at most R, by the stable flag description in Lemma 72.2. The finite geometric sum for μ is therefore a two-sided inverse. Since $\Delta\mathbf{u} = \mathbf{u} \otimes \mathbf{u}$ and $\Delta\mathbf{e} = \mathbf{e} \otimes \mathbf{e}$, Proposition 72.3 shows that $\Delta\mu$ is the inverse of $\mathbf{u} \otimes \mathbf{u}$ under the product in two arguments, truncated at total rank R. That inverse is $\mu \otimes \mu$ by the separated sums. On a triangular block matrix β_θ is the sum of its values on the diagonal blocks, since its characteristic polynomial is their product. Consequently

$$\Delta(D_\theta \mu) = (D_\theta \mu) \otimes \mu + \mu \otimes (D_\theta \mu).$$

Multiply by $\Delta\mathbf{u}$ and apply Proposition 72.3 once more. This proves the formula for ΔC_θ, its value in positive bidegrees, and hence cuspidality. Its value in rank zero is zero since an empty matrix has no eigenvalues.

Both $\mathbf{e}$ and $\mathbf{u}$ are characters or zero in each rank. The formula defining μ and Lemma 72.2 give virtuality of μ. Theorem 71.4 gives the same for β_θ, so pointwise multiplication followed by parabolic induction proves virtuality of C_θ. □

73. The torus calculation for cuspidal characters

Definition 73.1. Retain K, Q, m, ι and G_n from Definitions 72.1 and 72.4. Let $K_n = \{a \in \overline{K} : a^{Q^n} = a\}$ for $1 \leq n \leq R$. Embed $T_n = K_n^\times$ in G_n by multiplication in a K-basis of K_n. For $x \in T_n$ define b_x in $0 \leq b_x < Q^n - 1$ by $\iota(x) = \exp(2\pi i b_x/(Q^n - 1))$. Put $\theta_x = \iota^{b_x}$ on the m-th roots of unity. For a monic irreducible $P \in K[X]$ of degree n with $P(0) \neq 0$, choose a root x and put $\tau_P = (-1)^n C_{\theta_x}$ in rank n, extended by zero in all other ranks.

Lemma 73.2. *The set K_n is a field of size Q^n containing K. A monic irreducible of degree e has the distinct roots $z, z^Q, \ldots, z^{Q^{e-1}}$ in K_e. One has $K_e \subset K_n$ if $e \mid n$, and $Q^n - 1 \mid m$. On G_n, $C_\theta(g) = 0$ whenever the characteristic polynomial of g has two distinct irreducible factors. Also $\mu(g) = 0$ unless g is diagonalizable over $\overline{K}$.*

Proof. The roots of $X^{Q^n} - X$ form a field by the Frobenius power identities, and the derivative is -1, as in Lemma 22.4. The elements of K are fixed by Q-powering since their nonzero elements form a group of order $Q - 1$. Thus the fixed field of Q-powering in the closure is exactly K by degree. For a root z of degree e the K-algebra $K[z]$ is a field of size Q^e, by the basis $1, z, \ldots, z^{e-1}$ from polynomial division. Its elements satisfy $a^{Q^e} = a$. The orbit of z under Q-powering thus has size at most e; its orbit polynomial has coefficients fixed by Q-powering and is divisible by the minimal polynomial. It follows that the orbit has size e. Iterating Q^e-powering gives $K_e \subset K_n$ when $e \mid n$. The embedding of the group T_n of order $Q^n - 1$ into G_n gives the asserted divisibility by the definition of m. This justifies the choices in Definition 73.1.

If there are at least two primary factors, split $V = A \oplus B$ over K so that the characteristic polynomials of the actions a, b on the two nonzero summands are coprime. Explicitly, factor the characteristic polynomial as $D_1 D_2$ with both nonconstant, coprime, and with disjoint root sets. The adjugate identity for $XI - g$, whose adjugate commutes with g coefficientwise, gives $D_1(g)D_2(g) = 0$ by substitution. Bézout polynomials for D_1, D_2, obtained by polynomial

division, split V as the sum of the two kernels, with zero intersection. Both kernels are nonzero, as seen after triangularizing over the algebraic closure by successively taking an eigenvector in the quotient. On maps $l : B \to A$ the operator $l \mapsto al - lb$ is invertible. Its kernel is zero, since $al = lb$ implies $f(a)l = lf(b)$ and one may choose f equal to one modulo the polynomial on A and zero modulo that on B. Thus every upper right block is obtained by conjugating $a \oplus b$ by an upper triangular shear. Its C_θ-value equals the block average, zero by Theorem 72.5.

For the last assertion, let D be the product of the distinct irreducible factors and set $S = \ker D(g)$. It is nonzero in positive rank by the eigenvector argument. Since D splits with distinct roots over $\overline{K}$, $S = V$ precisely when g diagonalizes there: interpolation in the roots gives the eigenspace projections when $D(g) = 0$. Every invariant subspace with diagonalizable restriction lies in S. If $S \neq V$, induct on dimension in $\sum_{W \subseteq V,\, gW = W} \mu(g|W) = 0$, which is $\mu * \mathbf{u} = \mathbf{e}$ in Theorem 72.5. All proper subspaces contributing to this sum lie in S. Their sum over $W \subseteq S$ is zero by the same identity on $S \neq 0$. This forces $\mu(g) = 0$. □

Proposition 73.3. *For $1 \leq n \leq R$ and every θ of Definition 72.4, the pointwise product on G_n satisfies*

$$\mu C_\theta = \mathrm{Ind}_{T_n}^{G_n}(\theta|_{T_n}).$$

At the identity, $(-1)^n C_\theta(1) = \prod_{j=1}^{n-1}(Q^j - 1)$.

Proof. Multiplication by $z \in K_n^\times$ is diagonalizable over $\overline{K}$, with minimal polynomial that of z over K. Its characteristic polynomial is a power of that polynomial: on a basis over $K[z]$ it is the direct sum of copies of multiplication by z on $K[z]$. Conversely, a diagonalizable matrix g with characteristic polynomial P^s, where $e = \deg P,\ n = es$, makes V into a vector space of dimension s over $K[z]$ through $K[X]/P$ acting by $X \mapsto g$. This uses $P(g) = 0$ by diagonalizability. Choosing a basis over this field identifies g with multiplication by any root z of P on K_n, by Lemma 73.2. Its centralizer consists of the invertible $K[z]$-linear maps. The elements of T_n conjugate to g correspond exactly to the roots of P. Thus on every other class both sides of the asserted identity vanish, by Lemma 73.2 and the induction formula.

Write $q' = Q^e$ and $S_\theta(z) = \sum_{h=0}^{e-1} \theta(z^{Q^h})$. On the current class the stable subspaces are exactly the $K[z]$-subspaces. If $b_{s,j}$ counts the j-dimensional ones in a space of dimension s over a field of size q', then

$$\sum_{j=0}^{s} b_{s,j}(-1)^j (q')^{j(j-1)/2} X^j = \prod_{j=0}^{s-1}(1 - (q')^j X).$$

Indeed, relative to a fixed hyperplane the subspaces inside it contribute $b_{s-1,j}$, and the others contribute $(q')^{s-j} b_{s-1,j-1}$. For the latter choose the intersection of dimension $j-1$, and lift a vector with last coordinate one modulo that intersection. With out-of-range counts zero this recurrence gives multiplication of the polynomial for $s-1$ by $1-(q')^{s-1}X$, starting at 1. Setting $X = 1$ and using $\mu * \mathbf{u} = \mathbf{e}$ recursively shows that $\mu(g) = (-1)^s (q')^{s(s-1)/2}$. On a j-dimensional $K[z]$-subspace the eigenvalue sum is $jS_\theta(z)$. Differentiating the polynomial at $X = 1$ therefore gives

$$C_\theta(g) = -S_\theta(z) \prod_{j=1}^{s-1}(1 - (q')^j).$$

The identity case has $z = 1,\ e = 1,\ s = n$, giving the degree.

Counting ordered bases gives $|\mathrm{GL}_s(K[z])| = (q')^{s(s-1)/2} \prod_{j=1}^{s}((q')^j - 1)$ as in Proposition 15.2. The product $\mu(g)C_\theta(g)$ is therefore $|\mathrm{GL}_s(K[z])| S_\theta(z)/(Q^n - 1)$. Each conjugate in T_n has $|\mathrm{GL}_s(K[z])|$ conjugators, by the centralizer just computed. This is exactly the value of the induced character. □

Theorem 73.4. *The function τ_P of Definition 73.1 is independent of the root chosen. It is the character of an irreducible cuspidal representation of G_n of degree $\prod_{j=1}^{n-1}(Q^j - 1)$. Different degree-$n$ monic irreducibles with nonzero constant terms give inequivalent representations.*

Proof. For two lifts θ, η, apply Lemma 72.2 to $C_\eta = (D_\eta \mu) * \mathbf{u}$ in the second argument of the pairing with C_θ, taking complex conjugates of the adjunction if needed. Every term involving two positive ranks is zero by Theorem 72.5. The zero first rank term also vanishes since β_η is zero there. The surviving term is $\langle C_\theta, \mu\beta_\eta\rangle_{G_n}$. Since μ has integer values by its flag sum, moving it to the first slot gives $\langle \mu C_\theta, \beta_\eta\rangle_{G_n}$. By Proposition 73.3 and the subgroup adjunction in Lemma 72.2, we obtain

$$\langle C_\theta, C_\eta\rangle_{G_n} = \frac{1}{|T_n|}\sum_{z\in T_n} \overline{\theta(z)} \sum_{h=0}^{n-1} \eta(z^{Q^h}) = \#\{0 \le h < n : \theta|_{T_n} = (\eta|_{T_n})^{Q^h}\}.$$

The inner sum lists all eigenvalues of multiplication by z with their multiplicities, even when z belongs to a proper subfield, by Lemma 73.2. The last equality is the finite geometric sum on T_n.

The map $x \mapsto \theta_x|_{T_n}$ bijects T_n with its complex characters and carries Q-powering to Q-powering, since both sides use the same exponents modulo $Q^n - 1$. For roots x, y of degree n, the count in the last display is consequently 1 if their minimal polynomials agree and 0 otherwise. In particular the norm squared for x is one. By Theorem 72.5 the signed function is an integer linear combination of irreducible characters. By the orthogonality proved in Lemma 22.1, its norm squared is the sum of the squares of those integer coefficients. Exactly one is 1 or -1, and the degree computed in Proposition 73.3 makes the sign positive. The inner product between two root choices now proves independence and inequivalence as asserted. Cuspidality follows from Theorem 72.5. □

74. Squarefree labels and the marked projection

Definition 74.1. Let $c \in K[X]$ be squarefree, monic of degree $r \le R$ with $c(0) \neq 0$. Write it as a product of distinct monic irreducibles $P_1, \dots, P_h$, and put $\Xi_c = \tau_{P_1} * \cdots * \tau_{P_h}$ in rank r. The empty product in rank zero is $\mathbf{e}$. Factorization exists by successively factoring reducible polynomials of smaller degree. Euclidean division and Bézout's identity give Euclid's lemma for an irreducible divisor, and hence uniqueness of the set of monic factors.

Theorem 74.2. *The character Ξ_c is irreducible, independent of the order of the factors, and distinct for distinct labels c of the same degree. If $n_P = \deg P$ its degree and its share in the regular representation satisfy*

$$d_c = \Xi_c(1) = \frac{\prod_{j=1}^{r}(Q^j - 1)}{\prod_{P|c}(Q^{n_P} - 1)},$$

$$\frac{d_c^2}{|G_r|} = Q^{-r}\frac{\prod_{j=1}^{r}(1 - Q^{-j})}{\prod_{P|c}(1 - Q^{-n_P})^2} \;\ge\; Q^{-r}\left(1 - \sum_{j=1}^{r} Q^{-j}\right).$$

The products indexed by P use only the monic irreducible divisors of c.

Proof. Each τ_P is a character by Theorem 73.4. Repeated induction in Lemma 72.2 realizes Ξ_c on a complex representation. For another list of distinct irreducibles forming c', pair its product with the product for c. Repeatedly apply the adjunction of Lemma 72.2 to the latter. In the resulting h matrix slots each factor belonging to c' is assigned entirely to one slot. Indeed $\Delta\tau_{P'} = \tau_{P'} \otimes \mathbf{e} + \mathbf{e} \otimes \tau_{P'}$ by Theorem 72.5. Applying Proposition 72.3 and iterating gives a sum over all such assignments; the factors assigned to a slot are multiplied there in their original order. The other side in slot i is τ_{P_i}, of positive rank. Empty slots therefore give zero. If a slot receives two or more factors, the pairing in it vanishes by splitting off its first factor with the same adjunction and using cuspidality. For the remaining assignments each slot receives exactly one factor. The product of pairings in these slots is one if every polynomial matches and zero otherwise by Theorem 73.4. Distinctness of the factors gives exactly one matching assignment for equal sets and none for unequal sets. The rank zero convention agrees with these conclusions. By Lemma 22.1, the inner product of a

character with itself is the sum of the squares of its irreducible multiplicities. It is one here. The other pairings give both independence of the order and inequivalence.

Let $n_i = \deg P_i$. The stabilizer of a flag with successive dimensions n_i has order $Q^{\sum_{i<j} n_i n_j} \prod_i |G_{n_i}|$: the diagonal blocks are arbitrary invertible matrices and the strictly upper blocks are free. Transitivity on flags follows by choosing a basis successively. Each flag contributes the product of the dimensions from Theorem 73.4 at the identity. Multiplying by the flag count and substituting $|G_j| = Q^{j(j-1)/2} \prod_{\nu=1}^{j} (Q^\nu - 1)$ from the proof of Proposition 73.3 gives the asserted degree, since $\binom{r}{2} = \sum_i \binom{n_i}{2} + \sum_{i<j} n_i n_j$. Squaring and dividing by $|G_r|$ gives the second identity. Its denominator product is at most one. Multiplying the bounds $0 \le 1 - Q^{-j} \le 1$ inductively gives $\prod (1 - Q^{-j}) \ge 1 - \sum Q^{-j}$. □

Corollary 74.3. *For $p = g_i$ in Definition 22.3, the characters Ξ_c supply the representations of Proposition 22.5. Let e_i denote its projection onto the good tuples at p, formed from these representations. For $2 \le r < p$ one has the sharper estimate*

$$0 \le [1](1 - e_i) \le f \left(\frac{A+1}{p} + \frac{2}{p-1} \right) + \exp\!\left(-\frac{f}{2r(2r \log p + 1)} \right).$$

On the entire group $\mathcal{G}$ of Proposition 15.2, let e_0^{good} be the sum of the projections for the tuples good over both primes. It is a central self-adjoint idempotent. For all sufficiently large arithmetic scales, $0 \le [1](1 - e_0^{\mathrm{good}}) \le b^{-8}$.

Proof. Take $K = \mathbf{F}_p$, $R = r$ in Theorem 74.2. Since $\sum_{j=1}^{r} p^{-j} \le r/p$, its degree bound implies the hypothesis of Proposition 22.5. To obtain the sharper estimate, use $\sum_{j=1}^{r} p^{-j} \le 1/(p-1)$. For a uniformly chosen degree-r monic polynomial, the probability of a zero or a forbidden root is at most $(A+1)/p$. The chance of divisibility by the square of some monic positive degree polynomial is at most $\sum_{j=1}^{\lfloor r/2 \rfloor} p^{-j} \le 1/(p-1)$, by the count in Lemma 22.4. Bound the probability that an f-tuple fails either test in any slot by f times their sum. Failure to contain a primitive member has probability at most the exponential term, as in Proposition 22.5. On the retained tuples the product of the degree shares is at least $p^{-rf}(1 - 1/(p-1))^f$, and $1 - (1 - 1/(p-1))^f \le f/(p-1)$. This gives the displayed bound. Finally the two-prime projection is the tensor product of the two one-prime projections by distributivity. Their identity coefficients t_0, t_1 lie in $[0,1]$ by Lemma 22.1 and the normalized regular trace. Its complementary share is $1 - t_0 t_1 \le (1 - t_0) + (1 - t_1)$. Each summand is at most $b^{-8}/2$ eventually by Proposition 22.5, now with its hypotheses proved. □

75. Determinant twists and hyperplane averages

Proposition 75.1. *Let c be a label of degree r in Definition 74.1. For $u \in K^\times$ write $c^{[u]}(X) = u^r c(X/u)$ and $\chi_u = \theta_u|_{K^\times}$, using the exponent at rank one in Definition 73.1. The map $u \mapsto \chi_u$ is a group isomorphism from $K^\times$ onto its complex characters, and*

$$\Xi_{c^{[u]}}(g) = \chi_u(\det g)\, \Xi_c(g).$$

In rank one $\Xi_{X-u} = \chi_u$.

Proof. The exponent on $K^\times = T_1$ identifies the cyclic group and its dual as in Theorem 73.4. The formula for C_θ in Proposition 73.3 on a one-dimensional space gives the rank-one assertion. Suppose first $c = P$ is irreducible of degree d and $x \in K_d^\times$ is one of its roots. The roots of $P^{[u]}$ are the conjugates of ux, distinct and numbering d, so this is its minimal polynomial by Lemma 73.2. Write $u_d = \iota^{-1}(\exp(2\pi i/(Q^d - 1)))$ and $D = (Q^d - 1)/(Q - 1)$. Since $u_1 = u_d^D$, the exponent of ux at rank d equals that of x plus D times the rank-one exponent of u, modulo $Q^d - 1$. Hence

$$\theta_{ux}(z) = \theta_x(z)\, \chi_u(z^D) \qquad (z \in K_d^\times).$$

Here $z^D \in K^\times$, since its Q-th power equals itself.

At matrices whose characteristic polynomial has two different irreducible factors, both cuspidal characters vanish by Lemma 73.2. At a matrix of characteristic polynomial P_0^s, with $\deg P_0 = e$, $es = d$, every eigenvalue z belongs to $K_e^\times \subset K_d^\times$. The product of its d successive Frobenius powers equals the product of the e roots of P_0 raised to the s-th power, hence is $\det g = z^D$, with no condition on diagonalizability. Every invariant subspace uses a submultiset of these eigenvalues, by its triangular block matrix. In the defining sum $C_{\theta_{ux}}(g) = \sum_W \beta_{\theta_{ux}}(g|W)\mu(g|W)$ the last display thus multiplies every summand for x by the same scalar $\chi_u(\det g)$. This proves the twist identity for an irreducible factor. For a general label take the flag product of Definition 74.1: the product of the determinants on the successive quotients of each invariant flag equals $\det g$. This gives the identity on the whole class function. □

Theorem 75.2. *Let $\ell : K^r \to K$ be a nonzero linear functional and put $H_\ell = \{g \in G_r : g|_{\ker \ell} = 1\}$. In $\mathbb{C}[G_r]$ set*

$$B_{\ell,u} = \sum_{g \in H_\ell} \chi_u(\det g)^{-1}[g].$$

For a label c with $c(u) \neq 0$ one has $B_{\ell,u}\, e_{\Xi_c} = 0$, where e_{Ξ_c} is the central projection of Lemma 22.1 for the representation in Theorem 74.2. In particular, for the group $\mathcal{G}$ and $A = M_\omega$ of Definition 22.3, any pure group-algebra tensor with one factor $B_{\ell,u}$ satisfying $\chi_u^A = 1$ annihilates e_0^{good}.

Proof. The matrices in H_ℓ are exactly the invertible maps $I + a\ell$, $a \in K^r$, since their difference from the identity factors through the one-dimensional quotient by $\ker \ell$. Choose a basis of that kernel extended by a vector of ℓ-value one. Such a matrix has diagonal blocks I_{r-1} and $t \in K^\times$, and arbitrary upper right column. In any representation, $|H_\ell|^{-1}B_{\ell,u}$ acts as the projection onto vectors transforming by $\chi_u \circ \det$ under H_ℓ. Translation of the averaging sum proves the transformation law, and the average fixes each vector with that law. Its trace on the representation with character Ξ_c is

$$\frac{1}{Q-1} \sum_{t \in K^\times} \overline{\chi_u(t)}\, (\Delta\Xi_c)(I_{r-1}, t).$$

By Proposition 72.3 and Theorem 72.5, in this bidegree exactly one degree-one irreducible factor of c must go into the second slot, all others going to the first. Thus the expression is a sum over the linear factors $X - v$ of c of a constant times $\langle \chi_u, \chi_v \rangle_{K^\times}$, using Proposition 75.1. This is zero if $c(u) \neq 0$, by the geometric sum in Lemma 7.1. The trace of a projection is its dimension of image, so the operator vanishes. On all other irreducibles e_{Ξ_c} acts as zero by Lemma 22.1. Decomposing the faithful regular representation as in that lemma proves the group-algebra identity. In the two-prime tensor, $\chi_u^A = 1$ implies $u^A = 1$ by the group isomorphism. Each good tuple avoids that root in the chosen factor, by Definition 22.3; each tensor summand of Corollary 74.3 is therefore annihilated. □

Corollary 75.3. *Suppose $K = \mathbf{F}_p$, $9 \mid p - 1$, $r = 3^a$, $a \geq 1$. For a primitive label c of degree r and any two prescribed labels there is a primitive P avoiding both such that*

$$[g]e_{\Xi_P} = [g]e_{\Xi_c} \quad \textit{for every } g \in G_r \textit{ with reducible characteristic polynomial.}$$

Here $[g]e$ denotes the coefficient of $[g]$ in e.

Proof. By Lemma 22.6 at least one of the three primitive labels $c^{[u]}$, $u^3 = 1$, avoids both prescribed labels. Select such a P. Proposition 75.1 gives $\Xi_P = \chi_u(\det)\Xi_c$ and $\chi_u^3 = 1$. As the label is irreducible, Ξ_c vanishes on a matrix with two distinct irreducible characteristic factors by Lemma 73.2. In the remaining reducible case the characteristic polynomial is D^s for one irreducible D, $s > 1$ dividing 3^a. Its determinant is the s-th power of the product of the roots of D in $K^\times$, hence a cube, and the twist factor is one. Thus the characters agree at every stated matrix. The same matrices have inverses with reducible

characteristic polynomial: an invertible matrix with irreducible characteristic polynomial acts as multiplication by a degree-r field generator on K_r, by Lemma 73.2 and its proof. Its inverse also generates that same field, and hence has irreducible characteristic polynomial. Applying this argument to the inverse proves the needed implication. Finally the dimensions agree at the identity by the twist law. The formula for central projections in Lemma 22.1 now proves equality of their coefficients. □

76. Flag minors for the compact row weights

Definition 76.1. Let $n \geq 1$ and let $\lambda_1 \geq \cdots \geq \lambda_n$ be integers. Put $\kappa_i = \lambda_i - \lambda_n$. Draw κ_i boxes in row i, starting at the left, and read its columns in order from left to right, with respective heights $d_1 \geq \cdots \geq d_m > 0$; the empty diagram is allowed. For $I = (i_1 < \cdots < i_d) \subset \{1, \ldots, n\}$ put

$$M_I(X) = \det(X_{i_a,b})_{1 \leq a,b \leq d} \qquad (X \in \mathrm{GL}_n(\mathbb{C})).$$

Let V_λ be the complex span of the functions $(\det X)^{\lambda_n} \prod_{j=1}^{m} M_{I_j}(X)$, where $|I_j| = d_j$. If the entries of each I_j are written from top to bottom in column j, call the filling semistandard when its entries weakly increase from left to right in each row. The group acts on functions by $(\rho(g)F)(X) = F(g^{\mathsf{T}}X)$.

Proposition 76.2. *The action in Definition 76.1 preserves V_λ. The functions indexed by semistandard fillings form a basis of V_λ.*

Proof. The order of transposes gives $\rho(g)\rho(h) = \rho(gh)$. A d-row minor of $g^{\mathsf{T}}X$ is the sum over d-row subsets of X of their minors times the corresponding minor of g^{T}. To verify this expansion, expand the determinant by multilinearity in its rows. Equal chosen row indices give zero, and ordering the remaining indices gives the coefficient determinant. The common determinant power contributes a scalar. This proves stability.

Omit the common nowhere vanishing determinant factor for the basis calculation. Consider adjacent columns I, J of respective lengths $a \geq b$ with $I_s > J_s$ for some $1 \leq s \leq b$. Regard the last $a - s + 1$ entries of I and the first s entries of J as $a + 1$ independent row vectors of length a, using only their first b coordinates in the second minor. Alternate the product of the two minors on these $a+1$ vectors. The result is zero, since every alternating multilinear form on $a + 1$ vectors in dimension a vanishes by expansion in a basis. Group the permutations by the choice of vectors put into the first column. Up to the common positive factor $(a - s + 1)!\, s!$, each choice occurs once with sign, since permutations within either group have their signs canceled by the determinant signs. Substituting the original rows of X therefore writes the original product as a signed sum of the other choices.

Every new first column has received at least one of $J_1, \ldots, J_s$, all strictly smaller than $I_s, \ldots, I_a$. If a row is repeated its minor vanishes. Otherwise sorting each column introduces only a sign and makes the first column lexicographically smaller than I. The bound on the two sets of exchanged entries ensures that no other choice gives the original first column. Reading all indexed columns from left to right gives a finite lexicographically ordered set of lists. Straightening thus terminates and proves spanning.

For independence, order the variables first by matrix column, and in a column give smaller row index higher priority. In the lexicographic order on exponent vectors for this list, the leading monomial of M_I is $\prod_{a=1}^{d} X_{i_a,a}$. The determinant chooses the smallest available row successively, which proves this assertion and its coefficient one. Leading monomials multiply, since comparison of two exponent vectors is invariant under adding a fixed vector. In a product the exponents in matrix column a recover the multiset of entries in row a of the filling. For a semistandard filling their sorted list determines that entire row, so all the leading monomials are distinct. The polynomials are independent also as functions on the invertible matrices. Indeed $X + tI$ is invertible for all but at most n scalars t, and polynomial continuation in t then extends any relation to every X. A polynomial zero at all

matrices has zero coefficients by induction in its variables using the one-variable root bound. This proves the basis statement. □

Lemma 76.3. *For a partition $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$, the number of semistandard fillings of its diagram with entries $1, \ldots, n$, strictly increasing down columns, is*

$$\det\left[\binom{\alpha_i + n - i}{n - j}\right]_{1 \leq i,j \leq n} = \prod_{1 \leq i < j \leq n} \frac{\alpha_i - \alpha_j + j - i}{j - i}.$$

Proof. The assertion is one in dimension one. Removing the entries equal to n leaves a diagram γ of length at most $n-1$ with $\alpha_i \geq \gamma_i \geq \alpha_{i+1}$ for $1 \leq i < n$. Indeed those entries form row suffixes, and a removed box has no box below it. Every box in row n is removed by strict column increase. Conversely, under these inequalities the boxes of α outside γ lie at the bottoms of distinct columns, so filling them with n extends each filling of γ uniquely.

Write $x_i = \alpha_i + n - i$ and $y_i = \gamma_i + n - 1 - i$. The ranges $x_{i+1} \leq y_i < x_i$ are independent and imply the required ordering of γ. Sum the determinant formula in rank $n-1$ over these ranges, independently in its rows. For $h \geq 0$ the binomial recursion telescopes to

$$\sum_{y=x_{i+1}}^{x_i - 1} \binom{y}{h} = \binom{x_i}{h+1} - \binom{x_{i+1}}{h+1}.$$

The resulting determinant is the one in the statement: in that n-row determinant subtract row $i+1$ from row i, working from top to bottom, and expand along its last column, equal after these operations to the last coordinate vector. This proves the first formula by induction. For the second, as in Proposition 56.2, use the leading coefficient $1/h!$ of $\binom{X}{h}$ and the Vandermonde identity. With the degrees and nodes both in decreasing order the result is $\prod_{i<j}(x_i - x_j) / \prod_{h=0}^{n-1} h!$, which is the stated product. □

Corollary 76.4. *For the odd row rank r of Definition 56.1, put $\rho_i^0 = (r+1)/2 - i$ for $1 \leq i \leq r$, and $\lambda_{i,\tau} = t_{i-1,\tau} - \rho_i^0$. These are decreasing in the weak sense and are integers. The dimension of the space in Definition 76.1 at τ is $\mathcal{D}_\tau$. The tensor product over the f infinite places has dimension $\mathcal{D}$.*

Proof. Adjacent t's decrease by an integer at least one by Proposition 56.2, and successive ρ_i^0 differ by one. The dimension by Proposition 76.2 and Lemma 76.3 for $\alpha_i = \lambda_{i,\tau} - \lambda_{r,\tau}$ has numerator differences $\lambda_{i,\tau} - \lambda_{j,\tau} + j - i = t_{i-1,\tau} - t_{j-1,\tau}$. This is exactly the product in Definition 56.1. Tensor products of bases multiply these dimensions. □

77. The compact row representation and its matrix coefficient

Definition 77.1. For the weight λ and column heights of Definition 76.1, let

$$\mathcal{T}_\lambda = (\det)^{\lambda_n} \otimes \bigotimes_{j=1}^{m} \bigwedge^{d_j} \mathbb{C}^n, \qquad v_\lambda = 1 \otimes \bigotimes_{j=1}^{m} (e_1 \wedge \cdots \wedge e_{d_j}).$$

The first factor is the line on which g acts by $(\det g)^{\lambda_n}$. On each exterior power use the inner product making its increasing wedges of standard basis vectors orthonormal, and give the tensor product the product inner product, conjugate linear in the first variable. Denote this representation by T. Write W_λ for the span of $T(k)v_\lambda$ for $k \in U(n) = \{k : k^* k = I\}$.

Theorem 77.2. *The $U(n)$-space W_λ is irreducible, isomorphic to V_λ, and extends to the rational $\mathrm{GL}_n(\mathbb{C})$-action there. Its upper unitriangular-fixed line is spanned by v_λ, on which a diagonal matrix t acts by $\prod_i t_i^{\lambda_i}$. Its unit vector v_λ has matrix coefficient*

$$m_\lambda(k) = \langle v_\lambda, T(k) v_\lambda \rangle = (\det k)^{\lambda_n} \prod_{j=1}^{m} \det k_{\{1,\ldots,d_j\},\{1,\ldots,d_j\}}.$$

Proof. The exterior-power matrix entries are minors by expanding the wedge in basis vectors. This also shows $T(g^{\mathsf{T}}) = T(g)^{\mathsf{T}}$ and $T(g^*) = T(g)^*$. Thus $T(k)$ is unitary and the coefficient formula follows by selecting the initial coordinate in each wedge. For any vector a in the tensor coordinates define the linear function of a with values in the space of functions on the group by

$$F_a(X) = a^{\mathsf{T}} T(X) v_\lambda.$$

It sends the standard tensors to the spanning products in Definition 76.1 and intertwines $T(g)$ and $\rho(g)$ by the transpose identity.

Every complex $U(n)$-invariant subspace of $\mathcal{T}_\lambda$ is $\mathrm{GL}_n(\mathbb{C})$-invariant. Indeed it is finite-dimensional and closed. Differentiation of $T(\exp(tA))$ for $A^* = -A,\ t$ real, shows it is stable under the infinitesimal matrix $dT(A)$. The entries of T are rational functions regular on the group, so dT is complex linear. Every complex matrix is a complex linear combination of skew-Hermitian matrices by its Hermitian and skew-Hermitian parts. The subspace is therefore invariant under $T(\exp A) = \exp(dT(A))$ for all A. For the equality, differentiation of the group law gives the constant-coefficient differential equation on $t \mapsto T(\exp(tA))$; multiplying by $\exp(-t\, dT(A))$ gives derivative zero and value I at zero. The elementary shears $I + aE_{ij}$ and nonzero diagonal scalings are exponentials by $E_{ij}^2 = 0$ for $i \neq j$ and the complex logarithm of a nonzero scalar. Row reduction expresses every invertible matrix in these generators: adding a row with nonzero pivot when needed and rescaling permits elimination without row exchanges. This proves the invariance assertion.

In particular the orbit under $\mathrm{GL}_n(\mathbb{C})$ spans W_λ. It is stable under coordinate conjugation, since conjugation fixes v_λ and conjugates each representing matrix to the matrix at the conjugate element. Consequently the kernel of $a \mapsto F_a$, the bilinear annihilator of W_λ, is its Hermitian orthogonal complement. Restricting to W_λ now identifies it equivariantly with V_λ.

Every $F \in V_\lambda$ satisfies $F(Xb) = \prod_i b_{ii}^{\lambda_i} F(X)$ for b upper triangular invertible, since the first d columns are multiplied on the right by the first d diagonal block of b. If F is also fixed by $\rho(u)$ for all upper unitriangular u, its values are determined by $F(I)$. In fact every matrix with all initial principal minors nonzero is a product of a lower unitriangular matrix and an upper triangular invertible matrix: eliminate below each successive pivot; the next pivot equals the ratio of consecutive leading minors. On these matrices the two transformation laws determine F. They form a dense set, since each initial principal minor of $X + tI$ is a monic polynomial in t. Thus at most a line is fixed. The nonzero vector v_λ is fixed by these upper unitriangular matrices as they preserve the initial wedges; its diagonal weight is the one stated.

Finally every nonzero invariant subspace contains an upper unitriangular-fixed vector. In the full tensor model the diagonal action is diagonal with finitely many integer weights. Any invariant subspace is the direct sum of its intersections with the weight spaces. Indeed choose a single positive diagonal element separating the finitely many monomials (avoid the finitely many proper real hyperplanes in the logarithms of its entries), then use polynomial interpolation on its distinct eigenvalues. A finite union of such hyperplanes is proper since the product of their nonzero linear polynomials is a nonzero polynomial and cannot vanish everywhere, by the argument of Proposition 76.2. Choose a vector of lexicographically maximal weight in the subspace. For $i < j$, $dT(E_{ij})$ raises a weight by $+1$ in coordinate i and -1 in coordinate j, as follows by conjugating E_{ij} by diagonal matrices. Its value on the chosen vector is zero. Exponentiating shows all upper elementary shears fix this vector, and these generate the upper unitriangular group by successive elimination. If W_λ had a proper nonzero $U(n)$-invariant subspace, it and its nonzero orthogonal complement would both give fixed vectors, contradicting the one-dimensional fixed space just proved. This establishes irreducibility. □

Proposition 77.3. *For invariant probability measure* $\mathrm{d}k$ *on* $U(n)$*,*

$$\int_{U(n)} |m_\lambda(k)|^2 \,\mathrm{d}k = \frac{1}{\dim W_\lambda}.$$

In particular, for the compact row representations of Corollary 76.4 the product of these integrals over the infinite places is $1/\mathcal{D}$*.*

Proof. To construct the measure, give n-by-n complex matrices probability density proportional to $\exp(-\operatorname{tr}(X^*X))$ times Lebesgue measure. It is normalizable by the Gaussian integral in Lemma 4.2 and is invariant under left and right unitary multiplication by real orthogonal change of variables. The singular matrices have measure zero: a nonzero polynomial in independent complex coordinates has length-area product measure zero on its zero set, by induction in the number of variables, first excluding the common zero set of the nonzero leading coefficient and then the finitely many roots in each remaining fiber. On invertible matrices let $k = X(X^*X)^{-1/2}$. For a positive Hermitian matrix the positive inverse square root is defined on its eigenspaces by the positive real inverse square roots of its eigenvalues. Orthogonal diagonalization follows by taking an eigenvector and passing to its invariant complement, and positivity gives positive eigenvalues. This matrix function is continuous locally: on a fixed positive interval containing the nearby eigenvalues, expand $z^{-1/2}$ into its convergent binomial series about a positive number larger than the upper endpoint, uniformly on the interval. Such nearby bounds follow by bounding the change of the quadratic form on unit vectors by the matrix operator norm. The scalar expansion follows by differentiating the power series solution to $2(1-t)f'(t) = f(t)$, $f(0) = 1$. The image k is unitary and the map is equivariant on the left and right, since unitary conjugation carries each eigenspace to the conjugate eigenspace. The image probability is therefore bi-invariant. For uniqueness, integrate $F(xy)$ on the product of this measure in x and any left-invariant probability in y: integrating in the two orders gives equality of their integrals for every bounded Borel F.

On W_λ average the projection onto each unit vector $T(k)v_\lambda$. By left invariance the resulting operator commutes with the group. It is scalar, since any eigenspace of a commuting complex matrix is invariant and the space is irreducible by Theorem 77.2. Its trace is one, so its scalar is $1/\dim W_\lambda$. Evaluating the quadratic form of the average at v_λ proves the formula. Multiply over the places and use Corollary 76.4. □

78. Definite automorphic forms at the row level

Definition 78.1. Let $F = E^{\mathfrak{c}}$, and let H be the unitary group of $\sum_{a=1}^{r} \bar{x}_a x_a$ on E^r. Write H_f for its finite adelic group, $H(F)$ for its rational isometries, K^0 for the stabilizer of the standard lattice at every finite place, and $K(g)$ for its subgroup reducing to the identity on the chosen factors over g_0, g_1. Write $K_\infty = \prod_{\tau\in\Phi} U(r)$. For the weight of Corollary 76.4 put $W = \bigotimes_\tau W_{\lambda_\tau}$, $v = \bigotimes_\tau v_{\lambda_\tau}$; let $R_\infty(k)$ act on W by $\bigotimes_\tau T_\tau(\bar{k}_\tau)$. This uses conjugation on the compact argument, as for the column convention in Proposition 13.5. Let $\mathcal{A}^+$ be the functions $L : H_f \to W$ satisfying

$$L(\gamma h u) = R_\infty(\gamma_\infty) L(h) \qquad (\gamma \in H(F),\ h \in H_f,\ u \in K(g)).$$

Here γ_∞ is the tuple of its matrices at the chosen complex embeddings. On each compact group take the probability measure of Proposition 77.3.

Proposition 78.2. *The space* $\mathcal{A}^+$ *is linearly isomorphic to the continuous scalar functions on* $K_\infty \times H_f$ *that are invariant on the left by* $H(F)$*, invariant on the right by* $K(g)$*, and fixed by the operator*

$$(\mathcal{P}f)(k,h) = \mathcal{D} \int_{K_\infty} \langle R_\infty(k)v, R_\infty(t)v\rangle f(t,h)\,\mathrm{d}t.$$

This operator is the orthogonal projection in the compact variable onto the space of functions $k \mapsto \langle R_\infty(k)v, w\rangle$ with $w \in W$. The isomorphism sends L to $f_L(k,h) = \langle R_\infty(k)v, L(h)\rangle$.

Proof. Complex conjugation preserves the compact group probability, since its image is again invariant and the probability is unique by Proposition 77.3. The average of the orthogonal projections onto $R_\infty(k)v$ therefore factors over τ as the tensor product of the averages computed in that proposition. For every $w \in W$ this gives

$$\int_{K_\infty} R_\infty(t)v \, \langle R_\infty(t)v, w\rangle \, \mathrm{d}t = w/\mathcal{D}.$$

In particular the coefficient map is injective, and the displayed kernel fixes each of its functions. Its image on any continuous f is the coefficient function of $\mathcal{D} \int R_\infty(t)v f(t,h)\,\mathrm{d}t$ at fixed h. It is self-adjoint since its kernel is conjugate symmetric on exchanging k, t. The same formulas apply on the compact L^2 space by Cauchy–Schwarz, proving the orthogonal projection assertion.

Unitarity and the defining rule for L give $f_L(\gamma_\infty k, \gamma h u) = f_L(k,h)$. Continuity follows from local constancy on finite cosets and continuity of the compact coefficient. Conversely, for an invariant f fixed by $\mathcal{P}$ use the vector integral just given to define $L(h)$. Substitution $t = \gamma_\infty t'$ and invariance of measure give the defining rule of Definition 78.1. Since $\mathcal{P}f = f$ the coefficient recovers f. The vector average recovers L in the opposite direction as well. □

Theorem 78.3. *For the two marked primes in the arithmetic construction one has*

$$K^0/K(g) \simeq \mathcal{G}, \qquad \dim \mathcal{A}^+ = |\mathcal{G}|\, \mathfrak{m}_r\, \mathcal{D}.$$

Let each e_i of Corollary 74.3 act by right translation through its factor $J_i = \mathrm{GL}_r(\mathbf{F}_{g_i})^f$ of $\mathcal{G}$. Write R_i for its action and $\delta_i = [1](1 - e_i)$. Then

$$\operatorname{rank}(1 - R_i) = \delta_i \dim \mathcal{A}^+, \qquad (1 - \delta_0 - \delta_1)\dim \mathcal{A}^+ \le \dim(e_0^{\mathrm{good}} \mathcal{A}^+) \le \dim \mathcal{A}^+.$$

At all sufficiently large scales $\delta_0 + \delta_1 \le b^{-8}$.

Proof. At a marked rational prime the field splits completely by Lemma 13.2. On each pair an integral isometry is specified by an arbitrary matrix in $\mathrm{GL}_r(\mathbb{Z}_{g_i})$ on the chosen component, with inverse transpose on its partner. Reduction onto $\mathrm{GL}_r(\mathbf{F}_{g_i})$ is surjective by lifting the entries of an invertible matrix as in Proposition 13.5. The other places impose the same subgroup in numerator and denominator. This gives the quotient.

The double cosets of $H(F)$ by K^0 are finite by Theorem 23.2 in rank r. For a representative h, its stabilizer $\Gamma_h = H(F) \cap hK^0h^{-1}$ is finite, since it acts faithfully on its positive definite integral lattice with trace metric as in Lemma 13.4. Any finite-order element of $H(F) \cap hK(g)h^{-1}$ is the identity. At a chosen component of g_0 its conjugate belongs to the torsion-free principal congruence kernel proved in Proposition 13.5. The embedding of E into that completion is injective. Thus Γ_h acts freely on the $|\mathcal{G}|$ cosets of $K(g)$ in K^0 over h. Counting orbits and summing gives

$$|H(F)\backslash H_f / K(g)| = \sum_{[h]} |\mathcal{G}|/|\Gamma_h| = |\mathcal{G}|\, \mathfrak{m}_r.$$

Evaluation at these finer representatives takes arbitrary values in W, by triviality of their stabilizers and the transformation rule for L. Use Corollary 76.4 for its dimension. The sum of squared norms of these values gives a Hilbert norm independent of the representative choices by unitarity. Right translations act unitarily in this norm.

For any $s \in J_i$, choose a lift $\widetilde{s} \in K^0$ whose reduction is trivial in J_{1-i}. If right translation by s fixes a finer class $[h]$, then $h\widetilde{s} = \gamma h u$ for $\gamma \in H(F),\ u \in K(g)$. Thus $\gamma \in \Gamma_h$ and its conjugate has trivial reduction at the other marked prime. The same torsion argument at g_{1-i} forces $\gamma = 1$, then $s = 1$. Each J_i-orbit of the finer class set is therefore free. Evaluate L at the points $h\widetilde{t}$, $t \in J_i$, over one representative per orbit. Right translation just permutes these entries regularly, since different lifts of the same product differ by $K(g)$. On each orbit

the action is a direct sum of $\dim W$ copies of the regular representation. Its normalized trace on $1 - e_i$ equals the identity coefficient δ_i by the proof of Proposition 22.5. This trace equals relative rank since it is a projection. Finally R_0, R_1 commute and their product is the two-prime projection of Corollary 74.3. The codimension of the intersection of two images is at most the sum of the codimensions, by mapping to the direct sum of the quotient spaces. The quantitative bound follows from Proposition 22.5 and Theorem 74.2. □

Corollary 78.4. *With C, J_c, E_2 as in Definitions 68.1 and 69.1,*

$$E_2 \dim \mathcal{A}^+ = |\mathcal{G}|(CJ_c)^2.$$

For any vector space whose dimension is at most that of the marked definite subspace $e_0^{\mathrm{good}}\mathcal{A}^+$, the left side with its dimension in place of $\dim \mathcal{A}^+$ is bounded above by this same right side.

Proof. Insert the dimension from Theorem 78.3 in $E_2 = (CJ_c)^2/(\mathcal{D}\mathfrak{m}_r)$. The remaining assertion uses positivity and the subspace inclusion. □

79. A finite reduction of the marked trace comparison

Definition 79.1. Use the two-prime finite group $\mathcal{G}$ and the labels of Corollary 74.3. Let $\mathcal{E}_0 \subset \mathcal{G}$ be the elements whose matrix at some chosen coordinate over g_0 has irreducible characteristic polynomial. For a label c at one place write $e(c) = e_{\Xi_c}$, and for a tuple use $e(\mathbf{c}) = \bigotimes_w e(c_w)$. In a representation at one coordinate, $e(c_w)$ means the identity in every other factor.

Let H_1, H_2 be finite-dimensional unitary representations of $\mathcal{G}$. Let $\mathcal{B}$ be a unital subalgebra of $\mathrm{End}(H_1) \times \mathrm{End}(H_2)$ whose elements commute with the respective $\mathcal{G}$-actions. Write B_j for the component of $B \in \mathcal{B}$ on H_j. Let $(A_i)_{i \in I}$ be a commuting family in $\mathcal{B}$ whose members and their adjoints commute with one another on $H_1 \oplus H_2$. A joint eigensystem is a tuple $\omega = (\omega_i)$ with a nonzero joint eigenspace on this direct sum; $H_{j,\omega}$ denotes the corresponding joint eigenspace on H_j. Say a label at w occurs on $H_{j,\omega}$ if its one-coordinate projection has nonzero image there.

Lemma 79.2. *The direct sum in Definition 79.1 is the orthogonal sum of the finitely many joint eigenspaces. For every joint eigensystem ω a polynomial in finitely many A_i is the projection Π_ω onto it on both spaces. Each $H_{j,\omega}$ is $\mathcal{G}$-invariant.*

Proof. A normal complex matrix has an orthonormal eigenbasis. Indeed it has an eigenvalue by Lemma 16.2. For an eigenvector with eigenvalue z, use $\|(A^* - \bar{z})v\| = \|(A - z)v\|$, obtained by expanding both squares and using normality. Its orthogonal complement is invariant under both maps, and induction proves the claim. Diagonalize a nonscalar member of the family. The other operators and their adjoints preserve every eigenspace by commutation. Repeat on the proper summands so obtained until all restrictions are scalar, which terminates by decreasing dimension along every branch. Collect equal joint eigenvalues. For every $\omega' \neq \omega$ select $i(\omega')$ where they differ. Then

$$\Pi_\omega = \prod_{\omega' \neq \omega} \frac{A_{i(\omega')} - \omega'_{i(\omega')} I}{\omega_{i(\omega')} - \omega'_{i(\omega')}}.$$

This acts as one on the indicated block and zero on all others. The empty product works for a single block. Commutation with $\mathcal{G}$ gives its invariance assertion. □

Theorem 79.3. *In Definition 79.1 assume that*

$$\mathrm{tr}_{H_1}(B_1 s) = \mathrm{tr}_{H_2}(B_2 s) \qquad (B \in \mathcal{B},\ s \in \mathcal{E}_0),$$

where s acts by the respective representations. Suppose also that for each ω and each good tuple $\mathbf{c}$ there is a chosen coordinate w over g_0 at which c_w is primitive and one of the three labels $c_w^{[u]}$, $u^3 = 1$, occurs on neither $H_{1,\omega}$ nor $H_{2,\omega}$. Then for every $B \in \mathcal{B}$ the restrictions

of B_1, B_2 to $e_0^{\mathrm{good}} H_1, e_0^{\mathrm{good}} H_2$ have equal trace and equal characteristic polynomial. For the occurrence hypothesis it suffices that on each of the two spaces at most one primitive label occurs at each such coordinate for each eigensystem.

Proof. The ranks and primes in the construction satisfy $r = 3^a$, $a \geq 1$, $9 \mid g_0 - 1$ at sufficiently large x by Proposition 10.5 and Theorem 11.4. For fixed $\omega, \mathbf{c}$, choose the indicated w and a missing cubic alternative $P = c_w^{[u]}$. Replace the projection in this single coordinate in $e(\mathbf{c})$ by $e(P)$, calling the resulting product e'. It annihilates both ω-spaces, since $e(P)$ does so and the other coordinate factors commute with it. The coefficients of $e(\mathbf{c}) - e'$ vanish at every group element whose w-matrix has reducible characteristic polynomial: the coefficient at that element is the product of the unchanged coefficients times the difference of the two at w. This difference vanishes by the proof of Corollary 75.3, which applies to each of the three alternatives.

Summing $e(\mathbf{c}) - e'$ over the good tuples gives a group algebra element T_ω supported on $\mathcal{E}_0$. On both ω-spaces it acts exactly as e_0^{good}. Now $\Pi_\omega \in \mathcal{B}$ by Lemma 79.2. Apply the trace hypothesis to $B\Pi_\omega$ and extend linearly in s to T_ω. On each space $\Pi_\omega T_\omega = \Pi_\omega e_0^{\mathrm{good}}$, since all finite group actions commute with Π_ω. Summing over ω yields equality of the full traces of $B_j e_0^{\mathrm{good}}$. These are precisely the traces on the indicated ranges by commutation and the idempotent identity.

At $B = 1$ the dimensions agree. Apply the trace equality also to every positive power of B, all of which lie in $\mathcal{B}$. Over $\mathbb{C}$, triangularization by successive invariant eigenlines shows that the trace of B^a is the power sum of the eigenvalues with algebraic multiplicity. The polynomial identity in the proof of Theorem 71.4 determines every elementary symmetric coefficient recursively from the power sums: the coefficient of degree a on the differentiated side gives a nonzero integer multiple of the a-th elementary symmetric function. This proves equality of characteristic polynomials. Finally every good tuple has a primitive coordinate over g_0. The three alternatives there are distinct and primitive by Lemma 22.6; two singleton occurrence sets exclude at most two of them. □

Corollary 79.4. *Suppose a principal-level space H_1 on the indefinite form of Definition 15.1 and the definite space $H_2 = \mathcal{A}^+$ satisfy Theorem 79.3 for a common algebra $\mathcal{B}$ and commuting family as specified there. Put $d_0 = \dim(e_0^{\mathrm{good}} H_1)$. Then*

$$(1 - b^{-8})|\mathcal{G}|\,\mathfrak{m}_r\mathcal{D} \leq d_0 \leq |\mathcal{G}|\,\mathfrak{m}_r\mathcal{D}, \qquad d_0 E_2 \leq |\mathcal{G}|(CJ_c)^2$$

at sufficiently large scales.

Proof. Take the identity in Theorem 79.3. Theorem 78.3 bounds the resulting common dimension on both sides. Corollary 78.4 gives the last inequality. □

80. Elliptic tests on the definite group

Theorem 80.1. *Let $k \in U(n)$ have eigenvalues $z_1, \ldots, z_n$. For distinct eigenvalues the character of W_λ is*

$$\chi_\lambda(k) = \frac{\det[z_j^{\lambda_i + n - i}]_{i,j=1}^n}{\det[z_j^{n-i}]_{i,j=1}^n}.$$

At repeated eigenvalues use its continuous Laurent-polynomial extension. For every k,

$$\chi_\lambda(k) = \dim W_\lambda \int_{U(n)} m_\lambda(x^{-1}kx)\,\mathrm{d}x.$$

The character of $k \mapsto T(\bar{k})$ is $\overline{\chi_\lambda(k)}$.

Proof. Every unitary matrix diagonalizes in an orthonormal basis by the normal-matrix proof of Lemma 79.2. For a partition α of length at most n, let $S_\alpha(z)$ be the sum of the monomials of all semistandard fillings of its diagram, with one factor z_i for each entry i.

For the diagram κ in Definition 76.1, the basis of Proposition 76.2 diagonalizes the torus action. Each row index in its minors contributes the corresponding diagonal coordinate, so the character equals $(\prod_i z_i)^{\lambda_n} S_\kappa(z)$.

For distinct nonzero variables write $\Delta_n(z) = \det[z_j^{n-i}] = \prod_{j<l}(z_j - z_l)$ by the Vandermonde calculation in Proposition 56.2. We prove by induction that $\Delta_n S_\alpha = \det[z_j^{a_i}]$ with $a_i = \alpha_i + n - i$. It holds in one variable. In the removal of entries n in Lemma 76.3, write $s = z_n$ and $u_i = \gamma_i + n - 1 - i$ for the remaining diagram γ. Removed entries contribute $s^{\sum_i a_i - \sum_i u_i - (n-1)}$. Summing the determinants from the induction hypothesis in their independent rows therefore gives

$$\Delta_{n-1}(z_1, \dots, z_{n-1})\, S_\alpha(z) = s^{\sum_i a_i - (n-1)} \det\left[\sum_{b=a_{i+1}}^{a_i - 1} (z_j/s)^b \right]_{i,j<n}.$$

Evaluate each geometric sum and multiply by $\prod_{j<n}(z_j - s)$. The result is $s^{\sum_i a_i}$ times the determinant with entries $(z_j/s)^{a_i} - (z_j/s)^{a_{i+1}}$. This is $\det[z_j^{a_i}]_{i,j\le n}$: take s^{a_i} out of each row, then subtract the next row successively from top to bottom, leaving the last column the last coordinate vector. This proves the formula for α, and multiplication by the determinant power gives $\lambda_i + n - i$ in place of $\kappa_i + n - i$. The tableau expression is a Laurent polynomial on the nonzero coordinates and therefore also gives the continuous value at repetitions.

For the integral, write $m_\lambda(x^{-1}kx) = \langle T(x)v_\lambda, T(k)T(x)v_\lambda \rangle$. The average of the projections on $T(x)v_\lambda$ is $I/\dim W_\lambda$ by Proposition 77.3. Taking the trace after composition with $T(k)$ gives the identity. The eigenvalues of $\bar{k}$ are $\bar{z}_i$; the tableau polynomial has integer coefficients, giving the last assertion. □

Proposition 80.2. *In Definition 78.1 set $K = K(g)$ and choose a representative h_c for each class at level K. Let $\varphi : H_f \to \mathbb{C}$ have finite support modulo right K and be invariant on both sides by K. Define*

$$(\mathcal{T}_\varphi L)(h) = \sum_{yK \in H_f/K} \varphi(y) L(hy).$$

Writing χ_W for the character of R_∞, one has

$$\operatorname{tr}_{\mathcal{A}^+} \mathcal{T}_\varphi = \sum_c \sum_{\gamma \in H(F)} \varphi(h_c^{-1}\gamma h_c)\, \chi_W(\gamma_\infty).$$

At each c at most $|\operatorname{supp}(\varphi)/K|$ rational elements have a nonzero coefficient. Suppose φ is supported on local matrices which, at some chosen component above g_0, belong to $\mathrm{GL}_r(\mathbb{Z}_{g_0})$ and reduce to a matrix with irreducible characteristic polynomial. Then each contributing γ has irreducible characteristic polynomial over E and distinct eigenvalues at every infinite place. Its compact factor is the product over τ of the conjugates of the alternant ratios in Theorem 80.1.

Proof. Right invariance of L makes the operator independent of representatives in its sum. Left K-invariance of φ makes its output right K-invariant by reindexing, and its rational transformation law follows from that for L. Evaluate in the direct sum of copies of W constructed in Theorem 78.3. For the c-th diagonal block, yK contributes precisely when $h_c y = \gamma h_c k'$ for $\gamma \in H(F)$, $k' \in K$. Such a γ is unique by triviality of $H(F) \cap h_c K h_c^{-1}$ from that theorem. Conversely its coset $h_c^{-1}\gamma h_c K$ is uniquely determined by γ. The operator on that copy of W is $\varphi(y) R_\infty(\gamma_\infty)$, giving the trace and the finite count.

The characteristic polynomial at the chosen place equals the image of the rational characteristic polynomial, by conjugation and the determinant definition. For a monic polynomial over a discrete valuation field with integral coefficients and irreducible reduction of unchanged degree, a factorization into two positive-degree monic polynomials is impossible. In fact the minimum coefficient valuation of a product is the sum of the two minima: divide the factors by their minimum powers of a uniformizer and use that the product of their

nonzero residue polynomials is nonzero. Each monic factor has minimum at most zero, so for a monic integral product both minima must be zero. Reducing the factors would factor the reduction in their original positive degrees. Thus factorization over E is also impossible by the embedding of E into this local field. In characteristic zero a monic irreducible polynomial is relatively prime to its lower degree, nonzero derivative. Its roots are therefore distinct at each embedding. The eigenvalues have modulus one because the matrices are unitary at the infinite places. Apply Theorem 80.1 and multiply the traces on the tensor factors of W. □

81. The field of an elliptic term

Theorem 81.1. *Let E/F be the CM extension of Definition 10.1 with its conjugation, and let $\gamma \in \mathrm{GL}_r(E)$ have irreducible characteristic polynomial. Assume that at every embedding of E in the chosen CM type all complex eigenvalues of γ have absolute value one, and that γ preserves some nonsingular hermitian form h on E^r. Put $L = E[\gamma]$. It is a degree-r field over E. The involution of E extends to an involution $a \mapsto a^*$ on L with $\gamma^* = \gamma^{-1}$. Its fixed field K is totally real, and $L = K(i)$ is totally imaginary quadratic over K. In particular $[K : F] = r$.*

For any identification $E^r = L$ intertwining γ with multiplication by that element, the nonsingular hermitian forms preserved by γ are exactly

$$h_a(u, v) = \mathrm{Tr}_{L/E}(au^*v), \qquad a \in K^\times.$$

Here $\mathrm{Tr}_{L/E}$ is the trace of multiplication on the E-space L. The automorphisms of this space commuting with γ are the multiplications by $z \in L^\times$; they send h_a, by pullback, to h_{az^*z}. The sign on the eigenline at an embedding $\sigma : L \to \mathbb{C}$ over a chosen embedding of E is $\operatorname{sign}\sigma(a)$. There is one such eigenline for each of the real embeddings of K.

Proof. Write P for the characteristic polynomial. The adjugate argument of Lemma 73.2 gives $P(\gamma) = 0$. The kernel of evaluation $E[X] \to E[\gamma]$ is generated by a nonconstant divisor of P, by division by a monic polynomial of least degree in that kernel. Irreducibility gives $L \simeq E[X]/P$. The action on any nonzero vector gives an injective E-linear map $L \to E^r$, since nonzero field elements act invertibly. Equality of dimensions gives the identification in the statement. A linear operator commuting with γ commutes with every polynomial in γ, hence acts on L as multiplication by its value at 1.

The adjoint for h sends γ to γ^{-1} and scalars to their conjugates. Existence and uniqueness of the adjoint follow from the invertible Gram matrix. Since $P(0) \neq 0$, γ^{-1} is a polynomial in γ. Thus the adjoint restricts on the commutative algebra L to the field involution specified on its generators. For any complex embedding σ of L over the chosen type, $\sigma(\gamma)$ is a root of the image of P, so $\sigma(\gamma^{-1}) = \overline{\sigma(\gamma)}$. The same intertwining with complex conjugation holds on E and therefore on L. For an embedding over the other type compose with complex conjugation to get the same identity. Every embedding of K into $\mathbb{C}$ extends to L, by taking successive roots of minimal polynomials in $\mathbb{C}$ as in Lemma 16.2. It thus has real image. For $z \in L$ the elements $(z + z^*)/2$, $(z - z^*)/(2i)$ belong to K, and $i^* = -i$. This proves $L = K \oplus iK$ and also $E = F \oplus iF$. Comparing degrees gives $[K : F] = r$. Each embedding of K has two extensions to L, interchanged by complex conjugation since the two send i to opposite roots of -1. Precisely one extends the chosen type on E.

Given any other preserved hermitian form, its adjoint on L is the same by the equations on E, γ. Let $f(v) = h(1, v)$. Then $h(u, v) = f(u^*v)$. The pairing $(a, v) \mapsto \mathrm{Tr}_{L/E}(av)$ identifies L with its E-linear dual: if $a \neq 0$, test at $v = a^{-1}$ to get the nonzero trace r. The dimensions agree. Hence $f(v) = \mathrm{Tr}_{L/E}(av)$ for a unique a. Conjugating the entries of the multiplication matrix and changing to the starred basis gives $\overline{\mathrm{Tr}_{L/E}\, v} = \mathrm{Tr}_{L/E}(v^*)$. The hermitian condition consequently says $a^* = a$, using nondegeneracy of the trace pairing. Every nonzero such a gives a nonsingular form with γ unitary, by $\gamma^*\gamma = 1$, and multiplication by z gives the displayed pullback law.

At an embedding $\tau : E \to \mathbb{C}$, evaluation at all the roots of τP identifies $L \otimes_{E,\tau} \mathbb{C}$ with $\mathbb{C}^r$. Its matrix in the power basis is a Vandermonde at distinct roots by separability, as in Proposition 80.2. In those coordinates trace is summation, and $\sigma(u^*) = \overline{\sigma(u)}$. The orthogonal diagonal entries of the form are therefore precisely $\sigma(a)$. □

Corollary 81.2. *With L/K as in Theorem 81.1, fix signs $\varepsilon_\sigma \in \{\pm 1\}$ at the real embeddings of K. Suppose $d \in K^\times$ has these signs and there is a finite idele z of L with $z^*z = d$. Here a finite idele has nonzero components at every finite place and integral unit components at all but finitely many places. Multiplication $h_a \mapsto h_{da}$ changes precisely the specified eigenline signs. All finite forms and isometries with their commuting matrix γ are carried to each other by the single finite adelic multiplication by z. In particular if finite lattice frames are chosen on the first form they can be chosen on the second with the same framed Gram matrix and the same matrix of γ.*

Proof. At each infinite place the sign of $\sigma(da)$ is the product of the two signs, by Theorem 81.1. At the finite places the identity $h_a(zu, zv) = h_{da}(u, v)$ follows by the same trace calculation after scalar extension. Thus z^{-1} carries frames on h_a to frames on h_{da} with the same Gram matrix. It commutes with γ, so the framed matrix of γ is unchanged at every place. Since z is a finite idele these are legitimate adelic changes of frames. The inverse change uses d^{-1}, z^{-1}. □

82. Ideals in the fields of elliptic elements

Definition 82.1. For a finite extension $P/\mathbb{Q}$ of degree D, let O be its ring of algebraic integers. Use all real embeddings and one embedding from each nonreal conjugate pair to form $P_\infty = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, with the ordinary product of length and area. Put $e_i = 1$ at a real and $e_i = 2$ at a complex coordinate. A nonzero fractional ideal is a nonzero O-submodule of P contained in $m^{-1}O$ for some positive integer m. Those contained in O are called integral. Products of ideals are the additive spans of pairwise products. For an integral ideal $\mathfrak{a} \neq 0$ its norm is $N\mathfrak{a} = [O : \mathfrak{a}]$. For $a \in P$ write $N_{P/\mathbb{Q}}a, \operatorname{Tr}_{P/\mathbb{Q}} a$ for the determinant and trace of multiplication.

Lemma 82.2. *The image of O is a full lattice of some covolume $V_P > 0$ in P_∞, and $D = r_1 + 2r_2$. For every $a \neq 0$,*

$$|N_{P/\mathbb{Q}}a| = \prod_i |\sigma_i(a)|^{e_i}.$$

Every nonzero integral ideal $\mathfrak{a}$ has finite index and lattice covolume $V_P N\mathfrak{a}$.

Proof. Each minimal polynomial in characteristic zero has distinct roots since it is relatively prime to its derivative. Adjoining a finite list of generators of P gives exactly D embeddings into $\mathbb{C}$, by extending at each stage by sending a generator to every root of the transported minimal polynomial. Some rational linear combination of the generators has distinct values under all embeddings: the equality for a pair cuts out a proper linear condition on its coefficients, and a finite product of the resulting nonzero linear polynomials cannot vanish on all rational tuples by the root argument in Proposition 76.2. This element has degree D and generates P. Multiplying it by an integer clears the denominators in the rescaled monic polynomial, so we may take an integral generator u. The embedding matrix on $1, u, \ldots, u^{D-1}$ is a nonsingular Vandermonde. In the complex embedding coordinates every multiplication matrix diagonalizes with its conjugates as diagonal entries. This gives the norm formula and trace as the sum of embeddings. Conjugate embeddings pair up and replacing each conjugate pair by real and imaginary coordinates gives a real nonsingular matrix.

Algebraic integers form a ring: for integral elements a, b their powers below the degrees of their two monic equations span $\mathbb{Z}[a, b]$ as a finite $\mathbb{Z}$-module. Multiplication on this module and the adjugate determinant trick of Lemma 71.2 give a monic integer equation

for each member. A rational algebraic integer is an integer by clearing denominators in its monic equation. Consequently the trace of an algebraic integer is an integer. Writing $L_0 = \mathbb{Z}[u]$, we have $L_0 \subset O \subset \{x \in P : \mathrm{Tr}_{P/\mathbb{Q}}(xL_0) \subset \mathbb{Z}\}$. The right side is a lattice by the invertible trace matrix in the power basis, the product of the transpose of the embedding matrix and that matrix. It contains L_0 with finite index. It follows that O is a full lattice as well; the free-basis argument for subgroups of integer lattices appears in Lemma 12.1. Any nonzero ideal containing a contains aO, a full sublattice of index $|N_{P/\mathbb{Q}}a|$ by the multiplication determinant. The assertion on covolumes then follows by counting the cosets in a fundamental region, as in Lemma 20.2. □

Proposition 82.3. *Every nonzero fractional ideal is uniquely a product of integer powers of nonzero prime ideals with only finitely many nonzero exponents. An ideal is integral exactly when all exponents are nonnegative. Norm extends multiplicatively to fractional ideals, with $N((a)) = |N_{P/\mathbb{Q}}a|$. At a prime $\mathfrak{p}$, the local ring $O_\mathfrak{p}$, obtained by inverting $O \setminus \mathfrak{p}$, is a discrete valuation ring. Its valuation on $P^\times$ is the exponent of $\mathfrak{p}$ in the ideal of the element.*

Proof. A nonzero prime ideal is maximal since its quotient is a finite domain by Lemma 82.2, hence a field. Every nonzero integral ideal contains a product of nonzero prime ideals, allowing the empty product for O. Otherwise choose a counterexample maximal by inclusion: every proper enlargement has strictly smaller finite index. It is not prime, so some a, b outside it have product in it. The two larger ideals formed by adjoining a or b have product contained in it. Using the assertion for both gives a contradiction.

For a nonzero prime $\mathfrak{p}$ set $\mathfrak{p}^{-1} = \{x \in P : x\mathfrak{p} \subset O\}$. It is fractional, since a nonzero integer belongs to $\mathfrak{p}$ by its finite index. Fix $0 \neq a \in \mathfrak{p}$ and choose a product of as few primes as possible contained in (a). One factor must lie in $\mathfrak{p}$ by primality, hence equals it. The product of the remaining factors contains some $b \notin (a)$ by minimality, so $b/a \in \mathfrak{p}^{-1} \setminus O$. Now $\mathfrak{p} \subset \mathfrak{p}\mathfrak{p}^{-1} \subset O$. If the first inclusion were equality, multiplication by b/a would preserve the full integer lattice $\mathfrak{p}$. Its integral matrix satisfies a monic polynomial by the adjugate identity, contradicting $b/a \notin O$. Thus $\mathfrak{p}\mathfrak{p}^{-1} = O$.

For a proper nonzero ideal $\mathfrak{a}$ take a maximal ideal $\mathfrak{p}$ containing it, possible by finiteness of the quotient. Then $\mathfrak{a}\mathfrak{p}^{-1}$ is integral and strictly contains $\mathfrak{a}$, by the same lattice argument using $b/a \notin O$ above. Iterate at decreasing positive indices and multiply back the invertible primes. This proves factorization. If two products coincide, any prime in one contains the product in the other, hence contains and equals a factor there. Cancel the invertible factors successively to get uniqueness. Clearing a rational integer denominator proves the assertion for fractional ideals and its integrality test.

For any invertible integral ideal J, choose $x \in J,\ y \in J^{-1}$ with $xy \notin \mathfrak{p}$, since 1 is a sum of such products. Multiplying y by a suitable integral coefficient makes $xy \equiv 1 \bmod \mathfrak{p}$. Then multiplication by x induces $O/\mathfrak{p} \simeq J/\mathfrak{p}J$, with inverse multiplication by y, by the displayed congruence on both modules. This gives $[J : \mathfrak{p}J] = |O/\mathfrak{p}|$; successive indices show norm multiplicativity and extension to integer exponents. The statement for principal ideals follows from Lemma 82.2 after clearing denominators.

Write $v(x)$ for the indicated exponent. The element x lies in $O_\mathfrak{p}$ exactly when $v(x) \geq 0$: its denominator ideal $\{b \in O : bx \in O\}$ has prime exponents $\max(0, -v_\mathfrak{q}(x))$, by the integrality test. When $v(x) \geq 0$ this ideal is not contained in $\mathfrak{p}$, and conversely a denominator outside $\mathfrak{p}$ has exponent zero there. Units correspond to $v(x) = 0$. Any element of $\mathfrak{p} \setminus \mathfrak{p}^2$ has value one, and such elements exist by the index just proved. Thus every element of the local ring is a power of such a uniformizer times a unit, and any nonzero ideal there is generated by an element of least valuation. □

Definition 82.4. The ordinary class group is the group of nonzero fractional ideals modulo principal fractional ideals. For a totally real field P, let $\mathcal{C}_4(P)$ be the group of fractional ideals with exponent zero at every prime over 2, modulo the principal ideals (a) with $\sigma_i(a) > 0$ for every real embedding and $a \equiv 1 \bmod 4O_\mathfrak{p}$ for every $\mathfrak{p} \mid 2$.

Lemma 82.5. *There are finitely many integral ideals of bounded norm. The two class groups in Definition 82.4 are finite, and each of their classes has an integral representative, prime to* 2 *in the case of* $\mathcal{C}_4(P)$.

Proof. An ideal of norm j contains jO, since its index-j additive quotient is annihilated by j. Ideals of that norm are thus among finitely many subgroups of O/jO. Given an integral ideal $\mathfrak{a} \neq 0$, the cube argument of Lemma 13.4 on its real embedding lattice of covolume $V_P N\mathfrak{a}$ gives $0 \neq a \in \mathfrak{a}$ with each real and imaginary coordinate bounded by $C_P(N\mathfrak{a})^{1/D}$. Indeed two points of a cube with volume larger than the covolume are congruent modulo the lattice, by integration over its fundamental parallelepiped. Their difference has the asserted size. By the norm formula, $|N_{P/\mathbb{Q}}a| \leq C'_P N\mathfrak{a}$. The integral ideal $(a)\mathfrak{a}^{-1}$ then has norm at most C'_P by Proposition 82.3 and represents the inverse ordinary class. Clearing denominators shows all ordinary classes occur this way.

An element x integral at every prime over 2 can be written $b/m,\ b \in O,\ m$ a positive odd integer. Its denominator ideal in Proposition 82.3 has odd index: the norm of a prime over an odd rational prime is odd, since its finite residue field has this characteristic. Every residue characteristic is a rational prime since the quotient is a finite field. The denominator ideal contains its own index, which gives such an m. Thus x has a well-defined residue in $O/4O$. A unit at all primes over 2 has invertible residue by the same fact for x^{-1}. A residue of one is equivalent to the local congruences of Definition 82.4, by factorization of $4O$ and the just constructed odd denominator. On the group of principal ideals prime to 2, the further ray relation has finite index: the signs and invertible residue of a generator take only finitely many joint values, and two generators with the same values have quotient totally positive and congruent to one. Together with finiteness of ordinary classes this proves finiteness of $\mathcal{C}_4(P)$. Finally a fractional ideal prime to 2 can have its denominators cleared by a positive odd integer, by applying the denominator argument to a finite integer generating set obtained from Lemma 82.2. Multiplying by the square of that integer clears denominators and is positive and congruent to one modulo 4, so preserves its signed ray class. □

83. Counting ideals with signs and a residue condition

Lemma 83.1. *In the notation of Definition 82.1 put*

$$\mathcal{H} = \{y \in \mathbb{R}^{r_1+r_2} : \sum_i e_i y_i = 0\}, \qquad l(u) = (\log|\sigma_i u|)_i \quad (u \in O^\times).$$

The image of l *is discrete and spans* $\mathcal{H}$. *The same holds on any finite-index subgroup* U *of* $O^\times$. *One can choose in* U *units whose logarithms are a real basis of* $\mathcal{H}$; *the group* U_0 *they generate has finite index in* U *and its logarithms form a lattice with a bounded half-open fundamental parallelepiped.*

Proof. The norm of a unit has absolute value one by Proposition 82.3, so its logarithm vector lies in $\mathcal{H}$. Bounded logarithms imply bounded coordinates in the lattice $O \subset P_\infty$ of Lemma 82.2. Only finitely many units satisfy a given such bound; this applies also to the kernel of l.

For $y \in \mathcal{H}$, multiply each archimedean coordinate of O by $\exp(y_i)$. Its real covolume is unchanged by the defining equation of $\mathcal{H}$, using two real dimensions per complex place. By the cube argument of Lemma 82.5 there is $0 \neq a \in O$ with all $|\sigma_i(a)|\exp(y_i) \leq C$, for a fixed $C > 1$. Their product with the indicated multiplicities is the positive integer $|N_{P/\mathbb{Q}}a|$, between 1 and C^D. This gives a positive lower bound for each factor, depending only on C, D. The possible principal ideals (a) form a finite set by Lemma 82.5. Fix one generator per ideal; then a equals one of these generators times a unit. It follows that y lies within a fixed bounded distance of the negatives of the logarithms of units. Their linear span must be all of $\mathcal{H}$, since multiples of a vector orthogonal to a proper span would violate this

bound. If a subgroup has index h, the h-th power of every unit lies in it by counting cosets in the abelian quotient. Its logarithms therefore have the same span. Choose a basis from these logarithms and choose its unit lifts. Integer combinations are uniquely represented in this basis, with a half-open parallelepiped for the quotient. Every coset of U_0 in U has a representative whose logarithm is in the bounded parallelepiped. Finiteness of units with bounded logarithm proves the index assertion, including when $\mathcal{H} = 0$. □

Lemma 83.2. *Choose a parallelepiped $P_0 \subset \mathcal{H}$ as in Lemma 83.1. For a point of P_∞ with all coordinates nonzero put*

$$t(x) = \left(\prod_i |x_i|^{e_i} \right)^{1/D}, \qquad y_i(x) = \log |x_i| - \log t(x).$$

Let $\mathcal{R}$ be the set with $0 < t(x) \le 1$ and $y(x) \in P_0$, with either all real signs allowed or all real signs prescribed positive. Its ordinary real volume v is finite and positive. For every full lattice Λ in P_∞, its translate $z + \Lambda$, and $Y \ge 1$,

$$\#\big((z + \Lambda) \cap Y^{1/D}\mathcal{R}\big) = \frac{vY}{\operatorname{covol} \Lambda} + O_{\mathcal{R},\Lambda}(1 + Y^{1-1/D}).$$

Proof. The region is parametrized by $t \in (0, 1]$, $y \in P_0$, one sign for each real place, and an angle for each complex place. A real coordinate is $\pm t \exp y_i$, a complex coordinate $t \exp(y_i + i\theta_i)$. The parameter maps on the closures, with angles in $[0, 2\pi]$, are Lipschitz by boundedness of their first derivatives on compact boxes. The interior contains a nonempty open set, since t, y are recovered by the formulas above and the angles by continuous local arguments at nonzero coordinates. This also shows the boundary is contained in the images of finitely many faces of these parameter boxes, including $t = 0$. Indeed away from those faces the parameter map and its local inverse give an open neighborhood within the region. The closures cover all limit points by compactness of the parameter boxes. Thus the region is bounded and has the stated positive, finite volume. The number of continuous real parameters is $1 + (r_1 + r_2 - 1) + r_2 = D$.

Write $T = Y^{1/D}$. Divide each boundary parameter face into $O_{\mathcal{R}}(1+T^{D-1})$ boxes of side at most $1/T$ in its at most $D - 1$ coordinates. Their images after dilation by T have uniformly bounded diameters. Tile by translates of a fixed half-open fundamental parallelepiped of Λ, one for each point of $z + \Lambda$, using these points as corners of the tiles. The discrepancy between the count times the covolume and the volume is bounded by the total volume of tiles whose closures meet the boundary. The closures of all these tiles lie within their fixed diameter of the bounded diameter images just obtained. Disjoint interiors and a uniform upper bound on the volumes of these neighborhoods give the asserted error, uniformly in the translate. This proof also handles $D = 1$, where the boundary consists of finitely many points. □

Theorem 83.3. *In any ordinary class of the degree-D field P, the number of nonzero integral ideals of norm at most $Y \ge 1$ is*

$$c_P Y + O_P(1 + Y^{1-1/D}), \qquad c_P > 0,$$

with the same constant for every class. For totally real P, in each class of $\mathcal{C}_4(P)$ the number of integral ideals prime to 2 of norm at most Y is $c_{P,4}Y + O_P(1 + Y^{1-1/D})$, again with a common positive constant.

Proof. For the ordinary class choose an integral ideal $\mathfrak{b}$ in its inverse class by Lemma 82.5. The ideals being counted correspond to $0 \ne a \in \mathfrak{b}$ with $|N_{P/\mathbb{Q}} a| \le Y N\mathfrak{b}$, modulo all units, by $\mathfrak{a} = (a)\mathfrak{b}^{-1}$ and Proposition 82.3. Choose U_0 and P_0 for $U = O^\times$ in Lemma 83.1. For a unit u, $y(ua) = l(u) + y(a)$. Thus each U_0-orbit has precisely one representative with $y(a) \in P_0$, since its logarithms are exactly the basis lattice and l is injective on U_0. Multiplication of a nonzero element by units is a free action, so exactly $[U : U_0]$ such representatives

occur per ideal. Apply Lemma 83.2 to $\Lambda = \mathfrak{b}$, $z = 0$ with all signs allowed, using $YN\mathfrak{b}$ for its argument. Its covolume is $N\mathfrak{b}\,V_P$ by Lemma 82.2. This gives leading constant $v/(V_P[U : U_0])$, independent of $\mathfrak{b}$.

For a signed ray class choose an integral $\mathfrak{b}$ in its inverse class prime to 2. Now $\mathfrak{a}\mathfrak{b} = (a)$ with a totally positive, $a \in \mathfrak{b}$, and $a \equiv 1 \bmod 4O$. This equivalence follows from Definition 82.4 and the local congruence criterion in Lemma 82.5. These last two finite conditions specify one coset of $4\mathfrak{b}$. Indeed $\mathfrak{b} + 4O = O$, since any maximal ideal containing both would contradict the prime factorization; choose $a_0 \in \mathfrak{b}$ congruent to one. Also $\mathfrak{b} \cap 4O = 4\mathfrak{b}$ by taking prime exponents. The unit group preserving the generator conditions is $U = \{u \in O^\times : u > 0 \text{ at every real place},\ u \equiv 1 \bmod 4O\}$, of finite index by the finite residue and sign maps. Use its U_0, P_0 in the positive-sign region for the coset $a_0 + 4\mathfrak{b}$, still with argument $YN\mathfrak{b}$. This time the covolume is $4^D N\mathfrak{b}\,V_P$ and the leading constant is $v/(4^D V_P[U : U_0])$. The error is uniform over classes in each assertion by their finiteness. □

Corollary 83.4. *The series $\zeta_P(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}$ over nonzero integral ideals converges absolutely for $\Re s > 1$ and has a simple pole of positive residue at 1. If P is totally real and χ a nontrivial complex character of $\mathcal{C}_4(P)$, the series*

$$L_4(s, \chi) = \sum_{(\mathfrak{a},2)=1} \chi([\mathfrak{a}])(N\mathfrak{a})^{-s}$$

over integral ideals is holomorphic on $\Re s > 1 - 1/D$ by continuation. Both series have their Euler products over the allowed prime ideals in $\Re s > 1$.

Proof. For the full sum use the first count of Theorem 83.3 and add over all classes. For the character sum the main terms in the second count cancel: multiplication by a class on which $\chi \neq 1$ preserves the complete class sum, so that sum is zero. The respective counting functions are $cY + O_P(1 + Y^{1-1/D})$, $c > 0$, and $O_P(1 + Y^{1-1/D})$. Integrate each against $sY^{-s-1}\,\mathrm{d}Y$ on $[1, \infty)$ as in Proposition 3.3. The errors give holomorphic functions for $\Re s > 1 - 1/D$ by Lemma A.3. The first main term gives $cs/(s-1)$. Absolute convergence to the right of one follows from the unrestricted count, and expanding finite geometric products there recovers the series by unique ideal factorization in Proposition 82.3. □

84. Split prime classes of an elliptic field

Lemma 84.1. *Let K be a totally real number field and $L = K(i)$. Suppose L contains $\xi = \exp(2\pi i/12)$ in a fixed complex embedding, with * denoting complex conjugation over K. Then $O_L = O_K \oplus O_K\xi$. For every prime $\mathfrak{p}$ of O_K, with residue field κ of size $Q = N\mathfrak{p}$, the primes of L above it are either two of norm Q (the split case) or one of norm Q^2 (the inert case). All have ramification index one. If Q is odd the split case occurs exactly when $Q \equiv 1 \bmod 4$.*

Let $K_\mathfrak{p}$ be the fraction field of the completion of $(O_K)_\mathfrak{p}$ and let $L_\mathfrak{p} = L \otimes_K K_\mathfrak{p}$. In the split case every element of $K_\mathfrak{p}^\times$ has the form z^*z; in the inert case precisely the elements of even valuation have this form. In both cases a unit has a unit preimage.

Proof. The identity $\xi - \xi^* = i$ of Lemma 20.1 holds in the chosen embedding and hence in L. For $x \in O_L$, $b = (x - x^*)/i$ and $a = x - b\xi$ are fixed by * and integral, so belong to O_K. Conversely all $a + b\xi$ with integral coefficients are integral by Lemma 82.2. Independence follows from $\xi \neq \xi^*$. Thus $O_L = O_K[X]/(P)$, with $P = X^2 - (\xi + \xi^*)X + 1$ of discriminant -1. Modulo $\mathfrak{p}$ this gives two distinct linear factors or one irreducible quadratic. The prime ideals above $\mathfrak{p}$ correspond to the field factors of $\kappa[X]/(\bar{P})$, with the stated residue sizes. Their product is $\mathfrak{p}O_L$, since their intersection is the kernel of reduction and distinct maximal ideals are comaximal. This proves the ramification assertion by Proposition 82.3. At an odd prime, $i = 2\xi - (\xi + \xi^*)$ and 2 is invertible, so the same algebra is $\kappa[X]/(X^2 + 1)$. By the cyclicity proof in Lemma 22.4, -1 has a square root in this finite field precisely when its multiplicative group has order divisible by four.

Put $A = \widehat{(O_K)_\mathfrak{p}}$, with uniformizer π as in Proposition 82.3. Taking compatible residues gives the complete algebra $A[X]/(P)$ because $1, X$ are a basis. At each power $\mathfrak{p}^j$ this is the product of the quotients by the j-th powers of the primes above $\mathfrak{p}$, by the factorization just proved and the ordinary Chinese remainder argument: comaximal ideals have comaximal powers by the binomial expansion of $1 = (u+v)^n$, and two comaximal ideals have intersection equal to their product. Localizing at a prime leaves each of its prime-power quotients unchanged since the denominators are invertible there. The complete algebra is therefore the product of the completed integer rings above $\mathfrak{p}$. By Lemma 18.1 its split factors are A, A, interchanged by *; its inert factor is a discrete valuation ring with uniformizer π. In the split case the norm of $(t, 1)$ is t.

In the inert residue extension the conjugation interchanges the roots of $\bar{P}$. It is the Q-th power map, which fixes κ and has at most Q fixed elements by the root bound. Cyclicity in the group of order $Q^2 - 1$ makes $w \mapsto w^{Q+1}$ onto the group of order $Q - 1$. The additive map $w \mapsto w + w^Q$ onto κ is nonzero since its polynomial has degree $Q < Q^2$; its image is a nonzero κ-subspace of κ. Given a unit $t \in A$, first lift a residue with norm $\bar{t}$. If the norm of a lift z_j is t modulo π^j, multiply by $1 + \pi^j c$. Its norm changes by the factor $1 + \pi^j(c + c^*)$ modulo π^{j+1} for $j \geq 1$. Surjectivity of residue trace corrects the next digit. The successive lifts converge to a unit of exact norm t. Finally $\pi^* = \pi$, so extracting a power of π gives exactly the even valuations. □

Theorem 84.2. *For L/K as in Lemma 84.1, let $C = \mathcal{C}_4(K)$ be the signed ray group of Definition 82.4. Write $\chi_4(1) = 1$, $\chi_4(3) = -1$ on $(\mathbb{Z}/4\mathbb{Z})^\times$. On fractional ideals prime to 2 put $\omega(\mathfrak{a}) = \chi_4(N\mathfrak{a} \bmod 4)$, reducing positive rational norms with odd numerator and denominator. This descends to C, and*

$$\ker \omega = \langle [\mathfrak{p}] : \mathfrak{p} \nmid 2,\ \mathfrak{p} \text{ splits in } L \rangle.$$

Proof. Norm multiplicativity is Proposition 82.3. If $a \equiv 1 \bmod 4$ at all dyadic primes, write $a = 1 + 4c/m$ with $c \in O_K$ and m a positive odd integer by the denominator argument of Lemma 82.5. In an integer basis of O_K, the multiplication matrix of a is the identity modulo 4 over the ring of rationals with odd denominator. Its determinant $N_{K/\mathbb{Q}}a$ is one modulo 4. For a totally positive this determinant is the positive ideal norm of (a) by Lemma 82.2 and Proposition 82.3. Thus ω descends. By Lemma 84.1 its value is 1 at the odd split primes and -1 at the odd inert primes. The subgroup S on the right is contained in its kernel.

For real $\sigma > 1$, Corollary 83.4 gives the Euler product for $\zeta_L(\sigma)$. Taking the real logarithm and using $-\log(1-u) = \sum_{j \geq 1} u^j/j$ for $0 \leq u < 1$ gives

$$\log \zeta_L(\sigma) = 2 \sum_{\mathfrak{p} \nmid 2 \text{ split}} (N\mathfrak{p})^{-\sigma} + O_K(1) \quad (\sigma \downarrow 1).$$

Indeed the contribution at exponent one from inert primes is bounded by $\sum_\mathfrak{p} (N\mathfrak{p})^{-2} \leq \zeta_K(2)$. The higher powers at primes of L are bounded by $2\sum_\mathfrak{P} (N\mathfrak{P})^{-2} \leq 2\zeta_L(2)$. The finitely many dyadic factors are bounded, by factorization of $2O_L$. The positive-residue pole of Corollary 83.4 now forces divergence of the displayed split-prime sum as $\sigma \downarrow 1$.

Let λ be a character of C/S, viewed on C. If $\lambda \neq 1, \omega$, both λ and $\lambda\omega$ are nontrivial and their series L_4 are bounded near one by Corollary 83.4. For $\sigma > 1$, multiplying their absolutely convergent Euler products, with $Q = N\mathfrak{p}$ in each factor, gives

$$L_4(\sigma, \lambda) L_4(\sigma, \lambda\omega) = \prod_{\mathfrak{p} \nmid 2 \text{ split}} (1 - Q^{-\sigma})^{-2} \prod_{\mathfrak{p} \nmid 2 \text{ inert}} (1 - \lambda([\mathfrak{p}])^2 Q^{-2\sigma})^{-1}.$$

The logarithm of the absolute value of the second product has absolute value at most $2\sum_\mathfrak{p} Q^{-2}$, by the convergent logarithmic power series and $|\lambda| = 1$. The first product tends to infinity by the split-prime sum. This contradicts boundedness. Thus every character of C/S is 1 or ω. Both are trivial on $\ker \omega/S$. Characters of a finite abelian group separate its elements by Lemma 7.1, and C is finite by Lemma 82.5. Hence $\ker \omega/S$ is trivial. □

85. Changing the real signs of elliptic trace forms

Theorem 85.1. *Let L/K satisfy the hypotheses of Lemma 84.1. A family $s_\sigma \in \{1,-1\}$, indexed by the real embeddings of K, occurs as the signs of an element $d \in K^\times$ of the form $d = z^*z$ at all finite places for a finite idele z of L if and only if $\prod_\sigma s_\sigma = 1$. When this product equals one, d may be chosen with $d \equiv 1 \bmod 4$ at all primes over 2, and with its fractional ideal supported on split primes away from 2.*

Proof. Suppose first that $\prod_\sigma s_\sigma = 1$. The translated full lattice $1+4O_K$ in $K_\infty = \mathbb{R}^{[K:\mathbb{Q}]}$ has a point α in the open orthant with signs s_σ. Indeed its bounded translated fundamental tiles cover this real space by Lemma 82.2, and the point with coordinates Ts_σ for sufficiently large T lies farther from each coordinate hyperplane than a tile diameter. The norm of α is positive by the product of its real embeddings, and is one modulo 4 by the determinant argument of Theorem 84.2. Thus $\omega((\alpha)) = 1$. The same theorem gives integers $e_\mathfrak{p}$, zero except at finitely many odd split primes, such that $[(\alpha)] = [\prod_\mathfrak{p} \mathfrak{p}^{e_\mathfrak{p}}]$ in $\mathcal{C}_4(K)$. By Definition 82.4 their quotient is (b) with b totally positive and congruent to one at the indicated modulus. Then $d = \alpha/b$ has the required signs, is one modulo 4 at the dyadic primes, and has ideal $\prod \mathfrak{p}^{e_\mathfrak{p}}$. At each nonsplit prime d is a unit and at each split prime it is a local norm by Lemma 84.1. That lemma gives unit preimages except where d has nonzero valuation. The choices together are a finite idele.

Conversely, let $d = z^*z$ at the finite places. We first choose $u \in L^\times$ so that $d' = d/(u^*u)$ is one modulo 4 at each dyadic prime of K. For this it suffices to approximate z at all primes $\mathfrak{P}$ of L above 2: if each $u/z_\mathfrak{P}$ belongs to $1 + 4\widehat{(O_L)_\mathfrak{P}}$, the product with its conjugate is likewise congruent to one. On the base field this is the congruence in K, since the ramification indices are one by Lemma 84.1. For the approximation, choose $m \geq 0$ with all $2^m z_\mathfrak{P}$ integral. For each $\mathfrak{P}$ their residues modulo arbitrarily high powers are represented by elements of O_L: completion preserves the successive residue rings, and elements outside $\mathfrak{P}$ have inverses modulo $\mathfrak{P}^j$ by comaximality. The Chinese remainder argument in Lemma 84.1 specifies all these residues by one $c \in O_L$. Choose the powers strictly above the valuations of $2^m z_\mathfrak{P}$, with the differences in valuation at least $v_\mathfrak{P}(4)$. Then $c \neq 0$ and $u = c/2^m$ satisfies the required relative congruences.

The element d' is still a norm at every finite place and has the signs of d, since $\sigma(u^*u) > 0$ at every real embedding of K. At every odd inert prime the valuation of d' is even by Lemma 84.1. All other odd prime factors have $\omega = 1$, so $\omega((d')) = 1$ by Proposition 82.3 and Theorem 84.2. Hence the positive rational number $|N_{K/\mathbb{Q}} d'|$ is one modulo 4. Also $N_{K/\mathbb{Q}} d' \equiv 1 \bmod 4$ by the multiplication determinant computation in that theorem, which uses only the dyadic congruence for the signed determinant. An odd rational number and its negative have distinct residues modulo 4. Consequently $N_{K/\mathbb{Q}} d' > 0$, which is precisely the required product of signs. □

Corollary 85.2. *Use the fields and the unitary elliptic element γ of Theorem 81.1, and fix a real sign on each of its r eigenlines at each infinite place of the chosen CM type. Relative to the signs of h_a, suppose an even number of the prescribed signs differ. There is a rational γ-invariant trace form $h_{a'}$ with exactly the prescribed signs and a finite change of frames commuting with γ that preserves both the finite Gram matrix and the finite matrix of γ. In particular, starting with positive signs everywhere, this applies to any prescribed selection of j negative eigenlines at each place, for any $0 \leq j \leq r$, when the number of infinite places is even.*

Proof. The field L contains E, which contains ξ by Lemma 20.1. The real embeddings of K correspond to these eigenlines by Theorem 81.1. The quotients of the prescribed signs by the initial ones have product one. Apply Theorem 85.1 to these quotients and take $a' = da$. The frame change is given by Corollary 81.2. In the last assertion the number of sign changes is jf where f is the even number of infinite places. For the arithmetic data

of Proposition 10.3, $f = \varphi(M)U$ is even since $6 \mid M$ and negation pairs the distinct units modulo M. □

86. Finite framed classes with prescribed eigenline signs

Definition 86.1. Fix a polynomial P occurring in Theorem 81.1, write $L = E[X]/(P)$, $y = X \bmod P$, and let m_z denote multiplication by z on L or its scalar extensions. Use the resulting fields L/K and order the extensions of each embedding in the chosen CM type. Write $\mathbb{A}_{E,f}$ for the finite adelic ring, formed with the completed integral rings at almost all places, and $L_f = L \otimes_E \mathbb{A}_{E,f}$. The basis of L over E identifies this module with the finite adelic product of the completions of L. Indeed, at almost all primes a fixed integral power basis spans the full integer ring locally, since the two full integer lattices have finite index after clearing denominators by Lemma 82.2. At a remaining prime the decomposition follows by taking compatible residues in the prime-power factorization as in Lemma 84.1, using the ramification exponents in Proposition 82.3.

Choose a nonsingular finite adelic hermitian matrix J_f and a compact open subgroup K_0 of $G_f = \{g : g^* J_f g = J_f\}$, where the star on matrices is conjugate transpose. Given a sign assignment s on the embeddings of K, consider pairs (a, x), where $a \in K^\times$ has signs s and $x : \mathbb{A}_{E,f}^r \to L_f$ is an invertible finite adelic frame with $x^* h_a x = J_f$. Take their equivalence classes under

$$(a, x) \;\longmapsto\; (a/(w^*w),\, m_w x k) \qquad (w \in L^\times,\ k \in K_0).$$

Write $Y_x = x^{-1} m_y x$, $T(K) = \{u \in L^\times : u^* u = 1\}$ and $T_x = x^{-1} m_{T(K)} x \subset G_f$.

Proposition 86.2. *Definition 86.1 parametrizes precisely the rational simultaneous isomorphism classes, up to right K_0 on the frame, of triples consisting of a nonsingular hermitian form on an r-dimensional E-space, a preserving operator of characteristic polynomial P, and a finite adelic frame with Gram matrix J_f and the prescribed real eigenline signs of the form. The rational centralizer preserving the form is $T(K)$, and its subgroup preserving the frame modulo K_0 is identified with $T_x \cap K_0$. This latter group is finite.*

Proof. The cyclic module identification in Theorem 81.1 carries the operator to m_y. Its adjoint induces the unique involution on L extending conjugation and sending y to y^{-1}, so this is the same involution for all triples. The theorem identifies every form obtained this way with exactly one h_a on the fixed model. Its choices of intertwining isomorphism differ by m_w, $w \in L^\times$. Transporting the form by m_w changes a to $a/(w^*w)$, and transports the frame to $m_w x$, by the pullback rule in the theorem. The remaining choice of frame modulo level is right K_0. Conversely each pair gives a triple with the required signs, so these changes give the full equivalence relation. The same pullback rule gives the centralizer and the frame stabilizer as stated.

If $u \in T(K)$ and $x^{-1} m_u x \in K_0$, the finite vector $u = m_u(1)$ belongs to a compact subset of L_f, by continuity of evaluation and compactness of $x K_0 x^{-1}$. Every compact subset of the finite adeles of L is contained in $m^{-1}\widehat{O}_L$ for one positive integer m, where $\widehat{O}_L$ is the product of the completed integer rings: the open sets $(j!)^{-1}\widehat{O}_L$ form an increasing cover by the restricted product definition. Thus $mu \in O_L$ by Proposition 82.3. All its complex embeddings have absolute value m since $u^* u = 1$ and L/K is CM by Theorem 81.1. The integral lattice of Lemma 82.2 has only finitely many points in this bounded region. □

Theorem 86.3. *For sign assignments s, t in Definition 86.1 with $\prod_\sigma s_\sigma = \prod_\sigma t_\sigma$, their framed classes are in bijection. Representatives can be paired with exactly the same Y_x and exactly the same subgroup T_x of G_f. Hence any right-level invariant weight depending only on the simultaneous K_0-conjugacy class of (Y_x, T_x), including a factor $1/|T_x \cap K_0|$, has the same sum on the two sets whenever it is nonnegative or absolutely summable. The bijection also respects every right translation on frames before quotienting by K_0.*

Proof. Use Theorem 85.1 to choose $d \in K^\times$ with signs t_σ/s_σ, and a finite idele z with $z^*z = d$. The map

$$(a, x) \longmapsto (da, m_z^{-1}x)$$

preserves J_f by Corollary 81.2. It commutes with the left rational changes in Definition 86.1 because all multiplications in L_f commute, and it commutes with right multiplication on the frame by associativity. It has inverse given by d^{-1}, z^{-1}. Both m_y and each $m_u,\ u \in T(K)$, commute with m_z. Their conjugates by the new frame are therefore their old conjugates, term by term. Changing the representative by k conjugates all these matrices by the same k. Proposition 86.2 gives finite stabilizer orders. Reindexing a nonnegative or absolutely summable family along a bijection proves the last sum identity. □

87. Norms and involutions on idele classes

Definition 87.1. For an abelian group A written additively with involution σ, set $d = 1 - \sigma,\ N = 1 + \sigma$ and define

$$A^0 = \ker d/NA, \qquad A^{-1} = \ker N/dA, \qquad h(A) = |A^0|/|A^{-1}|$$

when the quotients are finite. In multiplicative notation the maps are $d(x) = x/\sigma x,\ N(x) = x\sigma x$.

Lemma 87.2. *For a short exact sequence $0 \to A \to B \to C \to 0$ compatible with involutions, finiteness of the two quotients for any two groups implies finiteness for the third and $h(B) = h(A)h(C)$. For finite A, $h(A) = 1$. For finitely generated A, $h(A) = 2^{r_+ - r_-}$, where $r_\pm$ are the respective dimensions of the two eigenspaces on $A \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. The endomorphisms d, N in Definition 87.1 have both composites zero. In the following cyclic sequence the last term is followed by the first:

$$A^0 \to B^0 \to C^0 \to A^{-1} \to B^{-1} \to C^{-1} \to A^0.$$

The unnamed connecting arrows take a representative c killed by $D = d$ or $D = N$, lift it to $b \in B$, and use the class of $Db \in A$. Changing the lift changes this by DA; changing c by the image of the other endomorphism can be lifted by that endomorphism and contributes zero. This proves well-definedness. At B, if a cycle becomes a boundary in C, subtract a lift of that boundary using the same endomorphism to obtain a cycle in A. At C vanishing of the connecting class means $Db = Da$ for some $a \in A$, so $b - a$ is a cycle lifting c. At A, if a cycle becomes a boundary Db in B, the image of b in C is a cycle with that connecting class. Consecutive arrows have zero composite by construction. These arguments prove exactness at all six positions. Finiteness follows by bounding each remaining term by its kernel and image. Alternating multiplication of their orders gives the stated quotient rule.

For a finite group the two identities $|\ker D|\,|DA| = |A|$ for $D = d, N$ give $h(A) = 1$. For finitely generated A first remove its finite torsion subgroup using the exact sequence just proved. In a free lattice the sum of its integral $+1$ and -1 eigensubgroups has finite index because it contains $2A$, via $2x = (x + \sigma x) + (x - \sigma x)$, and their intersection is zero. The individual contributions for free rank one with the two involutions are 2 and $1/2$ by direct substitution in the definition. The torsion-free groups have bases by the subgroup argument in Lemma 12.1; the rational eigenspaces are spanned by clearing denominators. Multiplication gives the formula. □

Lemma 87.3. *At an unramified quadratic extension of nonarchimedean local fields K'/K, all units of O_K are norms of units and every norm-one unit has the form $z/\sigma z$ for a unit z. The same conclusions hold for the split algebra $K \times K$. For $\mathbb{C}/\mathbb{R}$, the norm image is $\mathbb{R}_{>0}$ and every norm-one element is of the form $z/\bar{z}$.*

Proof. In the unramified case one may take the same uniformizer π in both fields. On the quadratic residue field the norm is surjective on units and the trace surjective on the additive

group by Lemma 84.1 and its proof. Lift a residue unit of the required norm. To improve a lift z correct through π^m to one correct through π^{m+1}, multiply by $1+\pi^m y$, $m \geq 1$. Its norm modulo π^{m+1} is $1+\pi^m \mathrm{Tr}(\bar{y})$, where $\bar{y}$ here means reduction. Trace surjectivity realizes the needed correction. The successive lifts converge to a unit of the prescribed norm by completeness. Over any quadratic field extension a norm-one element $c \neq -1$ equals $(1+c)/(1+\sigma c)$. For $c=-1$ take any nonzero $z = a - \sigma a$. At the unramified place divide this solution z by the corresponding power of the base uniformizer to make it a unit while preserving $z/\sigma z$. In the split case the norm of $(u,1)$ is u, and a norm-one pair (c, c^{-1}) is the quotient of $(c,1)$ by its conjugate. At a real place the norm is the squared complex absolute value, and the quotient construction above applies to every point of the unit circle. □

Theorem 87.4. *Let M/P be a quadratic extension of number fields unramified at every finite place. Let σ be its nonidentity automorphism and r the number of real places of P becoming complex. For any number field A write $J_A = \mathbb{A}_A^\times$ for its ideles and $C_A = J_A/A^\times$. Then both quotients in Definition 87.1 for C_M are finite,*

$$h(C_M) = 2, \qquad (C_M)^\sigma = C_P, \qquad (C_M)^0 = C_P/N_{M/P}C_M.$$

Here $(C_M)^\sigma$ denotes fixed classes under the natural embedding of C_P; the superscript 0 is the quotient from that definition.

Proof. Let $J_M^0 = M_\infty^\times \prod_{w \text{ finite}} O_{M,w}^\times$; this superscript denotes the subgroup of finite units in this proof. The ideal map from finite valuations has this kernel, is onto the fractional ideals, and sends the principal ideles onto principal ideals by Proposition 82.3. It gives an exact sequence

$$1 \longrightarrow J_M^0/O_M^\times \longrightarrow C_M \longrightarrow \mathrm{Cl}(M) \longrightarrow 1.$$

The last group is finite by Lemma 82.5. For J_M^0 compute the two endomorphisms separately at each place of P. The finite factors contribute trivial quotients by Lemma 87.3, including for the infinite product because solutions to the unit equations can be chosen at every place inside the product. At a split infinite place the two factors are exchanged and both quotients are trivial by the split formula of that lemma. A real place becoming complex gives quotient orders two and one. Thus $h(J_M^0) = 2^r$.

By Lemma 83.1, the logarithms of units span the weighted-sum-zero hyperplane of the real space on the archimedean places of M, have finite kernel and form a full lattice in that hyperplane. The real representation of σ on this space permutes coordinates. Each split infinite place of P gives one pair and hence one dimension in both eigenspaces. Each of the r other places gives one fixed coordinate. Passing to the weighted-sum-zero hyperplane removes one fixed dimension, because the sum maps onto the trivial real representation. The difference of the two dimensions is $r-1$. The rational eigenspaces of the unit lattice have the same dimensions by solving the rational linear equations for the two eigenvalues and extending scalars. Therefore $h(O_M^\times) = 2^{r-1}$ by Lemma 87.2. Applying that lemma twice in the displayed sequence and $1 \to O_M^\times \to J_M^0 \to J_M^0/O_M^\times \to 1$ gives the asserted finiteness and $h(C_M) = 2$.

If $[x] \in C_M$ is fixed, $x/\sigma x$ is a rational norm-one element of $M^\times$. The algebraic quotient construction in Lemma 87.3 gives $y \in M^\times$ with $y/\sigma y = x/\sigma x$. Now x/y is a fixed idele, hence lies in J_P by the local field embeddings and the diagonal descriptions at split places. Conversely every idele of P is fixed, and $J_P \cap M^\times = P^\times$, by invariance under σ. This proves the fixed-class identification and its norm-quotient consequence. □

88. A quadratic norm principle

Lemma 88.1. *In Theorem 87.4 one has*

$$[C_P : N_{M/P}C_M] \leq 2.$$

Proof. Let $H = C_P/N_{M/P}C_M$, which is finite by Theorem 87.4. At a finite place of P all units are local norms by Lemma 87.3, and so is their full product. At infinity every tuple positive at the real places and arbitrary at the complex places is a norm by the same lemma. From an ideal prime to 2 construct a class in H by taking the uniformizer at each prime to its ideal exponent, and one elsewhere, including infinity. The class is independent of uniformizers and multiplicative. If the ideal is generated by a totally positive $z \equiv 1$ at the prescribed modulus 2, these finite valuations, including the zero ones above 2, agree with the principal idele of z; their ratio differs only by finite units and infinite norms. Thus this map factors through the signed ray class group C_2 of Definition 219.1. It is onto: for any idele t choose $z \in P^\times$ of the same valuations as t at the primes above 2 and the same signs at the real places. Such a z exists by the open approximation of Lemma 89.1. Dividing t by z gives a representative of the required kind after removing the finite units and infinite norms.

Every class in H has square one, since the norm of a base idele embedded in J_M is its square. Hence H is a finite vector space over $\mathbb{F}_2$. Its characters $\xi : H \to \{1, -1\}$ are obtained by assigning either sign on a basis. Their average at a class is one at the identity, zero elsewhere by pairing opposite character values. For real $\sigma > 1$ write $L_\sigma = \log(1/(\sigma - 1))$ and $Q_\mathfrak{p} = N\mathfrak{p}$. At primes of P away from 2, let $[\mathfrak{p}]_H$ be the class just defined. For $\xi \neq 1$, its pullback to C_2 is nontrivial, so its ideal series is holomorphic at one by Proposition 219.2. The real logarithm of its absolute Euler product differs from the sum of its degree-one terms by at most $2\zeta_P(2)$, since $\sum_{k\geq 2} Q^{-k\sigma}/k \leq 2Q^{-2}$ for every prime norm $Q \geq 2$. Boundedness at one therefore gives

$$\sum_{\mathfrak{p}\nmid 2} \xi([\mathfrak{p}]_H) Q_\mathfrak{p}^{-\sigma} \leq O_\xi(1) \qquad (\sigma \downarrow 1).$$

For the trivial character the sum is $L_\sigma + O(1)$ by Corollary 83.4. Averaging these inequalities over all characters bounds the sum for $[\mathfrak{p}]_H = 1$ by $L_\sigma/|H| + O(1)$.

Every split prime lies in that class since a uniformizer on one branch has the prescribed norm. Apart from finitely many primes, the sum on split primes of P is $\frac{1}{2}L_\sigma + O(1)$. Indeed the logarithm of $\zeta_M(\sigma)$ equals $L_\sigma + O(1)$ by Corollary 83.4. At a split prime there are two primes of M of the same absolute norm, and at an inert prime the one absolute norm is $Q_\mathfrak{p}^2$. This follows from the residue algebra of the quadratic extension. The sum of all inert contributions and all higher prime powers is $O(1)$ by absolute ideal convergence at two. Comparison of the two bounds as $L_\sigma \to \infty$ gives $|H| \leq 2$. □

Theorem 88.2. *Let M/P be a quadratic number field extension unramified at every finite place. A nonzero $x \in P$ is the norm of an element of $M^\times$ if and only if it is a norm at each place of P. Equivalently it is positive at every real place becoming complex and has even valuation at every inert finite place.*

Proof. The norm of a rational element gives its local norms by taking each embedding. At a split place the norm is the product of the two components. At an inert place a base uniformizer has norm its square and all units are norms by Lemma 87.3. Conversely the valuation of every local norm there is even since conjugation preserves the valuation and the common uniformizer has valuation one in both fields. At a real place becoming complex use the positive norm assertion in the same lemma. These observations give the explicit local criterion.

If all local conditions hold, the local choices can be combined into $j \in J_M$ with $Nj = x$, choosing unit solutions where x is a finite unit. The class $[j]$ then lies in the norm kernel on C_M. Theorem 87.4 gives $h(C_M) = 2$. Lemma 88.1 bounds the order of its numerator by two. Its denominator is therefore one. By Definition 87.1 there exists $[k] \in C_M$ with $[j] = [k]/[\sigma k]$. For idele representatives this says $j = (k/\sigma k)y$ for some $y \in M^\times$. Taking norms gives $x = N_{M/P}y$, as required. □

Proposition 88.3. *Let P be any number field containing a_0 with $a_0^2 = 3$ and suppose that $i^2 = -1$ has no solution in P. Put $M = P(i)$ and let σ fix P and negate i. Then M/P is unramified at all finite places. Every local unit there is a norm of a unit. One has $O_M = O_P \oplus O_P\xi$ for $\xi = (a_0 + i)/2$.*

Proof. The element ξ satisfies $X^2 - a_0X + 1 = 0$ and $\xi^4 - \xi^2 + 1 = 0$, hence is integral. Its difference from its conjugate is i, a unit. For $x \in O_M$, both $b = (x - \sigma x)/i$ and $x - b\xi$ are fixed and integral, so belong to O_P. Subtracting conjugates also proves uniqueness. Over every completed integer ring the discriminant of this integral basis is -1. Consequently the quadratic polynomial has distinct residue roots over the algebraic closure at every finite prime. Lemma 18.1 gives either two copies of the base ring or the unramified quadratic integer ring. The product decomposition into the completions at the primes of M is the comaximal ideal argument of Lemma 84.1. The unit norm assertion follows from Lemma 87.3. □

Theorem 88.4. *Let P be any number field containing a square root of 3 and let $a \in P^\times$. If $i \notin P$, there exist $x, y \in P$ with $a = x^2 + y^2$ if and only if a is positive at every real embedding and is a norm from $P(i) \otimes_P P_\mathfrak{p}$ at every finite prime. If $i \in P$ every a has a representation.*

Proof. If $i \in P$, take $x = (a+1)/2$ and $y = (a-1)/(2i)$. Otherwise each element of $P(i)$ has the form $x + iy$ with norm $x^2 + y^2$. Proposition 88.3 gives unramifiedness at all finite primes. At a real embedding the norm image is precisely the positive reals, and at a complex place the norm is the product in a split algebra. Theorem 88.2 now gives both implications. □

89. Rational vectors of prescribed hermitian norm

Lemma 89.1. *For a number field M, its diagonal image is dense in the product of M_∞ and $M_\mathfrak{p}$ for any finite set of nonzero prime ideals $\mathfrak{p}$. The assertion also holds for any fixed finite number of coordinates over M.*

Proof. Let the targets be $\eta_\infty, \eta_\mathfrak{p}$. Choose a positive integer m clearing the denominators of all $m\eta_\mathfrak{p}$ in the corresponding completed integer rings, using powers of their residue characteristics. Choose positive integers $k_\mathfrak{p}$ so large that a congruence modulo $\mathfrak{p}^{k_\mathfrak{p}}$ after multiplication by m ensures the required finite precision. The argument on residues in Theorem 85.1 applies at these primes: every residue in a completion lifts to O_M by invertibility of the localization denominators, and the Chinese remainder argument in Lemma 84.1 combines the congruences. Thus elements b congruent to all $m\eta_\mathfrak{p}$ form one lattice coset $b_0 + \mathfrak{a}$, with $\mathfrak{a} = \prod \mathfrak{p}^{k_\mathfrak{p}}$. It tiles M_∞ by bounded translates of a fundamental parallelepiped by Lemma 82.2. Choose positive integers $n \to \infty$ congruent to one modulo $\mathfrak{a}$, for instance $n = 1 + jN\mathfrak{a}$ since $(N\mathfrak{a})O_M \subset \mathfrak{a}$. Choose b in the coset at bounded distance from $mn\eta_\infty$. Then $b/(mn)$ tends to η_∞ and meets every finite precision since $n \equiv 1$ and is a local unit there. The empty finite set uses $\mathfrak{a} = O_M$. Apply the scalar construction to each coordinate for the last assertion. □

Lemma 89.2. *Let F be a totally real number field containing $\sqrt{3}$, let $E = F(i)$ with conjugation *, and let h be a hermitian form on an E-space, linear in the second variable. Write $H(w) = h(w, w)$. Let u, v be two vectors, with $\alpha = H(u)$ totally positive and $\beta = H(v) \neq 0$. Assume $v_\mathfrak{p}(\beta) = 0$ whenever $v_\mathfrak{p}(\alpha) \neq 0$. Then the E-span of u, v contains a vector of norm one.*

Proof. Set $B = (h(u, v) + h(v, u))/2 \in F$, $\mu = \alpha\beta - B^2$. If μ is a square in F, including zero, choose $m \in F$ with $m^2 = \mu$ and put $M = F$. Otherwise take the quadratic field $M = F(m)$ with $m^2 = \mu$. We show $\alpha = x^2 + y^2$ in M. It is positive at every real embedding of M by restriction. When $i \notin M$ consider a finite prime of M above $\mathfrak{p}$ of F. The restriction of its valuation on F is a positive multiple of $v_\mathfrak{p}$: the contraction is a nonzero prime since the residue characteristic belongs to it, and local units of F remain units there, whereas a uniformizer has positive valuation by contraction and clearing a denominator outside $\mathfrak{p}$.

If α is a unit at $\mathfrak{p}$ it is consequently a unit upstairs and a local norm by Proposition 88.3. Otherwise β is a local unit and thus a local norm, and

$$\alpha\beta = B^2 + m^2 = (B + im)^*(B + im)$$

in the quadratic algebra over the completion of M. The product is nonzero, so $B + im$ is invertible there; its inverse is its conjugate divided by this norm. Dividing by the norm β proves the local norm condition for α. Theorem 88.4 supplies $x, y \in M$ in both cases $i \notin M$ and $i \in M$.

There are $p, q, r, s \in F$ satisfying

$$p^2 + q^2 + \mu(r^2 + s^2) = \alpha, \qquad pr + qs = 0.$$

For $M = F$ take $p = x, q = y, r = s = 0$. For $[M : F] = 2$ write $x = p + rm,\ y = q + sm$ and compare the coefficients of $1, m$. Let $C = p + iq,\ D = -s + ir,\ A = (C - BD)/\alpha$. The imaginary part of $\bar{C}D$ is $pr + qs = 0$, so $\bar{A}D$ lies in F. The cross terms in $H(Au + Dv)$ therefore sum to $2B\bar{A}D$. Writing $|z|^2 = \bar{z}z$ for $z \in E$, expansion gives

$$\begin{aligned}\alpha H(Au + Dv) &= |\alpha A + BD|^2 + (\alpha\beta - B^2)|D|^2 \\ &= p^2 + q^2 + \mu(r^2 + s^2) = \alpha.\end{aligned}$$

Division by α proves the assertion even if the two chosen vectors are linearly dependent. □

Theorem 89.3. *Let E/F be as in Lemma 89.2 and h a hermitian form on a finite-dimensional E-space. Suppose it represents 1 over $E \otimes_F F_v$ for every real and finite completion F_v of F. Then it represents 1 over E.*

Proof. Use a rational E-basis. At each real embedding choose a complex vector representing 1. Approximate these vectors by one rational vector u using Lemma 89.1 with $M = E$. The set of vectors with positive norm is open at each real place by continuity of the hermitian polynomial. Thus $\alpha = H(u)$ is totally positive. The primes of F at which $v_{\mathfrak{p}}(\alpha) \neq 0$ form a finite set by Proposition 82.3. At each of them choose a local vector representing 1. Having unit norm is an open condition there, since the units of a completed discrete valuation ring are the union of its nonzero residue classes. The finite algebra of E is a field or a product of the completions above $\mathfrak{p}$ by Lemma 84.1; here $\xi = (\sqrt{3} + i)/2$ is integral and gives the basis as in Proposition 88.3. Apply Lemma 89.1 with $M = E$, including all the primes above this finite set and the real-place complex coordinates already specified. We obtain a rational vector v with $\beta = H(v)$ totally positive and a unit at each prime in the set. Lemma 89.2 gives the required rational vector in their span. □

90. Rational classification of the hermitian spaces

Lemma 90.1. *Let F be a field of characteristic zero, and $D = F[X]/(X^2 + 1)$ with $i = X \bmod (X^2 + 1)$ and $\bar{i} = -i$. On a finite free D-module with nonsingular hermitian form h, any two vectors u, v of the same norm $a \in F^\times$ are carried to one another by an isometry. Each splits off a free line with nonsingular orthogonal complement.*

Proof. The algebra D is a quadratic field or a product of two fields. The four scalars $1, -1, i, -i$ are distinct in each field factor and have norm one. Choose one of them, η, so that

$$b = h(u - \eta v, u) = a - \bar{\eta}h(v, u)$$

is a unit of D. In each field factor a nonzero $h(v, u)$ forbids at most one choice; a zero forbids none. Thus at most two choices are forbidden. For $w = u - \eta v$, we have $h(w, w) = b + \bar{b}$ by equality of the norms of $u, \eta v$. The formula

$$Tz = z - wb^{-1}h(w, z)$$

sends u to ηv. In the expansion of $h(Tz, Tz')$ the coefficient of $h(z, w)h(w, z')$ is $-b^{-1} - \bar{b}^{-1} + h(w, w)/(b\bar{b}) = 0$, as in Lemma 59.1. It follows that T preserves h; the Gram identity

gives a left inverse for its square matrix, hence an inverse by taking determinants in each field factor. Multiplication by η^{-1} completes the isometry. The projection $z \mapsto ua^{-1}h(u,z)$ has image the line Du and kernel its orthogonal complement. Its two summands are free, by taking bases over each field factor, since the line has rank one at each factor. In such bases nonsingularity of the whole Gram matrix and of the scalar a gives nonsingularity on the complement. □

Theorem 90.2. *Let F be a totally real number field containing $\sqrt{3}$, $E = F(i)$, and let h, h' be nonsingular hermitian forms on E-spaces of the same finite dimension. If they are isometric over $E \otimes_F F_v$ at every finite place and have the same signature at every real embedding of F, there is an isometry over E. A rational isometry implies these local conditions.*

Proof. Over E, or over $\mathbb{C}$ at a real place of F, a nonzero hermitian form has some vector of nonzero norm. Otherwise expanding the norms of $x + y$ and $x + iy$ gives respectively zero real and imaginary parts of $h(x, y)$ for all x, y. Splitting off each nonzero diagonal line by the projection in Lemma 90.1 gives an orthogonal basis for a nonsingular form. At a real place rescale each basis vector by the inverse square root of the absolute value of its diagonal entry. This gives a diagonal of $+1$'s and -1's. The number of positive entries equals the maximal dimension of a positive definite subspace, since projection onto the positive coordinates is injective on any such subspace and those coordinates themselves give equality. The same holds for negative entries. This proves both invariance of signature and existence of local isometries between forms of equal signature.

We induct on the common dimension, the zero-dimensional assertion being immediate. Choose v for h' with norm $a \neq 0$. Then $a \in F^\times$, and $a^{-1}h$ represents 1 over all completions by the given and just constructed local isometries. Theorem 89.3 gives u with $h(u,u) = a$ rationally. At each completion any chosen isometry from h' to h sends v to a vector of norm a. Compose it with the isometry in Lemma 90.1 to send v to u. Its restriction is an isometry of the two local orthogonal complements. These are the scalar extensions of the rational complements, as is seen from the explicit line projections. Applying the induction hypothesis to the nonsingular complements and taking the orthogonal sum with $v \mapsto u$ gives the required rational isometry. Conversely a rational isometry extends to each completion by its matrix equation. □

Corollary 90.3. *Use the notation of Definition 86.1. For sign assignments s, t as in Theorem 86.3, suppose given rational hermitian spaces V_s, V_t, each with a finite frame of Gram matrix J_f, and having at each real place the numbers of positive and negative entries specified respectively by s, t. The bijection of that theorem identifies the framed conjugacy classes on the two specified rational unitary groups, subject to these eigenline signs and fixed P. Their finite matrices and embedded rational centralizers can be paired as in that theorem.*

Proof. The base field F of the construction is totally real, and $E = F(i)$ contains $\xi = \exp(2\pi i/12)$ by Lemma 20.1. Hence F contains $\sqrt{3} = \xi + \xi^*$. Given (a, x) of signs s, the trace form h_a and the specified V_s are locally isometric at every finite place by their two frames for J_f. At infinity the line signs of Theorem 81.1 give precisely the required signature. By Theorem 90.2 a rational isometry $b : (L, h_a) \to V_s$ exists. It carries the operator to $bm_y b^{-1}$ and the frame to bx. Another such b differs by a rational isometry of V_s, precisely the equivalence in its framed conjugacy classes. Conversely every framed class on V_s arises by Proposition 86.2. The finite matrix in the new frame is still $x^{-1}m_y x$ and the centralizer in finite coordinates is still T_x. The same argument for t composes with Theorem 86.3 to give the assertion. □

91. Counting rational operators in a moving lattice

Definition 91.1. Let V be a nonsingular rank-r hermitian space for the E/F of Definition 10.1, $r \geq 1$, $f = [F : \mathbb{Q}]$. Fix archimedean frames with Gram matrices S_τ diagonal with

entries $1, -1$, and a finite adelic frame with Gram matrix 1. Let $H = \prod_\tau U(S_\tau)$, and $G_f = U(1_r)(\mathbb{A}_{F,f})$, the finite isometry group of the identity matrix. Let K^0 preserve the standard integral finite lattice and $K_0 \subset K^0$ be compact open. Write $X = H \times G_f / K_0$, $\Gamma = U(V)(F)$. Compose the fixed frames on the right by $x \in X$, writing X_∞, X_f for the resulting frames, using any lift of the finite coordinate. For $\gamma \in \Gamma$ put

$$c_\infty(x,\gamma) = X_\infty^{-1}\gamma X_\infty, \qquad c_f(x,\gamma) = X_f^{-1}\gamma X_f, \qquad m(h) = \prod_\tau \|h_\tau\|_{\mathrm{op}}.$$

Use the ordinary Euclidean norm in each $\mathbb{C}^r$, and its sum of squares on their direct sum to measure $\|v\|_x = \|X_\infty^{-1} v\|$ for $v \in V$. Let $\Lambda_x = \{v \in V : X_f^{-1} v \in \widehat{O}_E^r\}$. The local finite matrix is defined up to K_0-conjugacy; Λ_x is independent of the lift.

For a compact set $\Omega \subset G_f$ invariant under K_0-conjugacy and real $R \geq 1$, let $N_R(x)$ count the $\gamma \in \Gamma$ with irreducible characteristic polynomial over E, $c_f(x,\gamma) \in \Omega$, and $m(c_\infty(x,\gamma)) \leq R$.

Lemma 91.2. *The image of Λ_x in the archimedean frame is a full real lattice of dimension $D = 2fr$ and covolume independent of x. Its shortest nonzero length λ_x is attained and is at most a constant independent of x. For all x and all γ with $c_f(x,\gamma) \in \Omega$ there is one positive integer $c = c(\Omega)$, independent of x, γ, such that $c\gamma\Lambda_x \subset \Lambda_x$.*

Proof. Apply Lemma 20.2 in a fixed rational basis. For a finite unitary matrix reducing the comparison of two frames to that same basis, the determinant over E has norm one to F. Its determinant over each $\mathbb{Q}_p$ therefore has absolute value one: over a splitting field that determinant is the product over all embeddings of the E-determinant, whose pairs over F have product one. This can also be seen by the embedding diagonalization in Lemma 82.2 applied to the finite separable algebra. Thus right movement of the finite frame by G_f leaves the covolume unchanged. At infinity every unitary matrix has determinant of complex absolute value one by taking the determinant of $h_\tau^* S_\tau h_\tau = S_\tau$. Hence its real determinant also has absolute value one. A bounded ball in a full lattice has finitely many points, and some ball contains a nonzero point, giving attainment and positivity. The cube argument of Lemma 82.5 bounds a shortest length in terms of dimension and covolume. The entries in the compact family Ω have denominators bounded by a positive integer c, by the increasing open cover used in Proposition 86.2. Thus $cX_f^{-1}\gamma X_f$ preserves the standard integral module by inclusion, giving the last assertion. □

Proposition 91.3. *With constants depending on the fixed data and Ω, one has uniformly for $R \geq 1$, $x \in X$*

$$N_R(x) \leq C R^{Dr(r+1)/2}.$$

If $N_R(x) > 0$ there is an E-basis $v_0, \ldots, v_{r-1}$ of V in Λ_x with $\|v_j\|_x \leq C(cR)^j$ for $0 \leq j < r$.

Proof. Choose $u \in \Lambda_x$ of shortest nonzero length λ_x by Lemma 91.2. For every $h_\tau \in U(S_\tau)$ the inverse is $S_\tau h_\tau^* S_\tau$ and has the same operator norm, since S_τ and its inverse are ordinary unitary matrices. Consequently $1 \leq \|h_\tau\|_{\mathrm{op}}$ by submultiplicativity, and every factor is at most $m(h)$. For a counted γ the vectors $c^j\gamma^j u$ are in Λ_x and have norm at most $(cR)^j\lambda_x$ by iteration. The vectors with $0 \leq j < r$ are E-independent. Indeed any relation would give a polynomial Q of degree less than r with $Q(\gamma)u = 0$. Its characteristic polynomial P is irreducible and annihilates γ by the adjugate identity used in Theorem 81.1. For nonzero Q, polynomial division gives $AQ + BP = 1$, so $u = 0$, a contradiction. These vectors therefore give the stated basis, using the uniform upper bound on λ_x.

Moreover the tuple $(c^j\gamma^j u)_{1 \leq j \leq r}$ determines γ, since the images of the basis $(\gamma^j u)_{0 \leq j < r}$ are given by this tuple. For every $a \geq 1$, at most $(1+2a)^D$ vectors of Λ_x have norm at most $a\lambda_x$. Indeed the open balls of radius $\lambda_x/2$ about them are disjoint in the enclosing ball of radius $(a + 1/2)\lambda_x$; compare their Euclidean volumes by dilation. Multiply these bounds for $a = (cR)^j$, $1 \leq j \leq r$, and use $1 + 2a \leq 3a$ to obtain the claimed exponent. □

Lemma 91.4. *For S any diagonal matrix of r signs, Haar volume on $U(S)$ satisfies*

$$\operatorname{vol}\{g : \|g\|_{\mathrm{op}} \le T\} \le C_S T^{4r^2} \qquad (T \ge 1)$$

for a constant depending also on its fixed normalization.

Proof. The invariant volume in Lemma 21.4 is finite on compact sets by its continuous density in local charts. Choose a relatively compact identity neighborhood containing all members of $U(S)$ within a fixed positive operator norm distance from identity. In the $2r^2$ real matrix coordinates, entries of the matrices being counted lie in a box of side $2T$. Divide it into boxes of side at most c_r/T, with number at most $C_r T^{4r^2}$; choose a regular subdivision with the least integer number of sides meeting this bound. Two group elements g, g' in a small box satisfy $\|g^{-1}g' - 1\|_{\mathrm{op}} \le T\|g' - g\|_{\mathrm{op}} \le 2rc_r$, since $\|g^{-1}\|_{\mathrm{op}} = \|g\|_{\mathrm{op}}$ as in Proposition 91.3. For small fixed c_r each nonempty box therefore lies, on the group, in one left translate of the chosen neighborhood. Adding the volumes proves the bound. □

92. Absolute convergence of the geometric trace

Lemma 92.1. *In Definition 91.1 use the product of invariant volume on H and counting measure on G_f/K_0. Let $J = r(r-1)/2$ and fix constants $C, c \ge 1$. For an ordered E-basis $B = (v_0, \dots, v_{r-1})$ of V write $W_R(B)$ for the set of $x \in X$ such that $v_i \in \Lambda_x$, $\|v_i\|_x \le C(cR)^i$ for every i. For $R \ge 1$ the following bounds hold with constants independent of R, B.*

$$\#\{(h(v_i, v_j))_{ij} : W_R(B) \ne \varnothing\} \ll R^{4frJ},$$
$$\operatorname{vol}_X W_R(B) \ll R^{4frJ + 4fr^2J}.$$

Proof. Write $T = (h(v_i, v_j))$. If $W_R(B) \ne \varnothing$, the finite gram matrix of the frame is the identity, giving $T_{ij} \in O_E$ by Proposition 82.3. At each real place $|\tau(T_{ij})| \le C^2(cR)^{i+j}$ by Cauchy–Schwarz with the diagonal sign matrix of operator norm one. In the full real lattice O_E of Lemma 82.2 there are $O_E(t^{2f})$ elements with each conjugate of absolute value at most $t \ge 1$, by the disjoint-ball argument of Proposition 91.3, with a fixed shortest length. Multiply these upper bounds over i, j, using $\sum_{i,j}(i + j) = 2rJ$.

In coordinates given by B, every possible finite lattice lies between O_E^r and $T^{-1}O_E^r$. The second inclusion holds because pairing its elements with each v_i is integral. The quotient between the two bounds has order $n_T = |N_{E/\mathbb{Q}} \det T|$, by applying the determinant of the integer endomorphism T to the full lattice O_E^r. Evaluation under all embeddings splits its rational determinant as a product of the determinants of τT, by the diagonalization in Lemma 82.2. There are at most n_T^r intermediate O_E-modules. Indeed at each prime the preimage of a submodule of the finite quotient is free of rank r over the local discrete valuation ring: project onto its first coordinate, lift a generator of that ideal, and continue on the kernel, as in the integer lattice proof of Lemma 12.1. Thus each local submodule has a generating tuple of length r in that local quotient. Its number of choices is at most the local quotient order to the power r. The local quotient orders multiply to n_T: the finite module is killed by $n_T O_E$, and the Chinese remainder idempotents for the powers of primes in this ideal decompose it and every submodule into the corresponding local factors. The same lattice determines its tuple of finite local lattices by the congruence and localization argument of Lemma 20.2. Hence it determines at most one coset modulo K^0. There are $[K^0 : K_0] < \infty$ lifts at level K_0, by compactness and openness.

At every real place put $M_\tau = X_\tau^{-1} B_\tau$, where B_τ is the matrix of these rational basis vectors in a fixed rational basis. Its columns have lengths at most $C(cR)^i$ in the indicated order and $M_\tau^* S_\tau M_\tau = \tau T$. The determinant bound of Lemma 5.1 gives

$$|\det M_\tau| \ll R^J, \qquad |\tau(\det T)| = |\det M_\tau|^2, \qquad n_T \ll R^{4fJ}.$$

This bounds the number of finite coordinates by $O(R^{4frJ})$.

For the real volume choose a reference archimedean frame X_∞^0 in the permitted region, with matrices M_τ^0. Then $(X_\tau^0)^{-1}X_\tau = M_\tau^0 M_\tau^{-1}$ has operator norm $\ll R^J/|\det M_\tau|$. Indeed row j of the adjugate of M_τ consists of signed minors using all columns except j, of size $O(R^{J-j})$. Multiply that row by column j of M_τ^0 and sum. The denominator is fixed for T, τ by its Gram identity. Moreover $\prod_\tau |\det M_\tau|^2 = |N_{F/\mathbb{Q}} \det T| \geq 1$, since $\det T$ is a nonzero integer of F. By Lemma 91.4, left translation by the reference frame and multiplication of the bounds at the f places give archimedean volume $O(R^{4fr^2J})$. The individual radii can be taken at least one by enlarging the fixed constant, using the determinant bound above. Multiplying by the finite count proves the result. □

Theorem 92.2. *Let D_0 be a measurable fundamental domain for Γ in X of Definition 91.1, with the measures of Lemma 92.1. Then*

$$\int_{D_0} N_R(x)\,\mathrm{d}x \ll R^{8f(r+1)^4} \qquad (R \geq 1).$$

Let ϕ be a bounded measurable finite test, invariant by K_0-conjugacy and supported in Ω. For a measurable $k : H \to \mathbb{C}$ with $|k(h)| \ll m(h)^{-B}$ and $B > 8f(r+1)^4$, the sum over the rational operators of irreducible characteristic polynomial can be integrated absolutely. For representatives of their rational conjugacy classes it satisfies

$$\int_{D_0} \sum_\gamma \phi(c_f(x,\gamma))\, k(c_\infty(x,\gamma))\,\mathrm{d}x = \sum_{[\gamma]} \int_{\Gamma_\gamma\backslash X} \phi(c_f(x,\gamma))\, k(c_\infty(x,\gamma))\,\mathrm{d}x,$$

where Γ_γ is the centralizer in Γ. When the finite test forces irreducibility, these are the full geometric sums. The bound applies to $B = r^8$ for the arithmetic parameters at all sufficiently large scales.

Proof. The construction of the fundamental domain in Lemma 21.4 applies here. Countably many finite coordinates follow from the lattice inclusions with bounded denominators and finite level index. Properness follows from bounded rational matrix entries at all places on intersections of compact sets, and the archimedean action is free. The same proof applies to any subgroup of Γ. In particular, the integral of a periodized nonnegative function on D_0 equals its integral on X.

Where $N_R(x) > 0$, Proposition 91.3 supplies a basis B with $x \in W_R(B)$ for fixed C, c. Two rational ordered bases of the same Gram matrix differ by a unique rational isometry, since the linear map sending basis to basis preserves the entries of the form. For each possible Gram matrix choose one such B_T. Hence the indicator of $N_R(x) > 0$ is bounded by $\sum_T \sum_{\delta\in\Gamma} \mathbf{1}_{W_R(B_T)}(\delta^{-1}x)$. Integrate and unfold these nonnegative indicators. By Lemma 92.1 their integral is $O(R^{8frJ+4fr^2J})$ with $J = r(r-1)/2$. The uniform point count of Proposition 91.3 costs the further factor $R^{fr^2(r+1)}$. Their total exponent is $f(2r^4+3r^3-3r^2) \leq 8f(r+1)^4$, proving the first bound.

On $2^a \leq m(c_\infty) < 2^{a+1}$, $a \geq 0$, bound the absolute sum by $\|\phi\|_\infty C 2^{-aB} N_{2^{a+1}}$. The integrated bounds sum as a convergent geometric series. Thus we can group by conjugacy and integrate termwise by Lemma A.2. For each fixed γ, the translates of D_0 by representatives of $\Gamma_\gamma\backslash\Gamma$ tile a fundamental domain for Γ_γ. The simultaneous conjugation and left translation law for c_f, c_∞ now gives the equality. If every element of Ω has irreducible characteristic polynomial at some completion, a proper factorization over E would persist there. Residue irreducibility of an integral matrix suffices by Proposition 80.2. Finally $f \leq x^{12}$ and $r \geq x^5$ for large scales by Propositions 10.5 and 12.2. Thus $8f(r+1)^4/r^8 \leq 128x^{-8} < 1$ eventually. □

93. Vanishing away from the elliptic set

Lemma 93.1. *Use H, X, Γ of Definition 91.1. For fixed $\gamma \in \Gamma$, let $Z = \Gamma_\gamma\backslash X$, $C(x) = c_\infty(x,\gamma)$. Right multiplication $x \mapsto xa$ by $a \in H$ on the archimedean coordinate preserves the*

measure of Z and satisfies $C(xa) = a^{-1}C(x)a$. The finite part $c_f(x,\gamma)$ modulo K_0-conjugacy stays fixed. For product measure on $Z \times H$ or $Z \times H^2$, the following changes preserve integrals:

$$(x,a) \longmapsto (xa, a^{-1}C(x)), \qquad (x,a,b) \longmapsto (xa, b, (ab)^{-1}C(x)).$$

In the first case the two factors $a, a^{-1}C(x)$ are cyclically interchanged, and in the second the three factors $a, b, (ab)^{-1}C(x)$ are cyclically permuted when the last factor is recovered from the new base point.

Proof. The rational centralizer commutes with γ, proving invariance of $C(x)$ on its quotient; the formulas follow from multiplication of the frames. Right H-translation preserves measure on X by Lemma 21.4, commutes with the left action, and sends a fundamental domain to another. The independence of the domain, as in Theorem 92.2, proves preservation on Z. In each product first replace x by $x' = xa$ with a, b fixed. Now $a^{-1}C(x) = C(x')a^{-1}$. For the first transformation invert a and translate it on the left at fixed x'. For the second put b first, and replace a by $b^{-1}C(x')a^{-1}$ at fixed x', b. These are measure preserving since the measure on H is invariant on both sides and under inversion: inversion exchanges left and right translations and has Jacobian of absolute value one at identity on its tangent space. Product integration as in Lemma A.2 proves the transformations for nonnegative and absolutely integrable tests. Multiplying the new factors and using $C(xa) = a^{-1}C(x)a$ identifies the recovered final factor as a in both cases. □

Theorem 93.2. *In Theorem 92.2 let γ have irreducible characteristic polynomial over E. Let $B > 8f(r+1)^4$ and let $e : H \to \mathbb{C}$ be measurable, with $|e(g)| \le C_0 m(g)^{-B}$ on H. Suppose*

$$\int_H |e(g)|\, m(g)^{B+1}\, \mathrm{d}g < \infty, \qquad \int_H e(a)e(a^{-1}g)\, \mathrm{d}a = e(g) \quad (g \in H).$$

If some eigenvalue of γ at an archimedean factor has absolute value different from one, then for every finite test ϕ as in that theorem

$$\int_{\Gamma_\gamma \backslash X} \phi(c_f(x,\gamma))e(c_\infty(x,\gamma))\, \mathrm{d}x = 0.$$

Proof. Write $Z, C(x)$ as in Lemma 93.1 and $\Phi(x) = \phi(c_f(x,\gamma))$. The preceding absolute convergence theorem gives $\int_Z |\Phi(x)| m(C(x))^{-B}\, \mathrm{d}x < \infty$ by taking the real archimedean test m^{-B} and the absolute finite test. At a place with the given eigenvalue λ, choose a nonzero left eigenrow v for the matrix γ_τ in the fixed archimedean frame. All eigenvalues there are simple by irreducibility and characteristic zero as in Theorem 81.1. Consequently a commuting invertible matrix acts by a nonzero scalar on the line spanned by v. For x represented by $h \in H$ in its real coordinates, set

$$D(x,a) = \log \|vh_\tau a_\tau\| - \log \|vh_\tau\|, \qquad l = \log|\lambda|.$$

The scalar just described cancels in the difference, making D well defined on Z. Direct cancellation gives $D(x,ab) = D(x,a) + D(xa,b)$, and $D(x, C(x)) = l$. By the inverse norm equality in Proposition 91.3, $|D(x,a)| \le \log m(a)$.

Put $I = \int_Z \Phi(x)e(C(x))\, \mathrm{d}x$ and define

$$T = \int_{Z \times H} \Phi(x)e(a)e(a^{-1}C(x))D(x,a)\, \mathrm{d}x\, \mathrm{d}a.$$

All the integrals used below are absolutely convergent. For example m is submultiplicative and at least one. Thus for $d = (ab)^{-1}C(x)$ one has

$$|e(d)| \le C_0 m(a)^B m(b)^B m(C(x))^{-B}.$$

Either $|D(x,a)|$, $|D(xa,b)|$, or their sum is at most $2m(a)m(b)$. This bounds the product with $|\Phi(x)e(a)e(b)|$ and $|e(d)|$ by a constant times the product of the three integrable functions

already specified on Z, H, H. The analogous two-variable bound follows by omitting b. A constant factor l also preserves integrability.

In the formula for I, replace $e(C(x))$ by its double convolution integrand using idempotence. For $b = a^{-1}C(x)$ we have $D(x,a) + D(xa,b) = l$. The transformation of the pair in Lemma 93.1 interchanges the two increment integrals. Hence $lI = 2T$.

In the first occurrence of e defining T, expand by idempotence and change variables from the intermediate factor and the original argument to the two successive factors a, b. By left invariance the measure on them is $\mathrm{d}a\,\mathrm{d}b$. The three kernel factors are now $e(a)e(b)e(d)$ and the increment is $D(x, ab)$. Split this increment as $D(x,a) + D(xa,b)$. The first term integrates to T by idempotence in the last two factors. The second has the same integral as the first by the triple transformation of Lemma 93.1, which leaves Φ unchanged. Therefore $T = 2T$, so $T = 0$. Since $l \neq 0$, the formula $lI = 2T$ gives the desired vanishing. □

94. Comparison of the elliptic geometric sums

Proposition 94.1. *In Theorem 92.2, consider a class $[\gamma]$ whose eigenvalues have absolute value one at every archimedean place. In the finite cosets G_f/K_0, let gK_0 run over representatives of the Γ_γ-orbits through its action in the fixed finite frame. Write $\gamma_{f,g}$ for its matrix in that frame moved by g, and $A_{\gamma,g}$ for the stabilizer of gK_0 in Γ_γ. The contribution to the integral at this finite orbit is*

$$\frac{\phi(\gamma_{f,g})}{|A_{\gamma,g}|} \int_H k(h^{-1}\gamma_\infty h)\,\mathrm{d}h,$$

where γ_∞ is in the fixed real frames. This formula holds whenever the finite weight is nonzero and the corresponding absolute orbital integral converges. Zero finite weights contribute zero. For a product test $k = \prod_\tau k_\tau$ with absolutely convergent individual orbital integrals, the real factor is the product of those integrals. At a given τ the matrix in each such integral can be taken diagonal, with each eigenvalue placed in a coordinate of the sign of its eigenline.

Proof. Use the decomposition over the countable discrete set of finite cosets in Definition 91.1. Above a single orbit the quotient by Γ_γ identifies with $A_{\gamma,g}\backslash H$, since any finite coset in the orbit can be carried to the chosen representative and the remaining changes are exactly its stabilizer. The group $A_{\gamma,g}$ is finite by Proposition 86.2, applied to the trace form and this finite frame. Its action on H by left translation is free, since the rational embedding at infinity is injective. The full real Haar integral is its order times the quotient integral, by the finite partition into translates of a measurable fundamental domain as in Lemma 21.4. The finite test is constant on the chosen slice, giving the formula. Product integration follows from Lemma A.2. The simultaneous diagonal description in Theorem 81.1 makes the eigenlines orthogonal; rescale each eigenvector by the reciprocal square root of its absolute diagonal entry and order the resulting vectors according to the sign matrix. The change is unitary for this hermitian matrix, and left translation in the real orbital integral gives the asserted diagonal representative. □

Theorem 94.2. *At all f real places let the fixed rational unitary group of Definition 91.1 have the same signature (p, q'), $p + q' = r$, f even. Identify all its finite groups with those of the definite comparison space using the fixed finite frames. Normalize the compact real groups to have mass one and take $K_0 = K(g)$ of Definition 78.1. Let ϕ be bi-K_0-invariant of compact support and assume each finite matrix in its support has irreducible characteristic polynomial at some completion. Use counting measure on G_f/K_0 and suppose $e = \prod_\tau e_\tau$ satisfies the hypotheses of Theorem 93.2 on the indefinite group.*

At place τ write $I_\tau(z_1, \ldots, z_r)$ for the integral of $e_\tau(u^{-1}\,\mathrm{diag}(z_1, \ldots, z_r)u)$ over $U(p, q')$ in coordinates of the fixed signs. Assume these integrals converge absolutely on distinct unit

circle arguments and satisfy

$$\frac{(-1)^{pq'}}{p!\,q'!}\sum_{\pi\in\mathfrak{S}_r} I_\tau(z_{\pi(1)},\dots,z_{\pi(r)}) = \overline{\chi_{\lambda_\tau}(\operatorname{diag}(z_1,\dots,z_r))},$$

where χ_{λ_τ} is the compact character of Theorem 80.1 for the row parameters. It suffices to assume this identity at the eigenvalues of the elliptic polynomials occurring with nonzero finite weight. Then the global geometric integral on the initial group equals the trace of $\mathcal{T}_\phi$ on the definite space $\mathcal{A}^+$ in Proposition 80.2.

Proof. By Theorem 92.2 the integral is an absolutely convergent sum over rational conjugacy classes. All with a nonunit-circle eigenvalue have zero contribution by Theorem 93.2. For each remaining polynomial P, order its roots $z_{\tau,1},\dots,z_{\tau,r}$ at each real place. Proposition 94.1 writes the contribution as a sum over the framed classes with those roots. At each such term $\phi(\gamma_{f,g})/|A_{\gamma,g}|$ is determined by the finite matrix and the embedded centralizer. Choosing q' of the eigenlines negative at each place changes fq' signs from the all-positive pattern. This number is even. Therefore Corollary 90.3 and Theorem 86.3 identify the framed classes for every such pattern with those of the definite group for P, preserving precisely this finite weight. There are no additional rational choices associated to the identification: the quotient by rational conjugation is the one in Proposition 86.2.

Permuting eigenvalues within equal-sign coordinates leaves I_τ unchanged, since the permutation matrix is unitary for the sign matrix and can be absorbed by left translation. Summing over all assignments of the negative lines thus gives $(p!q'!)^{-1}\sum_\pi I_\tau(z_{\tau,\pi(1)},\dots,z_{\tau,\pi(r)})$ at this place. The choices at different places are independent. Their finite sums distribute over the product, yielding $(-1)^{fpq'}\prod_\tau \overline{\chi_{\lambda_\tau}(\operatorname{diag}(z_{\tau,1},\dots,z_{\tau,r}))}$ by the assumed identity. The sign is one since f is even.

On the compact group, the integral of the test $\dim W_{\lambda_\tau}\,\overline{m_{\lambda_\tau}}$ on the same conjugates is exactly this conjugate character at τ by Theorem 80.1. Hence the definite geometric sum with this compact test gives the same finite weights and archimedean factors. To identify it with the trace, choose the finite double coset representatives h_c of Proposition 80.2. Their rational stabilizers are trivial by Theorem 78.3. Integration over each full compact real slice of mass one therefore produces exactly its sum of finite coefficients times the character of R_∞, the formula of Proposition 80.2. Absolute convergence on the initial group justifies the reindexing, and on the definite group the sum has finite support by that proposition. □

95. The compact-dual unitary measure

Definition 95.1. Let $S=\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_n)$, $\varepsilon_j=\pm1$, $n\ge1$. On the real space of skew-hermitian n-by-n matrices take the coordinate Lebesgue density $\mathrm{d}Z$, with the imaginary part of each diagonal entry and the two real coordinates of each strictly lower triangular entry. On $U(S)$ use the density $\mathrm{d}\nu_S$ obtained at g by applying this density to the skew-hermitian tangent coordinate $g^*S\,\mathrm{d}g$. Set

$$u_n=\prod_{j=1}^{n}\frac{2\pi^j}{(j-1)!},\qquad u_0=1,\qquad \mathrm{d}h_S=u_n^{-1}\,\mathrm{d}\nu_S.$$

Lemma 95.2. *The measures ν_S, h_S are invariant under left and right translations. For $Z^*=-Z$ put $A=SZ/2$, $D=1-A$. On $\det D\neq0$ the map $g=(1+A)D^{-1}$ is a smooth chart onto $\{g\in U(S):\det(g+1)\neq0\}$ and*

$$\mathrm{d}h_S(g)=u_n^{-1}|\det D|^{-2n}\,\mathrm{d}Z.$$

Its complement in $U(S)$ has measure zero.

Proof. Differentiating $g^*Sg=S$ makes $g^*S\,\mathrm{d}g$ skew-hermitian. Conversely the chart at zero below realizes every such tangent coordinate at identity, and translation does the same everywhere. The form is unchanged by left translation and is pulled back by right translation

by b to its congruence by b. For any invertible complex C, the real map $Z \mapsto C^*ZC$ on the skew-hermitian space has determinant of absolute value $|\det C|^{2n}$. For unitary C it preserves the real Euclidean norm $(\operatorname{tr} Z^*Z)^{1/2}$; hence it preserves every constant multiple of its Euclidean density, including the chosen one. For positive diagonal entries s_i the diagonal coordinate scales by s_i^2 and each of the two coordinates for $i > j$ scales by $s_i s_j$. Their product is $\prod_i s_i^{2n}$. For general C, diagonalize $C^*C > 0$ with a unitary matrix by the spectral argument of Lemma 79.2; dividing C by its positive square root on the right gives a unitary matrix. This proves the formula by composition. Since $|\det b| = 1$ for $b \in U(S)$, right invariance follows.

The identity $A^*S = -SA$ gives $(1+A)^*S(1+A) = D^*SD$. Thus the chart takes values in the group. Conversely $A = (g-1)(g+1)^{-1}$ satisfies $A^*S = -SA$ by multiplying the equation by $g+1$ and its adjoint and using $g^*Sg = S$. Then $Z = 2SA$ is its inverse coordinate and $D = 2(g+1)^{-1}$. These rational maps are smooth on their stated domains. From $g = 2D^{-1} - 1$ we get $\mathrm{d}g = D^{-1}S\,\mathrm{d}ZD^{-1}$. Also $g^*SD^{-1}S = (D^*)^{-1}$, since $D^* = S(1+A)S$ and A commutes with D^{-1}. The tangent coordinate is consequently $(D^{-1})^*\,\mathrm{d}ZD^{-1}$, giving the claimed density by the congruence determinant. Finally for each g only finitely many circle scalars t satisfy $\det(tg+1) = 0$, by the polynomial root bound and its nonzero constant coefficient. Integrating the exceptional indicator over the circle and $U(S)$ in either order by Lemma A.2, left invariance under $g \mapsto tg$ shows that the exceptional set has measure zero. □

Theorem 95.3. *For $S = 1_n$, $\nu_S(U(n)) = u_n$; thus h_S is the compact probability used in Proposition 77.3. For $0 \le k \le n$ the volume of the space of $n-k$ ordered orthonormal vectors with the quotient density of ν_{1_n} by ν_{1_k} is u_n/u_k. In particular $u_n = U_n$ in Definition 56.1.*

Proof. Consider the first-column map to the unit sphere in $\mathbb{C}^n$. It is onto by completion to an orthonormal basis: adjoin independent standard basis vectors and successively subtract projections onto the preceding normalized vectors, then normalize. Locally it has smooth sections by the same procedure, using a fixed list of independent columns at the point and the positivity of each normalizing length on a neighborhood. The stabilizer of the first column at the first standard vector is $U(n-1)$ in the remaining coordinates. At identity the differential sends a skew-hermitian matrix to its first column. The chosen coordinate density is the product of the ordinary tangent Euclidean density of that column on the sphere with the density on its $(n-1)$-dimensional square block. Thus the quotient density is the sphere's Euclidean surface density there. This identity holds at every point by left invariance and invariance of the sphere density under $U(n)$. In local product coordinates given by a section and right movement in the fiber, any change of section adds vertical components to lifts of base tangents and translates the fiber. These changes have determinant one for the product of the quotient and fiber densities, by block triangularity and the invariance in Lemma 95.2. Ordinary integration in charts, partitioning a finite cover of the compact sphere into disjoint measurable pieces, now gives the volume as the sphere area times $\nu_{1_{n-1}}(U(n-1))$.

The integral of $\exp(-\|z\|^2)$ on $\mathbb{C}^n$ is π^n by planar polar coordinates and multiplication of integrals. In radius ρ and surface coordinates the Jacobian is ρ^{2n-1}, because each tangent direction to the unit sphere is scaled by ρ and the radial unit normal is orthogonal to them. The radial integral is $(n-1)!/2$ after substitution $t = \rho^2$ and integration by parts as in Theorem 6.2. This gives sphere area $2\pi^n/(n-1)!$. Induction starting with the trivial group in dimension zero proves the stated value u_n. Stopping the column construction with k free columns gives the ordered frame quotient with fiber $U(k)$, and successive integration gives u_n/u_k. The displayed definition of u_n agrees factor by factor with U_n. By uniqueness of compact invariant probability in Proposition 77.3 the compact normalizations agree. □

96. Principal minors and the polar coefficient

Definition 96.1. For an integer weight $\alpha = (\alpha_1, \dots, \alpha_n)$ order the indices $i_1, \dots, i_n$ by decreasing weight and put $A_j = \{i_1, \dots, i_j\}$. For an invertible matrix M set

$$m_\alpha(M) = (\det M)^{\alpha_{i_n}} \prod_{j=1}^{n-1} \det M[A_j, A_j]^{\alpha_{i_j} - \alpha_{i_{j+1}}}.$$

The zero exponents make this independent of tie breaking. Let $t_1 > \cdots > t_n$ belong to $(n-1)/2 + \mathbb{Z}$ and let $S = \operatorname{diag}(\varepsilon_i)$ have signs ± 1. Write $\rho_i = (n+1)/2 - i$, $\mu_i = t_i + \rho_i$, and define

$$\Lambda_i = \mu_i - \#\{j > i : \varepsilon_j = \varepsilon_i\} + \#\{j < i : \varepsilon_j = \varepsilon_i\}.$$

Let $K = U(S) \cap U(n)$, the product of the unitary groups on the two sign subspaces, with probability measure $\mathrm{d}u$ from Theorem 95.3. Set $c_\Lambda = \int_K |m_\Lambda(u)|^2 \, \mathrm{d}u$. For $g \in U(S)$ write $g = kP$ with k ordinary unitary and P positive definite hermitian. Define its polar coefficient by

$$\psi(g) = \frac{1}{c_\Lambda} \int_K \frac{m_\Lambda(ku)\, \overline{m_\Lambda(u)}}{m_{-\mu}(u^* P u)} \, \mathrm{d}u.$$

Lemma 96.2. *Let $P > 0$ be hermitian with $P^{-1} = SPS$, and write $d(A) = \det P[A, A]$, $d(\varnothing) = 1$. Let $J_+ = \{i : \varepsilon_i = 1\}$, $A_j = \{1, \dots, j\}$ and $J = \{1 \le j < n : \varepsilon_j \ne \varepsilon_{j+1}\}$. Then*

$$d(A) = d(A^c) \ge 1, \qquad d(J_+) \le \prod_{j \in J} d(A_j),$$

$$d(J_+) \ge \frac{\|P\|_{\mathrm{op}} + \|P\|_{\mathrm{op}}^{-1}}{2}.$$

For any $c \ge 0$ such that $\mu_j - \mu_{j+1} \ge c$ on J, one has $m_{-\mu}(P) \ge d(J_+)^c$.

Proof. All principal blocks have positive determinant, and $\det P = 1$ since $\det P = \det P^{-1} > 0$. For a partition into two principal blocks A, B with cross block C, elimination gives $\det P = \det A \det(B - C^* A^{-1} C)$, and the lower diagonal block of P^{-1} is $(B - C^* A^{-1} C)^{-1}$, by multiplying the two triangular elimination factors. As $P^{-1} = SPS$, this proves $d(I) = d(I^c)$ for every subset I. Also $\det(B - C^* A^{-1} C) \le \det B$, since the conjugated positive matrix $B^{-1/2}(B - C^* A^{-1} C) B^{-1/2}$ has eigenvalues in $(0, 1]$. The argument applies to every principal submatrix. Thus $d(U \cup V) \le d(U) d(V)$ for disjoint sets and $1 \le d(I) d(I^c) = d(I)^2$. For nested $A \subset B$, taking complements gives $d(B \setminus A) \le d(A) d(B)$. Decompose J_+ into its maximal consecutive intervals. For each such interval with endpoints l, h its determinant is at most $d(A_{l-1}) d(A_h)$. All interior boundaries between opposite signs occur exactly once in the product over these intervals. The empty and full determinants are one. This proves the upper bound on $d(J_+)$.

The block diagonal part of P on the sign subspaces is $(P + P^{-1})/2$, of determinant $d(J_+) d(J_+^c) = d(J_+)^2$. The positive eigenvalues of P occur with their reciprocals since $P^{-1} = SPS$. If $T = \|P\|_{\mathrm{op}} > 1$, T and T^{-1} each supply a factor $(T + T^{-1})/2$ to that determinant, and all other factors are at least one. If $T = 1$ all factors are one. This proves the required lower bound. Finally μ is decreasing, so the distinct proper sets occurring for $-\mu$ are tails of this order. Complementarity and $\det P = 1$ give $m_{-\mu}(P) = \prod_{j<n} d(A_j)^{\mu_j - \mu_{j+1}}$. The exponents are positive, every base is at least one, and the exponents at the indicated boundaries are at least c. The upper bound on $d(J_+)$ yields the last assertion. □

Theorem 96.3. *The function ψ is continuous. It takes value one at identity and satisfies $\psi(g^{-1}) = \overline{\psi(g)}$. Under the gap condition of Lemma 96.2, if $T = \|g\|_{\mathrm{op}}$, it satisfies*

$$|\psi(g)| \le \left(\frac{T + T^{-1}}{2} \right)^{-c}.$$

If $c > B+1+4n^2$ for $B \geq 0$, then $\int_{U(S)} |\psi(g)|\, T^{B+1}\, \mathrm{d}h_S(g) < \infty$ and $0 < \int_{U(S)} |\psi|^2\, \mathrm{d}h_S < \infty$.

Proof. By the spectral argument of Lemma 79.2 the positive square root $P = (g^*g)^{1/2}$ exists and is unique. Indeed any positive square root commutes with its square and on each eigenspace has only the positive square root eigenvalue. It depends continuously on g: for any convergent sequence of positive definite squares near a fixed one, the roots are bounded and stay uniformly positive by the quadratic-form bounds on the squares. Every convergent subsequence of roots then has the unique root of the limit. Finite-dimensional compactness of bounded closed sets proves continuity. Since $S(g^*g)S = (g^*g)^{-1}$, uniqueness gives $SPS = P^{-1}$. Thus $P \in U(S)$ and $k = gP^{-1} \in K$. The polar decomposition is unique and $T = \|P\|_{\mathrm{op}}$. The integer weights in Definition 96.1 give continuous minors on invertible matrices, with a strictly positive denominator on each positive definite argument. The compact integration is continuous by uniform continuity on compact neighborhoods, and $c_\Lambda > 0$ since $m_\Lambda(1) = 1$ and every open identity neighborhood has positive smooth measure. At identity the numerator integrates to c_Λ.

Conjugating P by $u \in K$ preserves its sign-block determinants. Apply Lemma 96.2 in the denominator. Cauchy–Schwarz and left invariance give $\int_K |m_\Lambda(ku)m_\Lambda(u)|\, \mathrm{d}u \leq c_\Lambda$. The asserted inequality follows. For inversion write $g^{-1} = k^{-1}(kP^{-1}k^{-1})$ and substitute $u = kv$. The numerator is the complex conjugate of the original one at v. The denominator agrees, since $P^{-1} = SPS$, v commutes with S, and diagonal sign conjugation preserves every principal minor. This proves the symmetry.

Finally use $T \geq 1$ and the last bound in the weaker form $2^c T^{-c}$. On $2^j \leq T < 2^{j+1}$, Lemma 91.4 bounds the integral of $|\psi| T^{B+1}$ by a constant times $2^{j(B+1+4n^2-c)}$. Summation gives finiteness. Since $|\psi| \leq 1$, this also bounds its squared integral. Positivity of the latter follows from continuity and its value at identity. □

Corollary 96.4. *In Definition 56.1, reindex the row entries from 1 to r in decreasing order. Suppose the first $k-1$ entries have a common sign; the remaining signs are arbitrary. For all sufficiently large arithmetic scales take $B = r^8$. At each place put $d_\tau = (\int |\psi_\tau|^2\, \mathrm{d}h_{S_\tau})^{-1}$ and $e_\tau = d_\tau \overline{\psi_\tau}$. The product $e = \prod_\tau e_\tau$ is continuous and satisfies*

$$|e(g)| \ll m(g)^{-B}, \qquad \int_H |e(g)|\, m(g)^{B+1}\, \mathrm{d}g < \infty.$$

It satisfies $e(g^{-1}) = \overline{e(g)}$ and $\int_H e(a)e(a^{-1})\, \mathrm{d}a = e(1)$.

Proof. In the top $k-1$ positions there is no change of sign. The gap between the last top entry and the next is $1-k/2+W_\tau+2r^8$; all subsequent gaps are $2r^8$ by Definition 56.1. Here $W_\tau \geq 0$. The μ-gaps exceed the t-gaps by one. Since $k \leq r+1$ and $r \to \infty$ by Proposition 10.5, all sign-change gaps are at least $c = r^8 + 4r^2 + 2$ eventually. Apply Theorem 96.3 and multiply the pointwise bounds and the finite collection of integrals. At identity the convolution formula follows from inversion symmetry and $\int |e_\tau|^2 = d_\tau = e_\tau(1)$. □

97. Independence of the block splitting

Definition 97.1. Retain $S, K, \mu, \Lambda, c_\Lambda$ of Definition 96.1. Let B be the complex group of invertible matrices block diagonal for the signs, $\mathcal{T}$ its group of invertible diagonal matrices and $T = K \cap \mathcal{T}$. Upper and lower triangular matrices in B are defined by the given order of indices in each block. Write $\chi_\alpha(b) = \prod_i b_{ii}^{\alpha_i}$ for a triangular b. A complex matrix Y is strictly accretive if $(Y + Y^*)/2$ is positive definite. For $g \in \mathrm{GL}_n(\mathbb{C})$ and $a \in B$ with $a^{-1}g$ strictly accretive put

$$\Psi_a(g) = c_\Lambda^{-1} \int_K \frac{m_\Lambda(av)m_\Lambda(v^{-1})}{m_{-\mu}(v^{-1}a^{-1}gv)}\, \mathrm{d}v.$$

A strictly accretive matrix and each of its principal compressions are invertible, since a nonzero kernel vector would have quadratic form of real part zero. This also makes the denominator in the display nonzero for unitary v.

Lemma 97.2. *On the complex manifold $B/\mathcal{T}$, use the holomorphic volume form Θ obtained by wedging, in a fixed order, all strictly off-diagonal entries within blocks of $v^{-1}\,\mathrm{d}v$. Its restriction to K/T is a nonzero constant multiple of the oriented invariant probability density. For $b \in B$, where $v^{-1}bv = lu^{-1}$ admits a lower triangular l and a unipotent upper triangular u, put $w = vl = bvu$. This gives a holomorphic map $F_b : [v] \mapsto [w]$ on that open locus. Set $\delta_i = \#\{j > i : \varepsilon_j = \varepsilon_i\} - \#\{j < i : \varepsilon_j = \varepsilon_i\}$. Then*

$$F_b^*\Theta = \chi_\delta(u)\chi_\delta(l)^{-1}\Theta.$$

Proof. The quotient parametrizes ordered lines forming a basis inside each block. On the open set where one specified entry of each column is nonzero, scale the column to put that entry equal to one. These are complex charts with holomorphic transition maps. Under $v \mapsto vd$ with $d \in \mathcal{T}$ variable, $v^{-1}\,\mathrm{d}v$ changes by conjugation by d and the addition of a diagonal form. The factors d_{jj}/d_{ii} on the off-diagonal entries multiply to one. These forms also vanish on vertical diagonal directions, so their wedge defines Θ and is nonzero in every chart. On the compact cycle the columns are mutually orthogonal; their phases are the only ambiguity in their unit representatives. Choosing phases to make specified nonzero entries positive gives local smooth sections of $K \to K/T$. Tangent coordinates at identity modulo the diagonal are the off-diagonal coordinates of a skew-hermitian matrix, with each opposite pair negatives of complex conjugates. Their two-form wedge is a nonzero constant times the real two-coordinate density. Left translation proves that Θ restricts to a constant nonzero multiple of an oriented density everywhere. This orientation may be chosen at identity and translated since the diagonal isotropy acts by rotations on the pairs. Integration against that density after normalization is the pushforward of Haar probability from K: local sections identify the invariant density on K with base density times invariant fiber density, as in Theorem 95.3.

Elimination with the successive upper-left pivots in each block gives the unique l, u on the locus where these minors are nonzero; all their entries are rational with denominators products of such minors. Diagonal change of the representative conjugates l, u by the same diagonal matrix. Thus F_b is well defined and holomorphic. For $X = v^{-1}\,\mathrm{d}v$, $Y = w^{-1}\,\mathrm{d}w$ we have

$$Y = u^{-1}Xu + u^{-1}\,\mathrm{d}u = l^{-1}Xl + l^{-1}\,\mathrm{d}l.$$

Modulo upper triangular matrices the lower entries of Y are the image of the lower entries of X under conjugation by u. The determinant of this map is $\prod_{i>j,\ \varepsilon_i=\varepsilon_j} u_{jj}/u_{ii} = \chi_\delta(u)$. Indeed image terms from the elementary matrix at (i, j) occur only in rows at most i and columns at least j, so all off-diagonal corrections remaining below the diagonal have strictly smaller distance to it. The dual argument modulo lower triangular matrices, using l, gives determinant $\prod_{i<j,\ \varepsilon_i=\varepsilon_j} l_{jj}/l_{ii} = \chi_\delta(l)^{-1}$ on the upper entries. The two lists of entries depend separately on the corresponding lists of X. Multiplication of their determinants gives the formula. □

Theorem 97.3. *For fixed $g \in \mathrm{GL}_n(\mathbb{C})$, $\Psi_a(g)$ is independent of $a \in B$ whenever $a^{-1}g$ is strictly accretive. For $g \in U(S)$, if a is its block diagonal part, then $\Psi_a(g) = \psi(g)$ of Definition 96.1.*

Proof. Inside each sign block Λ is decreasing in the index order. For $i < j$ of the same sign, $t_i - t_j \geq j - i$ and $\rho_i - \rho_j = j - i$. The difference $\delta_i - \delta_j$ is twice the difference of their positions within that block, at most $2(j - i)$; now use $\Lambda = \mu - \delta$. Thus the leading sets defining m_Λ on block diagonal matrices can be ordered to intersect each block in a prefix. The determinant on such a prefix multiplies by the product of the corresponding diagonal entries under right upper triangular multiplication or left lower triangular multiplication. Telescoping the exponents in Definition 96.1 gives multiplier χ_Λ. For $m_{-\mu}$ on arbitrary matrices the defining sets are tails of the full order. Left upper or right lower triangular

multiplication therefore gives multiplier $\chi_{-\mu}$. These identities include the full determinant power, and allow vanishing numerator minors.

For $v \in B$ write $f_a(v)$ for the quotient of minors in Definition 97.1, continued from K by the same formula with v^{-1}. It is invariant under right diagonals and holomorphic where its denominator is defined and nonzero. For $w = bvu = vl$ as in Lemma 97.2, use $w = bvu$ on the first numerator and $w^{-1} = l^{-1}v^{-1}$ on the second. In the denominator use $w^{-1} = u^{-1}v^{-1}b^{-1}$ on the left and $w = vl$ on the right. The multiplication laws just proved give

$$f_a(w) = \chi_{\Lambda-\mu}(u)\chi_{\Lambda-\mu}(l)^{-1} f_{ab}(v).$$

Since $\Lambda - \mu = -\delta$, the Jacobian in Lemma 97.2 gives the identity of top forms $F_b^*(f_a\Theta) = f_{ab}\Theta$. The denominator at w is nonzero whenever the denominator at v for ab is nonzero, since all triangular diagonal factors are invertible.

Strict accretivity of Y is equivalent to that of Y^{-1}, since the hermitian part of Y^{-1} is the congruence of the hermitian part of Y by Y^{-1}. Thus admissible a are exactly the block matrices with $g^{-1}a$ strictly accretive. This is a convex open set, all of whose elements are invertible. Take the line segment between two such matrices. Subdivide it into finitely many intervals on each of which, writing the path as ab_s from its initial matrix a, all b_s are sufficiently close to identity. This is possible since the inverse is bounded along the compact segment. Uniformly for $v \in K$, all leading pivots of $v^{-1}b_s v$ are then nonzero by compactness and continuity. The maps F_{b_s} give a smooth deformation of K/T starting at the identity inclusion. Along this deformation $f_a\Theta$ is holomorphic in a neighborhood of the image: the denominator for ab_s is nonzero on K by strict accretivity and the invariance formula preserves every required denominator minor up to an invertible factor.

The integral of this form on the deformed cycle is constant. For detail, a holomorphic top form is closed by differentiation in complex charts. Pull back to the interval times the compact oriented manifold K/T and write the result $\omega_s + \mathrm{d}s \wedge \eta_s$, with ω_s of top real degree on the cycle. Closedness gives $\partial_s \omega_s = \mathrm{d}_{K/T}\eta_s$. The integral of this latter exact form is zero. Cover the compact manifold by finitely many coordinate balls whose smaller balls still cover it. Smooth nonnegative functions positive on the smaller balls and supported inside the larger ones exist by translating and rescaling the function $\exp(-1/(1-|x|^2))$ on the unit ball, extended by zero. Divide their sum into each function to obtain a partition of unity. The integral of the exterior derivative of each localized form is zero by integration of compactly supported coordinate derivatives and the fundamental theorem of calculus. Summing gives the assertion, including the zero-dimensional case where the integral is constant along a deformation in a discrete space. Thus the two endpoint integrals of $f_{ab_s}\Theta$ on K/T agree. Its normalized restriction is Haar integration for these diagonal-invariant functions by Lemma 97.2. Applying this to each subinterval proves independence.

For the polar decomposition $g = kP$ in Theorem 96.3, both k and $a = k(P + P^{-1})/2$ are admissible. Indeed $(P+P^{-1})/2 > 0$ commutes with P, so the remainder for the latter matrix is positive definite. This matrix is the block diagonal part of g, since $SPS = P^{-1}$ and k commutes with S. At $a = k$ the formula is exactly the polar integral, using $m_\Lambda(v^{-1}) = \overline{m_\Lambda(v)}$ for v unitary. Independence gives the claimed equality. □

98. The compact projection of the real coefficient

Lemma 98.1. *Use K, Λ of Definition 96.1. For $s \in \mathbb{C}^\times$ put $D(s) = \mathrm{diag}(s^1, \ldots, s^n)$ and $L = \sum_{i=1}^n i\Lambda_i$. For $v, j \in K$,*

$$s^{-L} m_\Lambda(vD(s)j^{-1})$$

extends to a polynomial in s, with value $m_\Lambda(v)m_\Lambda(j^{-1})$ at zero.

Proof. Since Λ is decreasing within each sign block by Theorem 97.3, its minor expression on a block matrix is the product of the expressions of the restricted weights on the two blocks. Indeed in each principal determinant in Definition 96.1 the matrix is a direct sum, and

combining successive equal subsets in each block telescopes the exponents to the restricted weight differences. For one block with indices in increasing order, let A be the first l indices. Multilinearity of the determinant in the columns gives

$$\det(vD(s)j^{-1})[A,A] = \sum_{\substack{I \text{ in the block} \\ |I|=l}} \det v[A,I]\, \det j^{-1}[I,A]\, s^{\sum_{i\in I} i}.$$

Terms with repeated indices cancel by alternation, and grouping permutations of a fixed set gives the second determinant. The least possible exponent belongs uniquely to $I = A$, whether or not its coefficient vanishes. Dividing by that power gives a polynomial with constant term the product of the two principal minors. The full block determinant, which can occur to a negative power, is the single monomial given by the full index set and has nonzero coefficient. Multiplying its factor and the nonnegative powers of the preceding polynomials accounts for exactly s^L by the same telescoping. The product of their constant coefficients gives the displayed value. □

Theorem 98.2. *For any $x, y \in U(S)$, the coefficient of Theorem 97.3 satisfies*

$$\int_K m_\Lambda(k^{-1})\psi(xky)\,\mathrm{d}k = c_\Lambda\,\psi(x)\psi(y).$$

Proof. Fix x, take a its block diagonal part and put $Z = a^{-1}x$. This remainder is strictly accretive by Theorem 97.3. For y sufficiently near identity in $U(S)$, ak is an admissible splitting of xky for every $k \in K$, since $k^{-1}Zky$ is strictly accretive uniformly by compactness. The identity matrix is also an admissible splitting of y. In the left side apply Definition 97.1 with ak; write $v \in K$ for the integral variable there, and set $j = kv$. Bi-invariance of compact probability gives that the left side equals $1/c_\Lambda$ times the integral over $j, v \in K$ of

$$\frac{m_\Lambda(vj^{-1})m_\Lambda(aj)m_\Lambda(v^{-1})}{m_{-\mu}(Z_jY_v)}, \qquad Z_j = j^{-1}Zj, \quad Y_v = v^{-1}yv.$$

By elimination starting with its last pivot, factor $Y_v = U_vC_v$ with U_v unipotent upper triangular and C_v lower triangular, now for the full order of indices. This factorization exists uniformly near identity because the trailing principal pivots are nonzero there; the factors are continuous rational functions there and both tend uniformly to identity as y tends to identity. The multiplication law for trailing minors in Theorem 97.3 gives $m_{-\mu}(Y_v) = \chi_{-\mu}(C_v)$.

Replace v in the integrand by $vD(s)$ with $|s| = 1$. This leaves its Haar integral invariant. We compute its circle mean first, holding j, v fixed. The factor $m_\Lambda((vD(s))^{-1})$ becomes $s^{-L}m_\Lambda(v^{-1})$ by diagonal multiplication. In the denominator write

$$D(s)^{-1}Y_vD(s) = U_v(s)\,(D(s)^{-1}C_vD(s)).$$

The second factor is lower triangular with the same diagonal as C_v. Here $U_v(s)$ has unit diagonal and upper entries $s^{j'-i}(U_v)_{ij'}$ for $i < j'$. It is a polynomial matrix with $U_v(0) = 1$. For $|s| \le 1$, $U_v(s)$ stays uniformly near identity with y sufficiently near identity. All principal minors of $Z_jU_v(s)$ are nonzero on this closed disk because these matrices are strictly accretive, uniformly over j, v. The circle integrand can therefore be written as the boundary values of a holomorphic function on a neighborhood of the disk: use Lemma 98.1 for $s^{-L}m_\Lambda(vD(s)j^{-1})$ and use the inverse of $m_{-\mu}(Z_jU_v(s))\chi_{-\mu}(C_v)$ for its varying denominator. The compact circle average is its value at zero, by the Cauchy formula in Lemma A.3. The resulting integrand is the product

$$\frac{m_\Lambda(aj)m_\Lambda(j^{-1})}{m_{-\mu}(Z_j)}\;\frac{m_\Lambda(v)m_\Lambda(v^{-1})}{m_{-\mu}(Y_v)}.$$

All compact interchanges are legitimate by continuity and the nonzero denominators. Integrating and dividing by c_Λ gives $c_\Lambda\Psi_a(x)\Psi_1(y)$, as asserted near identity.

For arbitrary y, use a real analytic path $y(t)$ from 1 to y. Explicitly write $y = kP$ as in Theorem 96.3, and diagonalize k in its two unitary blocks to write $k = \exp A$ with A

skew-hermitian and block diagonal, by choosing arguments of its eigenvalues. By positive spectral calculus $P = \exp H_0$ with H_0 hermitian, and $SH_0S = -H_0$ since $SPS = P^{-1}$. Thus $y(t) = \exp(tA)\exp(tH_0)$ belongs to $U(S)$ for real t, and its matrix entries are entire in complex t by their convergent exponential series. For real t_0, take as splittings of $xky(t)$ the block diagonal parts of $xky(t_0)$, fixed as t varies. The remainders at t_0 are uniformly strictly accretive for $k \in K$ by compactness, so the same holds on a complex disk about t_0. There the integral with these fixed splittings is holomorphic in t: the integrand is rational in the matrices with nonvanishing denominators, uniformly bounded on smaller compact disks and compact integration variables. Integrating its Cauchy power series term by term on a smaller disk as in Lemma A.3 proves holomorphy. On the real interval it gives precisely the required function by Theorem 97.3. The same argument applies to $\psi(y(t))$. The two real analytic functions agree near zero; successive overlapping disks and the identity theorem of Lemma A.3 give equality at $t = 1$. □

99. Orthogonality of the real kernel

Theorem 99.1. *Let $G = U(S)$ and suppose the moment bound of Theorem 96.3 holds for some $B \geq 0$. Write $d = (\int_G |\psi|^2 \,\mathrm{d}h_S)^{-1}$ and $e = d\,\overline{\psi}$. Then for every $x, y \in G$*

$$d \int_G \overline{\psi(z)}\, \psi(xzy)\, \mathrm{d}h_S(z) = \psi(x)\psi(y).$$

In particular $e(1) = d > 0$, $e(g^{-1}) = \overline{e(g)}$ and

$$\int_G e(z)e((zx)^{-1}y)\, \mathrm{d}h_S(z) = \frac{e(x^{-1})e(y)}{e(1)}.$$

There exists a continuous compactly supported $\beta : G \to \mathbb{C}$ such that, for every $y \in G$,

$$\int_G \beta(z)e(z^{-1}y)\, \mathrm{d}h_S(z) = e(y).$$

Proof. Write $m = m_\Lambda$ in this proof and $c = c_\Lambda$. If $w \in L^1(G)$ satisfies $cw(z) = \int_K m(k^{-1})w(k^{-1}z)\,\mathrm{d}k$, then

$$\int_G w(z)\psi(xzy)\, \mathrm{d}h_S(z) = \psi(x) \int_G w(z)\psi(zy)\, \mathrm{d}h_S(z).$$

Indeed replace cw by its compact average on the left, use $z = kz'$ and apply Theorem 98.2 to the middle k-integral at $x, z'y$. Divide by $c > 0$. All these integrals converge absolutely since ψ is bounded by one by Theorem 96.3 and m is bounded on K. The function $w(z) = \overline{\psi(z)} = \psi(z^{-1})$ is integrable by the same theorem and satisfies the required identity by Theorem 98.2 at $z^{-1}, 1$. Thus the displayed factorization holds for this w.

For $I(x, y) = \int_G \overline{\psi(z)}\psi(xzy)\, \mathrm{d}h_S(z)$ we have obtained $I(x, y) = \psi(x)I(1, y)$. In the integral $I(1, y)$ put $u = (zy)^{-1}$. Translation and inversion preserve the measure by Lemmas 95.2 and 93.1. Inversion symmetry of ψ then gives $I(1, y) = I(y, 1) = \psi(y)I(1, 1) = \psi(y)/d$. This proves the first identity. Since $e((zx)^{-1}y) = d\,\psi(y^{-1}zx)$, the second integral is $d^2 I(y^{-1}, x)$, giving the stated formula.

Choose a nonnegative continuous function η of $\mathrm{tr}(z^*z)$, equal to one near its value n at identity and zero above $n + 2$. This is compactly supported: the trace bound bounds all entries, and G is a closed subset of the matrices by its Gram equation. It is left K-invariant. The number $A = \int_G \eta|\psi|^2 \,\mathrm{d}h_S$ is positive by continuity and $\psi(1) = 1$. Thus $\beta = \eta\,\overline{\psi}/A$ is continuous, compactly supported and satisfies the same compact averaging identity for w, by left K-invariance. Apply the first displayed factorization of the proof to $w = \beta, x = y^{-1}$, with the last argument 1. Since $\int_G \beta(z)\psi(z)\, \mathrm{d}h_S(z) = 1$ and $e(z^{-1}y) = d\psi(y^{-1}z)$, the required formula follows. □

Proposition 99.2. *For the sign patterns and parameters of Corollary 96.4 use* $e = \prod_\tau d_\tau \overline{\psi_\tau}$ *on* H. *At any of the levels in Definition 91.1 let* $\mathcal{L} = L^2(\Gamma\backslash X)$, *using the quotient measure there and* $\mathrm{d}h_{S_\tau}$ *at each real place. Then*

$$(\mathcal{P}F)(x) = \int_H e(a)F(xa)\,\mathrm{d}a$$

defines an orthogonal projection on $\mathcal{L}$. *Its range consists precisely of the square-integrable functions fixed by convolution with this explicit real kernel. In the geometric integral of Theorem 92.2 for this* e, *all terms with a nonunit-circle eigenvalue give zero for all sufficiently large arithmetic scales.*

Proof. Products of the identities of Theorem 99.1 give $\int_H e(a)e(a^{-1}g)\,\mathrm{d}a = e(g)$, the symmetry under inversion, and a compactly supported reproducing function. All products are finite and the needed integrability follows from Corollary 96.4. The quotient measure is preserved by right H as in Lemma 93.1, whose proof applies to the whole rational group. For $F \in \mathcal{L}$, Cauchy–Schwarz with weight $|e(a)|$ and integration over the quotient give

$$\int_{\Gamma\backslash X} \left(\int_H |e(a)F(xa)|\,\mathrm{d}a \right)^2 \mathrm{d}x \le \|e\|_1^2 \|F\|_2^2.$$

This gives an almost everywhere defined bounded operator independent of representatives. Applying the same estimate twice justifies the iterated integral; explicitly $|e| * |e|$ has L^1-norm $\|e\|_1^2$ by left invariance. Hence $\mathcal{P}^2 = \mathcal{P}$ by the convolution identity. For the inner product with $F' \in \mathcal{L}$, the integral of the absolute product is bounded by $\|e\|_1 \|F\|_2 \|F'\|_2$. The changes $x \mapsto xa$, $a \mapsto a^{-1}$ and inversion symmetry give $\mathcal{P}^* = \mathcal{P}$. An idempotent fixes its range and a self-adjoint idempotent has range orthogonal to its kernel, proving the projection assertion. At $B = r^8$ the size requirement follows from Theorem 92.2. The actual kernel now satisfies all the hypotheses of Theorem 93.2, which gives the last claim. □

100. Finite rank of the compact type projections

Definition 100.1. Use the measure and the parameters of Theorem 99.1, with the moment bound there. Put $v_0 = \sqrt{d}\,\psi$ in $L^2(G, h_S)$. Write $\pi(g)f(z) = f(zg)$, and let W be the closed linear span of $\pi(g)v_0$, $g \in G$. The inner product is conjugate linear in its first variable. A compact dominant weight is $\nu \in \mathbb{Z}^n$ decreasing weakly in index order within each of the two sign sets. On K take the tensor product T_ν of the two compact representations of Theorem 77.2 for the restricted weights and their inherited order. A block of size zero supplies the trivial line. Put $s_\nu = \dim T_\nu$, $\chi_\nu = \operatorname{tr} T_\nu$, and write m_ν for the coefficient of the tensor product of their distinguished unit vectors. It agrees on K with the minor formula of Definition 96.1, by telescoping the two block factors as in Lemma 98.1.

Lemma 100.2. *For* $a, b \in W$,

$$d \int_G |\langle a, \pi(g)b\rangle|^2\,\mathrm{d}h_S(g) = \|a\|^2\|b\|^2, \qquad \langle v_0, \pi(g)v_0\rangle = \psi(g).$$

The action on W *is unitary, strongly continuous and irreducible.*

Proof. Right invariance of h_S gives unitarity. The second identity is Theorem 99.1 with $x = 1$. In particular $\|v_0\| = 1$, and $\|\pi(g)v_0 - v_0\|^2 = 2 - 2\operatorname{Re}\psi(g)$ gives continuity at identity by Theorem 96.3. Translation, linearity and approximation by finite sums of translates give strong continuity on W. For $u_i, v_i \in G$, $i = 1, 2$, change variables to $z = u_1^{-1}gv_1$ and use the two-sided identity of Theorem 99.1. It gives

$$d \int_G \overline{\psi(u_1^{-1}gv_1)}\,\psi(u_2^{-1}gv_2)\,\mathrm{d}h_S(g) = \psi(u_2^{-1}u_1)\psi(v_1^{-1}v_2).$$

These are the inner products of the corresponding translates, in reverse order for the u's. Expanding finite sums gives the norm identity there. For sequences of finite sums tending to a, b, subtract coefficients by changing one vector at a time. The finite-sum norm identity

makes the resulting coefficients Cauchy in L^2. They also converge uniformly pointwise to $\langle a, \pi(g)b\rangle$ by Cauchy–Schwarz and unitarity. On compact sets the two limits therefore agree in L^2; the norm balls exhaust G. This proves the identity. If a nonzero proper closed invariant subspace existed, take nonzero a in it and b in its invariant orthogonal complement. All the integrands would vanish, contradicting the positive right side. □

Theorem 100.3. *For a compact dominant weight let*

$$P_\nu = s_\nu \int_K \overline{m_\nu(k)}\, \pi(k)\, \mathrm{d}k.$$

The integral, taken on each vector of W, defines an orthogonal projection of finite rank. Its rank at $\nu = \Lambda$ is one. A positive rank requires

$$\sum_{i=1}^{n} (\nu_i - \Lambda_i) = 0, \qquad \sum_{i=1}^{j} (\nu_i - \Lambda_i) \geq 0 \quad (1 \leq j < n).$$

Proof. Compact integration of a continuous vector-valued function is the norm limit of finite sums: partition a finite cover by sets on which its oscillation is small and weight samples by the measures of the pieces. Uniform continuity and completeness give existence and the usual norm bound. To compute the compact convolution, average the rank-one orthogonal projector onto the distinguished line in each factor over conjugation by that factor. The average commutes with the representation; the eigenspaces of the self-adjoint average are invariant and irreducibility in Theorem 77.2 forces a scalar. Its trace is one, so the scalar is the reciprocal dimension, as in Proposition 77.3. The averages multiply for the two factors. Taking the coefficient between the distinguished vector and its translate gives $\int_K m_\nu(k) m_\nu(k^{-1}j)\, \mathrm{d}k = m_\nu(j)/s_\nu$. Inversion conjugates the coefficient. Integration using these identities shows $P_\nu^* = P_\nu = P_\nu^2$.

Put $F_\nu(x,y) = \int_K \overline{m_\nu(k)}\psi(xky)\, \mathrm{d}k$. For x, y in sufficiently small fixed neighborhoods of identity perform the change and elimination of Theorem 98.2. Choose any integers $o_1 < \cdots < o_n$ and put $D_o(s) = \mathrm{diag}(s^{o_i})$ and $N = \sum o_i(\nu_i - \Lambda_i)$. In the notation of that proof the circle integrand, apart from $1/c_\Lambda$, is

$$s^N H_\nu(s; v, j) \frac{m_\Lambda(aj) m_\Lambda(v^{-1})}{m_{-\mu}(Z_j U_v(s))\, m_{-\mu}(Y_v)}, \qquad H_\nu = s^{-\sum o_i \nu_i} m_\nu(v D_o(s) j^{-1}).$$

Here $(U_v(s))_{il} = s^{o_l - o_i}(U_v)_{il}$ for $i < l$. In the block minor expansion of Lemma 98.1 the least exponent on an index set of prescribed size still belongs to its initial segment. Thus H_ν is a polynomial with coefficients finite sums of products of functions of v and functions of j. A negative power of a full block determinant merely contributes separated nonzero determinant factors. The displayed denominator stays nonzero on the unit disk after shrinking the neighborhoods, by the strict accretivity argument in Theorem 98.2.

The circle mean vanishes if $N > 0$. If $N \leq 0$, put $h = -N$. The mean extracts coefficient h from the holomorphic factor after s^N. Expand $1/m_{-\mu}(Z_j(I+V))$ at $V = 0$ in the strictly upper triangular entries of V. It is rational and regular there, so its Taylor terms of total degree at most h are a finite sum of monomials in V with coefficients depending only on Z_j. This also follows by expanding each nonzero denominator about its constant term as a geometric series through degree h. Substitution of $V = U_v(s) - I$ makes all higher terms vanish through degree h since every entry is divisible by s. After multiplication by H_ν the selected coefficient is a finite sum of products of functions of (x, j) and functions of (y, v). The number L of products can be chosen solely from the dimensions, weights and o. Compact integration in j, v gives $F_\nu(x,y) = \sum_{l=1}^{L} A_l(x) B_l(y)$ in these neighborhoods. The zero case is included with $L = 0$.

Every $(L+1)$-rowed determinant $\det[F_\nu(x_i, y_j)]$ consequently vanishes there. For arbitrary arguments join each x_i, y_j to identity by its analytic path constructed in Theorem 98.2. Along these simultaneous paths each entry is real analytic: use the fixed splitting at a

real parameter value, uniform over the compact variable, and integrate the local Cauchy series as in that theorem. The identity theorem in Lemma A.3 extends the vanishing to the prescribed endpoints. By Lemma 100.2 the entries of the projected Gram matrix for any translates $\pi(y_i)v_0$ equal $s_\nu F_\nu(y_i^{-1}, y_j)$. Thus every $L+1$ such projected vectors are dependent: their Gram determinant is zero precisely when the linear combinations have a nonzero null vector, by the squared norm formula. They span a space of dimension at most L, which is closed since a finite orthonormal basis recovers the coordinates continuously. Density of the translates gives the rank bound. For Λ, Theorem 98.2 factors the Gram matrix and $s_\Lambda c_\Lambda = 1$ by the compact average above, so its rank equals one by evaluation at identity.

For positive rank, vanishing when $N > 0$ shows $\sum o_i(\nu_i - \Lambda_i) \le 0$ for every such o. Adding an arbitrary integer to each o_i gives equality of the total weights. Summation by parts gives the negative of $\sum_{j<n}(o_{j+1} - o_j)\sum_{i\le j}(\nu_i - \Lambda_i)$. Taking any single gap arbitrarily large proves the prefix inequalities. Conversely those inequalities and the vanishing total imply all these scalar inequalities. □

101. The compact character averages of the orbital integral

Proposition 101.1. *For a compact dominant weight as in Definition 100.1, the function*

$$B_\nu(g) = \int_K \overline{\chi_\nu(k)}\psi(g^{-1}kg)\,\mathrm{d}k$$

is nonnegative and integrable on G, and $d\int_G B_\nu(g)\,\mathrm{d}h_S(g) = \operatorname{rank} P_\nu$.

Proof. Let $b_1, \dots, b_m$ be an orthonormal basis of the range of P_ν, finite by Theorem 100.3. For $w_g = \pi(g)v_0$,

$$\langle w_g, P_\nu w_g\rangle = \sum_{i=1}^{m} |\langle b_i, \pi(g)v_0\rangle|^2 = s_\nu \int_K \overline{m_\nu(k)}\,\psi(g^{-1}kg)\,\mathrm{d}k.$$

Lemma 100.2 gives integral m/d for this nonnegative function. Replace g by ug and average in $u \in K$. On the right conjugate the compact variable by u. The conjugacy average of m_ν is χ_ν/s_ν, by Theorem 80.1 on each factor. The average on the left is consequently B_ν. Integrating it over G leaves the integral unchanged by left invariance and nonnegative iterated integration as in Lemma A.2. □

Definition 101.2. For diagonal coordinates $z = (z_1, \dots, z_n)$ put

$$Q(z) = \prod_{\substack{i<j\\ \varepsilon_i \ne \varepsilon_j}} (z_j - z_i), \qquad V(z) = \prod_{\substack{i<j\\ \varepsilon_i = \varepsilon_j}} (z_i - z_j).$$

An empty product is one. Write $Q(k)$ on K for the value on its two lists of eigenvalues in the respective sign positions. Up to a fixed sign $Q(z)$ is $\prod_{\varepsilon_i=+1,\varepsilon_j=-1}(z_i - z_j)$; it is symmetric in each list.

Lemma 101.3. *For a constant depending only on n and any $k \in K$, $g \in G$,*

$$|Q(k)|\,\|g\|_{\mathrm{op}}^2 \le C_n(1 + \|g^{-1}kg\|_{\mathrm{op}})^{n-1}.$$

The function Q is continuous on K.

Proof. With just one sign, G is compact unitary, $\|g\|_{\mathrm{op}} = 1$ and $Q = 1$. Otherwise let P_+, P_- be the monic characteristic polynomials of the two blocks of k, of degrees $p, q' > 0$. The map on polynomials of degree less than p given by multiplication by P_- modulo P_+ has determinant $R = \prod_{\varepsilon_i=+1} P_-(z_i)$. For distinct roots of P_+, evaluation at them identifies this map with a diagonal matrix, by division and polynomial interpolation. For repeated roots the identity follows by continuity in the roots, since division by a monic polynomial is polynomial in its coefficients. Hence R agrees with $Q(k)$ up to a fixed sign and is continuous.

The coefficients of the map and its adjugate are bounded in terms of n, since both input polynomials have roots on the unit circle. Apply the adjugate to the constant polynomial 1 to get B of degree $< p$ with $BP_- \equiv R \bmod P_+$. Then $H = BP_-$ has degree at most $n-1$ and bounded coefficients. Its value at every positive-block eigenvalue is R, and at every negative-block eigenvalue is zero. By block unitary diagonalization $H(k) = RE_+$ with $E_+ = (I+S)/2$.

Write $T = \|g\|_{\mathrm{op}}$. Since $g^{-1}Sg = Sg^*g$, $2\|g^{-1}E_+g\|_{\mathrm{op}} \geq T^2 - 1$. Evaluation of H at $g^{-1}kg$ bounds $|R|\,\|g^{-1}E_+g\|_{\mathrm{op}}$ by a constant times the right-hand power in the statement. Also $|R| \leq 2^{pq'}$. Adding these two estimates gives the bound for T^2. □

Theorem 101.4. *Set $N = 4n^2+1$ and assume the moment hypothesis of Theorem 99.1 and the sign-change gap of Lemma 96.2 with $c \geq (n-1)N$. The function*

$$H(k) = d\int_G Q(k)^N \psi(g^{-1}kg)\,\mathrm{d}h_S(g)$$

is continuous and conjugation invariant on K. Whenever the two blocks have disjoint spectra the orbital integral of ψ itself is absolutely convergent and equals $H(k)/(dQ(k)^N)$. These conclusions apply to the arithmetic rows of Corollary 96.4 at all sufficiently large scales.

Proof. Both $\|g^{-1}kg\|_{\mathrm{op}}$ and $\|g\|_{\mathrm{op}}$ are at least one because elements of $U(S)$ have determinant of modulus one. Combine Lemma 101.3 with Theorem 96.3. The modulus of the integrand, before its scalar d, is bounded by a constant depending on n, c times $\|g\|_{\mathrm{op}}^{-2N}$, uniformly in k. Its integral is finite by summing dyadic shells with Lemma 91.4, since $2N > 4n^2$. Dominated convergence from Lemma A.2 and continuity of the integrand give continuity of H. Left translation of g gives conjugacy invariance for all k. Division by the fixed nonzero $Q(k)^N$ gives the unregularized assertion, including at repeated roots within a single block. Finally the gap $r^8 + 4r^2 + 2$ used in Corollary 96.4 exceeds $(r-1)(4r^2+1)$ for all sufficiently large $r = n$. □

102. The Laurent polynomial on the compact Cartan

Definition 102.1. Let $\mathfrak{W}$ be the permutations preserving the two sign sets. Write $z^\alpha = \prod_i z_i^{\alpha_i}$, $(w\alpha)_i = \alpha_{w^{-1}i}$, and $A_\alpha(z) = \sum_{w\in\mathfrak{W}} \operatorname{sgn}(w) z^{w\alpha}$. Put $\delta_i = \#\{j > i : \varepsilon_j = \varepsilon_i\}$. For an exponent $\alpha \in \mathbb{Z}^n$ with distinct coordinates within each sign set there is a unique $w \in \mathfrak{W}$ such that $w\alpha = \nu + \delta$ with ν compact dominant: sort each block decreasingly and subtract δ. Put $X_\alpha = \operatorname{sgn}(w)\chi_\nu$ and $t(\alpha) = \operatorname{sgn}(w)\operatorname{rank} P_\nu$. For an exponent with a repeated coordinate within a block put $X_\alpha = t(\alpha) = 0$.

Lemma 102.2. *The χ_ν form an orthonormal system in $L^2(K, \mathrm{d}k)$. Their finite linear combinations are uniformly dense in the continuous class functions on K. On its torus $VX_\alpha = A_\alpha$.*

Proof. For two single-block modules average a map between them by

$$M \longmapsto \int T_\nu(k) M T_\lambda(k)^{-1}\,\mathrm{d}k.$$

Invariance shows that the average is a projection onto the space of intertwining maps. A nonzero intertwiner of irreducibles is an isomorphism by its kernel and image. An endomorphism is scalar since it has a complex eigenvalue and its eigenspace is invariant. Distinct highest weights cannot give isomorphic modules: by differentiating an intertwining equation their complexified actions intertwine as in Theorem 77.2, and the upper unitriangular-fixed lines have their specified diagonal weights. The trace of this averaging operator is therefore 1 for equal weights and 0 otherwise. Its trace in the elementary matrix basis is $\int \chi_\nu \overline{\chi_\lambda}$. Multiplying the two block identities gives orthonormality.

Expansion of the two block determinants gives A_α, and the ordinary Vandermonde product gives $V = A_\delta$. Theorem 80.1 on the two blocks gives $VX_\alpha = A_\alpha$, including

repeated coordinates by pairing the terms of opposite sign. Any continuous class function is determined by its restriction to the torus by unitary diagonalization. Approximate the restriction uniformly by trigonometric polynomials with the product of the one-dimensional Fejér kernels of Lemma 53.2. Indeed their nonnegative integrals are one and their mass outside any prescribed torus neighborhood of identity tends to zero, so uniform continuity gives the approximation. Average a polynomial approximant in $\mathfrak{W}$ to make it invariant. Multiplying it by V gives a finite alternating sum of monomials. A repeated coordinate forces zero coefficient by transposition, and collecting each other orbit expresses it in the basis $A_{\nu+\delta}$ with coefficient that of $z^{\nu+\delta}$. Divide by V where the same-block coordinates are distinct, and continue to the whole torus by continuity. Thus the symmetric approximants are restrictions of finite character sums, with the same uniform error on K. □

Theorem 102.3. *Under the hypotheses of Theorem 101.4 there is an alternating Laurent polynomial P for $\mathfrak{W}$, with integer coefficients, such that*

$$P(z) = V(z)H(\operatorname{diag} z), \qquad [z^\alpha]P = \sum_\beta [z^\beta]Q^N\, t(\alpha-\beta).$$

Here $[z^\alpha]$ denotes the Laurent coefficient. In particular $t(\Lambda+\delta)=1$. If $t(\gamma)\neq 0$, some permutation of its two blocks equals $\nu+\delta$ with $\sum_{i\le j}(\nu_i-\Lambda_i)\ge 0$ for all j and total sum zero. If a Laurent polynomial agrees with P on a nonempty open subset of the unit torus, it agrees everywhere.

Proof. Write $c_\beta = [z^\beta]Q^N$, integers invariant under $\mathfrak{W}$ by Definition 101.2. On the torus $\overline{Q}^N X_\alpha = \sum_\beta c_\beta X_{\alpha-\beta}$: multiply by V, alternate $z^\alpha \sum_\beta c_\beta z^{-\beta}$, and use Lemma 102.2. By block conjugation and continuity the identity holds on K. For fixed g multiply this identity after conjugation by $\psi(g^{-1}kg)$ and integrate over k. On integrating over g, the left side permits exchange by the uniform integrable bound in Theorem 101.4. Each compact average on the right is integrable by Proposition 101.1, including its possible sign. Consequently

$$\langle X_\alpha, H\rangle_{L^2(K)} = \sum_\beta c_\beta\, t(\alpha-\beta).$$

In particular all $b_\nu = \langle \chi_\nu, H\rangle$ are integers. For any finite set of distinct ν, expanding the squared norm of $H - \sum b_\nu \chi_\nu$ and using orthonormality gives $\sum |b_\nu|^2 \le \|H\|^2$. Thus only finitely many are nonzero. Subtracting their finite character sum gives a continuous class function orthogonal to all compact characters. Its squared integral is zero by uniform approximation in Lemma 102.2. It vanishes everywhere, since every nonempty open subset of the compact group has positive measure by the smooth density of Definition 95.1. Now $P = \sum_\nu b_\nu A_{\nu+\delta}$ has all the stated coefficients, as $[z^\alpha]P = \langle X_\alpha, H\rangle$. The assertions about t follow from Theorem 100.3. Finally an open torus set contains a product of open arcs. After clearing negative powers, hold all but one variable fixed in these arcs. A one-variable polynomial vanishing there has infinitely many roots and is zero. Repetition in all variables proves uniqueness. □

103. Compression along a signed unit vector

Lemma 103.1. *Put $\eta=\varepsilon_1$, $S'=\operatorname{diag}(\varepsilon_2,\dots,\varepsilon_n)$. In the first-column map $g\mapsto w = ge_1$ from $U(S)$, the range is $w^*Sw=\eta$ and the fiber has group $U(S')$. Use $h_S, h_{S'}$ of Definition 95.1 and mass one for the trivial group in dimension zero. After integrating the coordinate phases, the quotient measure in h_S is $(n-1)!\prod_{i\neq j}\mathrm{d}s_i$, where $s_i=|w_i|^2>0$ and s_j is determined by $\sum_i \varepsilon_i s_i=\eta$. Every choice of eliminated index j gives this formula.*

Proof. The complement of a vector of norm η for the signed form is nonsingular and has the residual signature: decompose orthogonally into this line and its complement, diagonalize the hermitian form on the complement and count positive and negative dimensions, as in Theorem 90.2. Thus w extends to a signed orthonormal frame. The stabilizer of e_1 has the

stated block form. Local smooth choices of frame follow by projecting a fixed basis of the complement at one point and performing successive signed orthogonalization; the pivots keep their nonzero signs in a neighborhood.

On the quadric define a density by taking twice ambient Lebesgue volume per unit change of w^*Sw. In local coordinates (t, y) with $t = w^*Sw$ this means twice the coefficient of $\mathrm{d}t\ \mathrm{d}y$ in ambient volume at $t = \eta$, times $\mathrm{d}y$. The derivative in a real coordinate of a nonzero entry can be made nonzero by a phase, so these coordinates cover the quadric. Changes of the y's give the ordinary density transformation. This density is invariant since G preserves t and has complex determinant of modulus one. At $w = e_1$ it is ordinary tangent coordinate measure, as the absolute derivative in the first real direction is 2. At identity the map on the skew-hermitian tangent coordinate in Definition 95.1 takes its first column multiplied by S, leaving a free lower square block. Signs do not change this coordinate density. Thus the quotient density for the two unnormalized measures is precisely the density just constructed. The product formula in local sections follows by the same block triangular change of coordinates as in Theorem 95.3. A countable chart cover suffices here: use the countable rational balls in matrix coordinates and partition into successive differences. For nonnegative integrands one can therefore integrate first over the fiber and then over the column.

For normalized measures multiply the quotient density by $u_{n-1}/u_n = (n-1)!/(2\pi^n)$. In planar polar coordinates ordinary complex volume is $\mathrm{d}s_i\ \mathrm{d}\theta_i/2$. Eliminating one s_j in favor of t has Jacobian of absolute value one. All phases range from 0 to 2π; their integration contributes $2\pi^n$ including the initial factor two. This gives the stated quotient density. A vanishing coordinate occupies a set of measure zero: near such a point eliminate a real coordinate belonging to a different nonzero entry, leaving two free real coordinates that have to vanish. The case $n = 1$ gives the usual probability on the circle. □

Theorem 103.2. *Let A be positive diagonal with signed eigenvalues $x_i = \varepsilon_i A_{ii}$ pairwise distinct. For a first column w, put $B = (g^*Ag)[2{:}n, 2{:}n]$ in any signed orthonormal completion. Write y for the signed eigenvalues of $S'B$, and $\mathcal{V}(x) = \prod_{i<j} |x_i - x_j|$. The image measure on y, ordered by decreasing magnitude within each sign, has density*

$$(n-1)!\frac{\mathcal{V}(y)}{\mathcal{V}(x)}\prod_l \mathrm{d}y_l$$

with positive coordinate volume. For roots of sign η there is one new magnitude between each consecutive pair of old magnitudes. For roots of the opposite sign there is one between each consecutive pair and one above the largest old magnitude if that sign occurs. These intervals vary independently, and their endpoints may be omitted. The empty list has Vandermonde product one.

Proof. Since $B > 0$, the matrix $S'B$ is similar to the hermitian matrix $B^{1/2}S'B^{1/2}$, whose signature is that of S' by the change of quadratic form coordinates. Its eigenspaces of distinct eigenvalues are orthogonal for S' since $(S'B)^*S' = S'(S'B)$. On a λ-eigenspace, $v^*Bv = \lambda v^*S'v$, so the restriction of S' is definite of the sign of λ. Choosing normalized bases there shows that B is congruent by $U(S')$ to the positive diagonal of the magnitudes $|y_l|$ in sign order. In particular its signed eigenvalues are independent of the completion.

In the basis g, the bottom square block of $g^{-1}SAg$ equals $S'B$. Its resolvent has first diagonal entry given both by its cofactor and by $e_1^*g^{-1} = \eta w^*S$. Thus, with z an indeterminate,

$$\frac{\prod_l (z - y_l)}{\prod_i (z - x_i)} = \sum_i \frac{u_i}{z - x_i}, \qquad u_i = \eta\varepsilon_i s_i, \quad \sum_i u_i = 1.$$

Multiply all signed eigenvalues by η to discuss their order, so the removed sign is positive and the u_i have the signs of these rescaled roots x_i. Between consecutive poles of the same sign the real rational function tends to infinities of opposite signs at the two ends. If negative poles occur, it is positive immediately to the left of the leftmost one and negative far to its

left, since it is asymptotic to $1/z$. These intervals account for all $n-1$ roots, exactly one simple root apiece. Conversely any points in those intervals give the residues

$$u_i = \frac{\prod_l (x_i - y_l)}{\prod_{k \neq i} (x_i - x_k)}.$$

In decreasing signed order the number of roots above x_i equals the number of poles above it for a positive pole, and is one less for a negative pole. The residues therefore have the required signs. Their sum is one by the leading term at infinity, using the partial fractions obtained by subtracting the displayed residues from the rational function. Thus the correspondence with interior s-coordinates is bijective.

Omit the last residue, which is determined by their sum. Its remaining derivative matrix has entries $-u_i/(x_i - y_l)$, $1 \le i, l \le n-1$. The absolute Cauchy determinant for the entries $1/(a_i - b_j)$ is $\mathcal{V}(a)\mathcal{V}(b)/\prod_{ij} |a_i - b_j|$. Indeed clearing the denominators leaves a polynomial alternating in each list, divisible by both Vandermonde products by setting two arguments equal and dividing off their difference. The degrees agree, so the quotient is constant. Multiplying the uncleared determinant by $a_m - b_m$ and setting $a_m = b_m$ reduces it to the determinant in size $m-1$, giving constant of modulus one by induction. Substitution of the residue formula now gives the absolute Jacobian $\mathcal{V}(y)/\mathcal{V}(x)$; all $|x_i - y_l|$ cancel. The change between u_i and s_i, any reorderings and changes of signs of the variables preserve the absolute Jacobian. Apply Lemma 103.1. In dimension one both coordinate lists of the density are empty after omission and the conclusion is immediate. □

104. Evaluation of the positive principal-minor integral

Definition 104.1. Let $b_1 > \cdots > b_n$ be integers, $\mu_i = b_i - i + 1$, and $A = \operatorname{diag}(a_i)$ with $a_i > 0$. For S as in Definition 95.1 put

$$J_S(\mu, A) = \int_{U(S)} \frac{\mathrm{d}h_S(g)}{m_{-\mu}(g^* A g)}, \qquad D(b) = \prod_{i<j} \frac{b_i - b_j}{j - i}.$$

The integrand is positive. For each sign σ, list its indices I_σ in their original order, and write a^σ for the list of magnitudes of that sign in decreasing order. For these two lists set $E_\sigma = \det[(a^\sigma_j)^{b_i}]_{i \in I_\sigma, j}$. The empty determinant is one.

Theorem 104.2. *When the a_i are distinct within each sign, the positive integral of Definition 104.1 is finite and equals*

$$J_S(\mu, A) = \frac{E_+ E_-}{D(b)\, \mathcal{V}((\varepsilon_i a_i)_i)}.$$

Proof. In dimension one the group is compact of mass one and the integrand is $a_1^{b_1}$. For higher dimension the trailing minors in Definition 96.1 give

$$m_{-\mu}(g^* A g) = (\det A)^{-\mu_1} m_{-\widehat{\mu}}(B), \qquad \widehat{\mu}_{i-1} = \mu_i - \mu_1 \quad (2 \le i \le n),$$

where $B = (g^* A g)[2{:}n, 2{:}n]$. Indeed the full determinant has modulus factor $|\det g|^2 = 1$; its exponent is $-\mu_1$. All shorter trailing minors have their successive weight differences, including the full determinant of B with exponent $\mu_1 - \mu_2$. On the first-column fiber, changing the remaining basis replaces B by $h^* B h$. Theorem 103.2 conjugates this positive form by $U(S')$ to the diagonal with signed eigenvalues y. Thus by invariance the fiber integral is $J_{S'}$ with these magnitudes and weights $\widehat{\mu}$. Its integer exponents in Definition 104.1 are $\widehat{b}_{i-1} = b_i - b_1 - 1$, still strictly decreasing. Nonnegative integration in Lemma 103.1 permits this step before finiteness is known.

Use the induction formula in the eigenvalue density of Theorem 103.2. The factor $\mathcal{V}(y)$ cancels, leaving one determinant in the magnitudes of each sign. In each determinant its j-th column depends only on one independent interval variable. Integrate by columns: expanding the determinant gives a finite sum of products of absolutely integrable powers. The finite

intervals have positive endpoints and all powers at infinity are $b_i - b_1 - 1 < -1$. This proves that the manipulations are absolutely integrable.

For the removed sign write its old magnitudes $a'_1 > \cdots > a'_m$. The entry with row $i \neq 1$ and interval (a'_{j+1}, a'_j) integrates to

$$\frac{(a'_{j+1})^{b_i - b_1} - (a'_j)^{b_i - b_1}}{b_1 - b_i}.$$

Extract the denominator once per row. The remaining determinant equals the m-rowed determinant on the a'_j with row exponents $b_i - b_1$ including its initial row of ones. Indeed in the latter matrix replace each column after the first by itself minus its predecessor, working from right to left, then expand in its first row. This includes $m = 1$.

For the opposite sign write its old magnitudes $h_1 > \cdots > h_e$, $h_0 = \infty$. The retained variables lie in (h_j, h_{j-1}). After taking the same row denominators out, the column entries are $h_j^{b_i - b_1} - h_{j-1}^{b_i - b_1}$, with $\infty^{b_i - b_1} = 0$. These differences have determinant $\det[h_j^{b_i - b_1}]$, by the same triangular column operation. If $e = 0$ its contribution is one. The original prefactor $(\det A)^{\mu_1} = (\prod a_i)^{b_1}$ puts back the missing power in every column of both determinants. Their product is E_+E_-. The remaining factor multiplying $1/\mathcal{V}((\varepsilon_i a_i)_i)$ is

$$\frac{(n-1)!}{D(\widehat{b}) \prod_{i>1}(b_1 - b_i)} = \frac{1}{D(b)}.$$

This uses cancellation of the common factors in all pairs with $i, j > 1$ and $\prod_{j=2}^{n}(j-1) = (n-1)!$. It completes the induction. □

105. From positive diagonal matrices to the elliptic torus

Lemma 105.1. *For the data of Definition 104.1 put*

$$F(a) = \int_G \frac{\mathrm{d}h_S(g)}{m_{-\mu}(g^* \operatorname{diag}(a) g)}, \qquad W(a) = \prod_{\substack{i<j \\ \varepsilon_i = \varepsilon_j}} (a_i - a_j) \prod_{\substack{i<j \\ \varepsilon_i \neq \varepsilon_j}} (a_i + a_j).$$

For $\operatorname{Re} a_i > 0$ *the integral is absolutely convergent and is holomorphic separately in all coordinates. With the block alternant of Definition 102.1 one has* $D(b)W(a)F(a) = A_b(a)$ *on this product of half-planes.*

Proof. Fix a positive diagonal A_0 with distinct entries. For any full-column-rank submatrix C of the columns of g, put $L = C^*C > 0$. If $\min_i \operatorname{Re} a_i \geq h > 0$, the matrix $L^{-1/2} C^* \operatorname{diag}(a) C L^{-1/2}$ has real part at least hI, in the sense of hermitian forms. Each eigenvalue has real part at least h by evaluation on its eigenvector. It follows that the modulus of its determinant is at least $h^{\dim L}$, since the determinant is the product of its eigenvalues with multiplicity. The same argument for the hermitian matrix with A_0 bounds its determinant above by $\|A_0\|_{\mathrm{op}}^{\dim L}$. Therefore the modulus of each trailing subdeterminant at a is bounded below by its value at A_0 times $(h/\|A_0\|_{\mathrm{op}})^{\dim L}$. All exponents of the proper minors of $m_{-\mu}$ are nonnegative. Its full determinant factor is $(\prod a_i)^{-\mu_1}$ and is uniformly bounded above and away from zero on compact subsets of the domain. Thus the absolute integrand there is bounded by a fixed multiple of the positive integrand at A_0, integrable by Theorem 104.2. Pointwise it is a rational holomorphic function with nonzero denominators. The Cauchy integral argument of Lemma A.3 gives separate holomorphy.

On real positive entries decreasing within each sign, $W(a)$ equals $\mathcal{V}((\varepsilon_i a_i)_i)$ and $A_b(a)$ is the product E_+E_- of Definition 104.1. Hence the identity holds on an open real box by Theorem 104.2. Keeping all other variables in their box intervals, continue it in one variable to its connected half-plane by Lemma A.3. Repeat with those already extended variables held fixed anywhere in their half-planes. This gives the asserted identity throughout, even when $W(a) = 0$. □

Theorem 105.2. *Use the hypotheses of Theorem 102.3, with $b_i = t_i + (n-1)/2 = \mu_i + i - 1$. Then*

$$P = \frac{d}{D(b)} Q^{N-1} A_b.$$

In particular for pairwise distinct unit z_i,

$$\int_G \psi(g^{-1}\operatorname{diag}(z) g)\, \mathrm{d}h_S(g) = \frac{A_b(z)}{D(b)\, V(z) Q(z)}.$$

Proof. First take z_i on the unit circle with $\operatorname{Re}(\varepsilon_i z_i) > 0$ and with all z_i distinct. Put $a_i = \varepsilon_i z_i$ and $A = \operatorname{diag}(a)$. For $h = g^{-1}\operatorname{diag}(z) g$ we have $Sh = g^* A g$, strictly accretive. Hence S is an admissible block splitting for h in Theorem 97.3. The diagonal multiplication of the numerator gives

$$\psi(h) = \frac{\sigma}{c_\Lambda} \int_K \frac{|m_\Lambda(v)|^2}{m_{-\mu}((gv)^* A (gv))}\, \mathrm{d}v, \qquad \sigma = \prod_i \varepsilon_i^{\Lambda_i}.$$

The integral of the absolute value over g, v is finite by Lemma 105.1, right invariance in g, and the definition of c_Λ. Integrating first in g gives $\sigma F(a)$.

In the sign substitution $z_i = \varepsilon_i a_i$, the two block determinants in A_b acquire together the factor $(-1)^{\sum_{\varepsilon_i = -1} b_i}$. The factors of VQ acquire, relative to $W(a)$, one minus sign for each $i < j$ with $\varepsilon_j = -1$. Indeed a pair of negative signs uses $z_i - z_j$ in V, and an opposite pair ending in a negative sign uses $z_j - z_i$ in Q. Thus the total sign in $A_b(z)/(VQ)(z)$ relative to $A_b(a)/W(a)$ is $(-1)^{\sum_{\varepsilon_i = -1}(b_i + i - 1)} = \sigma$. For the last equality use $b_i = \mu_i + i - 1$ and $\sum_{\varepsilon_i = -1} \mu_i = \sum_{\varepsilon_i = -1} \Lambda_i$; the earlier and later same-sign pairs cancel in Definition 96.1. By Lemma 105.1 this proves the orbital formula on the prescribed open set of the torus.

By Theorems 101.4 and 102.3, multiplication by dVQ^N gives P there. Both sides of the resulting identity are Laurent polynomials. It therefore holds everywhere by the uniqueness in Theorem 102.3. Dividing on the regular torus proves the last formula with the convergence asserted in Theorem 101.4. □

106. The formal degree and the elliptic character identity

Lemma 106.1. *Use b, N, Q, δ, Λ of Theorems 102.3 and 105.2. Put $u_i = \#\{j < i : \varepsilon_j \neq \varepsilon_i\}$ and $\kappa = b + (N-1)u$. Suppose κ is strictly decreasing within each sign in its index order. Then*

$$[z^\kappa] P = [z^\kappa](Q^{N-1} A_b) = 1.$$

Proof. Put $o_i = i$, with $\langle o, \beta \rangle = \sum i\beta_i$. In Q^j, for $j \geq 0$, the unique monomial with maximum $\langle o, \cdot \rangle$ is z^{ju}, with coefficient one. Indeed in each factor for $i < h$ the term z_h has strictly greater level than $-z_i$. The support of Q^j is invariant under $\mathfrak{W}$ by Definition 101.2. For $w \in \mathfrak{W}$, $\langle o, w\kappa \rangle \geq \langle o, \kappa \rangle$ with equality only for $w = 1$. In each sign block, interchanging a smaller preceding value with a larger following one strictly decreases its scalar product with the increasing indices. Repeatedly removing such inversions uniquely restores the strictly decreasing list.

By Definitions 96.1 and 102.1, $\Lambda_i + \delta_i = \mu_i + \#\{j < i : \varepsilon_j = \varepsilon_i\} = b_i - u_i$. Suppose β contributes to the convolution for $[z^\kappa]P$ in Theorem 102.3. Choose w so $w(\kappa - \beta) = \nu + \delta$ is sorted. The prefix inequalities and equal total in that theorem give $\langle o, \nu - \Lambda \rangle \leq 0$ by summation by parts. Since $w\beta$ occurs in Q^N, it follows that

$$\langle o, w\kappa \rangle \leq \langle o, \Lambda + \delta + Nu \rangle = \langle o, \kappa \rangle.$$

Thus $w = 1$; equality also forces $\beta = Nu$. Its coefficient is one and $t(\kappa - Nu) = t(\Lambda + \delta) = 1$. Conversely this term occurs, giving the first assertion. For a term $\beta + wb = \kappa$ in $Q^{N-1} A_b$, apply w^{-1}. Since $w^{-1}\beta$ occurs in Q^{N-1}, $\langle o, w^{-1}\kappa \rangle \leq \langle o, b + (N-1)u \rangle$. The same strictness gives $w = 1$, leaving only the leading monomial of Q^{N-1} with positive coefficient one. □

Theorem 106.2. *Under the hypotheses of Lemma 106.1, the reciprocal squared norm d of the unit-normalized-at-identity coefficient ψ is $D(b)$. With p positive and q' negative signs, $e = d\,\overline{\psi}$ and $\lambda_i = t_i - (n+1)/2 + i$, for pairwise distinct unit z_i we have*

$$\frac{(-1)^{pq'}}{p!\,q'!} \sum_{\pi \in \mathfrak{S}_n} \int_G e(g^{-1} \operatorname{diag}(z_{\pi(1)}, \dots, z_{\pi(n)}) g)\, \mathrm{d}h_S(g) = \overline{\chi_\lambda(\operatorname{diag} z)}.$$

Both assertions apply at every real place of Corollary 96.4 for all sufficiently large arithmetic scales, and their degrees multiply to $\mathcal{D}$.

Proof. Take the coefficient of z^κ in Theorem 105.2. Lemma 106.1 gives $1 = d/D(b)$. The product VQ is $(-1)^{pq'}$ times the full Vandermonde $\Delta(z) = \prod_{i<j}(z_i - z_j)$. Permuting the arguments of Δ gives the sign of that permutation. Alternating A_b over all permutations gives $p!q'!$ times the full alternant of b: in its monomial expansion each inner block permutation gives the same full alternating sum after changing the outer index, with the two inner signs cancelling. Hence after summation the unnormalized orbital formula gives, on multiplying by $(-1)^{pq'} d/(p!q'!)$, the full alternant divided by $\Delta(z)$. This is χ_λ by Theorem 80.1 since $b_i = \lambda_i + n - i$. Conjugating gives the identity for e. Absolute convergence follows from Theorem 101.4.

For the arithmetic parameters let $n = r$. Within a consecutive run of one sign the u_i are constant and the b_i strictly decrease. If $i < j$ of the same sign cross a change of sign, at least one intervening $b_h - b_{h+1}$ is at least $r^8 + 4r^2 + 1$ by the proof of Corollary 96.4, since the μ-gap is larger by one. The other gaps are positive. As $|u_i - u_j| \le r$ and $N - 1 = 4r^2$, this gives $\kappa_i > \kappa_j$ for sufficiently large r. The remaining convergence and regularization hypotheses hold by Theorem 101.4. Finally $D(b) = \mathcal{D}_\tau$ by Definition 56.1 because $b_i - b_j = t_i - t_j$. Take the product over the real places. □

Corollary 106.3. *For the arithmetic row parameters, finite frames, rational groups and elliptic finite tests in Theorem 94.2, assume the first $k-1$ row entries have a common sign at every real place. At all sufficiently large scales, the global geometric integral formed from the actual kernel $e = \prod_\tau \mathcal{D}_\tau \overline{\psi_\tau}$ equals the finite test trace on the definite space $\mathcal{A}^+$.*

Proof. Proposition 99.2 supplies the kernel hypotheses, and Theorem 106.2 supplies the required orbital identity and degrees in the specified measures. The compact highest weights are those of Corollary 76.4. Apply Theorem 94.2. □

107. Lattice minima on the indefinite quotient

Lemma 107.1. *Let L be a full lattice in a real Euclidean space of dimension $m \ge 1$. Write $\lambda_j(L)$ for the least radius of a closed ball containing j linearly independent lattice vectors. Let L^* be its Euclidean dual, defined by integral pairings with L. There is an ordered integral basis $v_1, \dots, v_m$ with $\|v_i\| \le C_m \lambda_i(L)$ and*

$$\lambda_{m+1-i}(L^*) \le C_m / \lambda_i(L) \qquad (1 \le i \le m).$$

If $\lambda_1(L) \ge 1/T$, $T \ge 1$, the lengths of this basis are bounded by $C_m \operatorname{covol}(L) T^{m-1}$.

Proof. Successively choose a shortest nonzero vector in the lattice orthogonally projected off the preceding vectors. Such vectors are primitive, and after lifting extend the preceding vectors to a primitive sublattice, by the basis extension and projection argument in Lemma 82.5. The lifted vectors thus form an integral basis. In Gram–Schmidt coordinates the resulting column matrix is DC, where $D = \operatorname{diag}(r_i)$ consists of the positive successive projected lengths and C is upper unitriangular. Reduce each entry of C above the diagonal to absolute value at most $1/2$ by subtracting integer multiples of earlier basis vectors, in decreasing row order in each column. This preserves all projected choices. In the projection after $i-1$ vectors, the reduced lift of the next projected vector has norm squared at most $r_{i+1}^2 + r_i^2/4$. It is nonzero so has norm at least r_i. Hence $r_{i+1} \ge \sqrt{3}\, r_i/2$. In the first i

columns every entry is bounded by a constant times r_i. Conversely among i independent vectors at least one projects nontrivially off the first $i-1$ vectors and hence has length at least r_i. This proves $r_i \le \lambda_i(L) \le C_m r_i$ and the basis bound.

The Euclidean dual basis has matrix $D^{-1}C^{-\mathsf{T}}$. The entries of C^{-1} are uniformly bounded in terms of m, by summing the finite power series in the strictly upper triangular $I-C$. In its j-th column the dual basis matrix has nonzero entries only in rows $h \ge j$, bounded there by C_m/r_h. Thus its last $m+1-i$ columns have lengths at most C'_m/r_i by the neighboring bound on the r_h. This proves the dual assertion. Finally $r_1 = \lambda_1(L)$, all $r_h \ge c_m r_1$, and their product is the covolume. Consequently every $r_i \le C_m \operatorname{covol}(L) T^{m-1}$, as required. □

Proposition 107.2. *In Definition 91.1 let $\lambda_j(x)$ be the j-th minimum of Λ_x for $\|\cdot\|_x$, where $m = 2fr$. There is a constant depending only on the fixed field and rank such that*

$$\lambda_i(x)\lambda_{m+1-i}(x) \le C.$$

For fixed arithmetic data and level, the image of $\{\lambda_1(x) \ge 1/T\}$ in $\Gamma\backslash X$, $T \ge 1$, is contained in a compact set. Its quotient volume is at most $C'(1+T)^K$ for fixed finite C', K.

Proof. In real frame coordinates write $\mathbf{S}$ for the direct sum of the sign matrices. The rational symmetric pairing $B(v,w) = \operatorname{Tr}_{E/\mathbb{Q}} h(v,w)$ is $2\langle v, \mathbf{S}w\rangle_{\mathbb{R}}$ in those coordinates. If L denotes the framed lattice, its B-dual is $\frac{1}{2}\mathbf{S}L^*$. Elements of this dual are rational vectors in the original coordinates by inverting the nonsingular rational pairing in a lattice basis. For such a w, each $h(v,w)$ with $v \in \Lambda_x$ belongs to the trace dual of O_E, by testing αv for all $\alpha \in O_E$. There is a fixed positive integer a sending this trace dual into O_E: invert the integral trace matrix and clear its nonzero determinant as in Lemma 12.1. The hermitian O_E-dual of Λ_x equals Λ_x. In fact its local completions are the ones in the finite frame by Lemma 20.2, and in that frame the Gram matrix is the identity, so the dual condition is precisely integrality at every place. Thus $(a/2)\mathbf{S}L^* \subset L$. Since $\mathbf{S}$ preserves Euclidean lengths, the first assertion follows from Lemma 107.1.

By Lemmas 91.2 and 107.1, on the indicated set there is an integral basis of lengths at most $R = C_0(1+T)^{m-1}$, increasing C_0 so $R \ge 1$. Choose r of these vectors giving an E-basis B_0 of the rational space; the full list spans it already over $\mathbb{Q}$. Let T_0 be their hermitian Gram matrix. It has integral entries, with all conjugates bounded by R^2. The lattice count in Lemma 92.1 gives at most $C_E(1+R)^{4fr^2}$ choices, with C_E also depending on r. All rational bases of the same Gram matrix lie in one Γ-orbit by their unique basis isometry. Fix one representative for each Gram matrix. The argument of that lemma bounds the number of possible finite coordinates for it by $[K^0 : K_0]|N_{E/\mathbb{Q}} \det T_0|^r$. For $M_\tau = X_\tau^{-1}(B_0)_\tau$, the column lengths are at most R, and $|\det M_\tau|^2 = |\tau \det T_0|$. Thus $|N_{E/\mathbb{Q}} \det T_0| \le R^{4fr}$.

At infinity fix one frame in the region, when it is nonempty. After translation by its inverse every other frame has operator norm at τ at most $C_r R^r/|\det M_\tau|$, by the column bound and the adjugate formula, exactly as in Lemma 92.1. These balls are compact since they are closed and bounded in a space of matrices and the signed unitary equation is closed. Their product volume is bounded by $C(1+R)^{4fr^3}$ by Lemma 91.4 since $\prod_\tau |\det M_\tau|^2 = |N_{F/\mathbb{Q}} \det T_0| \ge 1$. We have a finite union of compact sets of volume polynomial in R whose images cover the required set. The quotient volume of their image is at most their original volume by unfolding the sum of the translated indicators as in Theorem 92.2. Substitution of R proves the assertion. □

108. Small rational lifts at a marked split place

Lemma 108.1. *Fix the arithmetic data of Proposition 107.2 and let $m = 2fr$. For $\lambda = \lambda_1(x) > 0$ sufficiently small there exist a nonzero proper totally isotropic E-subspace $U \subset V$, and integral bases (a_i) of $\Lambda_x \cap U$ and (v_j) of $\Lambda_x \cap U^{\perp_h}$ as $\mathbb{Z}$-modules, such that*

$$\max_{i,j} \|a_i\|_x \|v_j\|_x \le C\,\lambda^{1/(2m)}.$$

Proof. Put $\epsilon = \lambda^{1/(2m)}$ and use the successive minima $\lambda_i = \lambda_i(x)$. The last minimum is bounded below by a positive constant because m independent vectors of length λ_m span a sublattice of covolume at least $\operatorname{covol}(\Lambda_x) > 0$; their parallelepiped has volume at most λ_m^m. The covolume is fixed by Lemma 91.2. Thus for sufficiently small λ there is a first $j < m$ with $\lambda_j/\lambda_{j+1} < \epsilon$. Otherwise $\lambda_m \le \lambda\epsilon^{-(m-1)}$ would tend to zero. We have $\lambda_j \le \lambda\epsilon^{-(j-1)} \le \sqrt{\lambda}$.

The $\mathbb{Q}$-span U of the vectors of length at most λ_j has dimension j by the strict gap. Every vector of length less than λ_{j+1} lies in U. Multiplication by each member of a fixed integral basis of O_E preserves the lattice and increases lengths by a fixed factor at most the largest absolute value of its conjugates. It therefore sends the generators of U into U. This proves that U is an E-subspace. In particular its rank over E lies strictly between zero and r. By Proposition 107.2, $\lambda_{m-j} \le C/\lambda_{j+1}$. Vectors of length at most λ_{m-j} pair to zero under the integral trace form $B = \operatorname{Tr}_{E/\mathbb{Q}} h$ with all the short generators of U, since the absolute value of the pairing is at most $2C\lambda_j/\lambda_{j+1} < 1$. The trace orthogonal complement of U equals its hermitian orthogonal complement: pair with every scalar multiple of a given member of U and use nondegeneracy of the field trace as in Lemma 82.2. These vectors span that complement by its rational dimension $m - j$. Likewise the generators of U pair to zero under B with one another by $2\lambda_j^2 < 1$. This makes U totally isotropic for h. The two saturated sublattices therefore have last minima at most λ_j and C/λ_{j+1}, respectively. Apply Lemma 107.1 in their real spans to get the two bases and the required product bound. □

Theorem 108.2. *At the level $K(g)$ of Definition 78.1 choose one factor w of the marked group $\mathrm{GL}_r(\mathbf{F}_p)^f$ for $p = g_0$ or $p = g_1$, using the same finite frames for the indefinite space. For $\lambda_1(x)$ as in Lemma 108.1, let U_w be the reduction in this frame of $\Lambda_x \cap U$ at w. It has dimension $\dim_E U$. For every matrix in*

$$N_w = \{I + L : \operatorname{im} L \subset U_w,\ L|_{U_w} = 0\}$$

there is a rational unitary element γ with $c_f(x, \gamma) \in K^0$ reducing to that matrix at w and to identity at all other marked factors. It satisfies

$$\max_\tau \|c_\infty(x,\gamma)_\tau - I\|_{\mathrm{op}} \le C\,\lambda_1(x)^{1/(2m)}$$

for a constant independent of x and of the matrix.

Proof. Write a, v for vectors in the respective integral bases of Lemma 108.1. The lattices obtained by intersecting either E-subspace with Λ_x are saturated: if a nonzero integral multiple of a lattice vector is in the subspace then the vector belongs to it. Their rational spans are the full subspaces. A saturated subgroup of a finite rank free abelian group extends from a basis to an integral basis of the group: its finitely generated torsion-free quotient is free by integer elimination as in Lemma 12.1, so lift a quotient basis. Consequently after completion at p the intersections are direct summands over $\mathbb{Z}_p$ and equal the intersections with the completed rational subspaces. The idempotents of the split algebra $O_E \otimes \mathbb{Z}_p$ give the same assertion in each factor, as in Lemma 13.2. In particular their local reductions span the full subspaces in the residue lattice, with unchanged dimensions. The chosen factors have residue field $\mathbf{F}_p$ by Theorem 11.4. The conjugate place $\bar{w}$ is the partner of w in the unitary group. The hermitian pairing identifies that partner's reduction perfectly with the dual of the w-lattice because the frame has Gram matrix I. The reductions of the v's at $\bar{w}$ thus give all linear functionals on the w-space vanishing on U_w: they vanish there, span the orthogonal space, and its dimension is $r - \dim_E U$. The reductions of the a's span U_w. Their rank-one tensors span the maps from the quotient to U_w by expanding in bases of the range and of the dual of the quotient.

Choose $\theta \in 2O_E$ equal to one modulo w and zero at every other prime of E above $g_0 g_1$, including $\bar{w}$. All these primes are odd and distinct, so the Chinese remainder argument of Proposition 82.3 applies. Write the prescribed L as a sum of the tensors just described with coefficients in $\mathbf{F}_p$; lift the coefficients to integers l_{av} between 0 and $p - 1$. In the rational

space put

$$Cy = \theta \sum_{a,v} l_{av}\, a\, h(v, y), \qquad T = C - C^{\dagger}, \qquad \gamma = I + T + \tfrac{1}{2}T^2,$$

where the dagger denotes adjoint for h, conjugate linear in the first entry. Its image for C is in U, and $C|_U = 0$. The adjoint has image in $U^{\perp_h}$ and kills $U^{\perp_h}$, since C has image in U. In particular $C^2 = (C^{\dagger})^2 = C^{\dagger}C = 0$; hence $T^2 = -CC^{\dagger}$ and $T^3 = 0$. Here $C^{\dagger}C = 0$ uses $U \subset U^{\perp_h}$. As $T^{\dagger} = -T$, multiplication shows that $\gamma^{\dagger} = I - T + T^2/2 = \gamma^{-1}$. The integral pairing and the choice of θ show that both $C, C^{\dagger}$ send Λ_x into $2\Lambda_x$. Thus γ and its inverse preserve Λ_x. At w, C reduces to L and $C^{\dagger}$ to zero by the conjugate scalar. At the other chosen factors both vanish. The partner action at $\bar{w}$ is the forced inverse transpose and represents the same single unitary factor.

In each real frame, the operator norm of $a\, h(v, \cdot)$ is at most $\|a\|_x \|v\|_x$ since its local sign matrix is unitary. The same holds with the vectors exchanged for its adjoint. The fixed scalar, the number of terms and their coefficients are bounded in terms of the fixed field, primes and rank. Lemma 108.1 gives the required bound on T, and consequently on $T + T^2/2$ for small $\lambda_1(x)$. □

109. Rapid decay at a cuspidal marked place

Definition 109.1. At $K_0 = K(g)$ use $Y = \Gamma\backslash X$ and the real kernel e of Corollary 106.3. For an irreducible label c in one chosen factor w over a marked prime, let $E_{w,c}$ be its central projection in the finite right action of $\mathcal{G} = K^0/K(g)$, given by Lemma 22.1. Let $\mathcal{H}_{w,c}$ consist of the bounded continuous functions $F : Y \to \mathbb{C}$ satisfying $R(e)F = F$, $E_{w,c}F = F$, where $R(e)F(x) = \int_H e(t)F(xt)\,\mathrm{d}t$. The local character is the cuspidal character of Theorem 73.4. Set $h(x) = \max(1, \lambda_1(x)^{-1})$.

Lemma 109.2. *For $F \in \mathcal{H}_{w,c}$ and x as in Theorem 108.2,*

$$|F(x)| \le \max_{u \in N_w} |F(x b_u^{-1}) - F(x)|,$$

where $b_u = c_\infty(x, \gamma_u)$ are the small lifts in that theorem. For every real $A \ge 0$ there is a constant C_A, depending only on the fixed data, such that

$$|F(x)|\, h(x)^A \le C_A \|F\|_\infty \qquad (x \in Y).$$

Proof. In the finite irreducible module, averaging the representation over N_w is the orthogonal projection onto its invariants: left and right invariance of the finite average give its idempotence and inversion its self-adjointness. Its trace is zero, by the proper-parabolic cuspidality in Theorem 73.4 evaluated at the two identity diagonal blocks in Definition 72.1. Thus this average acts by zero on the label. In every copy of it the same holds, so the average kills $E_{w,c}$ by the finite decomposition in Lemma 22.1. The unipotent group for any conjugate subspace has the same property. In particular $\sum_{u \in N_w} F(xu) = 0$. By Theorem 108.2, the rational translation γ_u carries $x b_u^{-1}$ to xu in X: the finite frames differ by the prescribed reduction at w and by the identity at all other marked factors. Subtracting the common value $F(x)$ before averaging proves the inequality.

Theorem 99.1 supplies a continuous compactly supported β on H with $\beta * e = e$ by taking the product over the real places. Absolute integration of the bounded F gives $R(\beta)F = F$. For b in a fixed relatively compact identity neighborhood use

$$F(xb^{-1}) - F(x) = \int_H (\beta(bt) - \beta(t)) F(xt)\,\mathrm{d}t.$$

The support of every difference lies in a fixed compact set C, and its L^1-norm tends to zero as $b \to 1$ by uniform continuity there. For $t \in C$ we have $h(x) \le D h(xt)$ with $D \ge 1$ independent of x, t. Indeed the norm of every framed lattice vector changes by at most $\max_\tau \|t_\tau^{-1}\|_{\mathrm{op}}$ under the new frame. The same inequality holds for $h_T = \min(h, T)$, $T \ge 1$.

Put $M_T = \sup_x h_T(x)^A |F(x)|$, finite by boundedness. The integral for the difference at each b_u multiplied by $h_T(x)^A$ is at most $D^A M_T \int |\beta(b_u t) - \beta(t)| \, \mathrm{d}t$. The quantitative estimate of Theorem 108.2 makes this at most $M_T/2$ for every u once $h(x) > H_A$, uniformly in F, T. On $h(x) \le H_A$ the corresponding value $h_T(x)^A |F(x)|$ is at most $H_A^A \|F\|_\infty$. Taking the supremum gives $M_T \le H_A^A \|F\|_\infty$, since otherwise both bounds are strictly smaller than M_T. Let T increase to infinity. □

Theorem 109.3. *The space $\mathcal{H}_{w,c}$ is finite dimensional, and every member lies in $L^1(Y) \cap L^2(Y)$. As a subspace of $L^2(Y)$ it equals the range of $\mathcal{P} E_{w,c}$, where $\mathcal{P}$ is Proposition 99.2. For any finite set of such marked factors and cuspidal labels, the sum of these spaces is also finite dimensional.*

Proof. On $2^j \le h(x) < 2^{j+1}$ the volume is bounded by a constant times 2^{jK} by Proposition 107.2. Choose $A > K + 1$ in Lemma 109.2. Both the integral and the squared integral of the absolute value converge by summing the resulting geometric bounds. Continuous functions have unique representatives in L^2 within this space: every nonempty quotient open set contains a chart of positive measure by the construction in Lemma 21.4.

By the decay with $A = 1$, outside $\{h \le 2C_1 + 1\}$ each F is bounded by $\|F\|_\infty/2$. The latter set is covered by the image of a compact subset of X by Proposition 107.2. The integral difference in Lemma 109.2 gives $|F(xb) - F(x)| \le \|F\|_\infty/2$ for all b in a single open identity neighborhood in H. Since the finite coordinate is discrete, the compact set can be covered by finitely many right neighborhoods of this kind at points $x_1, \ldots, x_s$. If $F(x_i) = 0$ for all i, both on this compact image and beyond it the supremum is at most $\|F\|_\infty/2$. Thus evaluation at these finitely many points is injective, which proves finite dimension.

The finite right action preserves the quotient measure and commutes with $\mathcal{P}$, since it acts on the finite coordinate. The bounded continuous forms satisfy both projections, hence lie in their joint Hilbert range. Conversely, bounded square-integrable functions are dense in $L^2(Y)$ by truncating their moduli, using Lemma A.2. Project such a bounded function first by $E_{w,c}$ and then by $\mathcal{P}$. The result is bounded by the finite group sum and $\|e\|_1$. It has a continuous representative since it is fixed by $R(\beta)$. More explicitly, pull a bounded measurable representative back to X and integrate it against β on the right. Translation continuity in L^1 gives the continuous representative on each real component, and the finite coordinate is discrete. Changes on a quotient null set pull back to null sets in each component by unfolding with the countable rational group. These changes give zero after convolution at every point since the archimedean action is transitive on each component of X. The projection identities therefore hold for the continuous representative, first almost everywhere and then everywhere. Thus a dense subspace of the joint Hilbert range is in $\mathcal{H}_{w,c}$. This finite-dimensional subspace is closed (convergent vectors have convergent coordinates in an orthonormal basis). Equality follows, and a finite sum of the indicated spaces has dimension at most the sum of the separate dimensions. □

110. The trace on the marked space

Definition 110.1. Let $\mathcal{E}$ be the finite group algebra projection onto the sum of the types with at least one primitive label at a chosen factor over g_0, with no restriction on the other factors. Explicitly it is $1 - \prod_w (1 - \sum_c E_{w,c})$, where w ranges over the chosen factors over g_0 and c over the primitive labels there. Their characters are cuspidal by Theorem 73.4 and Definition 22.3. Let $\mathcal{H}$ be the range of $\mathcal{P}\mathcal{E}$ on $L^2(Y)$. Normalize the finite Haar measure by $\mathrm{vol}(K(g)) = 1$. The algebra of compactly supported bi-$K(g)$-invariant functions acts by $R(\phi)F(x) = \int \phi(t) F(xt) \, \mathrm{d}t$. It contains the finite group algebra by assigning $[s]$ the indicator of $sK(g)$ for $s \in K^0/K(g)$.

Proposition 110.2. *Use the arithmetic rows at sufficiently large scale and a finite test ϕ in Definition 110.1, with $\mathcal{E} * \phi = \phi * \mathcal{E} = \phi$. Let $u_1, \ldots, u_l$ be an orthonormal basis of $\mathcal{H}$.*

For every $x, y \in X$ the following identity holds:

$$\sum_{\gamma\in\Gamma} e\big(X_\infty(x)^{-1}\gamma X_\infty(y)\big)\phi\big(X_f(x)^{-1}\gamma X_f(y)\big) = \sum_{j=1}^{l}(R(\phi)u_j)(x)\,\overline{u_j(y)}.$$

The series on the left converges absolutely and locally uniformly.

Proof. The finite central projectors for different factors commute, and distinct labels at one factor give orthogonal projections by Lemma 22.1. Thus the range of the stated finite projection is the sum of their ranges, as can be seen by decomposing the identity into the products of each projection and its complement. Theorem 109.3 makes $\mathcal{H}$ finite dimensional, with continuous, bounded representatives. Any finite test is a finite combination of indicators of right level cosets, as its support is compact and the cosets are open and disjoint. Right integration defines a bounded operator on $L^2(Y)$: at the full finite group it is a weighted integral of right translations, preserving the quotient measure by translation invariance as in Lemma 93.1. In more detail, lift the functions to the full finite coordinate with right $K(g)$-invariance, using the product disintegration by $K(g)$ of mass one. Cauchy–Schwarz with weight $|\phi|$, followed by right translation in this full quotient, gives operator norm at most $\|\phi\|_1$. Right translation there preserves measure because the finite group is unimodular: the local conjugation Jacobian on its unitary Lie algebra has modulus one. Indeed the trace pairing $\mathrm{Tr}_{E_v/F_v}\,\mathrm{tr}(AB)$ on skew-hermitian tangent matrices is nonsingular. After splitting the quadratic algebra over a field extension, these matrices are identified with arbitrary matrices in the first factor and their negative adjoints in the second, so the pairing is twice the ordinary nonsingular matrix trace pairing. Nonsingularity descends by its determinant, and conjugation preserves the pairing, hence has determinant of absolute value one. In a local chart this gives its Haar scaling at identity by the change of variables in Lemma 95.2, using the rational Cayley chart over the local field and the additive Haar Jacobian of Lemma 20.2. Equivalently, on sufficiently small congruence neighborhoods the Cayley map is a conjugation-equivariant bijection with tangent lattices and preserves their relative indices. Almost all finite factors preserve their integral subgroup. Left quotienting by Γ preserves this translation identity by the fundamental-domain argument in Lemma 93.1. Associativity of R and the convolution product follows by left translation in the inner integral. The real and finite operators commute by separate variables. Thus $T = \mathcal{P}R(\phi)$ has $T = (\mathcal{P}\mathcal{E})T = T(\mathcal{P}\mathcal{E})$. On this finite-dimensional range it acts by $R(\phi)$, which preserves continuity since the finite test acts by finitely many right translates on level-invariant functions.

On compact subsets of $X \times X$, finite coordinates belong to finite sets. The finite test confines the entries of possible rational γ in a fixed rational basis to $c^{-1}O_E$ for one positive integer c, by the compact denominator argument of Lemma 91.2. If $m(X_\infty(x)^{-1}\gamma X_\infty(y)) \le R$, all their archimedean conjugates have size at most CR on these compact sets: every unitary factor has norm at least one and the outer frame matrices and their inverses range in a compact set. The lattice packing bound of Lemma 92.1 gives $O((1+R)^{2fr^2})$ possible matrices. The decay at $B = r^8 > 2fr^2$ by Corollary 96.4 and the parameter bounds in Theorem 92.2 now give absolute uniform convergence by dyadic shells. The sum is continuous and descends in both variables by reindexing rational matrices and using the two finite level invariances.

Call the resulting kernel $K(x, y)$. For each bounded function F supported on a compact part of Y, unfolding on Y gives $TF(x) = \int_Y K(x, y)F(y)\,\mathrm{d}y$: in the integral on X change variables from the translating elements to their action on x. The original integral of the absolute values is at most $\|e\|_1\|\phi\|_1\|F\|_\infty$, which justifies all rearrangements. The right side is continuous in x: a compact set in the quotient is covered by the images of finitely many relatively compact local charts, so local uniform convergence and local boundedness dominate on its support. The same operator has the finite expansion $TF = \sum_j R(\phi)u_j\,\langle u_j, F\rangle$ in L^2. Its two continuous representatives therefore agree everywhere. For each x this holds for

all bounded measurable tests on compact supports. Their integrals separate continuous functions of y: integrate the conjugate of a proposed difference on a small relatively compact open set where it is nonzero. The positive quotient measure there gives a contradiction. This proves the pointwise identity. □

Theorem 110.3. *Under the hypotheses of Proposition 110.2, suppose ϕ is supported on finite matrices having irreducible characteristic polynomial at some completion. For the groups and signs of Corollary 106.3 we have*

$$\operatorname{tr}_{\mathcal{H}} R(\phi) = \operatorname{tr}_{\mathcal{A}^+} R(\phi).$$

Here the definite action is the finite test action of Proposition 80.2.

Proof. Set $x = y$ in Proposition 110.2. The finite sum is integrable by Cauchy–Schwarz on Y; its integral is $\sum_j \langle u_j, R(\phi) u_j \rangle$. On the geometric side absolute termwise integrability on the diagonal follows from Theorem 92.2 and the support hypothesis. This is exactly the integral of Corollary 106.3, which identifies it with the definite trace. □

111. Hecke tests on the actual marked spaces

Lemma 111.1. *At a split unmarked place let L be the local field, O its discrete valuation ring with uniformizer ϖ, and $K_L = \mathrm{GL}_r(O)$. With K_L of measure one, the algebra of compactly supported bi-K_L-invariant functions on $\mathrm{GL}_r(L)$ is commutative and stable under $\phi^*(g) = \overline{\phi(g^{-1})}$. Their actions on both $\mathcal{H}$ and $\mathcal{E}\mathcal{A}^+$ are commuting normal operators, also across distinct such places.*

Proof. By the split unitary identification of Lemma 13.2, the group and level on one member of the local pair have the stated form. Every invertible matrix is in a double coset of a diagonal matrix with entries powers of ϖ. Multiply by a scalar to make the matrix integral, move an entry of minimum valuation to the first diagonal position by permutation of rows and columns, and scale its unit factor away. Its divisibility into every other entry permits integral elementary row and column elimination of its column and row. The remaining square block is nonsingular, so induction diagonalizes it. Remove the initial common scalar. Thus transpose preserves every double coset. It is a measure-preserving antiautomorphism: it interchanges left and right Haar measures, which agree by unimodularity as in Proposition 110.2, and preserves the mass-one subgroup. Its pullback reverses convolution by inversion of order in the group and a change of the integration variable. Since it fixes every bi-invariant function, convolution is commutative. Inversion and conjugation preserve compact support and level. Right integration on the full finite quotient and unitarity as in Proposition 110.2 give $R(\phi)^* = R(\phi^*)$, including for the definite space with its invariant norm from Theorem 78.3. Hence these operators commute with their adjoints. Distinct places act on separate coordinates and commute under iterated integration. Operations away from the marked primes commute with $\mathcal{E}$ and with the real projection, so preserve the two indicated spaces. □

Theorem 111.2. *Take the finite spaces $\mathcal{H}$ and $\mathcal{H}^+ = \mathcal{E}\mathcal{A}^+$ of Definition 110.1. Let $\mathcal{B}$ be any unital algebra of finite bi-$K(g)$ tests commuting with $\mathbb{C}[\mathcal{G}]$, supported at the chosen factors over g_0 on the kernels of reduction. Let (A_i) in $\mathcal{B}$ act normally and commute with one another and their adjoints on the two spaces. In particular one can use the spherical tests of Lemma 111.1 at split places away from the marked level. For every joint eigensystem and every good tuple over g_0, suppose one of its primitive coordinates has a cubic alternative occurring on neither joint eigenspace, in the sense of Definition 79.1. Then the characteristic polynomials and traces of each $B \in \mathcal{B}$ agree on $e_0^{\mathrm{good}}\mathcal{H}$ and $e_0^{\mathrm{good}}\mathcal{A}^+$. Thus the dimension d_0 of the good indefinite space satisfies*

$$(1 - b^{-8})|\mathcal{G}|\mathfrak{m}_r\mathcal{D} \le d_0 \le |\mathcal{G}|\mathfrak{m}_r\mathcal{D}, \qquad d_0 E_2 \le |\mathcal{G}|(C J_c)^2.$$

Proof. Fix an eigensystem ω of the direct sum. Finiteness was proved for these spaces in Theorems 109.3 and 78.3. Lemma 79.2 gives its projection on both spaces by a polynomial Π_ω in finitely many A_i. Evaluate this polynomial by convolution in $\mathcal{B}$. For every good tuple choose a missing primitive alternative P at its indicated coordinate w. Subtract from its tensor of label projectors the same tensor with its w-factor replaced by $E_{w,P}$, then sum these differences over all the good tuples. Denote the resulting finite group algebra element by T_ω. On both ω-spaces it acts by e_0^{good} since the subtracted tensor annihilates the space. By Corollary 75.3 the coefficients of each difference vanish unless the matrix at its indicated w has irreducible characteristic polynomial. Both tensors in the difference also carry a primitive label at w. Hence $\mathcal{E}T_\omega = T_\omega\mathcal{E} = T_\omega$ already in the finite group algebra, by the expression for $\mathcal{E}$.

For $B \in \mathcal{B}$, the test $\phi = B * \Pi_\omega * T_\omega$ consequently satisfies the two projection identities of Proposition 110.2. At every point of its support the matrix at some chosen g_0-coordinate remains integral with irreducible reduction: the $\mathcal{B}$-factors have identity reduction in all these coordinates, and convolution support lies in the product of the compact supports. In the finite union of cosets contributing to the product, use the coordinate supplied by the coefficient of T_ω. Irreducible reduction implies irreducibility over its local field by the monic factor argument in Proposition 80.2. Apply Theorem 110.3 to ϕ. Its definite trace is already its trace on $\mathcal{H}^+$ by the projection identities. As Π_ω commutes with the finite group algebra, on both spaces we have $\Pi_\omega T_\omega = \Pi_\omega e_0^{\mathrm{good}}$. Summing over ω gives equality of the traces of Be_0^{good}. The range of e_0^{good} lies in the range of $\mathcal{E}$ by Definition 22.3. Apply the trace equality also to the identity and to all positive powers of B. The power-sum argument of Theorem 79.3 gives equality of characteristic polynomials on the two good spaces. The dimension and energy estimates now follow from Theorem 78.3 and Corollary 78.4. □

112. Recovery of a primitive label

Lemma 112.1. *Let p be a prime, $r \geq 1$, and $A \geq 1$ with $p-1 \nmid A$. Let $T : \mathbf{F}_{p^r}^\times \to L^\times$ be an injective homomorphism into a field of characteristic zero. Write $\mu_A(L) = \{x \in L : x^A = 1\}$ and fix $\eta \in \mathbf{F}_p^\times$ with $\eta^A = 1$. For a primitive label c of degree r, with roots taken in $\mathbf{F}_{p^r}$, put*

$$Z_c = \{T(\eta x) : c(x) = 0\}.$$

The sets Z_c for different primitive c are disjoint. Their products satisfy

$$\Big(\prod_{z \in Z_c} z\Big)^A \neq 1.$$

Consequently, for a nonempty finite multiset B in $L^\times$, there is at most one primitive c satisfying both

$$B \subset Z_c \cup \mu_A(L), \qquad \frac{\prod_{\beta \in B} \beta}{\prod_{z \in Z_c} z} \in \mu_A(L).$$

The first inclusion refers to individual elements; the numerator of the quotient counts their multiplicities.

Proof. If two sets meet, equality $T(\eta x) = T(\eta x')$ implies $x = x'$, whose unique monic minimal polynomial over $\mathbf{F}_p$ is the label. By Lemma 22.6, the roots of c are the Frobenius conjugates of a generator of $\mathbf{F}_{p^r}^\times$, and their product generates $\mathbf{F}_p^\times$. The r images forming Z_c are distinct by injectivity. Thus their product is $T(\eta^r)$ times an element of exact order $p-1$. Raising to A kills the first factor and preserves a nontrivial value from the second since $p-1 \nmid A$. If B satisfies the product condition, its product also has nontrivial A-th power. At least one $\beta \in B$ lies outside $\mu_A(L)$. For every admissible label this same β must belong to Z_c. Disjointness proves uniqueness. □

Proposition 112.2. *For the actual finite spaces and commuting family in Theorem 111.2, take a marked $p = g_0$, $A = M_\omega$ of Definition 22.3 (the level exponent, independent of a Hecke eigensystem). Fix one of the two spaces, one joint eigensystem and one chosen coordinate over p. Suppose there are an algebraically closed characteristic-zero field L, an injective T and a scalar η as in Lemma 112.1, and a single invertible matrix M of positive size over L. Suppose for every primitive label c occurring there one can choose a list $\alpha_1, \dots, \alpha_a \in \mu_A(L)$, with repetitions allowed, such that*

$$\left(\prod_{z \in Z_c} (X - z) \prod_{j=1}^{a} (X - \alpha_j) \right)(M) = 0,$$

$$\det M = \prod_{z \in Z_c} z \prod_{j=1}^{a} \alpha_j.$$

Then at most one primitive label occurs there. It suffices also to allow a different matrix for each occurring label, provided all those matrices have the same characteristic polynomial. If the hypothesis holds for both spaces at all the indicated systems and coordinates, the dimension and trace conclusions of Theorem 111.2 follow.

Proof. Let B be the multiset of roots of the common characteristic polynomial. It is nonempty and contains no zero. For each root β, the matrix $M - \beta I$ is singular; choose a nonzero vector in its kernel. Applying the displayed annihilating polynomial of M to this vector proves that β is one of its roots. The product of the multiset is $\det M$ by the constant coefficient of $\det(XI - M)$. Hence both conditions of Lemma 112.1 hold. At the marked primes $p - 1 > A$ by Theorem 11.4 and its prime size bound, so $p - 1 \nmid A$. The lemma proves the assertion. Each good tuple has a primitive member, and by Lemma 22.6 it has three distinct cubic alternatives. The two spaces exclude at most two choices by uniqueness on each, giving a common absent alternative as required. □

Corollary 112.3. *In the arithmetic ranks $3 \mid r$, for every primitive label c and every $u \in \mathbf{F}_p^\times$ with $u^3 = 1$, the products of the roots of c and $c^{[u]}$ agree. Thus the determinant prescription in Proposition 112.2 can take the same value for all three distinct cubic alternatives with identical auxiliary scalars.*

Proof. The second list of roots, counted with multiplicity, is obtained by multiplying the first by u by Lemma 22.6. Their product is multiplied by $u^r = 1$. The same remains true after applying the fixed twist η, the homomorphism T and multiplication by the prescribed auxiliary scalars. □

113. A tame element at the marked prime

Lemma 113.1. *Let p be a prime and fix the valued algebraic closure K_p of Lemma 16.2, with valuation $v(p) = 1$. Every $\mathbb{Q}_p$-automorphism of K_p preserves its valuation. For $a \geq 1$, $L_a = \mathbb{Q}_p(\mu_{p^a - 1}) \subset K_p$ has degree a, valuation group $\mathbb{Z}$, and residue field of order p^a. Reduction identifies its indicated group of roots of unity with the whole multiplicative residue group. The field $U = \bigcup_{j \geq 1} L_{j!}$ has value group $\mathbb{Z}$ and the same residue field as K_p.*

Proof. On any finite extension in K_p, fix a $\mathbb{Q}_p$-basis and use the maximum of the absolute values of the coordinates. The given field absolute value is at most a constant times this maximum by the triangle inequality. It is continuous for the coordinate norm by this bound. It has positive minimum on the coordinate unit sphere, since that sphere is compact: a sequence in $\mathbb{Z}_p^m$ has a subsequence with all coordinates stabilizing modulo successively higher powers of p by diagonal extraction, and completeness gives convergence. The unit sphere is closed there. Scaling by a power of p now bounds the coordinate norm by a constant times the field absolute value. A $\mathbb{Q}_p$-automorphism of K_p acts by a bounded linear map on a finite normal extension containing any prescribed element, for instance the splitting

field of its minimal polynomial. Apply its norm bound to successive powers of that element and take real k-th roots as $k \to \infty$, and then apply the inverse automorphism. This proves valuation invariance.

Construct the field with p^a elements as in Lemma 73.2 and lift the monic minimal polynomial of a generator of its multiplicative group to a monic $P \in \mathbb{Z}_p[X]$. Its residue degree is a, since the generator and zero generate the entire field. It is irreducible over $\mathbb{Q}_p$ by the monic factor argument of Proposition 80.2. In the resulting field take the ring $O = \mathbb{Z}_p[X]/P$. Its coefficient module in the power basis is free and complete, and O/pO is the chosen field. Every element with nonzero reduction is a unit: lift an inverse modulo p and invert the remaining factor $1 - pz$ by its convergent geometric series. Every nonzero member of O is a power of p times such a unit by its basis coefficients. Under an embedding of the field in K_p all members of O have absolute value at most one: the power-basis generator satisfies the monic integral polynomial P, so a value of absolute value greater than one would give a unique largest term. Applying this also to inverses of units gives valuation zero on the units. The field is $O[1/p]$; it has the asserted value group and residue field.

Each nonzero residue class lifts to a root of $X^{p^a-1} - 1$. Its derivative there is a unit. Given a root modulo p^j, adding $p^j t$ changes the polynomial value modulo p^{j+1} by $p^j t$ times that derivative. This selects a unique digit at each step; completeness gives the root. All $p^a - 1$ roots are obtained since they are distinct. The lift of a multiplicative generator has powers of degrees $0, \dots, a-1$ linearly independent over $\mathbb{Q}_p$: scale a putative dependence to integral coefficients with one unit, and reduce modulo p. The residue of the generator has minimal polynomial of degree a, a contradiction. Thus the constructed field equals L_a. If $a \mid b$ then $p^a - 1 \mid p^b - 1$, so $L_a \subset L_b$. Any integral element of K_p has all conjugates integral by valuation invariance and the extension of embeddings from Lemma 16.2. The coefficients of its monic minimal polynomial are therefore in $\mathbb{Z}_p$. Its residue is algebraic over $\mathbf{F}_p$ and belongs to a finite field: adjoining it has dimension at most that polynomial's degree. The unique subfield of order p^a in the residue field, if present, consists of the roots of $X^{p^a} - X$ by Lemma 73.2. The residues of the $L_{j!}$ therefore contain all residues, and the value group of their union is still $\mathbb{Z}$. □

Theorem 113.2. *Let w be a chosen split place above $p = g_0$, embed E into its completion $\mathbb{Q}_p$, and take its algebraic closure $\overline{E}$ inside K_p. For $r \geq 1$ put $m = p^{r!} - 1$. Identify the field with $p^{r!}$ elements inside the residue field, and fix any injective homomorphism T from its multiplicative group to $\mathbb{C}^\times$. There are an automorphism s of $\overline{E}/E$ preserving the chosen valuation and acting trivially on its residue field, and $\xi \in \overline{E}$ with*

$$\xi^m = p, \qquad T\big(\overline{s(\xi)/\xi}\big) = \exp(2\pi i/m).$$

The bar here denotes reduction of a valuation unit. For every $h \mid m$, the ratio for $\xi^{m/h}$ has image $\exp(2\pi i/h)$ under T. In particular the same s tests all the degrees dividing $r!$.

Proof. By Lemma 113.1 the roots of $X^m - 1$ lie in U and reduce isomorphically to the residue group of order m. Its image under T consists of all complex m-th roots of unity by injectivity and their number. Choose ζ in this group of local roots reducing to the specified generator under T. Take an m-th root ξ of p in K_p. Its valuation is $1/m$. The powers $1, \xi, \dots, \xi^{m-1}$ are independent over U, since their nonzero multiples by coefficients in U have distinct fractional parts of their valuations. A sum with a unique term of least valuation cannot vanish by the ultrametric inequality. Hence $X^m - p$ is its minimal polynomial over U. The map fixing U and sending ξ to $\zeta\xi$ is an automorphism of $U(\xi)$ by evaluation at this other root, with inverse given by ζ^{-1}. Extend to K_p by the algebraic-closure argument of Lemma 16.2. It preserves the valuation by Lemma 113.1 and is trivial on residues because it fixes their representatives in U. The elements algebraic over E form a stable algebraically closed subfield, and $E \subset \mathbb{Q}_p$ is fixed. Restriction gives s with the claimed inertia action.

Taking (m/h)-th powers gives the smaller ratios. If $a \mid r!$ then $p^a - 1 \mid m$, which gives the last assertion. □

114. A logarithm estimate for compressed matrices

Definition 114.1. Use the ordinary hermitian inner product on $\mathbb{C}^n$. Let $\|X\|_2 = (\operatorname{tr} X^*X)^{1/2}$ denote the Hilbert–Schmidt norm in this section. If P is an orthogonal projection, $[P, X] = PX - XP$. For $\|X\|_{\mathrm{op}} < 1$ put

$$L(X) = \sum_{l \geq 1} \frac{(-1)^{l+1}}{l} X^l, \qquad l_P(X) = \operatorname{tr}_{\operatorname{im} P} L\big((PXP)|_{\operatorname{im} P}\big).$$

A zero-dimensional trace is zero. The compressed determinant $\det((I + PXP)|_{\operatorname{im} P})$ is $\exp l_P(X)$. Indeed any complex matrix can be put in upper triangular form by choosing an eigenline and inducting on the quotient. In that basis both the convergent series and its exponential have the scalar functions on the diagonal. Each scalar $\exp(\sum_{l \geq 1} (-1)^{l+1} z^l / l) = 1 + z$ for $|z| < 1$ by differentiation and evaluation at zero.

Lemma 114.2. *In the orthogonal decomposition of P and $I - P$, write $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. If $\|X\|_{\mathrm{op}} \leq \delta \leq 1/2$, then*

$$\Big| l_P(X) - \operatorname{tr}(PL(X)) - \tfrac{1}{2} \operatorname{tr}(BC) \Big| \leq 3\delta \|B\|_2 \|C\|_2.$$

Proof. Write $(X^l)_{ij}$ for the four blocks. Multiplying on the left gives $(X^l)_{11} = A(X^{l-1})_{11} + B(X^{l-1})_{21}$ and $(X^l)_{21} = C(X^{l-1})_{11} + D(X^{l-1})_{21}$. Iterating twice gives, for $l \geq 2$,

$$(X^l)_{11} - A^l = \sum_{i=0}^{l-2} \sum_{j=0}^{l-2-i} A^i B D^j C (X^{l-2-i-j})_{11}.$$

Each term has absolute trace at most $\|B\|_2 \|C\|_2 \delta^{l-2}$. Indeed $|\operatorname{tr}(YZ)| \leq \|Y\|_2 \|Z\|_2$ by entrywise Cauchy–Schwarz, and multiplication on either side by a matrix of operator norm s multiplies the Hilbert–Schmidt norm by at most s, by summing squared column lengths and then applying the same fact to adjoints. All four compressions have operator norm at most that of the original matrix. There are $l(l-1)/2$ terms. At $l = 2$ the difference is exactly BC. The logarithmic series in Definition 114.1 now give the quadratic term and a remainder bounded by $\|B\|_2 \|C\|_2$ times

$$\sum_{l \geq 3} \frac{l-1}{2} \delta^{l-2} = \frac{\delta(2-\delta)}{2(1-\delta)^2} \leq 3\delta.$$

The last bound follows by monotonicity of the nonnegative series after division by δ up to $\delta = 1/2$; at $\delta = 0$ the claim is immediate. □

Theorem 114.3. *Let P, Q be orthogonal projections and $\|X\|_{\mathrm{op}} \leq \delta \leq 1/2$. Suppose Y is a matrix and $c \geq 0$ satisfies*

$$\|P - Q\|_2,\ \|Y\|_2 \leq c, \quad \|[P, X] - Y\|_2 \leq \delta c, \quad \operatorname{tr}([P, Q]Y) = 0.$$

Then

$$\Big| l_P(X) - \operatorname{tr}(QL(X)) + \tfrac{1}{4} \operatorname{tr}(Y^2) \Big| \leq 12\delta c^2.$$

Proof. Put $J = [P, X]$, $E = Q - P$, and use the two blocks of Lemma 114.2. Since J has off-diagonal blocks $B, -C$, we have $\operatorname{tr}(J^2) = -2 \operatorname{tr}(BC)$ and $2\|B\|_2 \|C\|_2 \leq \|J\|_2^2$. The lemma therefore gives error at most $3\delta \|J\|_2^2 / 2$ with $Q = P$, $Y = J$. Also $\|J\|_2 \leq (1 + \delta)c \leq 3c/2$. Substituting Y^2 for J^2 in the trace costs at most $\delta c(\|J\|_2 + \|Y\|_2)/4$, by cyclicity and the same trace bound.

The diagonal blocks of E have signs $E_{11} \leq 0$, $E_{22} \geq 0$. Their absolute traces sum to $\operatorname{tr} P + \operatorname{tr} Q - 2 \operatorname{tr}(PQ) = \|P - Q\|_2^2$, using $P^2 = P$, $Q^2 = Q$. Since $\|L(X)\|_{\mathrm{op}} \leq 2\delta$ by its series, their contribution to $\operatorname{tr}(EL(X))$ has modulus at most $2\delta c^2$, by diagonalizing those

hermitian blocks. Write E_{off} for the remainder. Then $\operatorname{tr}(E_{\mathrm{off}}X) = -\operatorname{tr}([P,Q]J)$ by block multiplication. The trace hypothesis and $\|[P,Q]\|_2 \le \|E\|_2$ bound this by δc^2. For $l \ge 2$ the derivation rule gives $\|[P,X^l]\|_2 \le l\delta^{l-1}\|J\|_2$. The Hilbert–Schmidt norm of the off-diagonal blocks of a matrix equals that of its commutator with P. Thus the off-diagonal part of $L(X) - X$ has norm at most $2\delta\|J\|_2$, by the geometric series. Pairing with E_{off} costs at most $3\delta c^2$. Adding the bounds gives at most $(27/8 + 5/8 + 2 + 1 + 3)\delta c^2 \le 12\delta c^2$. □

115. The moving principal minors near identity

Lemma 115.1. *Let $g \in U(S)$, $X = g - I$, $\delta = \|X\|_2 < 10^{-3}$. Let B_l, B_u retain respectively the strictly lower and strictly upper entries of X whose two indices have the same sign, setting other entries to zero. Put $Y = (I - B_l)g(I - B_u) = I + Z$. Then $\|Z\|_2 \le 4\delta$ and $\psi(g) = \Psi_1(Y)$, in the notation of Definition 97.1. If R projects onto any initial interval of indices, including the entire set, then*

$$l_R(Z) = l_R(X).$$

Proof. Expanding the product gives

$$Z = X - B_l - B_u - B_lX - XB_u + B_lB_u + B_lXB_u.$$

The Hilbert–Schmidt product bounds of Lemma 114.2 give $\|Z\|_2 \le 3\delta + 3\delta^2 + \delta^3 \le 4\delta$. All the intermediate matrices $g, (I-B_l)g, Y$ and their adjoints are strictly accretive because their operator distances to identity are less than one. For any such admissible left multiplication by a unit lower triangular block matrix L, Theorem 97.3 gives $\Psi_1(LM) = \Psi_L(LM) = \Psi_1(M)$. In the second equality $m_\Lambda(Lv) = m_\Lambda(v)$ by the triangular minor law in that theorem. Also $\Psi_1(M^*) = \overline{\Psi_1(M)}$, since its numerator is $|m_\Lambda(v)|^2$ and principal determinants conjugate under adjoint. This proves invariance for right unit upper triangular block multiplication whenever the arguments are accretive, by applying left multiplication to the adjoint. Now $\psi(g) = \Psi_1(g)$ by Theorem 97.3. Apply the two invariances to the stated Y.

An initial principal block of Y is the product of the corresponding three blocks, by lower triangularity on the left and upper triangularity on the right. Its determinant is the same as that of the initial block of g. The identity also holds on replacing X, B_l, B_u simultaneously by sX, sB_l, sB_u, $0 \le s \le 1$. The norm bounds on the whole path justify the logarithms in Definition 114.1. Their exponentials agree, and their difference is continuous, takes values in $2\pi i\mathbb{Z}$ and is zero at $s = 0$. This proves the claimed equality of logarithms. □

Theorem 115.2. *In Lemma 115.1 let P be the coordinate projection onto a final interval and $k \in K$. There exists $r_P(k) \in \mathbb{C}$ with*

$$l_{kPk^*}(Z) = \operatorname{tr}(PL(X)) + \tfrac{1}{4}\|[P,X]\|_2^2 + r_P(k),$$
$$|r_P(k)| \le 600\delta(\|[P,X]\|_2^2 + \|[P,k]\|_2^2).$$

In particular the exponential of the first line is the actual principal determinant of k^*Yk on that interval.

Proof. Put $R = I - P$, $Q = kPk^*$, $A = [P,X]$, $a = \|A\|_2$ and $v = \|Q - P\|_2 = \|[P,k]\|_2$ by unitary invariance. Write A_o for its opposite-sign entries and $A_c = A - A_o$. Since P is diagonal, $[P,B_l]$ and $[P,B_u]$ are the corresponding masks of A. Their sum is A_c, the diagonal of A being zero. In the expression for Z in Lemma 115.1, each quadratic commutator has norm at most $2\delta a$ by the product rule, and the cubic one at most $3\delta^2 a$. Therefore

$$\|[P,Z] - A_o\|_2 \le 7\delta a, \qquad \|[Q,Z] - A_o\|_2 \le 8\delta(a+v).$$

The second inequality uses $\|Z\|_{\mathrm{op}} \le 4\delta$. The trace of $[Q,P]A_o$ is zero: Q, P preserve the sign blocks while A_o exchanges them. Use Theorem 114.3 with first projection Q, second P, comparison matrix A_o, size 4δ and $c = 2(a+v)$. It gives

$$l_Q(Z) = \operatorname{tr}(PL(Z)) - \tfrac{1}{4}\operatorname{tr}(A_o^2) + e_1, \qquad |e_1| \le 384\delta(a^2 + v^2).$$

Use the same theorem on Z with both projections R, comparison $-A_o$ and $c = 2a$; its error is at most $192\delta a^2$. On X with both projections R use $-A$, size δ and $c = a$; its error is at most $12\delta a^2$. By Lemma 115.1 the two resulting compressed logarithms coincide and the full logarithms have equal trace. Subtract these equations from the full traces and insert in the formula for $l_Q(Z)$. This gives

$$l_Q(Z) = \operatorname{tr}(PL(X)) + \tfrac{1}{4}\operatorname{tr}(A^2 - 2A_o^2) + e_2, \qquad |e_2| \le 588\delta(a^2 + v^2).$$

We have $\operatorname{tr}(A^2 - 2A_o^2) = \operatorname{tr}(SASA)$, since $SAS = A_c - A_o$ and mixed products have zero trace. The signed unitary identity $g^* = Sg^{-1}S$ gives exactly $A^* = Sg^{-1}Ag^{-1}S$, using $P^* = P$, $SP = PS$ and $[P, g^{-1}] = -g^{-1}[P, g]g^{-1}$. The geometric inverse series gives $\|g^{-1} - I\|_{\mathrm{op}} \le 2\delta$. Hence $\|A^* - SAS\|_2 \le (4\delta + 4\delta^2)a \le 5\delta a$. Taking its trace product with A costs at most $5\delta a^2$. After the factor $1/4$ this is within the stated total. Finally restriction of Y to the subspace after orthogonal compression by Q is unitarily identified with the compression of k^*Yk by P; their determinant is $\exp l_Q(Z)$ by Definition 114.1. □

116. Phase and quadratic decay near identity

Lemma 116.1. *For a real weight α put $E_\alpha(X) = \sum_{i,j} |\alpha_i - \alpha_j|\,|X_{ij}|^2$. If $i_1, \dots, i_n$ are in decreasing weight order and P_j is the projection onto the first j such coordinates, then*

$$E_\alpha(X) = \sum_{j<n} (\alpha_{i_j} - \alpha_{i_{j+1}}) \|[P_j, X]\|_2^2.$$

For integral α and ordinary unitary k, $|m_\alpha(k)|^2 \le \exp(-E_\alpha(k)/2)$.

Proof. The squared norm of the commutator counts $|X_{ab}|^2$ once when exactly one of a, b belongs to the projection. The weights for which this happens sum to $|\alpha_a - \alpha_b|$ by telescoping. For a fixed $P = P_j$ put $B = (I - P)kP$. On $\operatorname{im} P$ the squared absolute determinant of PkP is $\det(I - B^*B)$, since $k^*k = I$. The eigenvalues of B^*B are between zero and one, so $\det(I - B^*B) \le \exp(-\operatorname{tr} B^*B)$ by $1 - s \le e^{-s}$. Also $\|Pk(I - P)\|_2^2 = \|B\|_2^2$: both are $\operatorname{rank} P - \|PkP\|_2^2$ by unitarity. Thus $\operatorname{tr} B^*B = \|[P, k]\|_2^2/2$. Take the powers in Definition 96.1 and multiply, recalling $|\det k| = 1$. □

Theorem 116.2. *Use the real coefficient of Definition 96.1. Let $L_0 \ge 0$ bound every $2|t_i|$, and let T satisfy*

$$T \ge \max\{1, n^2, n^2 \log(n + L_0 + 1)\}.$$

Suppose $g \in U(S)$, $X = g - I$, $\delta = \|X\|_2$ and $\delta T \le 10^{-8}$. With $q_X = E_\mu(X)/4$ and $D_\Lambda = \operatorname{diag}(\Lambda_i)$, the phase $\exp(\operatorname{tr}(D_\Lambda L(X)))$ has modulus one and

$$\bigl|\psi(g)\exp\bigl(-\operatorname{tr}(D_\Lambda L(X))\bigr) - e^{-q_X}\bigr| \le \tfrac{1}{10}e^{-q_X} + e^{-100T}.$$

Proof. Lemma 115.1 writes $\psi(g) = \Psi_1(Y)$. In $m_{-\mu}(k^*Yk)$ the proper minors are the final intervals after position j, occurring with power $\mu_j - \mu_{j+1}$. The full determinant has power $-\mu_1$ and its logarithm is $\operatorname{tr} L(X)$ by that lemma. Insert Theorem 115.2 for each proper minor. Weighted summation of the projected traces gives $-\operatorname{tr}(\operatorname{diag}(\mu)L(X))$: the coefficient in position i is $-\mu_1 + \sum_{j<i}(\mu_j - \mu_{j+1}) = -\mu_i$. A final projection and its complementary initial projection have commutators of the same norm. Lemma 116.1 therefore sums the quadratic terms to q_X and bounds the sum of the errors by $600\delta(4q_X + E_\mu(k))$.

For $k \in K$, the inequality $E_\mu(k) \le E_\Lambda(k) + 2n^2$ follows from $|\mu_i - \Lambda_i| \le n$ and $\sum_{ij} |k_{ij}|^2 = n$ by unitarity. The same individual weight bound and $\|L(X)\|_{\mathrm{op}} \le 2\delta$ show that replacing the phase weight μ by Λ costs at most $2n^2\delta$ in the exponent. The signed unitary equation implies $L(X)^* = -SL(X)S$: near identity the convergent logarithm of the inverse is minus the logarithm by substitution of the scalar power series, and $g^* = Sg^{-1}S$. Each diagonal entry is therefore purely imaginary, proving the phase assertion. Writing $z(k) = E_\Lambda(k)/2$, the integrand in the formula for the phase-corrected ψ is

$$c_\Lambda^{-1}|m_\Lambda(k)|^2 e^{-q_X + r(k)}, \qquad |r(k)| \le 3000\delta\bigl(q_X + z(k) + n^2\bigr).$$

This follows directly by exponentiating the sums above, so no choice of a logarithm of the whole minor product is needed.

The compact dimension computation in Proposition 77.3 and Theorem 77.2 gives $c_\Lambda^{-1} = s_\Lambda$ for the two blocks. By the dimension formula of Proposition 76.2 and Lemma 76.3, each block contributes a product of factors $(\Lambda_i - \Lambda_j + d_{ij})/d_{ij}$, where $d_{ij} \ge 1$ is the distance of the two indices within the block. Since $\Lambda_i = t_i + (n+1)/2 - i - (\mu_i - \Lambda_i)$, every factor is at most $L_0 + 4n + 1$. There are at most $n^2/2$ factors and $L_0 + 4n + 1 \le (n + L_0 + 1)^3$. In particular $\log s_\Lambda \le 3T$.

Since $c_\Lambda^{-1}|m_\Lambda|^2 \,\mathrm{d}k$ is a probability, the portion of the integrated error from $|r(k)| \le 1/20$ is at most $e^{-q_X}/10$, using $|e^r - 1| \le e^{|r|} - 1 \le 1/19$ there. On its complement put $s = q_X + z(k)$. The bound for r and $\delta T \le 10^{-8}$ imply $s + n^2 > (5000/3)T$, hence $s > 1600T$. They also give $|r(k)| \le s/2$, using $n^2 \le T$ and $T \ge 1$. Lemma 116.1 bounds the absolute error integrand on this set by

$$2\exp\bigl(3T - s + |r(k)|\bigr) \le e^{-100T}.$$

Integrate against compact probability. Adding the two bounds proves the theorem, including $X = 0$ by the value at identity in Theorem 96.3. □

117. The common compact block of a period

Definition 117.1. Let H be a group, let S, S' be diagonal sign matrices of sizes $n, m \ge 1$, and let $j = (g, g') : H \longrightarrow U(S) \times U(S')$ be a homomorphism. Use parameters, weights and coefficients $(\mu, \Lambda, \psi), (\mu', \Lambda', \psi')$ as in Definition 96.1, with the moment hypothesis of Theorem 99.1 for both. Identify a coordinate space V isometrically with a coordinate subspace in each of the two positive sign spaces. Write $D = \operatorname{diag}\Lambda$, $D' = \operatorname{diag}\Lambda'$. Assume that D, D' vanish on the two copies of V, that $-D$ has spectrum in $[B, 3B]$ on $V^\perp$, for $B \ge 1$, and that $D' \ge 0$. If both signs occur in S', also assume $D' \le 3BI$. Suppose that $j(H)$ contains the simultaneous copies of every $u \in U(V)$, extended by identity on both complements. Set $Z_s = g(s) - I$, $Z'_s = g'(s) - I$. Require

$$\operatorname{tr} Z_s = \operatorname{tr} Z'_s, \qquad \phi(s) = \operatorname{tr}(DZ_s) - \operatorname{tr}(D'Z'_s) \in i\mathbb{R}, \qquad \phi(st) = \phi(s) + \phi(t).$$

Put $\theta(s) = \psi(g(s))\,\overline{\psi'(g'(s))}e^{-\phi(s)}$. Write $o(g)$ for the matrix obtained from g by retaining only entries between opposite signs. All norms with subscript 2 in this section are Hilbert–Schmidt norms.

Lemma 117.2. *In Definition 117.1, if* $\max(\|Z_s\|_{\mathrm{op}}, \|Z'_s\|_{\mathrm{op}}) < 1$, *then*

$$\operatorname{tr}\bigl(DL(Z_s)\bigr) - \operatorname{tr}\bigl(D'L(Z'_s)\bigr) = \phi(s), \qquad L(Z) = \sum_{b \ge 1} (-1)^{b-1} Z^b / b.$$

If s_0 maps to one of the simultaneous copies of $u \in U(V)$, then $\theta(s_0 s) = \theta(s)$ for all s.

Proof. Apply the linear functional $\ell(A, A') = \operatorname{tr}(DA) - \operatorname{tr}(D'A')$ to $(j(s) - I)^b$ in the direct product matrix algebra. For $b \ge 1$ the binomial coefficients with alternating signs sum to zero. As $\phi(s^a) = a\phi(s)$, the result equals $\phi(s)\sum_{a=0}^{b} (-1)^{b-a}\binom{b}{a} a$. This sum is the derivative at one of $(t-1)^b$, hence is one for $b = 1$ and zero for $b > 1$. Summing the absolutely convergent logarithm series proves the identity.

For $k = u \oplus I$ on either factor the nested minors in Definition 96.1 give $m_\Lambda(k) = 1$ with the weight of that factor. Indeed every proper minor with positive exponent contains either all or none of V. After including the full determinant, the total exponent of $\det u$ is its weight zero, by telescoping. If $V = 0$ the assertion holds directly. For a compact k the polar formula reduces to $m_\Lambda(k)$ by compact coefficient orthogonality in Proposition 77.3 applied to its two sign blocks. Thus the unit vector v_0 in Lemma 100.2 satisfies $\langle v_0, \pi(k)v_0\rangle = 1$, so $\|\pi(k)v_0 - v_0\|^2 = 0$. Left multiplication by k fixes the coefficient on the whole group. This holds in both factors. Finally $DZ_{s_0} = D'Z'_{s_0} = 0$, so $\phi(s_0 s) = \phi(s)$. □

Proposition 117.3. *Under Definition 117.1, suppose* $0 \le r \le 1/100$ *and* $\|o(g(s))\|_2^2 + \|o(g'(s))\|_2^2 \le r^2$. *Some* $s_0 \in H$ *in the simultaneous compact block satisfies*

$$\theta(s_0 s) = \theta(s), \qquad \max(\|Z_{s_0 s}\|_2, \|Z'_{s_0 s}\|_2) \le 4r.$$

Proof. Suppress s in the notation. Taking the i-th diagonal entry of $(I+Z)S(I+Z)^* = S$ gives

$$-2\operatorname{Re} Z_{ii} = \sum_j \varepsilon_i \varepsilon_j |Z_{ij}|^2.$$

Multiply by $-D_{ii}$ and add the corresponding primed identity multiplied by D'_{ii}. The left sides sum to zero by the imaginary phase condition. Moving the terms with opposite signs shows that the sum of all squared entries weighted by these nonnegative row weights is at most $6Br^2$: only opposite-sign entries occur on the right, each counted twice with row weight at most $3B$. In particular the bottom rows of Z relative to $V \oplus V^\perp$ have squared norm at most $6r^2$.

Write $g = \left(\begin{smallmatrix} A & F \\ C & H_0 \end{smallmatrix}\right)$ in this splitting and $S = I \oplus S_0$. Thus $\|C\|_2^2 + \|H_0 - I\|_2^2 \le 6r^2$. From $g^* S g = S$ we have $A^*A = I - C^* S_0 C$ and $A^*F = -C^* S_0 H_0$. If $V \ne 0$, A is invertible since $\|C\|_2^2 < 1$. Write $A = uP$ with $P = (A^*A)^{1/2}$ and u unitary, as in Theorem 96.3. Multiplication by $(I+P)^{-1}$ gives $\|P - I\|_2 \le \|C\|_2^2$, and

$$\|F\|_2 \le \frac{1+\sqrt{6}\, r}{\sqrt{1-6r^2}} \|C\|_2 \le \tfrac{11}{10}\|C\|_2.$$

The last inequality follows on squaring for $r \le 1/100$. Take s_0 over the simultaneous element u^*; for $V = 0$ take identity and omit the empty corners. For $\widetilde{g} = g(s_0 s)$, summing its four squared block differences gives $\|\widetilde{g} - I\|_2^2 \le (1 + (11/10)^2 + 6r^2)6r^2 \le 14r^2$. The trace equality at $s_0 s$ now controls its second factor. Indeed the unweighted row identity for either matrix h gives $\|h - I\|_2^2 = -2\operatorname{Re}\operatorname{tr}(h - I) + 2\|o(h)\|_2^2$. Multiplication by these compact blocks preserves the norm of $o(h)$. Thus $\|Z'_{s_0 s}\|_2^2 \le 14r^2 + 2r^2$. Lemma 117.2 gives the coefficient equality. □

118. Norm balls and decay on the period subgroup

Lemma 118.1. *Let* H *be a locally compact group with a right invariant Borel measure* ν, *finite on compact sets. Let* $g_i : H \to U(S_i)$ *be continuous homomorphisms of sizes* $n_i \ge 1$, $1 \le i \le l$. *Put* $d = 2\sum_i n_i^2$ *and* $C_\delta = \{s : \max_i \|g_i(s) - I\|_{\mathrm{op}} \le \delta\}$. *For* $y_i \in U(S_i)$, $R \ge 1$ *and* $0 < \delta \le 1$,

$$\nu\{s : \max_i \|g_i(s) y_i\|_{\mathrm{op}} \le R\} \le (1 + 2R^2/\delta)^d \, \nu(C_\delta).$$

If $l = 1$, $g = g_1$ *is proper,* $n = n_1$, *and* $c > 4n^2$, *then for every* $y \in U(S_1)$

$$\int_H \left(\frac{2}{\|g(s)y\|_{\mathrm{op}}} \right)^c \mathrm{d}\nu(s) \le 2^c 3^{2n^2} \frac{c}{c - 4n^2} \, \nu(C_1) < \infty.$$

Proof. Use the maximum of the operator norms in the real vector space of matrix tuples of dimension d. From the tuples in the sublevel set choose a maximal family having pairwise distances greater than δ/R. The balls of radius $\delta/(2R)$ around the chosen tuples have disjoint interiors and are contained in the ball of radius $R + \delta/(2R)$. Lebesgue volume scales as the d-th power of the radius by scalar change of variables. Thus any such family has size at most $(1 + 2R^2/\delta)^d$; successive choices stop after finitely many steps. Choose a preimage t of each center. Every s in the sublevel set has a center within δ/R. For every i,

$$g_i(st^{-1}) - I = (g_i(s)y_i - g_i(t)y_i)(g_i(t)y_i)^{-1}.$$

A signed unitary matrix h satisfies $h^{-1} = S_i h^* S_i$, so $\|h^{-1}\|_{\mathrm{op}} = \|h\|_{\mathrm{op}}$. The right side has norm at most δ, giving a cover by the right translates $C_\delta t$. Right invariance gives the first bound, also when the sublevel set is empty.

In positive dimension $\|h\|_{\mathrm{op}} \ge 1$ follows by applying the preceding inverse identity to $hh^{-1} = I$. For $t \ge 1$ write $t^{-c} = \int_t^\infty cR^{-c-1}\,\mathrm{d}R$. The integrals on H and in R can be interchanged here. Indeed put $u = R^{-c}$ and approximate t^{-c} from below by $2^{-j}\sum_{a=1}^{2^j} \mathbf{1}_{\{t^{-c}\ge a2^{-j}\}}$. These functions increase to t^{-c}; the corresponding step functions of u increase to the sublevel indicator except at its endpoint, or equivalently give the same integral there. Lemma A.2 gives the identity by passage to the limit (at atoms the exceptional u's are countable on each finite sublevel set). The first bound with $\delta = 1$ gives the result on integrating $(3R^2)^{2n^2}cR^{-c-1}$ from one to infinity. The inverse image C_1 is compact by properness since the target ball is closed and bounded in a finite-dimensional matrix space. □

Lemma 118.2. *In Definition 96.1 let $h \ge 0$ and assume $\mu_j - \mu_{j+1} \ge h$ at every change of sign. Then for every $g \in U(S)$*

$$|\psi(g)| \le (1 + \|o(g)\|_2^2)^{-h/4}.$$

Proof. Use $g = kP$ from Theorem 96.3. Let $a = g - o(g)$ and $O = o(g)$. The diagonal sign blocks of $g^*Sg = S$, after multiplication by their signs, give $a^*a = I + O^*O$. The determinant on the right is the product of $1 + s_j$ over the nonnegative eigenvalues of O^*O. It is at least $1 + \sum_j s_j = 1 + \|O\|_2^2$. Since k preserves the two sign subspaces, $|\det a|$ is the product of the two sign-block determinants of P. These determinants coincide by Lemma 96.2; denote their common value by d_+. Hence $d_+^4 \ge 1 + \|O\|_2^2$. The same lemma bounds each denominator in the polar integral below by d_+^h, unchanged under conjugation of P by K. Cauchy–Schwarz bounds the integral of the absolute numerator by c_Λ as in Theorem 96.3. This proves the assertion including the one-sign case. □

119. A lower bound for a real period

Theorem 119.1. *Give the group H of Definition 117.1 a locally compact topology and a right invariant Borel measure ν, finite on compact sets and positive on nonempty open sets. Assume g, g' are continuous and g is proper. Let $\beta \ge 10^{12}$ satisfy*

$$n + m + \max\{\max_i 2|t_i|, \max_j 2|t'_j|\} \le \beta, \qquad \|g'(s)\|_{\mathrm{op}} \le \beta + 1 \quad \text{when } \|g(s) - I\|_{\mathrm{op}} \le 1.$$

Put $N = (n+m)^2$, $T = N\log\beta$, and suppose that the successive μ-gaps at sign changes in each factor are at least $h = 10^{22}T^3$. With $C = \nu\{s : \|g(s) - I\|_{\mathrm{op}} \le 1\}$, one has

$$\mathrm{Re}\int_H \theta(s)\,\mathrm{d}\nu(s) \;\ge\; \beta^{-15N}C > 0.$$

If $g'(H)$ is ordinary unitary, the exponent $15N$ can be replaced by $11N$.

Proof. The first-factor decay of Theorem 96.3 and $|\psi'| \le 1$ give, for $0 \le a \le h$,

$$|\theta(s)| \le \min\{1, (2/\|g(s)\|_{\mathrm{op}})^a\}.$$

For $\|g(s)\|_{\mathrm{op}} \ge 2$ this follows by decreasing the decay exponent to a, and otherwise the bound by one suffices. Lemma 118.1 with $a = 8N > 4n^2$ gives integrability of the continuous function θ. It also gives $C < \infty$, and the interior of this set contains identity, so $C > 0$.

Write $r = 10^{-9}/T$ and $e_0 = \exp(-100T)$. If $\|o(g(s))\|_2^2 + \|o(g'(s))\|_2^2 \le r^2$, use the representative $s_0 s$ of Proposition 117.3. Each matrix difference then has Hilbert–Schmidt norm at most $4r$, so its product with T is at most 10^{-8}. The size of T meets the lower bounds in Theorem 116.2 for both factors. By that theorem and Lemma 117.2, $\theta(s)$ is the product of a number within $A/10 + e_0$ of A and the conjugate of a number within $A'/10 + e_0$ of A', where $A, A' \in [0,1]$ are the corresponding positive Gaussians. Here the two logarithmic phases cancel $e^{-\phi(s_0 s)}$ exactly. Expansion bounds $|\theta(s) - AA'|$ by $0.21AA' + 4e_0$.

In particular $\operatorname{Re}\theta(s) \ge |\theta(s)|/2 - 6e_0$. On the complementary region Lemma 118.2 in the two factors gives

$$|\theta(s)| \le (1+r^2)^{-h/4} \le \exp(-hr^2/8) \le e_0.$$

We used $\log(1+r^2) \ge r^2/2$, obtained by integration of $1/(1+t)$ for $0 \le t \le r^2 \le 1$.

The positive part of $|\theta|/2 - \operatorname{Re}\theta$ is thus bounded by $6e_0$ everywhere and also by $3|\theta|/2$. Taking the geometric mean and using the decay estimate with $a = 16N$ bounds it by

$$3e^{-50T}(2/\|g(s)\|_{\mathrm{op}})^{8N}.$$

The last inequality of Lemma 118.1 bounds its integral by $3\beta^{-50N}16^{8N}C$. Indeed $8N/(8N - 4n^2) \le 2$ and $2^{8N}3^{2n^2} \cdot 2 \le 16^{8N}$.

Let U_0 consist of the elements for which both operator norms of the differences from identity are at most $\delta = \beta^{-4}$. Their Hilbert–Schmidt norms are at most $\sqrt{\beta}\,\delta$, since the squared norm is the sum of the squared singular values. Multiplication by $T \le \beta^2 \log\beta$ therefore gives at most 10^{-8}; indeed $\beta^{-3/2}\log\beta$ is decreasing on this range. Each weight difference $|\mu_i - \mu_j|$ and its primed analogue is at most 2β by Definition 96.1. Thus the two Gaussian exponents $E_\mu(Z_s)/4, E_{\mu'}(Z'_s)/4$ are each at most β^{-6}. By Theorem 116.2 each coefficient has modulus at least $0.9\exp(-\beta^{-6}) - e_0 > 3/4$. It follows that $|\theta| \ge 1/2$ on U_0.

Apply Lemma 118.1 to the pair with $R = \beta + 1$, both translating matrices equal to identity. The set defining C lies in this sublevel set. Its covering factor is at most β^{14N}, since $1 + 2(\beta+1)^2\beta^4 \le \beta^7$ and $d \le 2N$. If the second image is ordinary unitary, take $R = 2$ and use $1 + 8\beta^4 \le \beta^5$, giving β^{10N}. Write $u = 14$ or 10, respectively. Then

$$\operatorname{Re}\int_H \theta \,\mathrm{d}\nu \ge \tfrac{1}{4}\beta^{-uN}C - 3\beta^{-50N}16^{8N}C \ge \beta^{-(u+1)N}C.$$

For the final inequality $16^8 < 10^{10} \le \beta$, $3\beta^{-49N} \le \beta^{-uN}/8$, and $\beta^N \ge 8$. This proves both bounds. □

120. A signed interpolation basis at the Bessel point

Lemma 120.1. *Fix $\nu \ge 0$, put $l = 2\nu + 2$, and choose decreasing integers $0 = x_0 > x_1 > \cdots > x_l$. On $\mathbb{C}^l$, numbered $1, \dots, l$, let $S = \operatorname{diag}((-1)^{i-1})$ and $[v, w] = v^* S w$. Put*

$$c_i^2 = \prod_{\substack{0 \le j \le l \\ j \ne i}} |x_i - x_j|, \quad c_i > 0, \qquad P(z) = \prod_{i=1}^{l}(1 - x_i z)^{1/2} = \sum_{a \ge 0} p_a z^a,$$

taking the square roots equal to one at zero. Define real vectors $v_0, v_1, \dots, v_{l-1}$ by

$$(v_0)_i = c_0/c_i, \qquad (v_j)_i = c_i^{-1}\sum_{a=0}^{j-1} p_a x_i^{j-a} \quad (1 \le j < l).$$

Then $[v_0, v_0] = 1$, $[v_0, v_j] = 0$ for $j \ge 1$, and $[v_j, v_k] = -\mathbf{1}_{j+k=l}$ for $1 \le j, k < l$. For $T_x = \operatorname{diag}(x_i)$ one has

$$T_x v_0 = c_0 v_1, \qquad T_x v_j = v_{j+1} - p_j v_1 \quad (1 \le j < l-1).$$

Proof. Let $F(t) = \prod_{i=0}^{l}(t - x_i)$. Multiplication by $F(t)$ shows that $\sum_i (F'(x_i)(t - x_i))^{-1} = 1/F(t)$: the difference of the numerators has degree at most l and vanishes at all $l+1$ nodes. A nonzero polynomial of degree at most l has at most l roots by successive division by its linear factors. Comparing the coefficients of t^{-d-1} at infinity gives for $d \ge 0$, with $0^0 = 1$,

$$\sum_{i=0}^{l} \frac{x_i^d}{F'(x_i)} = [z^{d-l}]\prod_{i=1}^{l}(1 - x_i z)^{-1}.$$

A coefficient of negative index is zero. Since $F'(x_i) = (-1)^i c_i^2$, the signed moments from indices $i \ge 1$ are $0^d/c_0^2$ minus the coefficient on the right. This proves the assertions involving v_0, since $p_0 = 1$ and $1 \le j < l$. For $j, k \ge 1$ apply the formula to each monomial of degree

$j+k-a-b$ in their inner product. All those degrees are positive. If $a \geq j$ or $b \geq k$ and the other index is nonnegative, then $j+k-a-b-l<0$, since $j,k<l$. Consequently the sum for $-[v_j, v_k]$ equals the unrestricted coefficient of degree $j+k-l$ in

$$P(z)^2 \prod_{i=1}^{l} (1-x_i z)^{-1} = 1.$$

The identities for T_x follow by multiplying each entry by x_i and isolating the last summand of v_{j+1}. □

Proposition 120.2. *Let X be a coordinate space of dimension $k-1$, $k \geq 1$, with standard positive form. Extend the space of Lemma 120.1 by X, and write $S_1 = I_X \oplus S$. In the target use the complex space of Definition 15.1 in real-place coordinates with $[z,z]=-1$. Its middle positive space is $X \oplus \mathbb{C}u$ with unit u. Denote its Gram matrix by J, with hyperbolic pairs e_i, e_i^*, $1 \leq i \leq \nu$. Let $f_i = e_i$ for $1 \leq i \leq \nu$ and $f_{\nu+1} = z$. There exists a specified isometry Q from S_1 to J which is identity on X and satisfies*

$$Qv_0 = u, \qquad Qv_j = f_j \ (1 \leq j \leq \nu+1), \qquad Qv_{l-j} = -e_j^* \ (1 \leq j \leq \nu).$$

For $b \geq 8(\nu+1)$ and $K_0 = \max_i |x_i| \leq 2b^2$,

$$1 + \max(\|Q\|_{\mathrm{op}}, \|Q^{-1}\|_{\mathrm{op}}) \ \leq \ b^{5(\nu+1)}.$$

All operator norms here use the standard positive Euclidean norm on each coordinate space.

Proof. The Gram matrix computed in Lemma 120.1 is invertible, so the l prescribed vectors form a basis. The target vectors have the same Gram matrix, including the minus sign on each dual and the negative square of z. Consequently the stated prescriptions uniquely define Q and $Q^*JQ = S_1$.

Each $c_i \geq 1$ by integrality and distinctness, and $c_0 \leq K_0^{l/2}$. On $|z| = 1/(2K_0)$ the product $P(z)$ has modulus at most $(3/2)^{l/2}$. Indeed each factor has modulus at most $(3/2)^{1/2}$ and its Taylor series there is obtained from the branch taking value one at zero. This branch exists on $|x_i z| < 1$, for instance by exponentiating half the convergent logarithm series of $1-x_i z$. For $j \geq 1$, $c_i (v_j)_i$ is the coefficient of z^{j-1} in $x_i P(z)/(1-x_i z)$. The coefficient estimate in Lemma A.3 bounds $|(v_j)_i|$ by $(2K_0)^j (3/2)^{\nu+1}$. The same uniform upper bound at $j = l-1$ covers v_0. The norm of the l-dimensional block of Q^{-1} is at most l times the maximum absolute entry by Cauchy–Schwarz on the rows. Thus that norm is at most

$$l(4b^2)^{2\nu+1}(3/2)^{\nu+1} = 12(\nu+1)24^{\nu} b^{4\nu+2}.$$

For $\nu \geq 2$ we have $24 \leq b$; for $\nu = 0, 1$ one has $24^{\nu} \leq (3/2) b^{\nu}$. In all cases the last expression is at most $(9/4) b^{5\nu+3} \leq b^{5(\nu+1)} - 1$, using $b \geq 8$. The identity block on X obeys the same bound. Finally $Q^{-1} = S_1 Q^* J$ and S_1, J are ordinary unitary, so the two norms in the statement are equal. □

121. The character on the real Bessel subgroup

Definition 121.1. Use the matrix J, vectors f_j and positive middle space $P_0 = X \oplus \mathbb{C}u$ of Proposition 120.2. Use the successive blocks

$$e_1, \ldots, e_\nu, \quad P_0 \oplus \mathbb{C}z, \quad e_\nu^*, \ldots, e_1^*.$$

Let $\mathcal{U} \subset U(J)$ consist of the upper block triangular elements with identity diagonal blocks, the real group of Definition 20.3 in the scaled basis. Embed $K_0 = U(P_0)$ in $U(J)$ by identity on all the other coordinates, writing $\iota(k)$ for the image. Set $H = \{(\iota(k)n, k) : n \in \mathcal{U},\ k \in K_0\}$. For a matrix $y \in U(J)$ put $\sigma(y) = \sum_{j=1}^{\nu} f_j^* y f_{j+1}$, where a star on a column in a matrix product denotes its ordinary conjugate transpose. Use product measure on H in the coordinates (n,k), with $\mathrm{d}n$ as in Proposition 20.5 at this place and compact dual measure $\mathrm{d}k$ on K_0 from Theorem 95.3.

Lemma 121.2. *The group H is closed, its first projection is proper, and its measure is right invariant, finite on compact sets and positive on every nonempty open set. On its first projection σ is additive under multiplication. For $A = \sum_{j=1}^{\nu} f_{j+1} f_j^*$ and $C = A + JA^*J$, every $(y,k) \in H$ satisfies*

$$\operatorname{tr} C(y - I) = \sigma(y) - \overline{\sigma(y)}, \qquad \operatorname{tr}(y - I) = \operatorname{tr}(k - I).$$

Proof. The embedding of the compact group normalizes $\mathcal{U}$ by block diagonal conjugation. Hence $(n,k) \mapsto (\iota(k)n, k)$ is a continuous bijection onto a subgroup with continuous inverse. Its first projection is the closed matrix group specified by upper block triangularity, identity on the one-dimensional diagonal end blocks and middle diagonal block $k \oplus 1_z$, $k \in K_0$. The second coordinate is read continuously from that middle block. Thus the first projection is a homeomorphism onto a closed subgroup, hence is proper. Local finiteness and positivity follow from the smooth positive factors of the product measure. Right translation by $(\iota(k_0)n_0, k_0)$ sends (n,k) to $(\iota(k_0)^{-1} n \iota(k_0) n_0, k k_0)$. Conjugation here preserves the unipotent measure, as each matrix row coordinate of Lemma 20.4 changes by a unitary on the middle positive coordinates and its central imaginary coordinate is fixed. The factor measures are right invariant by Proposition 20.5 and Theorem 95.3.

For y_1, y_2 in the first projection, an entry in row e_j and the next flag column f_{j+1} of $y_1 y_2$ is the sum of their two corresponding entries. Only the diagonal at e_j and the entry in column f_{j+1} can contribute. At $f_{\nu+1} = z$ this uses that each middle block fixes z. Thus $\sigma(y^{-1}) = -\sigma(y)$, also for $\nu = 0$ by the empty sum. Now $\operatorname{tr}(Ay) = \sigma(y)$. Since $y^{-1} = Jy^*J$, cyclicity of the trace gives $\operatorname{tr}(JA^*Jy) = \overline{\operatorname{tr}(Ay^{-1})}$. Subtract the identity contribution, which is zero. The second trace formula follows by adding the traces of the diagonal blocks. □

Theorem 121.3. *Suppose a weight D of Definition 96.1 on the S_1-space vanishes on X. Assume its remaining diagonal entries $\lambda_1 > \cdots > \lambda_l$, in the alternating signs of Lemma 120.1, belong to $[-3B, -B]$ with $B \geq 1$. Fix an integer $a_0 \geq 0$ and on P_0 take $D' = a_0 uu^*$. Choose Q of Proposition 120.2 with $x_i = \lambda_i - a_0$. Define*

$$j(y,k) = (Q^{-1} y Q, k), \qquad \phi(y,k) = \sigma(y) - \overline{\sigma(y)}.$$

Then the trace, weight and compact block conditions of Definition 117.1 hold. If the parameter and gap bounds of Theorem 119.1 hold for these weights and β, its lower bound with exponent $11(r+k)^2$, $r = k + 2\nu + 1$, holds for this actual graph.

Proof. The matrices $Q^{-1}yQ$ preserve S_1. Let $D_0 = QDQ^{-1}$, which satisfies $D_0^* J = J D_0$. Since D restricts to $a_0 I + T_x$ on the alternating block, the transitions in Lemma 120.1 give, for $\nu \geq 1$,

$$D_0 e_j = a_0 e_j + f_{j+1} - p_j f_1 \quad (1 \leq j \leq \nu).$$

For every ν one also has $D_0 u = a_0 u + c_0 f_1$, and $D_0 X = 0$; when $\nu = 0$, $f_1 = z$. In Lemma 121.2, C is J-self-adjoint and $Ce_j = f_{j+1}$, since $Je_j = e_j^*$ is perpendicular in Euclidean coordinates to all the frame vectors. Hence $D_0 - C$ preserves every initial isotropic flag span and, by J-self-adjointness, its J-orthogonal complement. These are precisely the successive proper partial sums of the ordered blocks in Definition 121.1. Thus $D_0 - C$ is upper block triangular, also for $\nu = 0$ where there is a single block. The product with $y - \iota(k)$ is strictly upper block triangular, with trace zero.

The ordinary compression of $D_0 - C$ to P_0 equals D'. Indeed C vanishes on P_0, and f_1 has zero projection there. As $\iota(k) - I$ is supported on P_0, the preceding trace calculation and Lemma 121.2 give

$$\operatorname{tr}\big(D(Q^{-1}yQ - I)\big) = \operatorname{tr}(D'(k - I)) + \sigma(y) - \overline{\sigma(y)}.$$

The unweighted traces agree by the same lemma. This phase is imaginary and additive. An ordinary unitary on X, extended by identity, commutes with the change of basis on its complement and thus occurs in both coordinates of j at $n = I$. The other weight bounds follow from the assumptions and $D' \geq 0$ on the one-sign second factor. Continuity and

properness follow from Lemma 121.2 and conjugation by Q. The second matrix always has operator norm one, so all group and measure hypotheses of Theorem 119.1 are satisfied. □

122. A Bessel function at each real place

Proposition 122.1. *Use the sufficiently large scales of Proposition 10.5, a place $\tau \in \Phi$ and the row of Definition 56.1. Write $a_\tau = a$ for $\tau \in A$ and $a_\tau = 0$ otherwise. Use the positive signs on the first $k-1$ entries and alternating signs beginning with positive on the remaining $m = 2\nu + 2$. Put $\beta = b^3$. For this row and the compact $U(k)$-weight $D' = a_\tau uu^*$, the hypotheses of Theorem 121.3 and all bounds of Theorem 119.1 hold, uniformly on the interval $t \le v \le t^2$ of Proposition 12.2. Moreover $3B + a_\tau \le 2b^2$ with $B = W_\tau$.*

Proof. Number the first r coordinates starting at zero in this calculation. For $0 \le i < k-1$ there are i previous positive signs out of $k+\nu$ positive signs in total. In Definition 96.1 the compact correction subtracted from μ_i is $k+\nu-1-2i$; this equals $(k/2-1-i)+(r-1)/2-i = \mu_i$ by $r = k+2\nu+1$. Hence $\Lambda_i = 0$. For all other indices $|\Lambda_i - t_i| \le 3r/2$ by the same formula. Their consecutive t_i decrease by $2r^8$, so the corresponding Λ_i decrease strictly. They lie between $-W_\tau - m\,2r^8 - 3r/2$ and $-W_\tau - 2r^8 + 3r/2$. By the bounds in Proposition 56.2, $W_\tau \ge w(\tau) \ge xb\exp(-t^2)$, which exceeds every fixed power of x eventually since $t^2 = o(\log b)$. As $mr^8 = O(x^{41})$, these entries lie in $[-3B, -B]$.

Proposition 12.2 and Definition 56.1 give $B \le 2b + N_1$, $a_\tau \le a \le 2xb$ and $N_1 = O_M(x^{11}(1+\log x))$. Thus $3B + a_\tau \le 2b^2$ and $8(\nu+1) \le b$ eventually. In the compact factor take $t'_i = \Lambda'_i + (k+1)/2 - i$ in decreasing order of Λ', numbered from one. These entries are distinct and yield D', since with one sign the compact correction is $k+1-2i$. The bounds above imply $r + k + \max(2\max|t_i|, 2\max|t'_i|) \le b^3$. There are no sign changes in the compact factor. In the first factor every sign-change μ-gap is at least r^8 eventually by Corollary 96.4. Furthermore

$$\frac{r^7}{((r+k)^2\log(b^3))^3} \;\ge\; \frac{r}{12^3 x^{3\epsilon}} \;\longrightarrow\; \infty$$

by Proposition 10.5. These facts give the required gap and moment bounds in both factors; the second group is compact. □

Theorem 122.2. *With the data of Proposition 122.1, let $G_1 = U(J)$, and choose an ordinary unitary isometry p from S_1 to J. Put $K(y) = \psi(p^{-1}yp)$, $e_J(y) = d\,\overline{K(y)}$ with $d = \mathcal{D}_\tau$ from Theorem 106.2. Give G_1 the measure transported from h_{S_1} by p. Write $\pi = \mathrm{Sym}^{a_\tau} P_0$ for the unitary representation on symmetric tensors, also writing $\pi(k)$ for its action. There exist $x \in G_1$ and a bounded continuous function $W : G_1 \to \pi$ with $1 + \|x\|_{\mathrm{op}} \le b^{5(\nu+1)}$ and $W(x) \ne 0$. For every $(s,k) \in H$ and $y \in G_1$ one has*

$$W(sy) = \exp(\sigma(s) - \overline{\sigma(s)})\,\pi(k)W(y),$$

$$W(y) = \int_{G_1} e_J(a)W(ya)\,\mathrm{d}a, \qquad \|W(y)\| \le \int_H |K(x^{-1}sy)|\,\mathrm{d}\nu_H(s,k).$$

Here ν_H is the graph measure of Definition 121.1. With $c = 8r^2+1$ there is also the uniform bound

$$\int_H (2/\|x^{-1}sy\|_{\mathrm{op}})^c\,\mathrm{d}\nu_H(s,k) \le 16^c b^{33(r+k)^2}\|W(x)\|.$$

Proof. Such a p is obtained by taking the orthonormal sums and differences of each pair e_j, e_j^*, together with the middle coordinates and a permutation matching the signs. Put $x = Qp^{-1}$ with Q from Theorem 121.3, giving the stated size by Proposition 120.2. Equip the tensor power of P_0 with its product inner product and take the subspace fixed by permutations. Its compact action is unitary. The unit vector $v_0 = u^{\otimes a_\tau}$ has coefficient $\langle v_0, \pi(k)v_0\rangle = (u^*ku)^{a_\tau}$, where tensor power zero is the trivial line. This equals $m_{\Lambda'}(k) = \psi'(k)$ by the minor formula of Definition 96.1 and compact orthogonality as in Lemma 117.2. Its conjugate is the

coefficient at k^{-1}. This compact action pairs with the conjugate symmetric power specified in Definition 13.3: the ordinary tensor inner product gives a linear pairing with the conjugate space, invariant under the two actions.

For $s_H = (s, k)$, abbreviate $\phi(s_H) = \sigma(s) - \overline{\sigma(s)}$. Define

$$W(y) = \int_H K(x^{-1}sy)e^{-\phi(s_H)}\pi(k^{-1})v_0 \, \mathrm{d}\nu_H(s_H).$$

The integrand has norm $|K(x^{-1}sy)|$. Put $g(s_H) = Q^{-1}sQ$, $Y = p^{-1}x^{-1}yp$ and $C_1 = \nu_H\{\|g(s_H) - I\|_{\mathrm{op}} \le 1\}$. The norm of $g(s_H)Y$ is $\|x^{-1}sy\|_{\mathrm{op}}$. By Theorem 96.3 the integrand is bounded by $(2/\|g(s_H)Y\|_{\mathrm{op}})^c$, since $c \le r^8$ eventually. Lemma 118.1 bounds the integral of this majorant by $16^c C_1$, uniformly in y, as $c > 8r^2$. On every compact set of y another integrable majorant is $(2/\|g(s_H)\|_{\mathrm{op}})^c$ times $\sup \|Y^{-1}\|_{\mathrm{op}}^c$, by submultiplicativity. Thus W is bounded and continuous by Lemma A.2.

At $y = x$, pairing with v_0 gives exactly $\int_H \theta \, \mathrm{d}\nu_H$ for the graph in Theorem 121.3. Its real part is at least $b^{-33(r+k)^2}C_1 > 0$ by Theorem 119.1. Cauchy–Schwarz gives the asserted bound on the majorant in terms of $\|W(x)\|$. Right translation in H in the defining integral, using Lemma 121.2, proves the first transformation formula, including its phase sign. For the reproducing formula, substitute the definition of $W(ya)$ in the integral over G_1. The iterated absolute integral is at most $16^c C_1 \|e_J\|_1 < \infty$ by the uniform bound already proved and Corollary 96.4. Interchange the integrals and apply Theorem 99.1 in the form $\int e_J(a)K(za) \, \mathrm{d}a = K(z)$ to $z = x^{-1}sy$. This yields $W(y)$. □

123. The norm and right translates of the Bessel vector

Lemma 123.1. *On a product of two countably exhausted finite-measure spaces, with their product measure on the product sigma algebra, a nonnegative measurable function has equal product and iterated integrals in either order. An integrable complex or finite-dimensional vector-valued function also has equal integrals in either order.*

Proof. Restrict first to finite-measure factors. For indicator functions of measurable rectangles the two iterated integrals give the product of the two measures by the definition of product measure. The sets for which both section integrals are measurable and give the product measure comprise a family containing the whole space, closed under relative complements of nested members and under countable disjoint unions by Lemma A.2. To see that it contains every product-measurable set, let $\mathcal{L}$ be the smallest family with these closure properties containing the rectangles. For a fixed rectangle A the sets B with $B \cap A \in \mathcal{L}$ form such a family containing the rectangles, hence contain $\mathcal{L}$. Fix next $B \in \mathcal{L}$ and repeat with the roles interchanged. This proves closure of $\mathcal{L}$ under finite intersections. Differences and disjointification now give closure under arbitrary countable unions, proving the assertion about measurable sets. Increasing simple functions and then increasing finite-measure exhaustions give the result for nonnegative functions by Lemma A.2. For complex functions separate the positive and negative parts of their two real coordinates; absolute integrability makes the resulting integrals finite. Apply the same reasoning in each coordinate of a vector. □

Lemma 123.2. *Use $G_1, \mathcal{U}, H$ from Definition 121.1 and the Haar measures used in Theorem 122.2. There exists a positive continuous density $a(y)$ relative to $\mathrm{d}y$ on G_1 such that $\int_{\mathcal{U}} a(ny) \, \mathrm{d}n = 1$. Write $\mathrm{d}\bar{y} = a(y) \, \mathrm{d}y$. For every nonnegative Borel function h and $x_0 \in G_1$ one has*

$$\int_{G_1} \int_H h(x_0^{-1}sy) \, \mathrm{d}\nu_H(s, k) \, \mathrm{d}\bar{y} = \int_{G_1} h(y) \, \mathrm{d}y.$$

Integrals against $\mathrm{d}\bar{y}$ of nonnegative left $\mathcal{U}$-invariant functions are independent of the choice of such a density and invariant under right translation. One can choose $\mathrm{d}\bar{y}$ left invariant under $\iota(K_0)$.

Proof. Put $\eta(y) = \exp(-\|y\|_2^2)$ and $A(y) = \int_{\mathcal{U}} \eta(ny)\,\mathrm{d}n$. The inclusion $\mathcal{U} \hookrightarrow G_1$ is proper because it is a closed matrix subgroup. Up to conjugation by the ordinary unitary matrix p in Theorem 122.2, Lemma 118.1 applies to it. On compact sets of y, submultiplicativity gives $\|ny\|_2 \geq \|n\|_{\mathrm{op}} / \sup \|y^{-1}\|_{\mathrm{op}}$. For $t \geq 1$ the exponential of a negative constant times t^2 is bounded by a constant times t^{-8r^2}, as follows from the exponential series. The covering lemma therefore gives an integrable majorant. Positivity of measure on open sets and Lemma A.2 show that A is finite, continuous and positive. It is left $\mathcal{U}$-invariant by right invariance of $\mathrm{d}n$. The density $a = \eta / A$ has the required orbit integral one. Both numerator and denominator are left $\iota(K_0)$-invariant: compact multiplication preserves the Euclidean norm and compact conjugation preserves $\mathrm{d}n$ by Lemma 121.2.

The unipotent measure is invariant under inversion. Indeed the product density in Proposition 20.5 is bi-invariant, so its pullback by inversion is also bi-invariant and agrees at identity since the derivative there is minus identity. The matrix groups are exhausted by compact norm balls, and all the measures in question are locally finite. By Lemma 123.1, for any of the stated densities,

$$\int_{G_1} a(y) \int_{\mathcal{U}} h(ny)\,\mathrm{d}n\,\mathrm{d}y = \int_{G_1} h(w) \int_{\mathcal{U}} a(n^{-1}w)\,\mathrm{d}n\,\mathrm{d}w = \int_{G_1} h(w)\,\mathrm{d}w.$$

Apply this first to $h(x_0^{-1}\iota(k)\,\cdot)$ and then integrate in k, using the compact mass one of Theorem 95.3. This gives the graph formula. For two densities a_1, a_2 and an invariant h, insert $1 = \int_{\mathcal{U}} a_2(ny)\,\mathrm{d}n$ in $\int h(y) a_1(y)\,\mathrm{d}y$ and use the unipotent formula to get $\int h(w) a_2(w)\,\mathrm{d}w$. A right translate of a_1 also has orbit integral one. Right invariance of the group volume now proves invariance of the quotient integral. □

Theorem 123.3. *Put $D = \dim \pi$ and let W be the function of Theorem 122.2. There is a positive constant α such that $F = \alpha W$ satisfies*

$$\int \|F(y)\|\,\mathrm{d}\bar{y} < \infty, \qquad \int F_i(y)\overline{F_j(y)}\,\mathrm{d}\bar{y} = \delta_{ij}$$

in every orthonormal basis of π. For every $\varphi \in L^1(G_1) \cap L^2(G_1)$,

$$\int \left\| \int_{G_1} \varphi(g) F(yg)\,\mathrm{d}g \right\|^2 \mathrm{d}\bar{y} \leq \frac{D}{d} \int_{G_1} |\varphi(g)|^2\,\mathrm{d}g.$$

Proof. The graph bound on W integrates to at most $\|K\|_1$ by Lemma 123.2; this is finite by Corollary 96.4. Since W is bounded and nonzero on a neighborhood of x, its squared norm has finite positive integral q. Put $\alpha = \sqrt{D/q}$. The endomorphism $M = \int F(y) F(y)^*\,\mathrm{d}\bar{y}$ commutes with every $\pi(k)$ by compact equivariance and the invariant density of Lemma 123.2. It is scalar. Indeed torus invariance makes it diagonal on symmetric tensor monomials, whose distinct diagonal characters are indexed by the nonnegative exponent tuples with sum a_τ. Commutation with the derivative of the unitary rotation generated by $E_{ij} - E_{ji}$ makes the diagonal entries equal whenever one exponent unit moves from position j to i. The corresponding coefficient is nonzero when the exponent at j is positive, by differentiating the action on the tensor factors. All tuples are connected by these moves, including the single tuple when $a_\tau = 0$ or $k = 1$. As $\operatorname{tr} M = D$, this proves $M = I$.

Write $R(g)F(y) = F(yg)$, and use the quotient inner product with conjugate linearity in its first variable. These translates have the same left unipotent character, so their inner products are integrals of left unipotent invariant scalars. Lemma 123.2 gives a unitary right action on their closed span. Let $P = \int e_J(g) R(g)\,\mathrm{d}g$. More explicitly this defines a bounded operator on the span of the translates, with norm at most $\|e_J\|_1$, by Cauchy–Schwarz with the absolute kernel and quotient invariance. It extends to the closure; approximate the integral by finite sums there. To justify this approximation, $g \mapsto R(g)F$ is norm continuous: as $g \to 1$ its inner product with F converges to $\|F\|^2$ by dominated convergence against the integrable $\|F\|$ and boundedness and continuity of F. Its norm is constant. Translating

gives the same conclusion for each vector of the span, and approximation gives it on the closure.

By symmetry $P^{*}=P$. The second convolution identity of Theorem 99.1 gives $PR(g)P=K(g)P$: in the double integral the density at y is $\int e_J(z)e_J((zg)^{-1}y)\,\mathrm{d}z$. Absolute convergence follows from the squared L^{1}-norm of e_J, and that identity evaluates the density as $K(g)e_J(y)$. Theorem 122.2 gives $PF=F$. It follows that $\langle R(g)F,R(h)F\rangle = DK(g^{-1}h)$. This is also the Gram matrix of $\sqrt{D}\,\pi_1(g)v_1$ in the cyclic representation (π_1,v_1) of Lemma 100.2, transported by p. Thus $R(g)F\mapsto\sqrt{D}\,\pi_1(g)v_1$ extends by linearity and completion to an isometry of the two cyclic spans. For any unit vector v in the latter span, Cauchy–Schwarz and the integral identity in that lemma give

$$\left|\int \varphi(g)\langle v,\pi_1(g)v_1\rangle\,\mathrm{d}g\right|\le d^{-1/2}\|\varphi\|_{2}.$$

Taking the supremum over v gives the stated convolution bound after rescaling by $\sqrt{D}$. The Hilbert space integral agrees with the displayed pointwise integral by Lemma 123.1, first against bounded test vectors of finite support in measure and then by density. □

124. Truncation of the real Bessel function

Lemma 124.1. *Use G_1,b,r,d of Theorem 122.2. Put $c=8r^{2}+1$, $V_1=\mathrm{vol}\{\|g-I\|_{\mathrm{op}}\le 1\}$, and $I_j=\int_{G_1}\omega_j(g)\,\mathrm{d}g$, where $\omega_j(g)=(2/\|g\|_{\mathrm{op}})^{j}$. For all sufficiently large scales,*

$$0<V_1<\infty,\qquad I_c\le 16^{c}V_1,\qquad I_{2c}\le 16^{2c}V_1,\qquad dI_{2c}\le b^{30r^{2}}.$$

Proof. The first three bounds follow from Lemma 118.1 and positivity of the smooth volume near identity. On $\|g-I\|_{\mathrm{op}}\le b^{-4}$, Theorem 116.2 gives $|K(g)|\ge 1/2$ eventually. Indeed take $T=(r+k)^{2}\log(b^{3})$ as in Proposition 122.1. The Hilbert–Schmidt norm is at most $\sqrt{r}\,b^{-4}$, the largest weight difference for μ is at most $2b^{3}$, and $r,\log b$ are bounded by fixed powers of x, whereas b grows faster than every such power by Proposition 10.5. Thus both the Gaussian exponent and the product of the matrix difference norm with T tend to zero. The integral of $|K|^{2}$ is $1/d$ by Theorem 99.1. The small ball therefore has volume at most $4/d$. In Lemma 118.1 take $R=2,\delta=b^{-4}$, noting that the set defining V_1 lies in this sublevel set. This gives $dV_1\le 4(1+8b^{4})^{2r^{2}}\le 4b^{10r^{2}}$. Now $2c=16r^{2}+2$, so $4\cdot 16^{2c}b^{10r^{2}}\le b^{30r^{2}}$ for r,b sufficiently large. □

Theorem 124.2. *Retain F,α,D,x of Theorems 122.2 and 123.3. Put $n=\nu+1$, $R=b^{6n+2}$, $L=3r^{3}f$, where $f=|\Phi|$ is the number of real places, and*

$$S(y)=1+\inf_{(s,k)\in H}\|sy\|_{\mathrm{op}},\qquad E=\{y:S(y)\le R\}.$$

There is a bounded continuous vector $V(y)$ with the same left transformation formula as F, vanishing off E, with

$$\int\|V\|^{2}\,\mathrm{d}\bar{y}\le D,\qquad \int\|F-V\|^{2}\,\mathrm{d}\bar{y}\le b^{-10r^{2}}D.$$

There are a nonnegative Borel function A invariant under left translation by the first projection of H, and a positive finite Borel measure λ on G_1, satisfying

$$\max(\|F(y)\|,\|V(y)\|)\,b^{L\mathbf{1}_{F(y)\ne V(y)}}\le A(y)\le\lambda\{g:yg\in E\},$$

$$\lambda(G_1)\int A(y)\,\mathrm{d}\bar{y}\le b^{2r^{3}}D.$$

The support set E has compact image modulo $\mathcal{U}$.

Proof. The function S is invariant under the first projection. For $y'\to y$, submultiplicativity gives $S(y')-1\le(S(y)-1)\|y^{-1}y'\|_{\mathrm{op}}$ and the reverse bound with y,y' exchanged. This proves continuity. If $S(y)\le R$, some s satisfies $\|sy\|_{\mathrm{op}}\le R+1$. Its compact middle component is ordinary unitary, so $\|n_0y\|_{\mathrm{op}}\le R+1$ for some $n_0\in\mathcal{U}$. The group norm ball is

compact. Since E is closed and $\mathcal{U}$-invariant, its image is the image of its closed intersection with that compact ball, proving compactness.

Use c, ω_j, I_j, V_1 from Lemma 124.1. Write

$$t_s = x^{-1}sy, \quad H_2(y) = \int_H \omega_{2c}(t_s)\,\mathrm{d}\nu_H(s,k), \quad P = b^{33(r+k)^2}16^c.$$

The integral with exponent c is at most $P\|W(x)\|$ by Theorem 122.2. Since $\omega_{2c} \le 2^c\omega_c$, H_2 is finite, Borel and invariant by right translation in the graph integral when y is left translated by the first projection. By Lemma 123.2, $\int H_2\,\mathrm{d}\bar{y} = I_{2c}$.

If $\|t_s g - I\|_{\mathrm{op}} \le 1$, then $\|t_s g\|_{\mathrm{op}} \le 2$, so $S(yg) \le 1 + 2\|x\|_{\mathrm{op}} \le R$ by Theorem 122.2. Moreover $\|g\|_{\mathrm{op}} \le 2\|t_s\|_{\mathrm{op}}$, since a J-unitary has $\|t_s^{-1}\|_{\mathrm{op}} = \|t_s\|_{\mathrm{op}}$. The left translate $\{\|t_s g - I\|_{\mathrm{op}} \le 1\}$ has volume V_1. Integration on it therefore gives

$$\int_{\{g:yg\in E\}} \omega_c(g)\,\mathrm{d}g \ge V_1\, 2^{-c}\omega_c(t_s).$$

Multiply by $2^c\omega_c(t_s)/V_1$ and integrate in s, k. It follows that $H_2(y) \le \lambda_0\{g : yg \in E\}$ for the measure

$$\mathrm{d}\lambda_0(g) = 2^c P\,\|W(x)\|\,\omega_c(g)\,\mathrm{d}g/V_1.$$

Writing $p_0 = 150r^2$, Lemma 124.1 gives $\lambda_0(G_1) \le b^{p_0}\|W(x)\|$ eventually: $r + k \le 2r$ and $2^c 16^{2c} \le b^{2c}$ for sufficiently large b.

By Corollary 96.4 and the pointwise bound of Theorem 96.3, $|K(t)| \le \omega_j(t)$ whenever $0 \le j \le r^8$, also when $\|t\|_{\mathrm{op}} < 2$ by $|K| \le 1$. Use $j = 2c$ in the reproducing identity at x and apply Cauchy–Schwarz with measure $\omega_{2c}(t)\,\mathrm{d}t$. Then unfold with Lemma 123.2, using the invariance of $\|W\|$ under the graph, to obtain

$$\begin{aligned}\|W(x)\|^2 &\le d^2 I_{2c}\int_{G_1} \omega_{2c}(t)\|W(xt)\|^2\,\mathrm{d}t \\ &= d^2 I_{2c}\int H_2(y)\|W(y)\|^2\,\mathrm{d}\bar{y} \le d^2 I_{2c} b^{p_0}\|W(x)\|\,D/\alpha^2.\end{aligned}$$

Since $W(x) \ne 0$, it follows for $A = \alpha H_2,\ \lambda = \alpha\lambda_0$ that

$$\lambda(G_1)\int A\,\mathrm{d}\bar{y} \le D b^{2p_0}(dI_{2c})^2 \le D b^{360r^2} \le D b^{2r^3}$$

eventually. Also $\int A^2\,\mathrm{d}\bar{y} \le \lambda(G_1)\int A\,\mathrm{d}\bar{y}$.

Set $\zeta(y) = \max(0, \min(1, R - S(y)))$ and $V(y) = \zeta(y)F(y)$. The cutoff is continuous and invariant under the graph. If $F(y) \ne V(y)$, then $S(y) > R - 1$. In that case $\|t_s\|_{\mathrm{op}} > 2b$ for all s, since otherwise $S(y) \le 1 + 2b\|x\|_{\mathrm{op}} \le R - 1$ for b sufficiently large. From $f \le N_1 = O_M(x^{11}(1 + \log x))$ and $r \ge x^5$ in Propositions 10.5 and 12.2 one has $2c + L \le r^8$ eventually. Apply the pointwise bound with $j = 2c + L$ on this set, and with $j = 2c$ elsewhere, in the integral for W. This gives the required majorization, including that for V since $\zeta \in [0, 1]$. The norm of V is bounded by that of F, and the squared difference integrates to at most $b^{-2L}\int A^2\,\mathrm{d}\bar{y} \le b^{-10r^2}D$ because $2L - 2r^3 \ge 4r^3$ and $r \to \infty$. □

125. Right derivatives of the real Bessel functions

Lemma 125.1. *Retain ψ of Definition 96.1 in rank r, and put $H_0 = 1 + r + \max_i |t_i|$. There is a holomorphic extension of ψ near identity in the complex matrix space, of modulus at most two on $\|g - I\|_{\mathrm{op}} \le 1/(32rH_0)$. The function ψ is smooth on the whole real group.*

Proof. For $\delta = \|g - I\|_{\mathrm{op}} < 1$ the real part of the matrix is positive definite in the sense of Definition 97.1. Use the splitting $a = I$ there. Its numerator is $|m_\Lambda(k)|^2$, and the denominator is $m_{-\mu}(k^{-1}gk)$. At $\delta \le 1/(32rH_0)$, the logarithm series on every principal compression differs from zero by at most 2δ in operator norm, since $\sum_{j\ge1}\delta^j/j \le \delta/(1-\delta)$. Its trace, of absolute value at most $2r\delta$, exponentiates to the determinant by Definition 114.1 and its determinant calculation. In the product of minors the sum of the absolute proper exponents is the

variation of $-\mu$, at most $2H_0$; the absolute exponent of the full determinant is at most H_0. Hence the reciprocal denominator has modulus at most $\exp(6rH_0\delta) < 2$. Integrate with the probability $c_\Lambda^{-1}|m_\Lambda(k)|^2\,\mathrm{d}k$. The integral agrees with ψ on the real group by Theorem 97.3. Holomorphy follows by integrating the Taylor series of the rational integrands on compact interior polydisks, with the coefficient bound of Lemma A.3 successively in the variables. Around a general point g_0 use its fixed admissible block-diagonal splitting from Theorem 97.3. It stays admissible on a complex neighborhood; the compact set of all denominators stays disjoint from zero after shrinking that neighborhood. The identical argument yields a local holomorphic extension there, proving smoothness on the real group. □

Theorem 125.2. *Let F, π, D, d be as in Theorem 123.3 and $n = \nu + 1$. Index words w by ordered pairs of matrix coordinates (i, j) in G_1, and write $|w|$ for the length. There exist bounded smooth π-valued functions F_w with $F_\varnothing = F$, all obeying the left transformation rule of Theorem 122.2. If γ is a real differentiable group curve with $\dot\gamma(t) = \gamma(t)X$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}F_w(\gamma(t)) = \sum_{i,j} X_{ij}F_{(i,j)w}(\gamma(t)).$$

For all sufficiently large scales, all $|w| \le 2n^2$, and every integrable φ on G_1,

$$\int \left\| \int_{G_1} \varphi(g)F_w(yg)\,\mathrm{d}g \right\|^2 \mathrm{d}\bar y \le (b^5)^{2|w|}\frac{D}{d}\int_{G_1}|\varphi(g)|^2\,\mathrm{d}g.$$

Proof. The compactly supported reproducing function β_0 in Theorem 99.1 can be taken smooth. In its proof choose the real scalar cutoff η to be a smooth function of $\mathrm{tr}(g^*g)$ supported below $r + 2$ and positive at r; for instance take $\exp(-1/(r + 2 - t))$ for $t < r + 2$ and zero beyond. Repeated differentiation gives polynomials in $1/(r + 2 - t)$ times the same exponential, hence all derivatives vanish at the endpoint by the exponential series. The remaining factor is smooth by Lemma 125.1. Transport the function by p, writing B_0 for it on G_1. Then $B_0 * e_J = e_J$ and

$$F(y) = \int_{G_1} B_0(z)F(yz)\,\mathrm{d}z$$

by reproduction and Lemma 123.1. For any tangent matrix $X^*J + JX = 0$, replacing y by $y\exp(tX)$ transfers the derivative to $B_0(\exp(-tX)z)$. Thus every successive right derivative is continuous and bounded, given by the same integral with a smooth compactly supported derivative of B_0. This follows by differentiation under the integral, since on compact parameter sets the supports lie in one compact set and F is bounded. It also holds in quotient Hilbert norm for the right action R of Theorem 123.3. Indeed the norm of the integral with any integrable scalar kernel h is at most $\|h\|_1\|F\|_2$ by unitarity, and the difference quotients of a smooth compact kernel converge in L^1 to its derivative by uniform convergence on that compact support. They lie in the closed cyclic span. These smooth derivatives in a basis of tangent directions give ordinary smoothness in group coordinates, also seen directly by replacing $\exp(tX)$ by a local smooth matrix chart.

For any complex matrix Z, put $Z^\dagger = JZ^*J$ and define its derivative by complex linear extension from the real tangent space, using

$$Z = (Z - Z^\dagger)/2 + i(Z + Z^\dagger)/(2i).$$

The two matrices multiplying $1, i$ are tangent, each of operator norm at most $\|Z\|_{\mathrm{op}}$. The decomposition is unique by the involution, so this extension is complex linear. Define $F_{(i,j)w}$ as the derivative of F_w for the matrix unit E_{ij}. Smoothness, boundedness, the chain rule in the statement and the transformation under the graph follow from the preceding construction and commutation of left and right translations.

Take a word of length $l \ge 1$ in real tangent matrices $X_1, \dots, X_l$ of operator norms at most one. Its mixed derivative vector is the derivative, once in each variable at zero, of $R(g(t))F$ for $g(t) = \exp(t_1X_1)\cdots\exp(t_lX_l)$. By the Gram identity of Theorem 123.3 its

squared Hilbert norm is D times the derivative in all $2l$ real variables of $K(g(t)^{-1}g(s))$ at zero. The expression inside K is a product of $2l$ exponentials, with the first l in reverse order and with negative parameters. For complex parameters of moduli $\le u = (128lrH_0)^{-1}$ its difference from identity has operator norm at most $\exp(2lu) - 1 \le 4lu$. This follows by expanding the exponentials and bounding the sum of all nonconstant terms. Conjugation by the ordinary unitary p preserves the bound. Lemma 125.1 gives a holomorphic scalar of modulus at most two throughout this polydisk. Applying Lemma A.3 in each variable bounds the mixed derivative by $2u^{-2l}$; its factorial factor is one since each variable occurs once. The original word therefore has norm at most $\sqrt{2D}u^{-l}$. Expanding complex matrix units into their two tangent parts shows

$$\|F_w\|_2 \le \sqrt{D}(512lrH_0)^l \qquad (l = |w| \ge 1).$$

For $l \le 2n^2$, $512lrH_0 \le b^5$ eventually by Propositions 122.1 and 10.5: $H_0 \le b^3 + r + 1$, and r, n grow at most polynomially in x.

Under the cyclic isometry in Theorem 123.3 any such vector F_w corresponds to a vector of the same norm in the irreducible cyclic representation. For $\varphi \in L^1 \cap L^2$ and a unit test vector there, the two-vector formula of Lemma 100.2 and Cauchy–Schwarz bound the absolute integrated coefficient by $d^{-1/2}\|F_w\|_2\|\varphi\|_2$. Taking the supremum over unit vectors and squaring proves the desired bound. Length zero is Theorem 123.3; for $\varphi \in L^1$ outside L^2 the pointwise integral exists by boundedness, and the asserted extended-real inequality holds. □

126. Coordinates for two Bessel frames

Definition 126.1. In the coordinates of Definition 121.1 put $n = \nu + 1$, let $F_0 = (e_1, \dots, e_\nu, z)$, and let $L_0 = (e_1^*, \dots, e_\nu^*, u)$ be the indicated matrices of columns, with u the fixed unit coordinate in P_0. Write $P = e_n e_n^*$ on the separate coordinate space $\mathbb{C}^n$, $Q = I - P$, and $E_0 = (I_\nu \;\; 0)$. For $g \in G_1$ put $Z(g) = F_0^* J g F_0$. For any complex n-square matrix Z put

$$\Delta(Z) = |\det Z|^2 - |\det Z_{1:\nu,1:\nu}|^2, \qquad \Omega = \{Z : \Delta(Z) > 0\},$$

with empty determinant one. Write H_0 for the stabilizer of the individual columns of F_0 in G_1. For a ν-by-k complex matrix B and a skew-hermitian ν-square matrix T, let $u(B,T)$ fix z and on the ordered blocks e, P_0, e^* act by

$$\begin{pmatrix} I & B & T - BB^*/2 \\ 0 & I & -B^* \\ 0 & 0 & I \end{pmatrix}.$$

Lemma 126.2. *The map $(B, T, k_0) \mapsto u(B,T)\iota(k_0)$ parametrizes H_0 bijectively, smoothly in both directions. A left and right invariant measure $\mathrm{d}h_0$ there is $\mathrm{d}B\ \mathrm{d}T\ \mathrm{d}k_0$, with ordinary complex entry measures for B, real imaginary diagonal and ordinary complex strictly lower entries for T, and probability on $U(k)$.*

Proof. An isometry fixing z preserves its orthogonal complement. Fixing all e_i gives identity on their columns and on the bottom quotient by the hermitian pairing. The middle diagonal block is then some $k_0 \in U(k)$. After right multiplication by $\iota(k_0)^{-1}$, the first middle row block B is free, the middle last block is $-B^*$, and the last upper block C satisfies $C + C^* = -BB^*$, by multiplying the Gram identity. Thus $T = C + BB^*/2$ gives the claimed unique parameters. Conversely the displayed matrix in Definition 126.1 satisfies that identity. Multiplication of two unipotent matrices adds their B's and adds the T's with a skew-hermitian correction depending only on the two B's. Left and right translations preserve $\mathrm{d}B\ \mathrm{d}T$ by successive translations. Compact conjugation acts unitarily on the B coordinates and fixes T. These facts prove both invariances of the product measure. □

Proposition 126.3. *For $Z \in \Omega$ put $\xi = (Z^{-1})_{nn}$, $v = e_n^* Z^{-1}$, $s = (1 - |\xi|^2)^{1/2}$. Then $\Delta(Z) = |\det Z|^2 s^2$ and $s > 0$. Put*

$$A_Z = -P - Qv^*v(Q/2 + P), \qquad B_Z = Q + sP, \qquad Y_Z = (F_0 A_Z + L_0 B_Z)Z.$$

There is an isometry $\Gamma_Z \in G_1$ carrying F_0 to Y_Z, carrying Y_{Z^*} to F_0, and fixing the complement of the columns of F_0, L_0 pointwise. It depends real analytically on $Z \in \Omega$ and $Z(\Gamma_Z) = Z$. Every g with $Z(g) = Z$ can be written $h_1 \Gamma_Z h_2^{-1}$ with $h_1, h_2 \in H_0$.

Proof. Positive $\Delta(Z)$ implies $\det Z \neq 0$, and the cofactor formula gives $\Delta(Z) = |\det Z|^2(1 - |\xi|^2)$. In the Euclidean orthonormal columns (F_0, L_0) the Gram matrix has blocks $\left(\begin{smallmatrix} -P & Q \\ Q & P \end{smallmatrix}\right)$. Consequently $F_0^* J Y_Z = Z$. To compute the square Gram of $Y_Z Z^{-1}$, the first diagonal term is $-P$, the last is $s^2 P$, and the two mixed terms sum to $-v^*v + |\xi|^2 P$. Their sum is $-v^*v$, giving $Y_Z^* J Y_Z = -P$. The bases (F_0, Y_{Z^*}) and (Y_Z, F_0) therefore have equal Gram matrix. They are bases of the same subspace since $B_Z Z$ and its counterpart for Z^* are invertible. On this subspace the required isometry has matrix, in the basis (F_0, L_0),

$$\Gamma_Z = T_Z \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} T_{Z^*}^{-1}, \qquad T_Z = \begin{pmatrix} I & A_Z Z \\ 0 & B_Z Z \end{pmatrix}.$$

Its extension by identity on the positive complement is an isometry. This expression is real analytic, as the square root is evaluated at a strictly positive real number.

For the last assertion put $Y = gF_0$. Let C, L, w, D_0 be its row blocks on e, P_0, z, e^*, respectively, with w a single row. Then $D_0 = E_0 Z$, $w = -e_n^* Z$. Write $CZ^{-1} = (C_1, c)$, $LZ^{-1} = (L_1, u')$, and $v = (v_1, \xi)$, splitting off the last column or entry. The identity $Y^* J Y = -P$, multiplied by Z^{-*} and Z^{-1} on its two sides respectively, gives

$$C_1 + C_1^* = -v_1^* v_1 - L_1^* L_1, \qquad c = -v_1^* \xi - L_1^* u', \qquad (u')^* u' = s^2.$$

Here $Z^{-*} = (Z^{-1})^*$. Conversely these equations impose precisely that Gram identity. Choose $k_0 \in U(k)$ with $k_0 u = u'/s$; extend a unit vector to an orthonormal basis by repeated orthogonal projection and normalization of coordinate vectors. For Y_Z the corresponding free data are $L_1 = 0$, $u' = su$, $C_1 = -v_1^* v_1/2$. Applying $u(B, T)\iota(k_0)$ to Y_Z gives the desired $L_1 = -B^*$, $u' = sk_0 u$ and $C_1 = -v_1^* v_1/2 + T - BB^*/2$. Choose $B = -L_1^*$, $T = (C_1 - C_1^*)/2$. The intervening equation then also gives exactly the prescribed c, so $Y = h_1 Y_Z$. It follows that $g^{-1} h_1 \Gamma_Z$ fixes F_0. This proves the decomposition, including $\nu = 0$ with empty unipotent parameters. □

127. The measure on the space of frames

Lemma 127.1. *Let Ψ be a continuously differentiable map from an open subset of $\mathbb{R}^a$ to $\mathbb{R}^b$. Near a point where its derivative has rank b, choose $a - b$ coordinate functions u so that (u, Ψ) has invertible derivative. With Lebesgue densities on both spaces, the density on $\Psi^{-1}(t)$ is given locally by*

$$|\det D(u, \Psi)|^{-1} \, \mathrm{d}u.$$

This is independent of the complementary coordinate functions, including smooth replacements for which the derivative remains invertible. Integrating these densities in $\mathrm{d}t$ gives ambient Lebesgue integration there. Under invertible linear changes with absolute determinants a_0, b_0 in the ambient and target spaces, respectively, the density transforms by a_0/b_0 in the new coordinates.

Proof. Complete a nonzero b-row minor to an invertible matrix with coordinate rows. A continuously differentiable square map whose derivative is invertible has continuously differentiable local inverse. To verify this, translate and compose with the inverse derivative at the point. The map now has form $x + R(x)$, $R(0) = 0$, with derivative norm of R at most $1/2$ on a small closed cube in the maximum norm. The same Lipschitz bound holds by integrating along segments. For targets in the concentric half cube, iteration of $x \mapsto y - R(x)$

stays in the cube and has successively halved differences, giving the unique inverse. The inverse is Lipschitz with constant two. Substituting this bound in the derivative expansion yields derivative $(I + DR(x))^{-1}$ for the inverse, which is continuous. Differentiating this formula gives the smooth version.

For completeness, the local change of variables for these coordinates uses the absolute determinant. For an invertible linear map the volume factor is $|\det|$ by elimination into permutations, dilations and shears, using translated one-dimensional integration for a shear. Partition a compact cube inside a diffeomorphism chart into small cubes. On each subcube with center x, after subtracting the value at x and applying the inverse derivative at x, the map is identity plus a function vanishing at the center with Lipschitz constant $\eta \to 0$ uniformly under refinement. This follows from uniform continuity of the derivative and boundedness of its inverse on the compact cube. Its image lies in the concentric cube expanded by $1 + \eta$, and contains the one contracted by $1 - \eta$ by the iteration just proved. After the linear change back these give the two volume factors $|\det|(1 \pm \eta)^a$. Images of faces have zero a-volume by covering their Lipschitz images by $O(j^{a-1})$ cubes of side $O(1/j)$. Summing over disjoint interiors and passing to the limit gives the determinant integral for cubes. Finite unions, subdivision into cubes and the measurable-set closure argument of Lemma 123.1 extend the equality to Borel integrals in a countable collection of such charts.

Apply that change of variables to (u, Ψ) and integrate in t by Lemma 123.1. Comparing two admissible coordinate systems, the transition fixes t, so its Jacobian is precisely the Jacobian of its u coordinates along the fiber. This proves independence on each fiber, including the indicated point. Linear ambient and target changes have the stated ratio by the same determinant rule. □

Proposition 127.2. *Let $\mathcal{V} = G_1 F_0$, with matrices of full column rank. Give the space of all complex r-by-n matrices coordinate Lebesgue measure and the hermitian n-square matrices the real diagonal and complex strictly lower coordinate measures. On $\mathcal{V}$ write $\mathrm{d}\lambda(Y)$ for the fiber density of $Y \mapsto Y^*JY$ at $-P$ from Lemma 127.1. Put $u_j = U_j$ of Definition 56.1, including $u_0 = 1$. For $h \geq 0$ Borel on G_1,*

$$\int_{G_1} h(g)\,\mathrm{d}g = \frac{2^n u_k}{u_r} \int_{\mathcal{V}} \int_{H_0} h(g_Y q)\,\mathrm{d}h_0(q)\,\mathrm{d}\lambda(Y), \qquad g_Y F_0 = Y.$$

The inner integral is independent of the choice of the lift.

Proof. For every full column rank Y, Y^*J is surjective. The derivative $B \mapsto Y^*JB + B^*JY$ reaches an arbitrary hermitian C by choosing $Y^*JB = C/2$, column by column. Thus the density is defined near every frame. An isometry of J has determinant of modulus one and acts on each ambient column with real volume determinant one. It preserves the Gram map, hence preserves its level density by Lemma 127.1.

At identity, write $U = J\dot{g}$, an arbitrary skew-hermitian matrix in the density of Definition 95.1. Indeed ordinary unitary conjugation by p preserves this density as in Lemma 95.2. The derivative of $g \mapsto gF_0$ gives exactly the columns $UF_0 = J\dot{Y}$. In the coordinates with the F_0 indices first, these columns have an arbitrary skew-hermitian upper n-square block and arbitrary complex lower block. Every such tuple extends to a skew-hermitian U. These are exactly the tangent constraints for $Y^*JY = -P$ at F_0. The local inverse construction of Lemma 127.1, applied on a linear complement of the stabilizer directions in the Cayley chart of Lemma 95.2, therefore gives a local section of the frame map. Left translations give local sections on the entire orbit, open in the full-rank Gram level. Each g over such a section has a unique expression $g_Y q$ by Lemma 126.2. Two sections differ by an element of H_0 depending smoothly on Y; left invariance of $\mathrm{d}h_0$ proves the claimed independence and measurability.

We compute the ratio in these coordinates at identity. The ambient change $\dot{Y} \mapsto J\dot{Y}$ preserves real volume. If V is the upper n-square block in these coordinates, the Gram derivative is $V + V^*$. Passing to $V + V^*$ and $(V - V^*)/2$ has absolute determinant 2^n in their

free hermitian and skew-hermitian coordinates. On the diagonal each real part doubles; in a pair of complex entries the map $(a, b) \mapsto (a + \bar{b}, (a - \bar{b})/2)$ has real determinant of absolute value one. Thus $\mathrm{d}\lambda$ is 2^{-n} times the independent first-column-block coordinates of U.

The remaining independent skew-hermitian coordinates of U give its block on P_0, e^*. On the tangent of H_0 it is $\left(\begin{smallmatrix} \dot{k} & -\dot{B}^* \\ \dot{B} & \dot{T} \end{smallmatrix}\right)$ by Lemma 126.2. Their coordinate density is $u_k \, \mathrm{d}h_0$, since the compact probability at identity has density u_k^{-1} by Theorem 95.3. Choosing a different tangent lift of the frame only adds a linear function of the first coordinates to these remaining entries, a shear of determinant one. Finally the full group density is $1/u_r$ times all independent coordinates of U. This gives $2^n u_k/u_r$ at identity. Both densities are left invariant on G_1 by the preceding section comparison and invariance of the level density, so the same ratio holds everywhere. Integrating in countably many charts, made disjoint by successive differences, gives the formula by Lemma 123.1. □

128. The exact density of the real open cell

Theorem 128.1. *With the normalizations of Lemmas 126.2 and 95.2, write* $C_{\mathrm{op}} = u_k^2/(u_{k-1}u_r)$. *For every nonnegative Borel function* h *on* G_1,

$$\int_{Z(g)\in\Omega} h(g)\,\mathrm{d}g = C_{\mathrm{op}} \int_\Omega \Delta(Z)^{k-1} \iint_{H_0^2} h(q_1 \Gamma_Z q_2^{-1})\,\mathrm{d}h_0(q_1)\,\mathrm{d}h_0(q_2)\,\mathrm{d}Z,$$

where $\mathrm{d}Z$ *is ordinary Lebesgue measure on its complex entries.*

Proof. Use the frame density of Proposition 127.2 and the notation of Proposition 126.3. For Y in the ambient matrix space keep $Z = F_0^* JY$ as an independent coordinate. Its rows are exactly $D_0, -w$, so this step has absolute determinant one. For invertible Z replace the remaining rows C, L by

$$C = (C_1, c)Z, \qquad L = (L_1, u')Z.$$

At fixed Z the ambient real volume factor is $|\det Z|^{2(\nu+k)}$ since there are $\nu + k$ independent complex rows. In the constraint replace $Y^*JY + P$ by $Z^{-*}(Y^*JY + P)Z^{-1}$. Equivalently the original output is the congruence by Z of this new output. The determinant of this congruence on hermitian n-square matrices is $|\det Z|^{2n}$, by multiplying by i in the skew-hermitian calculation of Lemma 95.2. As Z itself is an independent coordinate the derivative is block triangular in this change. Thus the residual factor for the level density is $|\det Z|^{2(k-1)}$ by Lemma 127.1.

The three blocks of the new output are

$$C_1 + C_1^* + L_1^* L_1 + v_1^* v_1, \qquad c + L_1^* u' + v_1^* \xi, \qquad (u')^* u' - (1 - |\xi|^2).$$

Eliminating the first block leaves $2^{-\nu}\,\mathrm{d}T$, with $T = (C_1 - C_1^*)/2$, by the paired-entry calculation in Proposition 127.2. Eliminating the second gives a complex translation of c, with determinant one. For the last block, use $u' = s\omega$, where ω runs over the unit sphere in $\mathbb{C}^k$. Radius s contributes $s^{2k-1}\,\mathrm{d}s$ by Theorem 95.3, and the constraint uses $\mathrm{d}(s^2) = 2s\,\mathrm{d}s$. At the specified positive $s = (1 - |\xi|^2)^{1/2}$ the remaining factor therefore is

$$\frac{u_k}{2u_{k-1}} s^{2(k-1)}\,\mathrm{d}\omega,$$

where $\mathrm{d}\omega$ is probability on the sphere. The changes in the first two output blocks as L_1, u' vary do not affect these determinants, since those variables were held fixed when the first two blocks were eliminated. Together with $\mathrm{d}L_1\,\mathrm{d}Z$, these are all the free coordinates. All frames satisfying these equations at $Z \in \Omega$ lie in the orbit: the explicit $q_1 Y_Z$ in the proof of Proposition 126.3 realizes every solution. In that formula $B = -L_1^*$ is volume preserving, T is unchanged and $\omega = k_0 u$. The distribution of $k_0 u$ for compact probability is $\mathrm{d}\omega$ by the first-column calculation in Theorem 95.3.

Finally $\Delta(Z) = |\det Z|^2 s^2$, and $n = \nu + 1$ cancels $2^{-\nu-1}$ against 2^n in Proposition 127.2. The remaining constant is C_{op}. In its inner integral choose $g_Y = q_1 \Gamma_Z$. Different choices of

k_0 carrying u to ω give different lifts of the same Y, so that inner integral is unchanged. The H_0 measure is inversion invariant by bi-invariance and the derivative of inversion at identity, as in Lemma 123.2. Changing the second integration variable to its inverse gives the formula. □

Corollary 128.2. *Outside a set of Lebesgue measure zero, write $Z = l^* \operatorname{diag}(c_1, \dots, c_n) u$ with l, u upper unitriangular complex matrices and $c_i \neq 0$. Use ordinary complex measure on each independent upper entry of l, u and on each c_i. The first two factors of the measure on the right of Theorem 128.1 in these coordinates are*

$$C_{\mathrm{op}}(|c_n|^2 - 1)^{k-1} \prod_{j=1}^{n-1} |c_j|^{2(k-1)+4(n-j)} \, \mathrm{d}l \, \mathrm{d}u \prod_{j=1}^{n} \mathrm{d}c_j \qquad (|c_n| > 1).$$

Proof. The set where any initial principal determinant vanishes has zero Lebesgue measure. Each such polynomial is nonzero at the identity. More generally, for a nonzero polynomial in complex coordinates apply induction on the number of variables to its nonzero leading coefficient in the last variable. Off the null set of that coefficient there are finitely many roots in the last variable. Lemma 123.1 on bounded boxes and countable exhaustion prove the claim. Eliminate the first nonzero pivot c_1. The remaining part of the first column is c_1 times the conjugates of the independent first-row entries of l, and the remaining part of the first row is c_1 times those of u. The lower-right block is its Schur complement translated by their outer product times c_1. The first step thus has volume determinant $|c_1|^{4(n-1)}$. The initial determinants of the Schur complement are the remaining ones divided by c_1, by subtracting the first row multiples. Repeating gives unique factors, and the total Jacobian is $\prod_j |c_j|^{4(n-j)}$. The full and penultimate determinants are $\prod_j c_j$ and $\prod_{j<n} c_j$. Substitution in Δ gives the asserted power and range, also when $n = 1$. □

129. Integration of a convolution over nearby frame matrices

Lemma 129.1. *For $n_0 \in \mathcal{U}$ define $A(n_0)$ by $n_0 F_0 = F_0 A(n_0)$. This is a surjective homomorphism to the complex upper unitriangular group N_n, with kernel $\mathcal{U}_0 = \{u(B, T)\}$ of Definition 126.1. Its Haar integration with the normalization of Proposition 20.5 is*

$$\int_{\mathcal{U}} h(n_0) \, \mathrm{d}n_0 = \int_{N_n} \int_{\mathcal{U}_0} h(v \, s(A)) \, \mathrm{d}v \, \mathrm{d}A,$$

where $s(A)$ is the section described below, $\mathrm{d}v = \mathrm{d}B \, \mathrm{d}T$, and $\mathrm{d}A$ is Lebesgue measure in the strictly upper complex entries. Moreover $Z(n_0 y) = A(n_0)^{-*} Z(y)$.

Proof. The defining flag and the identity middle quotient in Definition 20.3 give $n_0 F_0 = F_0 A$ with A upper unitriangular. The action on these independent columns is multiplicative. The kernel is the intersection with H_0, which by Lemma 126.2 is $\mathcal{U}_0$ since the positive middle quotient is fixed. Write $A = \left(\begin{smallmatrix} a & v \\ 0 & 1 \end{smallmatrix}\right)$. Take $s(A)$ to fix the middle positive space and set its matrix on e, z, e^* to be

$$\begin{pmatrix} a & v & vv^*a^{-*}/2 \\ 0 & 1 & v^*a^{-*} \\ 0 & 0 & a^{-*} \end{pmatrix}.$$

The e, z, e^* Gram pairings are preserved: the two terms from the e, e^* pairing cancel the negative middle square on the last block, the mixed z, e^* terms cancel, and the e, e^* identity block is preserved by $a^*a^{-*} = I$. This section lies in $\mathcal{U}$ and maps to A.

On N_n, left or right multiplication adds to an entry a constant and terms from smaller distances above the diagonal, so $\mathrm{d}A$ is bi-invariant. Every n_0 has unique smooth coordinates $v_0 s(A)$ with $v_0 \in \mathcal{U}_0$. The product density is right invariant: multiplying on the right changes A by right translation and at fixed A changes v_0 by right translation, with factor $s(A) n_0 s(AA(n_0))^{-1}$ in the kernel. At identity the coordinate tangent entries are the strictly upper e, e entries and the e, z entries for A, the e, P_0 entries for B, and the e, e^* entries for T

with their skew constraint. They are exactly the independent row entries and the imaginary central entries in the successive layers of Lemma 20.4. Choosing the opposite half of the skew entries conjugates and changes sign, with absolute determinant one. The two densities agree at identity and hence everywhere by right invariance. Lastly use $n_0^{-1}F_0 = F_0A(n_0)^{-1}$ and J-unitarity in the definition of Z to obtain its transformation law. □

Proposition 129.2. *Let W_1, m_0 be nonnegative Borel functions on G_1 with W_1 left $\mathcal{U}$-invariant. Let $C \geq 0,\ R \geq 1$. Assume that $\int_{\mathcal{U}_0} m_0(vy)\,\mathrm{d}v \leq C$ for all y, and that $m_0(y)m_0(n_0y) > 0$ implies $\max_{ij}|A(n_0)_{ij}| \leq R$ for $n_0 \in \mathcal{U}$. Then*

$$\int_{G_1} m_0W_1\ \mathrm{d}g \leq C(\pi R^2)^{n(n-1)/2}\int W_1(y)\,\mathrm{d}\bar{y}.$$

Proof. Unfold the left integral by the unipotent formula in Lemma 123.2, taking the invariant W_1 outside the $\mathcal{U}$ integral. Fix y. If no translate has $m_0(n_0y) > 0$, the inner integral is zero. Otherwise fix one such translate n_1y. For every other one, $A(n_0)A(n_1)^{-1}$ has its entries bounded by R. This restricts $A(n_0)$ to a right translate of a product of complex disks in the free entries, of volume $(\pi R^2)^{n(n-1)/2}$. Integrate first over $\mathcal{U}_0$ in Lemma 129.1 and use the assumed bound C. This proves the inequality. □

Corollary 129.3. *Let $P_* \geq 8,\ c_* > 0$ and let $\mathcal{Q}$ be a Borel set of frame matrices with $\Delta(Z) \geq c_*$ and $\|Z\|_{\mathrm{op}}, \|Z^{-1}\|_{\mathrm{op}} \leq P_*$. For a nonnegative Borel W_1 invariant on the left under $\mathcal{U}$ and $\iota(K_0)$,*

$$C_{\mathrm{op}}c_*^{k-1}\int_{\mathcal{Q}}\int_{H_0} W_1(\Gamma_Z h^{-1})\,\mathrm{d}h_0(h)\,\mathrm{d}Z \leq P_*^{3n^2}\int W_1(y)\,\mathrm{d}\bar{y}.$$

Proof. Use the construction in Lemma 123.2 with $\mathcal{U}_0$ in place of $\mathcal{U}$ to get a density a_0 with unit $\mathcal{U}_0$ orbit integral, invariant on the left under $\iota(K_0)$. That proof applies since $\mathcal{U}_0$ is closed by Lemma 129.1 and compact conjugation preserves $\mathrm{d}v$ by Lemma 126.2. In particular $\int_{H_0} a_0(hy)\,\mathrm{d}h_0(h) = 1$, by compact normalization. Apply Theorem 128.1 to m_0W_1 with $m_0(y) = a_0(y)\mathbf{1}_{\mathcal{Q}}(Z(y))$. Both W_1 and Z are invariant under the left H_0-action, so this integral bounds the left side from above. If $m_0(y)m_0(n_0y) > 0$, Lemma 129.1 gives $A(n_0)^{-*} = Z(n_0y)Z(y)^{-1}$. Taking inverse adjoints bounds $\|A(n_0)\|_{\mathrm{op}} \leq P_*^2$. Also $\int_{\mathcal{U}_0} m_0(vy)\,\mathrm{d}v \leq 1$. Proposition 129.2 applies with $C = 1,\ R = P_*^2$. The factor $(\pi P_*^4)^{n(n-1)/2}$ is at most $P_*^{3n^2}$, since $\pi \leq 4 \leq P_*$ from the unit disk in its enclosing square. □

130. Uniform estimates from two bounded frames

Lemma 130.1. *Write $s(Z) = (1 - |(Z^{-1})_{nn}|^2)^{1/2}$ on Ω. Suppose $M \geq 8n,\ |Z_{ij}| \leq M^2,\ \Delta(Z) \geq M^{-1}$ and $P_* \geq M^{9n}$. If $\|Z' - Z\|_{\mathrm{op}} \leq P_*^{-6}$, then*

$$\Delta(Z') \geq \Delta(Z)/2, \qquad \max(\|Z'\|_{\mathrm{op}}, \|(Z')^{-1}\|_{\mathrm{op}}, s(Z')^{-1}) \leq P_*.$$

For any $Z' \in \Omega$ with the last three bounds and $P_* \geq 8$,

$$\|\Gamma_{Z'}\|_{\mathrm{op}} \leq P_*^8, \qquad \|\,\mathrm{d}\Gamma_{Z'}[H]\|_{\mathrm{op}} \leq P_*^{16}\|H\|_{\mathrm{op}}$$

for every real tangent perturbation H of the complex matrix.

Proof. The norms of Z, Z' are at most $nM^2 + 1 \leq M^4$. For either the full determinant or the initial minor, multilinearity and Lemma 5.1 bound the change by $nM^{4(n-1)}P_*^{-6}$, on replacing one column at a time. The two absolute determinants are at most M^{4n}; for the empty minor its change is zero. It follows that $|\Delta(Z') - \Delta(Z)| \leq 4nM^{8n}P_*^{-6} \leq 1/(2M)$. In particular $|\det Z'| \geq (2M)^{-1/2}$. The product of the singular values equals the absolute determinant by taking determinants in the polar decomposition as in Theorem 96.3. Thus the inverse norm is at most $M^{4(n-1)}(2M)^{1/2}$, and $s(Z')^{-1} = |\det Z'|/\sqrt{\Delta(Z')} \leq M^{4n}(2M)^{1/2}$. This proves the first assertions.

For the derivative normalize $\|H\|_{\mathrm{op}} = 1$; the zero direction is immediate. Use the matrices in Proposition 126.3 at Z' and at its adjoint. Differentiating an inverse gives $\mathrm{d}v$ of norm at most P_*^2, while $|\,\mathrm{d}s| \le |\xi|\,|\,\mathrm{d}\xi|/s \le P_*^3$. Thus $\|A_{Z'}\| \le 2P_*^2$, $\|\,\mathrm{d}A_{Z'}\| \le 2P_*^3$, $\|B_{Z'}\| \le 1$, $\|B_{Z'}^{-1}\| \le P_*$, $\|\,\mathrm{d}B_{Z'}\| \le P_*^3$, in operator norm. The triangular expression for $T_{Z'}$ gives $\|T_{Z'}\| \le 4P_*^3$, $\|\,\mathrm{d}T_{Z'}\| \le 4P_*^4$. Its inverse has upper right block $-A_{Z'}B_{Z'}^{-1}$ and lower right block $(Z')^{-1}B_{Z'}^{-1}$; its norm is at most $4P_*^3$. The derivative of this inverse has norm at most $64P_*^{10}$ by the inverse derivative identity. In the formula for Γ all these bounds also apply at the adjoint. Multiplication gives norm at most $16P_*^6$, and the product rule bounds its derivative by $16P_*^7 + 256P_*^{13} \le P_*^{16}$, since $P_* \ge 8$. The action on the orthogonal complement is constant identity. □

Proposition 130.2. *For $\rho \ge 8n$ put $\mathcal{B}_\rho = \{y : \|y^{-1}f_j\| \le \rho,\ 1 \le j \le n\}$, with Euclidean column norm. Use $S(y)$ from Theorem 124.2. Suppose $Z \in \Omega$, $|Z_{ij}| \le \rho^2$, and put $M = \rho^{9n}/\Delta(Z)$. Then $M \ge \rho$. If $y \in \mathcal{B}_\rho$ and $S(\Gamma_Z y) \le \rho$, then $\|y\|_{\mathrm{op}} \le M^{24n}$. The entry bound assumed here holds when $Z = Z(\gamma)$ and $\mathcal{B}_\rho \cap \gamma\mathcal{B}_\rho \ne \varnothing$.*

Proof. In the last assertion write $w, \gamma^{-1}w \in \mathcal{B}_\rho$. The cross Gram of $w^{-1}F_0$ and $w^{-1}\gamma F_0$ is $Z(\gamma)$. The form matrix J has norm one, so Cauchy–Schwarz gives the entry bound. Hence $\Delta(Z) \le |\det Z|^2 \le (n\rho^2)^{2n} \le \rho^{6n}$ by Lemma 5.1. This gives $M \ge \rho$ and $\Delta(Z) \ge M^{-1}$.

Choose $v \in \mathcal{U}$ with $\|v\Gamma_Z y\|_{\mathrm{op}} \le \rho$. Indeed an element of the graph with this bound exists by the definition of the infimum with the additive term one, and removing its compact factor preserves the norm. Write $A = A(v)$ and $\widetilde{Z} = Z(v\Gamma_Z) = A^{-*}Z$ by Lemma 129.1. The two column frames $(v\Gamma_Z y)^{-1}F_0$, $y^{-1}F_0$ have cross Gram $\widetilde{Z}$; their individual column norms are at most ρ. Thus $\|\widetilde{Z}\|_{\mathrm{op}} \le n\rho^2 \le M^3$. Left multiplication by a lower unitriangular matrix preserves each leading determinant, so $\Delta(\widetilde{Z}) = \Delta(Z)$. It follows for $U = Z, \widetilde{Z}, Z^*$ that

$$\|U\|_{\mathrm{op}} \le M^3, \qquad \|U^{-1}\|_{\mathrm{op}} \le M^{3(n-1)}\sqrt{M} \le M^{3n}, \qquad s(U)^{-1} \le M^{3n}\sqrt{M}.$$

Here the inverse and s estimates use the same singular value and determinant formulas as Lemma 130.1. Consequently $\|A\|_{\mathrm{op}} \le M^{3+3n} \le M^{6n}$, since $A^* = Z\widetilde{Z}^{-1}$. The inverse block formula in that lemma gives $\|T_{Z^*}^{-1}\|_{\mathrm{op}} \le M^{11n}$: its upper right block has norm at most $(1 + M^{6n})M^{3n}\sqrt{M}$, its lower right at most $M^{6n}\sqrt{M}$.

Now $\Gamma_Z^{-1}F_0 = Y_{Z^*}$ and $vF_0 = F_0A$. Hence $\|y^{-1}Y_{Z^*}\|_{\mathrm{op}} \le \rho\|A\|_{\mathrm{op}}$, whereas $\|y^{-1}F_0\|_{\mathrm{op}} \le \sqrt{n}\,\rho$. Multiplying the two-frame matrix by $T_{Z^*}^{-1}$ bounds y^{-1} on the span of the columns of F_0, L_0 by

$$(\sqrt{n}\,\rho + \rho M^{6n})M^{11n} \le M^{20n}.$$

This span contains the entire negative eigenspace of J: the complement is the positive space X in Proposition 120.2. Let P_- be its negative eigenspace projection, so $J = I - 2P_-$. For $z_0 = y^{-1}$, the identity $z_0 J z_0^* = J$ gives $z_0 z_0^* = J + 2z_0 P_- z_0^*$. Consequently $\|y\|_{\mathrm{op}}^2 = \|z_0\|_{\mathrm{op}}^2 \le 1 + 2M^{40n} \le M^{48n}$. This proves the claimed bound. □

131. Evaluation of a convolution at a frame matrix

Lemma 131.1. *Let $d_0 \ge 1$, $\eta > 0$, and let a finite tree of vector-valued functions on $[0,\eta]^{d_0}$ have depth d_0, root f, and at most m_0 children per vertex. Suppose each function is continuously differentiable and, at depths $< d_0$, each of its first coordinate derivatives is a linear combination of its children with scalar coefficients of modulus $\le K_0$. Coefficients may depend on the point and the coordinate. Allow an additional measure parameter ω with all functions measurable and these bounds holding at each ω. Then*

$$\int \|f(0,\omega)\|^2\,\mathrm{d}\omega \le (2/\eta + 2\eta m_0^2 K_0^2)^{d_0} \max_j \iint_{[0,\eta]^{d_0}} \|f_j(t,\omega)\|^2\,\mathrm{d}t\,\mathrm{d}\omega,$$

where the maximum is over every vertex, including the root.

Proof. For a vector-valued C^1 function v of one coordinate, write $v(0) = v(t) - \int_0^t v'(u)\,\mathrm{d}u$. Cauchy–Schwarz and averaging over $0 \le t \le \eta$ give

$$\|v(0)\|^2 \le (2/\eta)\int_0^\eta \|v\|^2 + 2\eta\int_0^\eta \|v'\|^2.$$

The squared derivative is at most $m_0K_0^2\sum_j \|v_j\|^2$ by the assumed expansion and Cauchy–Schwarz. Apply this inequality successively in the coordinates, integrating the ones already treated and holding the others at zero. At every step the sum of the nonnegative scalar multipliers of the new integrals increases by a factor at most $2/\eta + 2\eta m_0^2K_0^2$. At depth i of this iteration only vertices of tree depth at most i have been used. Integrate also in ω, using Lemma 123.1, and bound the final integrals by their maximum. This gives the formula. □

Theorem 131.2. *Use F, D, d of Theorem 123.3. Suppose $P_* \ge \max(8r, b^5)$, $c_* > 0$, and throughout $\|Z' - Z\|_{\mathrm{op}} \le P_*^{-6}$ suppose*

$$\Delta(Z') \ge c_*, \qquad \max(\|Z'\|_{\mathrm{op}}, \|(Z')^{-1}\|_{\mathrm{op}}, s(Z')^{-1}) \le P_*.$$

For $\varphi \in L^1(G_1)\cap L^2(G_1)$ supported where $\|y\|_{\mathrm{op}} \le P_*^{20}$, one has

$$C_{\mathrm{op}}c_*^{k-1}\int_{\substack{h\in H_0\\ \|h\|_{\mathrm{op}}\le P_*^{20}}} \left\|\int \varphi(y)F(\Gamma_Z h^{-1}y)\,\mathrm{d}y\right\|^2 \mathrm{d}h_0(h) \le P_*^{430n^2}\,\frac{D}{d}\|\varphi\|_2^2.$$

Proof. Vary Z in its $d_0 = 2n^2$ real matrix coordinates by $Z_t = Z + \sum_{a=1}^{d_0} t_aE_a$, where E_a are the ordinary real and imaginary matrix units. Use the cube of side $\eta = P_*^{-8}$. Its operator displacement is at most $2n\eta \le P_*^{-6}$, by the entry bound and Cauchy–Schwarz. Put $G(t,h) = \Gamma_{Z_t}h^{-1}$. For differentiation in any coordinate the right tangent matrix

$$T = G(t,h)^{-1}\partial_a G(t,h) = h\Gamma_{Z_t}^{-1}(\partial_a\Gamma_{Z_t})h^{-1}$$

has norm at most P_*^{64}, by Lemma 130.1 and the equality of the norms of each isometry and its inverse.

The root of the tree in Lemma 131.1 is the integral in the statement with Z_t in place of Z. Every vertex of depth l has form $\int \varphi_j(y)F_w(G(t,h)y)\,\mathrm{d}y$, $|w| = l$, with $|\varphi_j| \le |\varphi|$. For a coordinate derivative the tangent at the argument of F_w is $y^{-1}Ty$. Use Theorem 125.2 and expand its (i,j)-entry as $\sum_{a,b}(y^{-1})_{ia}T_{ab}y_{bj}$. A child has the new word $(i,j)w$ and the new kernel obtained by multiplying the parent kernel by $(y^{-1})_{ia}y_{bj}/P_*^{40}$. This multiplier has modulus at most one on the support. The corresponding scalar coefficient is $P_*^{40}T_{ab}$, bounded by $K_0 = P_*^{104}$. There are at most $m_0 = r^4$ children. The derivative passes under the integral because each derivative is bounded on the specified region by a constant times $|\varphi(y)|$, using boundedness of the finitely many functions of each depth and the smoothness on compact neighborhoods of the section. This also gives continuity of the partial derivatives by dominated convergence, including those at depth d_0 using smoothness at the next depth. The parameter h ranges over the measurable subset indicated in the statement.

For each vertex use on the whole group the nonnegative function

$$W_1(g) = \left\|\int \varphi_j(y)F_w(gy)\,\mathrm{d}y\right\|^2.$$

Its left invariances follow from the unitary transformation formula for F_w. Theorem 125.2 bounds its quotient integral by $P_*^{2l}(D/d)\|\varphi\|_2^2$ for $l \le d_0$. Corollary 129.3 on this cube then bounds its full integral in t, h, multiplied by $C_{\mathrm{op}}c_*^{k-1}$, by $P_*^{3n^2+2l}(D/d)\|\varphi\|_2^2$. Restriction of h decreases this nonnegative integral. Lebesgue measure of the entries of Z_t is exactly $\mathrm{d}t$.

Finally $m_0^2 \le P_*^8$ and $2/\eta + 2\eta m_0^2K_0^2 \le 2P_*^8 + 2P_*^{208} \le P_*^{210}$. Lemma 131.1 therefore gives total exponent at most $210(2n^2) + 3n^2 + 4n^2 \le 430n^2$, as required. □

132. The real open-cell second moment

Lemma 132.1. *For every $T_0 \ge 8r$ there exist a compact $\Theta \subset \mathcal{U}_0$ of positive finite volume and a compact $\mathcal{T} \subset G_1$ such that*

$$\Theta\{g : \|g\|_{\mathrm{op}} \le T_0\} \subset \mathcal{T} \subset \{g : \|g\|_{\mathrm{op}} \le T_0^{12}\}, \qquad \int_{\mathcal{U}_0} \mathbf{1}_{\mathcal{T}}(vy)\,\mathrm{d}v \le 2\,\mathrm{vol}\,\Theta$$

for every $y \in G_1$.

Proof. Use the coordinates $u(B,T)$ and measure in Lemma 126.2. Put $K_1 = \sqrt{r}\,T_0^2$, $L = 8r^2K_1$ and take Θ to be $\|B\|_2 \le L,\ \|T\|_2 \le L^2$ in Hilbert–Schmidt norms. It is compact with positive finite volume, also if a parameter space is zero-dimensional. Set $\mathcal{T}$ equal to its product with the closed group ball of radius T_0. That ball is compact, and the explicit block matrix bounds $\|u(B,T)\|_{\mathrm{op}} \le 5L^2$ on Θ. Since $L = 8r^{5/2}T_0^2 \le T_0^5$, the second inclusion follows.

If $\|u(B',T')\|_{\mathrm{op}} \le T_0^2$, reading the top middle block gives $\|B'\|_2 \le K_1$. The skew-hermitian part of the top last block gives $\|T'\|_2 \le K_1$. Multiplication on the right by such an element takes Θ inside the two balls of radii $L + K_1,\ L^2 + K_1 + LK_1$. Indeed the central correction is a half difference of the two products $B'B^*, BB'^*$. Thus the volume of their entire product set is at most $(1 + 2K_1/L)^{2r^2}\,\mathrm{vol}\,\Theta \le e^{1/2}\,\mathrm{vol}\,\Theta \le 2\,\mathrm{vol}\,\Theta$, by scaling of Lebesgue measure in real dimension $2\nu k + \nu^2 \le 2r^2$. If $v_1 y = w_1 g_1 \in \mathcal{T}$, $w_1 \in \Theta$, $\|g_1\|_{\mathrm{op}} \le T_0$, then any other $vy = wg \in \mathcal{T}$ satisfies $w^{-1}vv_1^{-1}w_1 = gg_1^{-1} \in \mathcal{U}_0$ of norm at most T_0^2. The set of such v's is contained in a right translate of the product just estimated. Right invariance gives the last inequality. If there is no v_1, the integral is zero. □

Theorem 132.2. *Let F be the actual normalized real Bessel function of Theorem 123.3. Let $\rho \ge b$, $\gamma \in G_1$, $\mathcal{B}_\rho \cap \gamma\mathcal{B}_\rho \ne \varnothing$, and $\Delta = \Delta(Z(\gamma)) > 0$. Put $M = \rho^{9n}/\Delta$. At all sufficiently large arithmetic scales there exist a compact set $\mathcal{T} \subset G_1$, a random $t \in H_0$ with a specified Borel probability law, and a positive finite measure λ_γ on G_1 satisfying*

$$\mathbb{E}_t \left\| \int_{\mathcal{T}} \varphi(y) F(\gamma t y) F(ty)^*\,\mathrm{d}y \right\|_2^2 \le \int |\varphi(y)|^2\,\mathrm{d}\lambda_\gamma(y),$$

$$\lambda_\gamma(G_1) \le M^{15000n^3} \frac{D^2}{d\,C_{\mathrm{op}}(\Delta/2)^{k-1}}.$$

Here $\|\cdot\|_2$ on endomorphisms is the Hilbert–Schmidt norm, and φ is any bounded Borel scalar function. For every t in the construction, if $y \in \mathcal{B}_\rho$ and $S(\gamma t y) \le \rho$, then $y \in \mathcal{T}$.

Proof. Write $Z = Z(\gamma)$, $\gamma = h_1\Gamma_Z h_2^{-1}$ by Proposition 126.3. By Proposition 130.2 the entries of Z are bounded by ρ^2 in modulus, $M \ge \rho$, and $\Delta \ge M^{-1}$. Set $T_0 = M^{24n}$, $P_* = M^{30n}$ and $c_* = \Delta/2$. Since b dominates every fixed power of r by Proposition 10.5, $P_* \ge \max(8r, b^5)$ and $T_0 \ge 8r$ eventually. Lemma 130.1 verifies the neighborhood hypotheses of Theorem 131.2. Use $\Theta, \mathcal{T}$ of Lemma 132.1. Let v have uniform probability on Θ, let k_0 have independent Haar probability on $U(k)$, and put $\omega = v\iota(k_0)$, $t = h_2\omega^{-1}$. These are continuous functions of the two variables and have values in H_0. If the frame and size conditions on y hold, then $\omega^{-1}y \in \mathcal{B}_\rho$ since $\omega F_0 = F_0$. Also $S(\Gamma_Z\omega^{-1}y) \le \rho$ by the invariance of S and $\gamma t = h_1\Gamma_Z\omega^{-1}$. Proposition 130.2 gives $\|\omega^{-1}y\|_{\mathrm{op}} \le T_0$, hence $\|v^{-1}y\|_{\mathrm{op}} \le T_0$ by compact unitarity. The first inclusion in Lemma 132.1 now gives $y \in \mathcal{T}$.

For each fixed ω, use the unitary left transformation of F in Theorem 122.2 to remove h_1 from the first vector in the product and t from the second. The matrices and scalar phases removed are independent of y, and preserve the Hilbert–Schmidt norm. In an orthonormal basis of π, the resulting squared norm is

$$\sum_{j=1}^{D} \left\| \int \varphi_j(y) F(\Gamma_Z\omega^{-1}y)\,\mathrm{d}y \right\|^2, \qquad \varphi_j = \mathbf{1}_{\mathcal{T}}\varphi\,\overline{F_j}.$$

Each test is bounded, supported on a compact set where $\|y\|_{\mathrm{op}} \le T_0^{12} \le P_*^{20}$. Also $\|\omega\|_{\mathrm{op}} = \|v\|_{\mathrm{op}} \le T_0^{12}$, since $\Theta \subset \mathcal{T}$. By Lemma 126.2 integrating this sum in the probability law is integration in the indicated subset of H_0 divided by $\operatorname{vol}\Theta$. Theorem 131.2 term by term thus proves the first inequality with

$$\mathrm{d}\lambda_\gamma(y) = \frac{P_*^{430n^2} D}{dC_{\mathrm{op}} c_*^{k-1} \operatorname{vol}\Theta}\, \mathbf{1}_{\mathcal{T}}(y) \|F(y)\|^2 \,\mathrm{d}y.$$

To bound its mass, if $y, n_0 y \in \mathcal{T}$ with $n_0 \in \mathcal{U}$, then $\|A(n_0)\|_{\mathrm{op}} \le \|n_0\|_{\mathrm{op}} \le P_*^{40}$. The first inequality uses $n_0 F_0 = F_0 A(n_0)$ and the Euclidean orthonormal columns of F_0. The second uses the two norm bounds and $\|y^{-1}\|_{\mathrm{op}} = \|y\|_{\mathrm{op}}$. Proposition 129.2 and Lemma 132.1 now give

$$\int_{\mathcal{T}} \|F(y)\|^2 \,\mathrm{d}y \le 2 \operatorname{vol}\Theta (\pi P_*^{80})^{n(n-1)/2} D \le \operatorname{vol}\Theta\, P_*^{42n^2} D.$$

Plugging this into the measure and using $P_*^{472n^2} = M^{14160n^3} \le M^{15000n^3}$ proves its asserted mass bound. □

133. Support at the last real pivot

Definition 133.1. Use $F_0, L_0, G_1, \iota, P_0, z, u$ of Definitions 121.1 and 126.1. For $p = (p_1, \dots, p_n) \in (\mathbb{C}^\times)^n$, put $a(c) = c/\min(1, |c|)$ and $s(c) = \sqrt{|a(c)|^2 - 1}$. The matrix $w(p)$ sends e_j to $p_j e_j^*$ and e_j^* to $e_j/\overline{p_j}$ for $1 \le j \le \nu$, fixing the middle space. On the ordered orthonormal pair u, z put

$$B(c) = \begin{pmatrix} \overline{a(c)} & s(c) \\ -s(c) & -a(c) \end{pmatrix}.$$

Extend $B(c)$ by the identity on the other coordinates, and put $g_p = w(p) B(p_n)$.

Lemma 133.2. *The map $p \mapsto g_p$ is continuous into G_1, and $Z(g_p) = \operatorname{diag}(p_1, \dots, p_\nu, a(p_n))$. If $\sigma \in G_1$ fixes all the e_j, e_j^* and $z^* J \sigma z = c$, then $|c| \ge 1$ and $\sigma = \iota(k_1) B(c) \iota(k_2)$ for some $k_1, k_2 \in U(P_0)$.*

Proof. In a hyperbolic pair the mixed pairing after applying $w(p)$ is $\overline{p_j}/\overline{p_j} = 1$ and both diagonal pairings are zero. The two columns of $B(c)$ have squared signed norms $1, -1$ and zero mixed pairing since $|a(c)|^2 - s(c)^2 = 1$. The diagonal frame pairings are now as stated and the coordinates are continuous everywhere on the indicated domain.

Since σ preserves the orthogonal complement of the fixed hyperbolic pairs, write $\sigma z = v - cz$, $v \in P_0$. Its signed norm gives $\|v\|^2 = |c|^2 - 1$. If $v \ne 0$, take k_1 sending u to $v/\|v\|$, by completing both unit vectors to orthonormal bases by successive orthogonal projections. If $v = 0$, take $k_1 = 1$. Then $B(c)^{-1} \iota(k_1)^{-1} \sigma$ fixes z and all hyperbolic coordinates. On their orthogonal complement P_0 it is an ordinary unitary k_2, giving the factorization. □

Proposition 133.3. *Put $R = b^{6n+2}$ and use S of Theorem 124.2. Suppose $S(y), S(g_p^{-1} y) \le R$. Write $d_j = \prod_{a=1}^{j} p_a$ for $0 \le j \le \nu$, with $d_0 = 1$. Then*

$$|d_j| \le R^{2j} \quad (1 \le j \le \nu), \qquad |d_\nu| \max(1, |p_n|) \le R^{2n}.$$

Moreover, with maxima over an empty index set omitted, put

$$L = \max\bigl(\{1\} \cup \{R^{2j}/|d_j| : 1 \le j \le \nu\}\bigr), \quad H = \max\bigl(\{1\} \cup \{|p_j|^{-1} : 1 \le j \le \nu\}\bigr).$$

Then $\|y\|_{\mathrm{op}} \le 1 + 2\sqrt{n}(1 + H) R (1 + nL)^n$.

Proof. Since S is one plus an infimum, choose $t, h \in \mathcal{U}$ with

$$\max(\|ty\|_{\mathrm{op}}, \|h g_p^{-1} y\|_{\mathrm{op}}) \le R.$$

Indeed the compact part of each element in the defining infimum is ordinary unitary, as in Theorem 124.2. For every $T \in G_1$, $T^{-1} = J T^* J$, so the norms of T and its inverse

agree. Therefore $m = (ty)(hg_p^{-1}y)^{-1} = tg_ph^{-1}$ has norm at most R^2. The frame columns are Euclidean orthonormal, so $X = Z(m)$ has norm at most R^2. By Lemma 129.1,

$$X = A(t)^{-*}\operatorname{diag}(p_1, \dots, p_\nu, a(p_n))A(h)^{-1}.$$

The right multiplication law here follows directly by applying h^{-1} to F_0. Each leading determinant thus gives the corresponding product of diagonal entries. Every j-row minor of X has absolute value at most R^{2j}: project the selected columns to the selected coordinates, then successively remove their projections onto earlier columns; the absolute determinant is the product of the residual lengths. This proves the two product bounds.

Write the displayed factorization as $X = L_0'D_0'U_0'$. For $i > j$, the entry $(L_0')_{ij}$ equals the determinant of the submatrix on rows $1, \dots, j-1, i$ and columns $1, \dots, j$ divided by d_j. Indeed only the first j inner indices contribute because U_0' is upper triangular, and the remaining lower-factor determinant is its last diagonal entry in this minor. The transposed argument gives $(U_0')_{ji}$ via the column-replaced minor with the same denominator. All these entries have modulus at most L, involving only $j \le \nu$. For either triangular factor $I + N$, strict triangularity gives $N^n = 0$ and $\|N\|_{\mathrm{op}} \le nL$ by entrywise Cauchy–Schwarz. The finite geometric inverse then has norm at most $(1 + nL)^n$. This bounds both $A(t)$ and $A(h)$.

Now $y^{-1}F_0 = (ty)^{-1}F_0A(t)$ and $y^{-1}g_pF_0 = (hg_p^{-1}y)^{-1}F_0A(h)$ both have norm at most $C = R(1 + nL)^n$. Hence the columns $y^{-1}e_j, y^{-1}z$ have norm at most C, and the columns $y^{-1}e_j^*$ have norm at most HC because $g_pe_j = p_je_j^*$. The negative eigenspace of J has orthonormal basis $z, (e_j - e_j^*)/\sqrt{2}$. For its ordinary orthogonal projection P_- this yields $\|y^{-1}P_-\|_{\mathrm{op}} \le \sqrt{n}(1+H)C$, by Cauchy–Schwarz in these n coordinates. Finally $J = I - 2P_-$ and $y^{-1}J(y^{-1})^* = J$ give $\|y^{-1}\|_{\mathrm{op}}^2 \le 1 + 2\|y^{-1}P_-\|_{\mathrm{op}}^2$. Taking square roots and using the inverse norm identity proves the bound. □

134. The real kernel at a common last pivot

Definition 134.1. Let V be the cut function of Theorem 124.2 and $D = \dim \pi$. For $g \in G_1$ for which the entries are absolutely integrable, set

$$T(g) = \int_{G_1} V(y)V(g^{-1}y)^* \, \mathrm{d}y \in \mathrm{End}(\pi).$$

Here vw^* carries h to $v\langle w, h\rangle$, with the inner product conjugate-linear in its first entry. Put

$$\omega(p) = \frac{\sqrt{C_{\mathrm{op}}}}{D} \max(0, |p_n|^2 - 1)^{(k-1)/2} \prod_{j=1}^{\nu} |p_j|^{k-1+2(n-j)},$$

$$\Xi(p) = \omega(p)T(g_p), \qquad J_n(p, q) = \operatorname{tr}(\Xi(p)^*\Xi(q)).$$

Here $p, q \in (\mathbb{C}^\times)^n$, g_p is from Definition 133.1, $k = \dim P_0 \ge 2$ at the scales of Proposition 122.1, and C_{op} is as in Theorem 128.1.

Theorem 134.2. *The integral $T(g_p)$ exists for every p. The maps Ξ and J_n are continuous. Each finite matrix $(J_n(p_i, p_j))_{i,j}$ is positive semidefinite. If $p_n = q_n = c = z^*J\sigma z$ for an isometry $\sigma \in G_1$ fixing each e_j, e_j^*, one has*

$$J_n(p, q) = \omega(p)\omega(q) \operatorname{tr}(T(w(p)\sigma)^*T(w(q)\sigma)).$$

For fixed $p_1, \dots, p_\nu, q_1, \dots, q_\nu$, put $p' = (p_1, \dots, p_\nu, c)$, $q' = (q_1, \dots, q_\nu, c)$. The function $c \mapsto J_n(p', q')$ extends by zero at $c = 0$ to a continuous function on $\mathbb{C}$, supported in

$$1 \le |c| \le R^{2n} / \max\left(\left|\prod_{j=1}^{\nu} p_j\right|, \left|\prod_{j=1}^{\nu} q_j\right|\right).$$

Proof. On any compact subset of pivot space all $|d_j|^{-1}$ for $j \le \nu$ and $|p_j|^{-1}$ for $j \le \nu$ are bounded. By Proposition 133.3 a single closed group norm ball contains every y for which the integrand is nonzero, uniformly over this subset. Such a ball is compact because the

ambient matrix space is finite dimensional, G_1 is closed there by $g^{*}Jg = J$, and every solution of that equation is invertible. It has finite measure by the smooth density of Lemma 95.2 in finitely many translated charts. The integrand is continuous in p, y by Lemma 133.2 and bounded in norm by $\sup_y \|V(y)\|^2$, which is finite by Theorem 124.2. Dominated convergence in Lemma A.2 now proves existence and continuity of $T(g_p)$. The prefactor is continuous, including at $|p_n| = 1$ since $k \geq 2$. By Corollary 128.2, $D^2\omega^2$ is exactly the density multiplying Lebesgue measure in the pivot and unitriangular coordinates on $|p_n| > 1$.

The trace is a finite sum of products of entries, so J_n is continuous. For scalars t_i its quadratic form is $\|\sum_i t_i \Xi(p_i)\|_2^2$, where $\|\cdot\|_2$ is the Hilbert–Schmidt norm; this gives the positivity including all repetitions of pivots. Write $\sigma = \iota(k_1)B(c)\iota(k_2)$ by Lemma 133.2. Since $w(p)$ commutes with these compact matrices, $w(p)\sigma = \iota(k_1)g_p\iota(k_2)$. The left compact action on V from Theorems 122.2 and 124.2 is π: the simple-root coordinates of a pure compact element vanish. Substitution $y = \iota(k_1)y'$ in the defining integral gives

$$T(w(p)\sigma) = \pi(k_1)T(g_p)\pi(k_2).$$

This proves absolute integrability there. Apply the same formula to q; both outer factors are unitary, giving the trace identity. Finally the prefactors vanish for $|c| \leq 1$. Outside the stated outer radius one of the two support intersections is empty by Proposition 133.3. These statements give the claimed extension, compact support and hence integrability on $\mathbb{C}$. □

135. The measure of a pair of real Bessel orbits

Theorem 135.1. *Let $S_0 = \{\iota(k_0)t : k_0 \in U(P_0),\ t \in \mathcal{U}\}$. Use the product measure $\mathrm{d}s_0 = \mathrm{d}k_0\,\mathrm{d}t$ given by Definition 121.1 on its first projection. For every nonnegative Borel function h on G_1 one has*

$$\int_{G_1} h(g)\,\mathrm{d}g = D^2 \int_{(\mathbb{C}^{\times})^n} \omega(p)^2 \int_{S_0^2} h(s_1 g_p s_2^{-1})\,\mathrm{d}s_1\,\mathrm{d}s_2\ \mathrm{d}p.$$

Here $\mathrm{d}p$ is ordinary Lebesgue measure in the n complex entries. For $|p_n| \leq 1$ the integrand on the right is assigned zero. The same identity holds for integrable complex h.

Proof. First the set where a leading minor of $Z(g)$ or $\Delta(Z(g))$ vanishes has measure zero. On the full-measure Cayley chart of Lemma 95.2, transported to J by an ordinary unitary change of basis, the product of the squared absolute values of the leading minors times Δ is a rational function of real coordinates with a nowhere vanishing denominator. Its numerator is a nonzero real polynomial. Indeed take g_p with all the first ν pivots one and $p_n = 2$. By Lemma 133.2 the product is nonzero, and replacing g_p by a scalar circle multiple preserves its nonvanishing. Some such multiple lies in the chart, as only finitely many scalars can give eigenvalue -1. A nonzero polynomial on a real coordinate space has null zero set by the successive slicing argument in Corollary 128.2, which applies unchanged to real variables. The chart density is absolutely continuous, proving the assertion.

Whenever $Z = Z(g)$ is invertible, put $v = gF_0Z^{-1}e_n$. Then $F_0^{*}Jv = e_n$, so the e_j^{*}-coordinates of v are zero and its z-coordinate is -1. Computing its signed norm in two ways gives

$$\|\operatorname{proj}_{P_0} v\|^2 - 1 = v^{*}Jv = -|(Z^{-1})_{nn}|^2.$$

Since $(Z^{-1})_{nn}$ is the penultimate leading determinant divided by $\det Z$, this gives $\Delta(Z) \geq 0$. Consequently $\Delta > 0$ and every leading minor is nonzero on a set of full Haar measure.

Use the decomposition $Z = l^{*}\operatorname{diag}(p_1, \ldots, p_n)u'$ of Corollary 128.2 on this set. For $|p_n| > 1$, the representative $s(l)^{-1}g_p s(u')$, where $s(\cdot)$ is the section of Lemma 129.1, has precisely this Z. By Proposition 126.3, it and Γ_Z belong to the same double coset of H_0. The inner integral in Theorem 128.1 is independent of the choice of representative there, by

left and right invariance from Lemma 126.2. This replacement has a measurable integrand since the new representative is continuous in its coordinates.

In the resulting formula the first factor on the left of g_p is $h_1 s(l)^{-1}$; the factor on its right is the inverse of $h_2 s(u')^{-1}$, where $h_1, h_2 \in H_0$. As h, l vary, $hs(l)^{-1}$ has measure $\mathrm{d}s_0$. Indeed $h = v_0\iota(k_0)$ with $v_0 \in \mathcal{U}_0$. Conjugation by k_0 preserves its measure in the B, T coordinates of Lemma 126.2. At $A = l^{-1}$ the lift $s(l)^{-1}$ differs from $s(A)$ on the left by an element of $\mathcal{U}_0$. Right invariance of the measure on this kernel absorbs the difference. Inversion preserves the entry measure on N_n: in the finite geometric inverse each entry has leading term minus itself plus a polynomial in entries of smaller distance above the diagonal. The asserted measure is now the disintegration of Lemma 129.1 and compact probability on $U(P_0)$. The same reasoning applies on the right. The remaining pivot density is $D^2\omega(p)^2$ by Corollary 128.2. This proves the identity for nonnegative functions. The four nonnegative parts of a complex integrable function give the final assertion. □

Corollary 135.2. *For a nonnegative Borel function $h(y, z)$ invariant under independent left multiplications by S_0 in its two arguments,*

$$\iint h(y,z)\,\mathrm{d}\bar{y}\,\mathrm{d}\bar{z} = D^2 \int_{|p_n|>1} \omega(p)^2 \int_{G_1} h(y, g_p^{-1}y)\,\mathrm{d}y\,\mathrm{d}p.$$

The integrations in the first ν pivots are over $(\mathbb{C}^\times)^\nu$. Absolute integrability on either side permits the same formula for complex h.

Proof. Choose the compact-invariant density a of Lemma 123.2. Then $\int_{S_0} a(sy)\,\mathrm{d}s_0(s) = 1$ for every y. On the left insert $a(y)a(z)$ against ordinary Haar measures. At fixed y change z to $g = yz^{-1}$; inversion preserves the bi-invariant density by the derivative at the identity, and left translation preserves it by Lemma 95.2. Use Theorem 135.1 in g. Set $y = s_1 y'$, giving $z = s_2 g_p^{-1} y'$. The independent invariances of h remove s_1, s_2. The two density integrals are one by the orbit identity for a. Lemma 123.1 justifies all changes of order for these nonnegative integrands. Applying this conclusion first to the modulus gives the integrable complex version. □

136. Haar measure in successive real floors

Definition 136.1. Number the hyperbolic pairs $1, \dots, \nu$ from outside to inside. Write V_j for the space remaining after removing the first $j-1$ pairs, and G_j for its isometries acting as the identity on the removed pairs. Thus $V_n = P_0 \oplus \mathbb{C}z$. For $j < n$ let η_j be the measure on the kernel of the j-th layer in Lemma 20.4, given by Lebesgue measure on its complex row and imaginary central entry. Let $w_j(c)$ be $w(p)$ acting on the j-th pair only with parameter c. Put $d_j = k - 1 + 2(n-j) = \dim_{\mathbb{C}} V_{j+1}$ and set

$$\alpha_j = (c \mapsto w_j(c)^{-1})_*(|c|^{2d_j}\,\mathrm{d}c), \qquad \theta_j = \alpha_j * \eta_j \quad (1 \le j < n).$$

Here $*$ is the image of the product of measures under group multiplication in the order written. Write κ for probability on $\iota(U(P_0))$. Define

$$\alpha_n = (c \mapsto B(c)^{-1})_* \left(\frac{u_k}{u_{k-1}} (|c|^2 - 1)^{k-1} \mathbf{1}_{|c|>1}\,\mathrm{d}c \right), \qquad \theta_n = \alpha_n * \kappa,$$
$$\mu_{\mathrm{up}} = \kappa * \theta_n * \theta_{n-1} * \cdots * \theta_1,$$
$$\mu_{\mathrm{all}} = \kappa * \theta_n * (\eta_{n-1} * \theta_{n-1}) * \cdots * (\eta_1 * \theta_1).$$

All pushforwards can be read by integrating nonnegative Borel tests on the indicated products with the displayed densities.

Theorem 136.2. *Put $\lambda = u_r/u_k$, where u_j are the compact volumes in Definition 95.1. The measures of Definition 136.1 satisfy*

$$\mu_{\mathrm{all}} = \lambda\,\mathrm{d}g, \qquad \mathrm{d}t * \mu_{\mathrm{up}} = \mu_{\mathrm{all}}.$$

Here $\mathrm{d}t$ *is Haar measure on* $\mathcal{U}$ *in Proposition 20.5. Thus every nonnegative Borel function* A *invariant under left* $\mathcal{U}$ *satisfies* $\int A\,\mathrm{d}\mu_{\mathrm{up}} = \lambda \int A\,\mathrm{d}\bar{y}$.

Proof. Conjugation by $h \in G_{j+1}$ preserves η_j. Indeed the row in Lemma 20.4 becomes lh^{-1} and the corner entry t stays fixed. The absolute real determinant of $l \mapsto lh^{-1}$ is $|\det h|^{-2} = 1$, since taking determinants in the hermitian identity gives $|\det h| = 1$. Also $w_j(c)$ commutes with G_{j+1} since each map fixes the space on which the other acts.

For fixed pivots, integrate the defining product for μ_{all} from inside to outside. If the inner product has current value $h \in G_{j+1}$, the next expression is $hv_j w_j(c_j)^{-1} v_j'$ with independent measures η_j on v_j, v_j'. Move v_j to the left of h using the conjugation just proved. By induction the integrand in the inner group has argument $t_{\mathrm{in}} g_{\mathrm{in}} t_{\mathrm{in}}'$ where the middle factor is the product of the inner Weyl matrices and $B(c_n)^{-1}$, and the others are the two inner unipotent products with compact factors. Since $w_j(c_j)$ commutes with t_{in}', this step inserts its inverse into the middle product and appends the new independent unipotents to the outer left and right. The initial inner expression is $k_1 B(c_n)^{-1} k_2$ with the two compact probabilities. This proves by induction that after all steps the two outer factors have measure $\mathrm{d}s_0$ on S_0. Indeed the left order is outside to inside followed by compact probability, and the right order is its reverse; the successive semidirect decomposition of Lemma 20.4 with the conjugation invariance above gives precisely the product measure of Proposition 20.5. The compact group preserves this measure by the same conjugation. Inversion also preserves $\mathrm{d}s_0$, by inversion on the compact group and the unipotent group together with that invariance.

The product in the middle is $g_{\bar{c}}$, where conjugation is componentwise. This follows from $w_j(c)^{-1} = w_j(\bar{c})$ and $B(c)^{-1} = B(\bar{c})$ by multiplying the matrices of Definition 133.1. Coordinatewise conjugation preserves the pivot densities. Their product is $D^2\omega(c)^2(u_k/u_{k-1})/C_{\mathrm{op}}$, and this last constant ratio is u_r/u_k by Theorem 128.1. Comparison with Theorem 135.1 proves the first identity.

For the second identity move only the left v_j's to the front of the product for μ_{all}. At each step all original factors preceding v_j belong to G_{j+1}, so the same change of variable applies at fixed preceding factors. Starting with $j = 1$ and continuing inward leaves the front product $v_1 \cdots v_\nu$ with measure $\mathrm{d}t$; the remaining expression is exactly the defining product for μ_{up}. These integrations all use Lemma 123.1. Finally apply the two measure identities to $a(y)A(y)$, with a of Lemma 123.2. On the left, $A(ty) = A(y)$ and $\int a(ty)\,\mathrm{d}t = 1$; on the right one obtains $\lambda \int a(y)A(y)\,\mathrm{d}y$. □

137. The real energy on sets of finite measure

Proposition 137.1. *Let* V, F *be as in Theorem 124.2. For* $B = V$ *or* F *and* $x \in G_1$ *set*

$$f_x^B(y) = \langle V(y), B(yx)\rangle, \qquad c_B(x) = \int f_x^B(y)\,\mathrm{d}\bar{y}.$$

For every Borel $E \subset G_1$ *of finite Haar measure define*

$$H_E(p) = \omega(p)^2 \int_E \int_{G_1} f_x^V(y)\overline{f_x^V(g_p^{-1}y)}\,\mathrm{d}y\,\mathrm{d}x.$$

On $|p_n| \le 1$ *take* $H_E = 0$. *Then* H_E *is integrable and*

$$\int H_E(p)\,\mathrm{d}p = D^{-2}\int_E |c_V(x)|^2\,\mathrm{d}x.$$

For fixed p, *let* $C(p)$ *be the upper bound for* $\|y\|_{\mathrm{op}}$ *in Proposition 133.3. If* $E \supset \{x : \|x\|_{\mathrm{op}} \le C(p)^2\}$, *then* $H_E(p) = \|\Xi(p)\|_2^2$.

Proof. The functions f_x^B are invariant under S_0 on the left by the common unitary transformation in Theorems 122.2 and 124.2. Their quotient integrals in absolute value are at most D: use Cauchy–Schwarz, the squared norm bounds in the latter theorem and right invariance in Lemma 123.2. For fixed p the integrand defining H_E can be nonzero only when both y

and yx satisfy the two support conditions of Proposition 133.3. Thus y belongs to a compact group norm ball and so does $x = y^{-1}(yx)$, the latter with radius $C(p)^2$. Boundedness gives absolute integrability for each p.

Apply Corollary 135.2 for each x to $h(y,z) = f_x^V(y)\overline{f_x^V(z)}$. Applied to $|h|$, the same identity bounds the integral, including its pivot density $D^2\omega^2$, by D^2 uniformly in x. Integration on E thus permits all changes of order by Lemma 123.1. This gives the stated energy identity. Under the last containment, integrate over all x for this particular p. Set $y' = yx$, so $\mathrm{d}x = \mathrm{d}y'$ at fixed y. Expanding in an orthonormal basis the resulting two integrals factor coordinate by coordinate to give $\omega(p)^2 \sum_{i,j} |T(g_p)_{ij}|^2$ by Definition 134.1. Both y and y' are confined to the same compact ball by the support condition. □

Theorem 137.2. *Use the preceding notation and $d > 0$ of Theorem 123.3. One has*

$$\int_{G_1} |c_F(x)|^2 \,\mathrm{d}x \le D^2/d, \qquad |c_V(x) - c_F(x)| \le Db^{-5r^2} \quad (x \in G_1).$$

For every Borel E of finite Haar measure $|E|$,

$$D^{-1}\left(\int_E |c_V(x)|^2 \,\mathrm{d}x\right)^{1/2} \le d^{-1/2} + b^{-5r^2}|E|^{1/2}.$$

In particular $D^{-2}\int_E |c_V|^2 \le (1 + b^{-10})/d$ whenever $b \ge 3$ and $d|E| \le b^{10r^2-22}$.

Proof. The previous proof gives $|c_F| \le D$. For a compact group norm ball A use $\varphi = \mathbf{1}_A\overline{c_F}$ in Theorem 123.3. It belongs to L^1 and L^2 since the ball has finite measure. Pair the resulting convolution with V on the quotient. The integral of the absolute scalar integrand is bounded by $D\|\varphi\|_1$, as follows from the same quotient Cauchy–Schwarz estimate. Lemma 123.1 identifies the pairing with $\int_A |c_F|^2$. Quotient Cauchy–Schwarz and the Schur bound give

$$\int_A |c_F|^2 \le \sqrt{D}\,(D/d)^{1/2}\left(\int_A |c_F|^2\right)^{1/2}.$$

This bounds the integral by D^2/d, also if it vanishes. Increasing the balls to the whole group and applying Lemma A.2 gives the first assertion. For the second use $\|V\|_2 \le \sqrt{D}$, right invariance and $\|V - F\|_2 \le b^{-5r^2}\sqrt{D}$ on the quotient by Theorem 124.2. The triangle inequality on E, obtained by expanding the square and applying Cauchy–Schwarz to the mixed term, now gives the finite-measure bound. In the indicated range the right side is at most $d^{-1/2}(1 + b^{-11})$, whose square is at most $(1 + b^{-10})/d$ since $2 + b^{-11} \le b$. □

138. Real floor quadrics and their measures

Definition 138.1. For $1 \le j \le n$ use the space V_j of Definition 136.1. Put $h = n - j$ and $m = k - 1$. In $P_0 \oplus \mathbb{C}z$ replace the two coordinates u, z by the hyperbolic pair $(u+z)/\sqrt{2}, (u-z)/\sqrt{2}$. This change is ordinary unitary. Write a vector in V_j as (d, t, A, w, B), with d, t the new pair, $A, B \in \mathbb{C}^h$ on the remaining hyperbolic pairs and $w \in \mathbb{C}^m$ on the positive complement. Its signed norm is

$$Q(v) = 2\,\mathrm{Re}(\bar{t}d + A^*B) + \|w\|^2.$$

For every $\gamma \in \mathbb{R}$, define $\delta_\gamma^{(j)}$ as the image of the product measure $\mathrm{d}t\,\mathrm{d}A\,\mathrm{d}w\,\mathrm{d}B\,\mathrm{d}a$ for $t \in \mathbb{C}^\times$, $a \in \mathbb{R}$ under the map retaining t, A, w, B and putting

$$d = \frac{\gamma - 2\,\mathrm{Re}(A^*B) - \|w\|^2}{2\bar{t}} + ita.$$

Here complex coordinates have ordinary two-dimensional Lebesgue measure. Each such image is supported on $Q = \gamma$.

Lemma 138.2. *For compactly supported continuous ϕ on V_j, the function $\gamma \mapsto \int \phi \, \mathrm{d}\delta_\gamma^{(j)}$ is continuous. For every nonnegative Borel function B_1 on $\mathbb{R} \times V_j$,*

$$\int_{V_j} B_1(Q(v), v) \, \mathrm{d}v = \frac{1}{2} \int_{\mathbb{R}} \int_{V_j} B_1(\gamma, v) \, \mathrm{d}\delta_\gamma^{(j)}(v) \, \mathrm{d}\gamma.$$

The measure at every γ is invariant under G_j; the same flat formula may be used with any one of the displayed hyperbolic pairs as its distinguished pair.

Proof. At fixed $t \neq 0, A, w, B$, write $d = t(x + ia)$ with $x, a \in \mathbb{R}$. The ordinary Jacobian is $|t|^2$ and $\mathrm{d}\gamma = 2|t|^2 \, \mathrm{d}x$. The discarded set $t = 0$ has zero ambient measure by Lemma 123.1. These facts prove the integral identity. If ϕ vanishes outside the coordinate box of radius L, the free complex coordinates in its flat formula have moduli $\leq L$ where it is nonzero, and $|a| \leq L/|t|$ since $a = \mathrm{Im}(d/t)$. The integral $\int_{|t| \leq L} |t|^{-1} \, \mathrm{d}t = 2\pi L$ by polar coordinates. Thus the parameter integrands are dominated by the integrable product of these indicators times $\|\phi\|_\infty$. Continuity follows from Lemma A.2.

Each G_j-isometry preserves Q and has complex determinant of modulus one. Its real Jacobian equals the squared modulus of this determinant, for instance by triangularizing a complex matrix by unitary changes of coordinates as in Lemma 95.2. For continuous compactly supported ϕ, insert $\mathbf{1}_I(\gamma)\phi(v)$ in the integral identity for any bounded interval I. After an isometry change in the ambient integral, the two continuous functions of γ thus have the same integral on every I. They agree at every real point by continuity. The same comparison works on exchanging two hyperbolic coordinate pairs, which preserves the ambient measure. Equality on continuous compact tests implies equality of the corresponding locally finite Borel measures here: on a bounded open set approximate its indicator from below by the continuous minima of one and a positive integer times distance to the complement. Monotone convergence and open exhaustion give equality on opens, and differences and countable disjoint unions on bounded boxes then give equality on their generated Borel sets, as in Lemma 123.1. □

Proposition 138.3. *For $j < n$, the map $g \mapsto g^{-1}e_j$ sends θ_j to $\delta_0^{(j)}$. At $j = n$, the map $g \mapsto g^{-1}z$ sends θ_n to $\delta_{-1}^{(n)}$. The same assertions hold after applying any fixed member of G_j to the resulting vector.*

Proof. At $j < n$ use e_j, e_j^* as the distinguished pair by Lemma 138.2. In the notation of Lemma 20.4 a point integrated by θ_j is $g = w_j(c)^{-1}H(l, t_0)$. The outside coordinates of $g^{-1}e_j$ are $c\bar{t}_0, c$, and the inner vector is $v' = cJ'l^*$, where J' is the hermitian matrix on V_{j+1}. This follows by triangular inversion and $t_0 + \bar{t}_0 = -lJ'l^*$. In particular

$$c\bar{t}_0 = -\frac{(v')^* J' v'}{2\bar{c}} - ic \, \mathrm{Im} \, t_0.$$

Changing the row l to v' multiplies its real measure by $|c|^{2d_j}$, because J' is ordinary unitary. This is exactly the pivot weight in Definition 136.1. Taking $a = -\mathrm{Im} \, t_0$ leaves ordinary real measure, which gives the required flat formula.

At the last floor $g = B(c)^{-1}\iota(k_0)$ and $g^{-1}z = (|c|^2 - 1)^{1/2}k_0^{-1}u - cz$ on $|c| > 1$. The vector $k_0^{-1}u$ has the probability of a uniformly distributed unit vector by Theorem 95.3. In ordinary coordinates $v' = v_+ - cz$, $Q(v') = \|v_+\|^2 - |c|^2$. The polar formula of that theorem in the k positive coordinates, with $\gamma = \|v_+\|^2 - |c|^2$, gives radial measure

$$\frac{1}{2} \frac{u_k}{u_{k-1}} (\gamma + |c|^2)^{k-1} \mathbf{1}_{\gamma + |c|^2 > 0} \, \mathrm{d}\gamma \, \mathrm{d}\omega \, \mathrm{d}c,$$

where $\mathrm{d}\omega$ is the unit-vector probability. For compact continuous tests the integral in ω, c is continuous in γ, since $k \geq 2$, c stays in a bounded set and the positive radial power tends to zero at the boundary. Comparing the two disintegrations on every interval as in Lemma 138.2 identifies this measure at $\gamma = -1$ with the required flat measure (after the

common factor $1/2$ is canceled). It agrees with θ_n by its definition. All fixed isometries preserve the resulting measures by the lemma. □

139. Fourier bounds on a bounded part of a real floor

Lemma 139.1. *Write $\psi(t) = \exp(t - \bar{t})$ for $t \in \mathbb{C}$. For $m \geq 0$, $\xi \in \mathbb{C}^{\times}$, $H \in L^1(\mathbb{C}^m) \cap L^2(\mathbb{C}^m)$, put $\widehat{H}_\xi(v) = \int \psi(\xi v^* w) H(w)\,\mathrm{d}w$. Then*

$$\int |\widehat{H}_\xi(v)|^2\,\mathrm{d}v \leq (\pi/|\xi|)^{2m} \int |H(w)|^2\,\mathrm{d}w.$$

Proof. For $\epsilon > 0$ integrate the squared transform against $\exp(-\epsilon\|v\|^2)$. Absolute integration of its expansion follows from the Gaussian integral and $\|H\|_1^2 < \infty$. In $2m$ real coordinates the integral of the Gaussian times $\psi(\xi v^* z)$ equals $(\pi/\epsilon)^m \exp(-|\xi|^2\|z\|^2/\epsilon)$. Indeed the linear form $2\,\mathrm{Im}(\xi v^* z)$ in those real coordinates has coefficient vector of squared length $4|\xi|^2\|z\|^2$; apply the one-dimensional Gaussian transform proved in Lemma 4.2 to each coordinate. Use $z = w - w'$ in the expanded squared transform. For each z, Cauchy–Schwarz gives $\int |H(w')H(w'+z)|\,\mathrm{d}w' \leq \|H\|_2^2$. The remaining integral of $\exp(-|\xi|^2\|z\|^2/\epsilon)$ is $(\pi\epsilon/|\xi|^2)^m$. This proves the bound uniformly in ϵ, and monotone convergence in Lemma A.2 as $\epsilon \downarrow 0$ proves the assertion. For $m = 0$ the measures have mass one. □

Theorem 139.2. *In Definition 138.1 let μ, ν' each be either ambient Lebesgue measure or a measure $\delta_\gamma^{(j)}$, with arbitrary real level in either slot. Write $\mathcal{Q}_L$ for the box of vectors whose flat coordinate entries d, t, A, w, B have modulus at most $L > 0$. Put*

$$M_{\mathrm{amb}}(L) = (\pi L^2)^{2h+2}, \qquad M_{\mathrm{quad}}(L) = 4\pi L^2 (\pi L^2)^{2h},$$

and denote the appropriate quantity by M_μ or $M_{\nu'}$. Take Borel $f \in L^2(\mu)$ vanishing off $\mathcal{Q}_L$, a shift $y_0 \in V_j$, and $L' > 0$, $\xi \in \mathbb{C}^{\times}$. Then

$$\int_{\mathcal{Q}_{L'}} \left| \int \psi(\xi[v + y_0, z]) f(z)\,\mathrm{d}\mu(z) \right|^2 \mathrm{d}\nu'(v) \leq M_{\nu'}(L') M_\mu(L) (\pi/|\xi|)^{2m} \int |f|^2\,\mathrm{d}\mu.$$

Here $[v, z] = v^* J z$ and $m = k - 1$.

Proof. For either measure separate the variables $p = (t, A)$, w, and the fibre. The fibre variables are (B, d) for ambient measure and (B, a) for a quadric. All parameter measures are products by Definition 138.1. At fixed p, w the fibre volume in $\mathcal{Q}_L$ is at most $m_{\mu,L}(p)$. For ambient measure take $(\pi L^2)^{h+1}$, and for a quadric take $(\pi L^2)^h 2L/|t|$, in both cases times the indicator that the entries of p have modulus at most L. The latter bound follows from $|a| \leq L/|t|$ in Lemma 138.2. Set its value zero at $t = 0$, a null set. Direct integration gives $\int m_{\mu,L}(p)\,\mathrm{d}p = M_\mu(L)$. This argument applies also to the target measure. In particular each box has finite measure since the remaining w range in a finite box. Thus $f \in L^1(\mu)$.

Use subscripts v, z to distinguish output and input coordinates. The pairing in the phase is a sum of three terms:

$$\bar{d}_v t_z + B_v^* A_z, \qquad w_v^* w_z, \qquad \bar{t}_v d_z + A_v^* B_z + [y_0, z].$$

At fixed p_v, p_z, w_z integrate f times the phase of the third term over the input fibre; call the resulting function $H_{p_v,p_z}(w_z)$. Its absolute value is bounded by the fibre integral of $|f|$. Cauchy–Schwarz on that fibre gives

$$|H_{p_v,p_z}(w_z)|^2 \leq m_{\mu,L}(p_z) \int_{\mathrm{fibre}} |f(z)|^2.$$

In the third term d_z may depend on w_z; throughout this fibre integral p_v is fixed, and the third term is independent of w_v. Thus the second phase produces precisely $\widehat{H}_{p_v,p_z,\xi}(w_v)$ of Lemma 139.1. The first term has phase of modulus one depending only on the output and p_z, so the absolute value of the full integral is at most $\int |\widehat{H}_{p_v,p_z,\xi}(w_v)|\,\mathrm{d}p_z$, uniformly in the output fibre. All these decompositions use Lemma 123.1.

Apply weighted Cauchy–Schwarz in p_z to square this last bound, using weight $m_{\mu,L}(p_z)$ and omitting its zero set, where the corresponding integrals vanish almost everywhere. For each fixed p_v, integration in w_v over the whole space and Lemma 139.1 bound the result by

$$M_\mu(L)(\pi/|\xi|)^{2m} \iint \frac{|H_{p_v,p_z}(w_z)|^2}{m_{\mu,L}(p_z)} \,\mathrm{d}w_z \,\mathrm{d}p_z \le M_\mu(L)(\pi/|\xi|)^{2m}\|f\|^2_{L^2(\mu)}.$$

The fibre estimate ensures the Fourier lemma applies for almost every p_z; w_z ranges in a bounded box which also gives the L^1 condition. On the output box integrate the uniform fibre bound first, at cost $m_{\nu',L'}(p_v)$, then integrate in p_v. Its integral $M_{\nu'}(L')$ gives the claimed factor. □

140. An exact negative moment for the real floor weights

Lemma 140.1. *Let $m = s + 1 + c$, $s, c \ge 0$ integers, and let Π be ordinary orthogonal projection of rank $s + 1$ on $\mathbb{C}^m$. For $T, H > 0$,*

$$\int_{\|w\|\le T} \left(\frac{H}{\|\Pi w\|^2}\right)^s \mathrm{d}w = \frac{H^s\pi^m}{s!(c+1)!}T^{2(c+1)}.$$

Values on the null kernel may be assigned arbitrarily.

Proof. Choose orthonormal bases in the range and kernel, so that $w = (w', w'')$ and $\Pi w = (w', 0)$. This ordinary unitary change preserves real volume. The ball in $\mathbb{C}^c$ above w', with $u = \|w'\|^2$, has volume $\pi^c(T^2 - u)^c/c!$. By Theorem 95.3, polar integration in $\mathbb{C}^{s+1}$ gives $\pi^{s+1}u^s \,\mathrm{d}u/s!$, including the angular probability. Their product times $H^s u^{-s}$ integrates over $0 < u < T^2$. Integrating $(T^2 - u)^c$ gives $T^{2(c+1)}/(c + 1)$, proving the formula, also for $s = 0$ or $c = 0$. □

Proposition 140.2. *Let $m = k - 1 = s + 1 + 3n$ and $s \ge 0$. Let E_+ denote the inclusion of the m orthonormal positive coordinates of Definition 138.1 into the full space. Let C be any complex matrix with m columns and at most n rows, and let $g, h \in G_1$. Call a pair (z, z') close if $\|z\| < 1/4$ and $\|z - z'\| < 1/4$, with ordinary Euclidean norms. Define*

$$W = \{w \in \ker C : (g^{-1}E_+w, h^{-1}E_+w) \text{ is close}\}.$$

Let $|\cdot|$ on subsets of $\mathbb{C}$ denote ordinary area, with outer area permitted for arbitrary subsets. There is an orthogonal projection Π of rank $s + 1$ such that the function $\phi(v) = 36/\|\Pi v\|^2$, valued $+\infty$ on the kernel, has the following properties.

1. $\left|\{v^*w : w \in W\}\right|^{-1} \le \phi(v)$, *with* $1/0 = +\infty$.
2. *The set of $\alpha \in \mathbb{C}$ such that for each $w \in W$ there exist close pairs $(z_1, z_2), (z_1', z_2')$ with*
$$(\alpha v)^*w = [z_1, z_1'] - [z_2, z_2']$$
has outer area at most $\phi(v)$.
3. *For integers $j \ge 0$, $n \ge 48$, and $L \ge \max(m, 4)$,*
$$\int_{\max_i |v_i|\le L^j} \phi(v)^s \,\mathrm{d}v \le L^{(2j+3)(3n+1)}.$$

Proof. Write P_- for the Euclidean projection onto the negative eigenspace of J, of rank n by Proposition 133.3. Stack the maps $C, P_-g^{-1}E_+, P_-h^{-1}E_+$. Their combined rank is at most $3n$, so their common kernel has dimension at least $s + 1$. Choose Π onto any $(s + 1)$-plane in it. For w in this plane the two images $g^{-1}E_+w, h^{-1}E_+w$ lie in the positive eigenspace, where Euclidean and signed squared norms agree. Since g, h are isometries, both Euclidean norms equal $\|w\|$. Consequently $\{w \in \operatorname{ran}\Pi : \|w\| < 1/8\} \subset W$ by the triangle inequality.

Put $a = \|\Pi v\|$. If $a > 0$, each scalar t with $|t| < a/8$ equals v^*w for $w = t\Pi v/a^2$ in this ball. Thus the image has area at least $\pi a^2/64$. It is measurable since W is open in the subspace $\ker C$ and a linear functional on that space is either zero or open onto $\mathbb{C}$. For any

close pair the second vector has norm $< 1/2$. Since J is ordinary unitary, the difference of the two pairings in the second assertion has modulus at most

$$\|z_1 - z_2\|\|z_1'\| + \|z_2\|\|z_1' - z_2'\| < 3/16.$$

It follows from the disk just constructed that $|\alpha|a/8 \le 3/16$, hence $|\alpha| \le 3/(2a)$. Its disk has area $9\pi/(4a^2)$. Both this and the first reciprocal-area bound are at most $36/a^2$, using $2 < \pi < 4$. When $a = 0$ the extended inequalities follow directly.

The entry box lies in the ball of radius $T = \sqrt{m}\,L^j$. Use Lemma 140.1 with $c = 3n, H = 36$. From the exponential series $(36\pi)^s/s! \le \exp(36\pi)$. As $L \ge 4$, $\log L \ge 1$, and $3n + 1 \ge 145 > 36\pi$. Thus $\exp(36\pi) \le L^{3n+1}$, and $\pi^{3n+1}/(3n+1)! \le L^{3n+1}$. Finally $T^{2(3n+1)} \le L^{(2j+1)(3n+1)}$. Multiplication gives the third bound. □

141. A pair of negative unit vectors at the last pivot

Lemma 141.1. *On $M = G_n$ the measure $\mathrm{d}m = \kappa * \theta_n$ of Definition 136.1 is bi-invariant and inversion invariant. For $t \ge 1$, with $c_k = \pi u_k/(k u_{k-1})$,*

$$\delta_{-1}^{(n)}\{v : \|v\| \le t\} = c_k((t^2-1)/2)^k,$$
$$v_2 := m\{h : \|h\|_{\mathrm{op}} \le 2\} = c_k(9/16)^k.$$

For $A \ge 1$ write $B_G(A) = \{g \in G_1 : \|g\|_{\mathrm{op}} \le A\}$. There is a nonnegative Borel function $a_A \le v_2^{-1}$ on G_1, supported in $B_G(2A)$, with $\int_M a_A(hg)\,\mathrm{d}m(h) = 1$ for all $g \in MB_G(A)$.

Proof. For a nonnegative Borel test F_1, first average $F_1(h)$ on the left over κ. This gives a function only of $v = h^{-1}z$, since two representatives of the same v differ on the left by a matrix fixing z, hence by $\iota(U(P_0))$. The function of v is Borel. Indeed for $v = v_+ - cz$ with $Q(v) = -1$ choose a unitary carrying u to $v_+/\|v_+\|$ when $v_+ \ne 0$, and the identity otherwise. A Borel choice is obtained by appending the coordinate vectors in order, subtracting their projections to the vectors retained and normalizing each nonzero residual. The inverse of the resulting compact rotation times the boost, in the order $(\iota(k_0)B(c))^{-1}$, is a Borel representative with $h^{-1}z = v$. Proposition 138.3 identifies the distribution of v under $\kappa * \theta_n$ with $\delta_{-1}^{(n)}$. Right translation acts on v by the inverse isometry, preserving that measure by Lemma 138.2. This proves right invariance. The product $\kappa * \alpha_n * \kappa$ is also invariant under inversion by $B(c)^{-1} = B(\bar{c})$, radiality of its density, and inversion invariance of compact probability. Thus it is left invariant as well.

For $v = v_+ - cz$ on the negative unit quadric, $\|v\|^2 = 2|c|^2 - 1$. In the radial formula of Proposition 138.3 integrate $(u_k/u_{k-1})(|c|^2-1)^{k-1}\,\mathrm{d}c$ from $|c| = 1$ to $\sqrt{(t^2+1)/2}$. In polar coordinates this gives the first formula. The norm of $B(c)$ on $|c| > 1$ is $|c| + \sqrt{|c|^2-1}$, since its two nontrivial squared singular values are $(|c| \pm \sqrt{|c|^2-1})^2$ by direct multiplication. Compact factors preserve this norm, which is at most two exactly when $|c| \le 5/4$. The same radial integration gives v_2. It also shows that m is finite on all norm balls.

The set $MB_G(A)$ is closed. For a convergent sequence $h_i b_i$ with $b_i \in B_G(A)$, compactness gives a subsequence with b_i convergent and then h_i has a limit in the closed subgroup M. On this set put $a_A(g) = \mathbf{1}_{B_G(2A)}(g)/N_A(g)$, where $N_A(g) = \int_M \mathbf{1}_{B_G(2A)}(hg)\,\mathrm{d}m(h)$, and set $a_A = 0$ off it. The denominator is left M-invariant, Borel, and finite since $\|h\|_{\mathrm{op}} \le 2A\|g\|_{\mathrm{op}}$ on its integration set. If $\|g\|_{\mathrm{op}} \le A$ it is at least v_2; invariance extends this to $MB_G(A)$. The asserted orbit integral and all the other bounds follow directly. □

Proposition 141.2. *Fix heads $p, q \in (\mathbb{C}^\times)^\nu$. Write $w_p = w((p,c))$, $P(p) = \prod_{j=1}^{\nu} |p_j|^{d_j}$ for d_j of Definition 136.1, and $A_p = C((p,c))$ for the bound in Proposition 137.1; it is independent of c. Put $X = A_pA_q$, $\delta = \delta_{-1}^{(n)}$. For $g, x \in G_1$ define functions of $v = h^{-1}z$ and $v' = \tau^{-1}z$, $h, \tau \in M$, by*

$$U_{g,x}(v) = \langle V(hg), V(hgx)\rangle,$$
$$W_{g,x}(v') = \langle V(\tau w_p^{-1} g), V(\tau w_q^{-1} gx)\rangle.$$

For every $\xi \in \mathbb{C}$, the integral $I_\xi = \int_{\mathbb{C}} \psi(\xi c) J_n((p,c),(q,c))\,\mathrm{d}c$ equals

$$\frac{u_k P(p)P(q)}{u_r D^2} \int_{B_G(X)} \int_{B_G(2A_p)} a_{A_p}(g) \iint \psi(\xi[v,v'])U_{g,x}(v)\overline{W_{g,x}(v')}\,\mathrm{d}\delta(v)\,\mathrm{d}\delta(v')\,\mathrm{d}g\,\mathrm{d}x.$$

The quadrics in the last integral may be restricted respectively to $\|v\| \le 2RA_p$ and $\|v'\| \le 2RA_p\|w_p\|_{\mathrm{op}}$.

Proof. The two functions on the quadrics are well defined by the compact equivariance of V and the commutation of both Weyl elements with $\iota(U(P_0))$. They are Borel by the representative in Lemma 141.1. By Theorem 134.2 and the middle radial density, I_ξ equals $u_k P(p)P(q)/(u_r D^2)$ times

$$\int_M \psi(\xi[z,\sigma z])\,\mathrm{tr}\big(T(w_p\sigma)^* T(w_q\sigma)\big)\,\mathrm{d}m(\sigma).$$

Indeed the middle Haar product integrates a pair of independent compact factors around $B(c)^{-1}$. These factors preserve the displayed trace by that theorem and preserve $[z,\sigma z]$. The change $c \mapsto \bar{c}$ replaces the inverse boost by $B(c)$. The ratio of the pivot and radial constants is $C_{\mathrm{op}} u_{k-1}/u_k = u_k/u_r$.

Expand the trace in its two independent integration variables y, y'. Its integrand is

$$\langle V(y), V(y')\rangle\,\overline{\langle V(\sigma^{-1}w_p^{-1}y), V(\sigma^{-1}w_q^{-1}y')\rangle}.$$

Whenever this product is nonzero, $\|y\|_{\mathrm{op}} \le A_p$, $\|y'\|_{\mathrm{op}} \le A_q$. In fact the middle double-coset decomposition of Lemma 133.2 reduces each pair to Proposition 133.3 by compact translations, which preserve both norm and S. It also gives $|[z,\sigma z]| \le R^{2n}/|\prod_j p_j|$. Thus σ lies in a norm ball in M, by the boost formula of Lemma 141.1. The integral of the absolute value of the expanded product is finite, as all three variables are bounded and V is bounded.

Set $y' = yx$, so $\|x\|_{\mathrm{op}} \le X$ on this support. Insert $1 = \int_M a_{A_p}(h^{-1}y)\,\mathrm{d}m(h)$ there and make the changes $y = hg$, $\sigma = h\tau^{-1}$ at fixed h. The measures are preserved by translation and inversion. Now $[z,\sigma z] = [h^{-1}z, \tau^{-1}z]$ and w_p, w_q commute with M. The two scalar factors become exactly the displayed $U_{g,x}, \overline{W_{g,x}}$. The change of order is justified also for their absolute values by the original absolute integral and the nonnegative insertion, using Lemma 123.1. Projection from h, τ to their negative unit vectors gives the two copies of δ by Proposition 138.3.

If $V(hg) \ne 0$, some $t \in \mathcal{U}$ satisfies $\|thg\|_{\mathrm{op}} \le R$ by Theorem 124.2. The vector $t^{-1}z$ equals z plus a linear combination of the outer e_j by Lemma 129.1. Hence $\|h^{-1}z\| \le \|h^{-1}t^{-1}z\| \le R\|g\|_{\mathrm{op}}$, where the first inequality is ordinary orthogonal projection onto V_n. Apply this also to $\tau w_p^{-1} g$ to bound the second quadric support. Lemma 141.1 confines g to the required norm ball, proving the last claims. □

142. Fourier decay at the last real pivot

Theorem 142.1. *Use the notation of Proposition 141.2, and let $T_1 = 2RA_p$, $T_2 = T_1\|w_p\|_{\mathrm{op}}$. Write $b_G(t) = \int_{B_G(t)} \mathrm{d}g$, $v(t) = c_k((t^2-1)/2)^k$ for $t \ge 1$, and $M_V = \sup_y \|V(y)\|$. For every $\xi \ne 0$ one has*

$$|I_\xi| \le C(p,q)(\pi/|\xi|)^{k-1},$$
$$C(p,q) = \frac{u_k P(p)P(q)}{u_r D^2} \frac{b_G(X) b_G(2A_p)}{v_2}\, 4\pi T_1 T_2\, M_V^4 \sqrt{v(T_1)v(T_2)}.$$

At every ξ, including zero, one also has

$$|I_\xi| \le \frac{u_k P(p)P(q)}{u_r D^2} \frac{b_G(X) b_G(2A_p)}{v_2}\, M_V^4 v(T_1) v(T_2).$$

Proof. All the constants are finite by Lemmas 95.2 and 141.1 and Theorem 124.2. For fixed g, x in Proposition 141.2, the two amplitudes have modulus at most M_V^2. Their squared integrals on the quadric are at most $M_V^4 v(T_1)$ and $M_V^4 v(T_2)$ by their respective supports and Lemma 141.1. The Euclidean norm balls of these radii are contained in the corresponding coordinate boxes of Definition 138.1, because the coordinate basis is ordinary orthonormal. Pair $U_{g,x}$ by Cauchy–Schwarz with the Fourier integral of $\overline{W_{g,x}}$ on the first box. Theorem 139.2 with both measures $\delta_{-1}^{(n)}$, $h = 0$, $m = k - 1$ and shift zero gives

$$\left| \iint \psi(\xi[v, v']) U_{g,x}(v) \overline{W_{g,x}(v')} \, \mathrm{d}\delta(v) \, \mathrm{d}\delta(v') \right| \le (\pi/|\xi|)^{k-1} (4\pi T_1 T_2) M_V^4 \sqrt{v(T_1) v(T_2)}.$$

Here the factor $4\pi T_1 T_2$ is the square root of the product of the two fibre bounds. Integrating in x, g and using $0 \le a_{A_p} \le v_2^{-1}$ gives the first assertion. The direct integral bound by the product of the two L^1 norms gives the second assertion. □

Corollary 142.2. *Define $J_\nu(p, q) = I_0$ on $(\mathbb{C}^\times)^\nu$ in each argument. This is a continuous positive semidefinite kernel. For all p, q, the Fourier transform at frequency ξ of the common-last-entry function integrating to $J_\nu(p, q)$ satisfies Theorem 142.1. Its constants are bounded uniformly when the two heads range in compact sets.*

Proof. For a compact set of heads the last variable is confined to a common bounded disk by Theorem 134.2 and the lower bounds on all first pivot moduli. Its integrand vanishes for $|c| \le 1$, and elsewhere it is continuous jointly in the heads and c. It is bounded on this compact range by continuity. Lemma A.2 gives continuity of the mean. For heads p_i and scalars s_i, the quadratic form of J_ν is $\int_{\mathbb{C}} \| \sum_i s_i \Xi((p_i, c)) \|_2^2 \, \mathrm{d}c$; the integral is finite and nonnegative by Theorem 134.2. The Fourier assertion is that theorem's common-last-entry function in Proposition 141.2. Its constants are locally bounded because all $A_p, A_q, P(p), P(q), \|w_p\|_{\mathrm{op}}$ are bounded on compact head sets, by their explicit formulas in Proposition 133.3 and Definition 133.1. □

143. Bounded representatives at partial real flags

Lemma 143.1. *Let $J = J^* = J^{-1}$ on a finite-dimensional complex Euclidean space decomposed as $E \oplus F$, with both spaces J-invariant and Euclidean orthogonal. Let $T^* J T = J$ and $B = \max(1, \|T|_E\|_{\mathrm{op}})$. Some isometry H acting identically on E satisfies*

$$\|TH\|_{\mathrm{op}} \le B + \sqrt{1 + B^2}.$$

Proof. Write $T_E : E \to V$ for $T|_E$, $J_E = J|_E$ and $\mathcal{V} = TF$. The projection onto TE along $\mathcal{V}$ is $T_E J_E T_E^* J$. Indeed this map is the identity on TE and zero on TF by $T^* J T = J$. Its complementary projection Q has norm at most $1 + B^2$. Write P for ordinary orthogonal projection onto $\mathcal{V}$. The self-adjoint operator $PJ|_{\mathcal{V}}$ on $\mathcal{V}$ has inverse $QJ|_{\mathcal{V}}$. To check the left inverse, for $y \in \mathcal{V}$ compute $QJPJy = Qy - QJ(I - P)Jy = y$. Here $J(\mathcal{V}^\perp) = TE$, since it is the signed orthogonal complement of TF. In finite dimension the left inverse is the inverse. Every eigenvalue of $PJ|_{\mathcal{V}}$ therefore has absolute value at least $1/(1 + B^2)$.

Diagonalize this operator in an ordinary orthonormal basis, as in Lemma 79.2. The numbers of either sign coincide with those of $J|_F$. Indeed in a diagonal signed basis the largest dimension of a subspace on which the form is positive equals the number of positive entries, by projecting to their span, and similarly for negative entries. The isometry T preserves these maximal dimensions. Send a unit signed eigenbasis of F to the matched eigenvectors in $\mathcal{V}$ divided by the square roots of the absolute eigenvalues. This signed isometry $U : F \to \mathcal{V}$ has ordinary norm at most $\sqrt{1 + B^2}$. Define $H|_F = T^{-1} U$ and $H|_E = I$. The two spaces are also signed orthogonal, so H is an isometry. On the two summands TH is T_E, U, which gives its norm bound by the triangle inequality. □

Theorem 143.2. *For $1 \le j \le \nu$ write $E = (e_1, \dots, e_j)$, $E' = (e_1^*, \dots, e_j^*)$, and $Q_j = \{s \in G_1 : sE = EA_s,\ A_s \in N_j(\mathbb{C})\}$, with N_j upper unitriangular. Let w flip these j*

pairs with parameters $p_i \in \mathbb{C}^\times$ as in Definition 133.1 and fix the other coordinates. Write $D_i = \prod_{l=1}^{i} p_l$, $D_0 = 1$. Suppose

$$s, t \in Q_j, \qquad \|sg\|_{\mathrm{op}},\ \|tw^{-1}g\|_{\mathrm{op}} \le R, \qquad L \ge \max(4, r, R, \max_{0\le i\le j} |D_i|^{-1}).$$

Then $|D_i| \le R^{2i}$ for $i \le j$, and some $h \in G_{j+1}$ satisfies

$$\|hg\|_{\mathrm{op}} \le L^{4j+2}, \qquad \|hw^{-1}g\|_{\mathrm{op}} \le L^{6j+3}.$$

Also $\|sE'\|_{\mathrm{op}} \le L^{4j+1}$. If s acts trivially on the intervening signed space V_{j+1} modulo span E*, then $\|g\|_{\mathrm{op}} \le L^{6j+4}$ and $\|w^{-1}g\|_{\mathrm{op}} \le L^{8j+5}$.*

Proof. As in Proposition 133.3, take $X = E^* J(swt^{-1})E$. It has norm at most R^2, since $swt^{-1} = (sg)(tw^{-1}g)^{-1}$. Its triangular factorization is $X = L_1 D U_1$, where $L_1 = A_s^{-*}$, $U_1 = A_t^{-1}$, $D = \mathrm{diag}(p_i)$. This gives the bound on the leading determinants. For $b \le i \le j$,

$$(L_1^{-1})_{ib} = (-1)^{b+i} \frac{\det X[\{1, \dots, i\} \setminus \{b\},\ 1{:}i-1]}{D_{i-1}}.$$

Indeed the signed cofactors form a row annihilating the first $i-1$ columns in the leading i rows, by a determinant with a repeated column. Their last entry after division is one. The corresponding row of L_1^{-1} has these properties, and the first $i-1$ entries are uniquely determined by the invertible leading minor. Transpose the argument to obtain the analogous formula for the entries of U_1^{-1}. All the numerator minors have modulus at most $R^{2(i-1)}$ by the column projection proof in Proposition 133.3. Bounding an operator norm by j times its largest entry now gives $\|A_s\|_{\mathrm{op}}, \|A_t\|_{\mathrm{op}} \le L^{2j}$.

Write $T = g^{-1}$. Then $TE = (sg)^{-1} E A_s$ and $TwE = (tw^{-1}g)^{-1} E A_t$ have norms at most L^{2j+1}. Since $wE = E'D$ and $|p_i|^{-1} = |D_{i-1}/D_i| \le L^{2j-1}$, we obtain $\|TE'\|_{\mathrm{op}} \le L^{4j}$ and $\|sE'\|_{\mathrm{op}} \le R\|TE'\|_{\mathrm{op}} \le L^{4j+1}$. On the span of E, E', T therefore has norm at most $2L^{4j}$. Apply Lemma 143.1 to this span and V_{j+1}, and take $h = H^{-1}$. Then $\|hg\|_{\mathrm{op}} = \|(hg)^{-1}\|_{\mathrm{op}} \le 6L^{4j} \le L^{4j+2}$. Since $|p_i| = |D_i/D_{i-1}| \le L^{2j+1}$, the norm of w and of its inverse is at most L^{2j+1}. It commutes with h, giving the second assertion.

Under the additional hypothesis, in the blocks E, V_{j+1}, E' the matrix s has diagonal A_s, I, A_s^{-*}, zero lower blocks and upper blocks B_1, C_1, D_1 in positions $12, 13, 23$. This follows by preserving E and its signed perpendicular, and by evaluating its mixed E, E' pairing. The signed inner products of middle and last columns give $B_1 = -A_s D_1^* J|_{V_{j+1}}$. The whole last block column has norm at most L^{4j+1} by the preceding bound on sE'. Thus $\|s\|_{\mathrm{op}} \le 6L^{6j+1} \le L^{6j+3}$ by adding the norm bounds of its six blocks. Multiplying sg by s^{-1}, and then multiplying g by w^{-1}, gives the two final inequalities. □

144. Uniform integrals on the successive real floors

Definition 144.1. Write $f_j = e_j$ for $1 \le j \le \nu$, $f_n = z$. For $0 \le j \le n$ let Q_j consist of those $s \in G_1$ with $sf_a - f_a \in \mathrm{span}(f_1, \dots, f_{a-1})$ for all $a \le j$. Thus $Q_0 = G_1$, and for $1 \le j \le \nu$ this agrees with Theorem 143.2. Say g is near at j when $Q_j g$ meets $B_G(R)$; write $\mathcal{N}_j$ for this set. For $1 \le j < n$ and $x \in G_1$ let $v_j(g, x)$ be the ordinary area of the set

$$\{[f_j, w] : [f_a, w] = 0\ (a < j),\ (g^{-1}w, (gx)^{-1}w) \text{ is close}\},$$

using Proposition 140.2.

Lemma 144.2. *Each $\mathcal{N}_j$ is closed, $v_j(g, x)$ is a positive finite Borel function invariant under $g \mapsto qg$ for $q \in Q_j$, and for every nonnegative Borel function H_1 invariant on the left by Q_j one has*

$$\int H_1(\sigma q g)\, \mathrm{d}\theta_j(\sigma) = \int H_1(\sigma g)\, \mathrm{d}\theta_j(\sigma) \qquad (q \in Q_{j-1},\ 1 \le j \le n).$$

If $g \in B_G(R)$, $\sigma \in G_j$, $\sigma g \in \mathcal{N}_j$, then $\|\sigma^{-1} f_j\| \le R^2$.

Proof. The flag conditions are closed matrix equations and specify a subgroup since the flag matrices are upper unitriangular. The product $Q_j B_G(R)$ is closed by the compact subsequence argument of Lemma 141.1. The close set in the defining subspace for w is open, convex, invariant under scalars of modulus at most one, and bounded by the condition on $g^{-1}w$. Its image by $w \mapsto [f_j, w]$ is a bounded open disk centered at zero with positive radius. Indeed e_j^* is an allowed direction with image one. Membership of a scalar t in this disk is an open condition on (g, x, t), since one can perturb a witnessing w by a multiple of e_j^* to realize any nearby scalar and the norm inequalities are strict. Integrating its Borel indicator in t proves measurability. Under $w \mapsto qw$, the first $j-1$ orthogonality equations are equivalent by triangularity of q^{-1}, and $[f_j, qw]$ equals $[f_j, w]$ on that subspace. The close pair is preserved on simultaneously replacing g by qg.

Any $q \in Q_{j-1}$ factors as hu, where $h \in G_j$ and u has upper block triangular form in the blocks consisting of the first $j-1$ vectors, V_j, and their duals, with identity middle block and upper unitriangular first block. In fact q preserves the span of the first vectors and its signed perpendicular and thus acts on their intermediate quotient as an isometry. Lift this isometry diagonally to h to obtain the factorization. The subgroup of such u is normalized by G_j and lies in Q_j, because the middle images are unchanged modulo the first $j-1$ vectors. Hence in the integral the left invariance of H_1 reduces q to h. For fixed g, $H_1(\sigma g)$ on G_j depends only on $\sigma^{-1} f_j$, since the stabilizer of f_j there is contained in Q_j. A Borel representative outside a null set is the triangular inverse parametrization at $j < n$ used in Proposition 138.3 when the dual coordinate $t \neq 0$. At $j = n$ use Lemma 141.1. The law of $\sigma^{-1} f_j$ is invariant under h^{-1} by Proposition 138.3, so the null exception also has zero measure after this change. This proves the integral identity. Finally choose $q' \in Q_j$ with $\|q' \sigma g\|_{\mathrm{op}} \le R$. Projecting $\sigma^{-1}(q')^{-1} f_j$ orthogonally onto V_j gives $\sigma^{-1} f_j$, since σ fixes the earlier frame vectors. The unprojected vector has norm at most $\|g\|_{\mathrm{op}} \|(q'\sigma g)^{-1}\|_{\mathrm{op}} \le R^2$. □

Theorem 144.3. *At sufficiently large arithmetic scales, uniformly in $g, x \in G_1$,*

$$\int_{\sigma g \in \mathcal{N}_j} v_j(\sigma g, x)^{-s} \, \mathrm{d}\theta_j(\sigma) \le R^{29n} \quad (1 \le j < n),$$

$$\theta_j\{\sigma : \sigma g \in \mathcal{N}_j\} \le R^{5k} \quad (1 \le j \le n).$$

Here $m = k - 1 = s + 1 + 3n$, as in Proposition 140.2.

Proof. If $\sigma g \in \mathcal{N}_j$ for some $\sigma \in G_j$, then $g \in \mathcal{N}_{j-1}$, because $G_j, Q_j \subset Q_{j-1}$. Otherwise both relevant integrals vanish by the support of θ_j. We can now take $g \in B_G(R)$ using the integral invariance of Lemma 144.2. Put $T = R^2,\ h = n - j$. The orbit vector $v = \sigma^{-1} f_j$ on the integration set has norm at most T.

For the first bound apply Proposition 140.2 to g, gx with no row constraints. If w belongs to its set W, then $\sigma E_+ w$ is signed orthogonal to the preceding f_a, and the close pair required for $v_j(\sigma g, x)$ is $(g^{-1}E_+ w, (gx)^{-1}E_+ w)$. Its scalar at f_j is $[v, E_+ w] = w_v^* w$, where w_v is the m-component of v in Definition 138.1. Thus $v_j(\sigma g, x)^{-1} \le \phi(w_v)$ for the projection there. By Proposition 138.3 and the fibre bound proved in Theorem 139.2, the integral is at most

$$M_{\mathrm{quad}}(T) \int_{\|w\| \le T} \left(\frac{36}{\|\Pi w\|^2} \right)^s \mathrm{d}w.$$

Indeed at each fixed w one integrates all remaining parameters in the box using $m_{\mu,T}(p)$ from that proof. By Lemma 140.1 and the exponential estimate in Proposition 140.2, the ball integral is at most $R^{6(3n+1)}$. Also $M_{\mathrm{quad}}(R^2) = 4\pi R^4 (\pi R^4)^{2h} \le R^{6+10h}$ as $R \ge 4$. Their product is at most R^{29n} when $n \ge 48$.

For the unweighted bound the identical fibre calculation applies to the constant one, including at the last floor by the negative quadric assertion of Proposition 138.3. The remaining ball has volume $\pi^m R^{4m}/m!$. Since $\pi^m/m! \le \exp(\pi) \le R^4$, the resulting bound is $R^{4m+10h+10} \le R^{5k}$ for the actual ranks $k \ge n^7,\ n \ge 48$ by Proposition 10.5. □

145. Recovery of a large position from its flag coordinates

Lemma 145.1. *Let $u \in \mathcal{U}$, $\nu \ge 1$, and put $Z_0 \ge 1$. Suppose that for each $a = 1, \dots, \nu$ there is a vector w_a with $[e_b, w_a] = \delta_{ba}$ for $1 \le b \le a$ and $\|w_a\|, \|uw_a\| \le Z_0$. With $Y_0 = \nu Z_0(1 + \nu Z_0)^\nu$ one has*

$$\|u^{\pm 1}\|_{\mathrm{op}} \le 12(1 + Y_0)^3.$$

Proof. Let W, W' have columns w_a, uw_a. Their bottom blocks, in the rows of $e_1^*, \dots, e_\nu^*$, are lower unitriangular matrices Z, Z'. For W this is the assumed pairing condition. For W', $u^{-1}e_b - e_b$ lies in the span of the earlier vectors and all their pairings with w_a vanish for $b \le a$. The norms of W, W' are at most $\sqrt{\nu} Z_0$ by columnwise Cauchy–Schwarz. Each strictly lower part of Z, Z' has norm at most νZ_0 by the entry bound, so the two inverse norms are at most $(1 + \nu Z_0)^\nu$ by the finite geometric series.

Write u in the three blocks e, V_n, e^* with diagonal A, I, A^{-*} and upper blocks B, C, D'. The identities $Z' = A^{-*}Z$ and $Z = A^*Z'$ give $\|A\|_{\mathrm{op}}, \|A^{-*}\|_{\mathrm{op}} \le Y_0$. Both WZ^{-1} and $W'Z^{-1}$ have norm at most Y_0, and the first has bottom block I. Multiplication by u sends the first matrix to the second. In the middle block this gives $\|D'\|_{\mathrm{op}} \le 2Y_0$. The mixed signed pairing of the middle and last columns of u gives $B = -A(D')^* J|_{V_n}$, so $\|B\|_{\mathrm{op}} \le 2Y_0^2$. The first block of the same matrix equation now gives $\|C\|_{\mathrm{op}} \le Y_0 + Y_0^2 + 2Y_0^3$. Summing the six block bounds proves the stated inequality for u, and $u^{-1} = Ju^*J$ gives the other one. □

Theorem 145.2. *Let $0 \le j \le \nu$, $p, q \in (\mathbb{C}^\times)^j$, and let w_p, w_q flip these first j pairs. Write $D_{p,i} = \prod_{a \le i} p_a$ for $1 \le i \le j$ and similarly for q. Let $g, x \in G_1$ satisfy*

$$S(g), S(gx) \le R, \qquad w_p^{-1} g,\ w_q^{-1} gx \in \mathcal{N}_j.$$

Choose $L \ge \max(4, r, R)$ also bounding the inverses $|D_{p,i}|^{-1}, |D_{q,i}|^{-1}$, and $T \ge 1$. Put

$$A = \max(R, r, 4, T, H_j), \qquad H_0 = 1, \quad H_j = L^{4j+1} \quad (j > 0).$$

If $\|x - I\|_{\mathrm{op}} > A^{16n}$, then there is an a with $j < a \le \nu$ and $v_a(g, x) < \pi/T^2$. When $j = \nu$ the assumptions imply $\|x - I\|_{\mathrm{op}} \le A^{16n}$.

Proof. Suppose all the indicated areas are at least π/T^2. As in Proposition 133.3, choose $u, v \in \mathcal{U}$ such that both ug and vgx lie in $B_G(R)$. Put $s = vu^{-1}$. For $a \le j$ take $w_a = ue_a^*$. Its first a dual coordinates have the required values by the unitriangular action of u^{-1} on the first frame. Its image under s is ve_a^*. The two near conditions give left multipliers in Q_j as required by Theorem 143.2 applied separately to g, p and gx, q. Consequently both norms are at most L^{4j+1}. No such vectors are needed when $j = 0$.

For $a > j$, the disk of attainable values in Definition 144.1 has radius at least $1/T$ by Lemma 144.2. Choose a witnessing vector w for a value c of modulus $1/(2T)$, and put $w_a = uw/c$. The first $a - 1$ dual coordinates are zero and the a-th is one by the flag action of u^{-1}. The close conditions give $\|g^{-1}w\| < 1/4,\ \|(gx)^{-1}w\| < 1/2$. Multiplying by ug and vgx shows that the norms of w_a and sw_a are at most RT. We can apply Lemma 145.1 with $Z_0 = \max(H_j, RT) \le A^2$. Since $\nu \le A$, $Y_0 \le A^{3+4\nu}$. Hence

$$\|x - I\|_{\mathrm{op}} \le 1 + R^2\, 12(1 + Y_0)^3 \le A^{16n}.$$

For the last inequality $n = \nu + 1,\ A \ge 4$, so $1 + 96A^{11+12\nu} \le A^{15+12\nu} \le A^{16n}$. The first inequality uses $x = (ug)^{-1}s^{-1}(vgx)$. This proves the contrapositive, including the empty range at $j = \nu$. □

146. Integration of the scalar-area weights along a real radical

Definition 146.1. For $0 \le j \le \nu$ let $N^j \subset Q_j$ be the subgroup acting identically on the middle quotient V_{j+1} with respect to the first j pairs. Thus $N^0 = \{1\}$. Give N^j the product

of the first j layer measures in Proposition 20.5, using unit mass at $j = 0$. For $a \le \nu$ put

$$a_a(g, y) = v_a(g, g^{-1}y), \qquad \mathcal{W}_j(g, y) = \prod_{a=1}^{j} a_a(g, y)^s.$$

The two images in the close-pair condition for $a_a(g, y)$ are $g^{-1}w, y^{-1}w$. For $t \in V_{j+1},\ \beta \in \mathbb{R}$ write $n_j(t, \beta)$ for the j-th unipotent element sending

$$e_j^* \longmapsto e_j^* + t + (i\beta - [t, t]/2)e_j,$$
$$z' \longmapsto z' - [t, z']e_j \qquad (z' \in V_{j+1}),$$

and fixing e_j and the previous pairs.

Lemma 146.2. *For $1 \le j \le \nu$, to integrate on N^j one may integrate at $u = u_0 n_j(t, \beta)$ or at $u = n_j(t, \beta)u_0$ against $\mathrm{d}u_0\ \mathrm{d}t\ \mathrm{d}\beta$, $u_0 \in N^{j-1}$. Also $Q_j = N^j G_{j+1} = G_{j+1}N^j$, and conjugation by Q_j preserves the measure on N^j. For $q \in Q_j$, $\mathcal{W}_j(qg, qy) = \mathcal{W}_j(g, y)$.*

Proof. The layer matrices are those of Lemma 20.4 with row $-t^*J|_{V_{j+1}}$ and imaginary corner coordinate β. This row change has absolute real determinant one since the Gram matrix is ordinary unitary. In the decomposition by the first $j - 1$ pairs the middle block of $u \in N^j$ ranges over exactly these n_j. The projection to that block is a homomorphism with kernel N^{j-1}, split by the indicated embedding. The layer product in outside-to-inside order gives $\mathrm{d}u_0\ \mathrm{d}t\ \mathrm{d}\beta$. Conjugation by any $h \in G_j$ preserves the measure on N^{j-1}, since in each of the earlier layers it acts on the vector variable as a signed isometry of complex determinant of modulus one and fixes the real central variable. This is the conjugation computation of Theorem 136.2. In particular, reversing these two factors has the same measure. The same argument shows that G_{j+1} preserves measure on N^j. Haar measure there is bi-invariant by the iterated semidirect products with these unit determinants, so conjugation by N^j also preserves it. The factorization of Q_j is obtained by extracting the middle block, as in Lemma 144.2. That lemma and $Q_j \subset Q_a$ for $a \le j$ give the last identity. □

Theorem 146.3. *Let $0 \le j \le \nu,\ g \in \mathcal{N}_j,\ y \in G_1$, and $L \ge \max(4, r, R)$, at sufficiently large arithmetic scales. Then*

$$\int_{N^j} \mathbf{1}_{\|g^{-1}uy - I\|_{\mathrm{op}} \le L} \mathcal{W}_j(g, uy)\, \mathrm{d}u \le L^{70nj}.$$

Proof. Simultaneously multiply g, y on the left by an element of Q_j, conjugating the integration variable by that element. Lemma 146.2 shows that the integral is unchanged, so assume $g \in B_G(R)$. If the indicated set is empty the bound holds. Otherwise translate y on the left by one of its admissible elements in N^j; right invariance in u then permits $\|g^{-1}y - I\|_{\mathrm{op}} \le L$. For any contributing u,

$$\|u\|_{\mathrm{op}} \le R^2(1 + L)^2 \le L^6.$$

In fact $u = g(g^{-1}uy)(g^{-1}y)^{-1}g^{-1}$, and the norm of a signed isometry equals that of its inverse.

We proceed by induction from $j = 0$, where the assertion is the bound by one for an indicator. For $j > 0$ write $u = u_0 n_j(t, \beta)$ by Lemma 146.2. The middle block obtained by removing $j - 1$ pairs is $n_j(t, \beta)$. Its e_j^* column gives $\|t\|, |\beta| \le L^6$ wherever the integrand is nonzero. Write t_+ for its component in the positive coordinate subspace $E_+\mathbb{C}^m$, identifying that subspace with $\mathbb{C}^m$. Use W, ϕ of Proposition 140.2 for the pair of isometries g, y with no row constraints. We claim $a_j(g, uy) \le \phi(t_+)$.

Indeed let $\alpha = [e_j, w]$ be attained in its defining disk, and let $z = E_+w_0$ with $w_0 \in W$. Direct subtraction gives

$$[g^{-1}w, g^{-1}z] - [(uy)^{-1}w, y^{-1}z] = [w, z - uz] = \bar{\alpha}\, t_+^* w_0.$$

For the last equality, $n_j(t, \beta)z = z - [t, z]e_j$; u_0 fixes both z and e_j modulo the span of the earlier e_a, and these discarded components pair to zero with w. The pairs formed by the

two images of w and by the two images of z are both close. Thus every α lies in the scalar set in part 2 of Proposition 140.2, giving the claim.

At fixed t, β apply the induction hypothesis to g and $n_j(t,\beta)y$, using $\mathcal{N}_j \subset \mathcal{N}_{j-1}$. The result is bounded by $L^{70n(j-1)}$ times the integral of $\phi(t_+)^s$ over the above t, β ranges. The Euclidean complement of E_+ in V_{j+1} has complex dimension $2(n-j)$. Its ball of radius L^6 has volume at most $(\pi L^{12})^{2(n-j)} \le L^{26n}$. By Lemma 140.1, the integral on the positive ball of radius L^6 is at most $L^{14(3n+1)}$; the exponential and factorial factors are bounded as in Proposition 140.2. The real central interval has length at most $2L^6 \le L^7$. Together these three bounds are at most L^{70n} for $n \ge 48$. Nonnegative integration by Lemma 123.1 completes the induction. □

147. Reciprocal weights on the opposite real flag

Theorem 147.1. *For $0 \le j \le \nu$, let w_p flip the first j pairs, with parameters $p_1, \dots, p_j \in \mathbb{C}^\times$. Assume $g \in \mathcal{N}_j,\ x \in G_1,\ L \ge \max(4, r, R)$ and $|p_1 \cdots p_i|^{-1} \le L$ for $1 \le i \le j$. At sufficiently large arithmetic scales,*

$$\int_{N^j} \frac{\mathbf{1}_{w_p u g \in \mathcal{N}_j}}{\mathcal{W}_j(w_p u g, w_p u g x)}\, \mathrm{d}u \le L^{125n^2 j} \prod_{a=1}^{j} |p_a|^{-2d_a - 1}.$$

Here $d_a = \dim_{\mathbb{C}} V_{a+1}$ as in Definition 136.1.

Proof. The integral is unchanged under $g \mapsto qg$ for $q \in Q_j$. For $q \in N^j$ this uses right invariance in u. For $q \in G_{j+1}$, conjugate u by q, commute q with w_p and use the invariance of both $\mathcal{N}_j$ and $\mathcal{W}_j$ under simultaneous left multiplication. These cases suffice by Lemma 146.2. Thus suppose $g \in B_G(R)$.

We use induction on j, the empty product giving the assertion at $j = 0$. Decompose $u = n_j(t,\beta)u_0$ in the inner-to-outer order of Lemma 146.2. If $w_p u g \in \mathcal{N}_j$, apply Theorem 143.2 at ug, with left multiplier u^{-1} and Weyl element w_p^{-1}, whose parameters are $\bar{p}_a$. Its proof bounds $\|u^{\pm 1}\|_{\mathrm{op}} \le L^{6j+3}$, since u^{-1} is trivial on the middle quotient and $g \in B_G(R)$. More explicitly its mixed middle and last blocks are bounded by L^{6j+1}, L^{4j+1} and its diagonal blocks by L^{4j+1}, as shown in the last paragraph of that proof. Also $|p_j| \le L^{2j+1}$ by its leading-minor estimate. The middle block of u on V_j is $n_j(t,\beta)$. Consequently

$$|p_j|\|t\|,\ |p_j||\beta| \le L^{8j+4} \le L^{12n}$$

on the integration set.

For fixed u_0, the image $u_0 E_+ w_0$ equals $E_+ w_0$ plus earlier $e_1, \dots, e_{j-1}$ with coefficients linear in w_0. Put these coefficients in the matrix C of Proposition 140.2; it has at most $j-1$ rows. Apply that proposition with isometries g, gx. If $w_0 \in W$, the vector $w = w_p n_j(t,\beta) u_0 E_+ w_0$ has zero earlier dual coordinates and j-th dual coordinate $-p_j t_+^* w_0$, where t_+ is the projection of t onto the indicated positive coordinates. Indeed $u_0 E_+ w_0 = E_+ w_0$; then n_j subtracts $[t, E_+ w_0] e_j$, and $w_p e_j = p_j e_j^*$. The two inverse images of w under $w_p u g, w_p u g x$ are $g^{-1} E_+ w_0, (gx)^{-1} E_+ w_0$, a close pair. Therefore part 1 of that proposition gives

$$a_j(w_p u g, w_p u g x)^{-1} \le \phi(-\bar{p}_j t_+).$$

Dependence of the projection in ϕ on u_0 causes no integration issue, since we use its bound on the integral at each fixed u_0 and the resulting scalar bound is uniform.

Write $w_{<j}$ for the flip on the first $j-1$ pairs. The two arguments of each earlier area $a_i,\ i < j$, are obtained from $w_{<j} u_0 g, w_{<j} u_0 g x$ by simultaneous left multiplication by $w_j(p_j) n_j(t,\beta) \in G_j \subset Q_{j-1}$. These earlier areas are unchanged by Lemma 146.2. Also the original near condition implies $w_{<j} u_0 g \in \mathcal{N}_{j-1}$. It remains to bound the integration in t, β of the new factor. Set $t' = -\bar{p}_j t,\ \beta' = |p_j|\beta$. The change gives the factor $|p_j|^{-2d_j - 1}$. Both

the norm of t' and the modulus of β' are bounded by $T = L^{12n}$ on the original set. Enlarge to these bounds. As in Theorem 146.3, the integral of $\phi(t'_+)^s$ is at most

$$L^{2(3n+1)}T^{2(3n+1)}(\pi T^2)^{2(n-j)}\,2T \le L^{125n^2}.$$

Indeed the exponent after $\pi \le L,\ 2 \le L$ is at most $(24n+2)(3n+1)+(24n+1)2n+12n+1 \le 125n^2$. Now integrate the remaining earlier factors in u_0 by the induction hypothesis, valid since $g \in B_G(R)$. All original integrands are nonnegative and measurable by Lemma 144.2. Their iterated integrals give the asserted bound. □

Corollary 147.2. *With the head and scale of Theorem 147.1, let A be nonnegative Borel, invariant under left N^j, and zero outside $\mathcal{N}_j$. For a positive sigma-finite Borel measure ρ' on G_1,*

$$\int \frac{A(y)\mathbf{1}_{w_p y\in\mathcal{N}_j}}{\mathcal{W}_j(w_p y, w_p y x)}\,\mathrm{d}(\,\mathrm{d}u * \rho')(y) \le L^{125n^2 j}\prod_{a=1}^{j}|p_a|^{-2d_a-1}\int A\,\mathrm{d}\rho'.$$

Here $\mathrm{d}u$ denotes the integration measure on N^j.

Proof. At $y = uy_0$, $A(y) = A(y_0)$. If this value is positive, $y_0 \in \mathcal{N}_j$, so the integral in u with $g = y_0$ is bounded by Theorem 147.1. Integrate in y_0 by Lemma 123.1. For a possibly infinite A the same argument first applies to its truncations and then Lemma A.2 gives the result. □

148. Successive averages of a real Bessel pairing

Definition 148.1. Fix $0 \le j_0 \le \nu$ and two heads of length j_0. Let $d_* = w_q^{-1}w_p$. On e_a for $a \le j_0$ its diagonal entry is $r_a = p_a/q_a$ and on e_a^* it is $1/\bar{r}_a$, by Definition 133.1. Set $r_a = 1$ for $j_0 < a \le n$. Let B be any bounded Borel function taking values in π and transforming on S_0 by the same rule as V in Theorem 124.2. Write $P_i = \kappa * \theta_n * \cdots * \theta_{i+1}$ for $0 \le i \le n$, with $P_n = \kappa$. Define

$$K_i(g,y) = \int \langle V(\sigma g), B(d_*\sigma y)\rangle\,\mathrm{d}P_i(\sigma).$$

In this section write $\ell(u) = \sigma(u)$ for the sum of the simple-root entries in Definition 121.1.

Lemma 148.2. *All K_i are bounded Borel and vanish unless $g \in \mathcal{N}_i$. For $1 \le i \le n$,*

$$K_{i-1}(g,y) = \int K_i(\sigma g, \sigma y)\,\mathrm{d}\theta_i(\sigma), \qquad K_n(g,y) = \langle V(g), B(d_* y)\rangle.$$

The iterated integrals are absolutely convergent for all g, y. For $i \ge j_0$, $K_i(g,y)$ is invariant under simultaneous left translation of both arguments by $Q_i \cap G_{j_0+1}$. For $1 \le i \le n,\ u, u' \in N^{i-1}$,

$$K_i(ug, u'y) = \psi(\ell(d_* u' d_*^{-1}) - \ell(u))K_i(g,y).$$

For $j_0 < i \le n$, $|K_i|$ is invariant under simultaneous left multiplication by Q_i.

Proof. The compact factor commutes with d_*, and both functions have the same compact unitary action. The formula for K_n follows, also with the modulus inside the integral. If this pairing is nonzero, $S(g) \le R$ implies that a left unipotent translate of g belongs to $B_G(R)$. In particular $g \in \mathcal{N}_n$. Use backward induction in the product measures for the integral of the absolute pairing. When $\sigma g \in \mathcal{N}_i,\ \sigma \in G_i$, one has $g \in \mathcal{N}_{i-1}$. By Theorem 144.3, each integration increases the uniform bound by at most R^{5k}. The root bound is $M_V \sup\|B\|$, and thus all the claimed integrals are finite. Lemma 123.1 gives the recurrence and Borel measurability.

For the simultaneous invariance begin at $i = n$. Here $Q_n = \mathcal{U}\,\iota(U(P_0))$, since after removing all outer pairs the flag condition on z means precisely fixing z on the middle quotient. On the indicated intersection d_* commutes with the multiplier, and both components undergo the same unitary action. In the induction step take $j_0 \le i-1,\ q' \in Q_{i-1} \cap G_{j_0+1}$ and factor $q' = hu$ with $h \in G_i,\ u \in N^{i-1} \cap G_{j_0+1}$ by Lemma 146.2. For $\sigma \in G_i$, conjugation

of u by σh stays in that intersection and in Q_i, so the induction hypothesis removes it from the integrand. The remaining integrand on G_i is invariant on the left by $G_i \cap Q_i$, since this stabilizer lies in G_{j_0+1}. It factors measurably through $\sigma^{-1} f_i$ outside a null set by the chart argument in Lemma 144.2. Right multiplication by h preserves that law by Proposition 138.3. This proves the claimed invariance down to $i = j_0$.

The measure P_i is carried by isometries fixing the first i frame vectors and their hyperbolic partners when present. For $i = n$ these are the compact middle rotations. Conjugating $u \in N^{i-1}$ by such a σ preserves that radical and all its simple-root entries. Indeed the only possible nonzero ones have indices $a \le i-1$, and both frame coordinates in each such entry are among the first i. For $a < i$ the ordinary e_a-row is fixed as well by the split block decomposition. Conjugation by d_* multiplies the a-th entry by r_a/r_{a+1}. Applying the separate Bessel transformation laws inside the integral therefore gives the displayed phase.

Finally suppose $j_0 < i < n$. For a simultaneous $u \in N^i$ the difference of the two phases in the integral involves only the first j_0 simple-root entries, because their multipliers r_a/r_{a+1} equal one for $a > j_0$. These entries are unchanged by σ as $j_0 + 1 \le i$. The phase is constant in the integral, so its modulus is unchanged. Each $q' \in Q_i$ is a product of such u and an element of G_{i+1}, which lies in $Q_i \cap G_{j_0+1}$. At $i = n$, use the separate phase law for N^ν and compact equivariance. This proves the last assertion. □

Proposition 148.3. *Let $j_0 < i \le \nu$ and $g, y \in G_1$. For $\sigma \in G_{i+1}$ write $v = \sigma^{-1} f_{i+1}$. There is a Borel amplitude $F(v)$ on the corresponding quadric satisfying $F(v) = K_{i+1}(\sigma g, \sigma y)$ almost everywhere for θ_{i+1}, and*

$$K_i(g, n_i(t,\beta)y) = \int \psi(-[t,v]) F(v)\, \mathrm{d}\delta_\gamma^{(i+1)}(v) \qquad (t \in V_{i+1},\ \beta \in \mathbb{R}).$$

Here $\gamma = 0$ for $i+1 < n$ and $\gamma = -1$ for $i+1 = n$.

Proof. The stabilizer of f_{i+1} in G_{i+1} belongs to $Q_{i+1} \cap G_{j_0+1}$, so Lemma 148.2 and the measurable lifts of Proposition 138.3 give F. Its integral in absolute value is finite by the same lemma. In the recurrence for K_i, conjugation by σ changes $n_i(t,\beta)$ into $n_i(\sigma t, \beta)$ by the signed isometry property. This is an element of N^i commuting with d_*, since $i > j_0$ and its action is confined to V_i. Its only possible simple-root entry is the e_i-coordinate of the image of f_{i+1}, which equals $-[\sigma t, f_{i+1}] = -[t,v]$ by Definition 146.1. Apply the independent translation rule in Lemma 148.2 to the second argument of K_{i+1}. The orbit law in Proposition 138.3 gives the formula. □

149. A Fourier estimate centered at an admissible translate

Definition 149.1. For $L \ge 1$, $0 \le i \le \nu$ write $\chi_i(g,y)$ for the indicator of the existence of $u \in N^i$ satisfying $\|g^{-1}uy - I\|_{\mathrm{op}} \le L$. This is a Borel indicator. Indeed if g_a, y_a converge and u_a are witnesses, then $g_a^{-1} u_a y_a$ lies in a compact set. Thus a subsequence of u_a converges in the closed subgroup N^i and its limit is a witness.

Lemma 149.2. *Take $1 \le i \le \nu$, $g \in \mathcal{N}_i$, $y \in G_1$, $L \ge \max(4, r, R)$. Choose $s \in Q_i$ with $sg \in B_G(R)$ and denote by $h \in G_{i+1}$ its action on the middle quotient. If $\sigma \in G_{i+1}$ and $\sigma g \in \mathcal{N}_{i+1}$, then $\|h\sigma^{-1} f_{i+1}\| \le R^2$. Suppose $\chi_{i-1}(g, n_i(t_0,\beta_0)y) = 1$. For every (t,β) with $\chi_{i-1}(g, n_i(t,\beta)y) = 1$, $\|h(t - t_0)\| \le L^6$. For each fixed t the admissible central coordinates β belong to an interval of length at most $2L^6$.*

Proof. Choose $s' \in Q_{i+1}$ with $s'\sigma g \in B_G(R)$. Then $s\sigma^{-1}(s')^{-1} = (sg)(s'\sigma g)^{-1}$ has norm at most R^2. Applied to f_{i+1}, its ordinary projection to V_{i+1} is $h\sigma^{-1} f_{i+1}$. In fact $(s')^{-1} f_{i+1} - f_{i+1}$ is in the span of the first i vectors, fixed by σ^{-1} and preserved by s. Also $s|_{V_{i+1}}$ agrees with h modulo the same span. This proves the first bound.

For an admissible (t,β) choose its witness $u_t \in N^{i-1}$ and use u_0 for the chosen base witness. In the expression

$$s\,(u_t n_i(t,\beta))(u_0 n_i(t_0,\beta_0))^{-1} s^{-1}$$

insert g, g^{-1} at the two extremes and yy^{-1} between the two middle factors. The two bounded positions and $sg \in B_G(R)$ give norm at most $R^2(1+L)^2 \le L^6$. On the middle quotient obtained by removing the first $i-1$ pairs, u_t, u_0 are the identity. The image of s there is $hn_i(t_s, \beta_s)$ for some t_s, β_s, by the layer splitting of Lemma 20.4. Multiplication in the layer adds the vector variables and sends the central coordinate of a product to $\beta + \beta' - \mathrm{Im}[t, t']$, by multiplying the corner entries of Definition 146.1. Conjugating by h sends t to ht and fixes the real coordinate. Thus the quotient of the displayed expression is $n_i(h(t-t_0), \beta - \beta_0 + b(t))$, for a real function $b(t)$ independent of β. This quotient is an ordinary diagonal block of the full matrix, hence has norm at most L^6. Its last column bounds both $\|h(t-t_0)\|$ and the modulus of the imaginary corner coordinate. The latter yields the claimed interval. □

Theorem 149.3. *For $j_0 < i \le \nu$ in Definition 148.1, $g, y \in G_1$ and $L \ge \max(4, r, R)$, one has*

$$\int \chi_{i-1}(g, n_i(t,\beta)y)|K_i(g, n_i(t,\beta)y)|^2 \,\mathrm{d}t\,\mathrm{d}\beta \le \pi^{2m} L^{38n} \chi_i(g,y) \cdot \int |K_{i+1}(\sigma g, \sigma y)|^2 \,\mathrm{d}\theta_{i+1}(\sigma).$$

Here $m = k-1$, and if the indicator on the right vanishes the right side is taken to be zero.

Proof. If there is an admissible pair on the left then $\chi_i(g,y) = 1$ by Lemma 146.2. If $g \notin \mathcal{N}_i$ the left side vanishes by Lemma 148.2. In the remaining case use h, t_0, β_0 of Lemma 149.2. By Proposition 148.3 the pairing has the form $\int \psi(-[t,v])F(v)\,\mathrm{d}\delta_\gamma^{(i+1)}(v)$, independent of β. It has amplitude zero unless $\sigma g \in \mathcal{N}_{i+1}$, so after the change $v' = hv$ its support lies in $\|v'\| \le R^2$. This change preserves the quadric measure by Lemma 138.2. The function $F(h^{-1}v')$ is bounded and has compactly bounded support, hence belongs to L^2. Its squared integral is the θ_{i+1}-integral on the right by Proposition 138.3.

Integrate the indicator first in β, at a cost of $2L^6$, and put $t' = h(t-t_0)$, a change of real determinant one. The resulting output range lies in $\|t'\| \le L^6$. The factor $\psi(-[ht_0, v'])$ has modulus one and can be absorbed in the amplitude without changing its squared norm. Apply Theorem 139.2 on V_{i+1} with ambient output, quadric input and frequency -1. The Euclidean balls are contained in the flat boxes of the same radii. The cost multiplying the squared amplitude is at most

$$2L^6 M_{\mathrm{amb}}(L^6) M_{\mathrm{quad}}(R^2)\pi^{2m} \le L^{7+26(n-i)+6+10(n-i-1)}\pi^{2m} \le L^{38n}\pi^{2m}.$$

This proves the assertion, including cases in which the integral on the left is zero. □

150. Squared norms of successive pairings on a ball

Definition 150.1. Use Definition 148.1 and its twist d_*. Write $\tau_i = \theta_i * \cdots * \theta_1$, with τ_0 unit mass at the identity. For $0 \le l \le \nu$ put $\rho_l = \eta_{l+1} * \cdots * \eta_\nu * \mu_{\mathrm{up}}$, so $\rho_\nu = \mu_{\mathrm{up}}$. With the radius L used to define χ_l, set

$$A_l = \iint \chi_l(g,y)|K_{l+1}(g,y)|^2 \,\mathrm{d}\rho_l(y)\,\mathrm{d}\tau_{l+1}(g),$$
$$Z_l(B, d_*; x) = \int \mathcal{W}_l(g, gx)|K_{l+1}(g, gx)|^2 \,\mathrm{d}\tau_{l+1}(g).$$

These are nonnegative integrals. The definition of Z_l is independent of L.

Lemma 150.2. *For $0 \le l \le \nu$ one has $\mathrm{d}u_{N^l} * \rho_l = \lambda\,\mathrm{d}y$. If h normalizes N^l and the change $u \mapsto huh^{-1}$ has Haar Jacobian $\delta(h)$, every nonnegative left N^l-invariant Borel function A satisfies*

$$\int A(hy)\,\mathrm{d}\rho_l(y) = \delta(h)\int A(y)\,\mathrm{d}\rho_l(y).$$

For $h \in G_{l+1}$ this Jacobian equals one. For $l = \nu$ and $h = d_*$ it is $\delta(d_*) = \prod_{a=1}^{j_0} |r_a|^{2d_a+2}$.

Proof. Outside-to-inside multiplication of the first l layer measures gives the measure of N^l by Lemma 146.2. Including the remaining layers gives $\mathrm{d}u_{\mathcal{U}} * \mu_{\mathrm{up}}$, which is $\lambda\,\mathrm{d}y$ by Theorem 136.2. To obtain a cutoff on N^l, divide $\zeta(y) = \exp(-\|y\|_2^2)$ by $\int_{N^l} \zeta(uy)\,\mathrm{d}u$. The denominator is positive, finite and left N^l-invariant. Indeed the matrix equations define a closed subgroup, and the polynomial ball estimate of Lemma 118.1 gives convergence locally uniformly in y, exactly as in Lemma 123.2. Denote the quotient by a_l; then $\int a_l(uy)\,\mathrm{d}u = 1$. Applying the convolution formula to Aa_l gives $\int A\,\mathrm{d}\rho_l = \lambda \int A(y) a_l(y)\,\mathrm{d}y$. Use it first for $A(h\,\cdot)$, change variables by left Haar invariance, and use the convolution formula again:

$$\int A(hy)\,\mathrm{d}\rho_l(y) = \lambda \int A(z) a_l(h^{-1}z)\,\mathrm{d}z = \int A(z) \int a_l(h^{-1}uz)\,\mathrm{d}u\;\mathrm{d}\rho_l(z).$$

Putting $u = hvh^{-1}$ makes the inner integral $\delta(h)$. The argument applies also at infinite integral by Lemma 123.1. For G_{l+1} the Jacobian is the one calculated in Lemma 146.2 (and is one trivially if $l = 0$). On layer a, conjugation by d_* sends the vector and central coordinate to $(\bar{r}_a\, d_*|_{V_{a+1}} t, |r_a|^2\beta)$, by evaluating on e_a^* in Definition 146.1. The middle isometry has complex determinant of modulus one; the complex vector space has dimension d_a. Multiplying the layer Jacobians, including the central real coordinate of each, proves the last assertion. □

Theorem 150.3. *At sufficiently large arithmetic scales take $L \geq \max(4, r, R)$. Suppose that B in Definition 148.1 has squared quotient norm at most $e^2 D$. For each twist one has*

$$\int_{\|x-I\|_{\mathrm{op}} \leq L} \sum_{l=j_0}^{\nu} Z_l(B, d_*; x)\,\lambda\,\mathrm{d}x \leq n\,\pi^{2m\nu} L^{70n^2} \lambda^2 D^2 e^2\,\delta(d_*), \qquad m = k - 1.$$

Proof. At $l = \nu$, Lemma 148.2 and Cauchy–Schwarz in the value space give

$$A_\nu \leq \int \|V(g)\|^2\,\mathrm{d}\tau_n(g) \int \|B(d_* y)\|^2\,\mathrm{d}\rho_\nu(y) \leq \lambda^2 D^2 e^2 \delta(d_*).$$

Indeed $\kappa * \tau_n = \mu_{\mathrm{up}}$ and $\|V\|$ is compact invariant. Theorem 136.2 converts each squared norm to its quotient integral with factor λ; Lemma 150.2 removes the twist from the second. Use Theorem 124.2 for the first quotient norm.

For $j_0 < i \leq \nu$ write $\rho_{i-1} = \eta_i * \rho_i$ in A_{i-1}. Theorem 149.3 bounds the inner integral over η_i by $\pi^{2m} L^{38n} \chi_i(g, y) \int |K_{i+1}(\sigma g, \sigma y)|^2\,\mathrm{d}\theta_{i+1}(\sigma)$. Here $\chi_i(\sigma g, \sigma y) = \chi_i(g, y)$ as $\sigma \in G_{i+1}$ normalizes N^i. For fixed σ, g, the function

$$y' \longmapsto \chi_i(\sigma g, y') |K_{i+1}(\sigma g, y')|^2$$

is left N^i-invariant: absorb the multiplier into the witness for the indicator and use the unit-modulus phase of Lemma 148.2. Hence Lemma 150.2 permits the replacement of σy by y' under ρ_i. The measure on σg is $\theta_{i+1} * \tau_i = \tau_{i+1}$. This proves $A_{i-1} \leq \pi^{2m} L^{38n} A_i$.

Finally substitute $y' = gx$ at fixed g in the integral of Z_l over the stated ball. Decompose $\lambda\,\mathrm{d}y' = \mathrm{d}u_{N^l} * \rho_l$ by Lemma 150.2. The modulus of $K_{l+1}(g, uy)$ equals that of $K_{l+1}(g, y)$. When this is positive, $g \in \mathcal{N}_{l+1} \subset \mathcal{N}_l$ by Lemma 148.2. Theorem 146.3 bounds the integral of the remaining indicator times $\mathcal{W}_l(g, uy)$ in u by $L^{70nl} \chi_l(g, y)$; if $\chi_l = 0$ it vanishes. The contribution is therefore at most $L^{70nl} A_l$. All integrals rearranged here are nonnegative, so Lemma 123.1 applies without a preliminary integrability assertion. Iterate the bound for A_l and sum, using $70nl + 38n(\nu - l) \leq 70n^2$ and $\nu + 1 = n$. □

151. Stopping the real integrals at the first small area

Definition 151.1. Fix B, d_* as in Definition 148.1, $0 \leq j \leq \nu$, and $g, x \in G_1$. Write

$$U_j(g, x) = \int |K_{j+1}(hg, hgx)|\,\mathrm{d}\theta_{j+1}(h).$$

For $j \le l \le \nu$ let

$$E_{j,l}(g,x) = \int \mathcal{W}_l(hg,hgx)|K_{l+1}(hg,hgx)|^2 \,\mathrm{d}(\theta_{l+1} * \cdots * \theta_{j+1})(h).$$

In particular $E_{0,l}(1,x) = Z_l(B,d_*;x)$ in Definition 150.1.

Proposition 151.2. *At sufficiently large arithmetic scales, for every g,x and j in Definition 151.1 one has*

$$\mathcal{W}_j(g,gx)U_j(g,x)^2 \le R^{29n(\nu-j)+5k}E_{j,\nu}(g,x).$$

Let $u>0$. Suppose that for every $h \in G_{j+1}$ for which $\langle V(hg), B(d_*hgx)\rangle \ne 0$, there is some $j<i\le\nu$ with $v_i(hg,x)\le u$. Then

$$\mathcal{W}_j(g,gx)U_j(g,x)^2 \le R^{29n(\nu-j)}u^s\sum_{l=j}^{\nu-1}E_{j,l}(g,x).$$

An empty sum is zero.

Proof. Lemma 148.2 gives $|K_i(g,gx)| \le U_i(g,x)$ for $0\le i\le\nu$. In each integral defining U_{i-1} the integrand vanishes unless $hg\in\mathcal{N}_i$. For $i\le\nu$ insert the reciprocal factors $v_i(hg,x)^{-s/2}$ and $v_i(hg,x)^{s/2}$. Cauchy–Schwarz and Theorem 144.3 give

$$U_{i-1}(g,x)^2 \le R^{29n}\int v_i(hg,x)^s|K_i(hg,hgx)|^2\,\mathrm{d}\theta_i(h).$$

For $i=n$ use Cauchy–Schwarz with unit weight on the support. The cost is at most R^{5k} by the same theorem. These inequalities hold also when a nonnegative right side is infinite, by applying Cauchy–Schwarz on increasing bounded sets. At each successive integration $h\in G_i\subset Q_a$ for $a<i$. The preceding factors $v_a(\,\cdot\,,x)$ are unchanged under this left multiplication by Lemma 144.2. Starting at $i=j+1$, iterating up to n and multiplying by $\mathcal{W}_j(g,gx)$ now yields the first inequality.

For the second, at each $i\le\nu$ split the preceding weighted integral according as its newly encountered area is at most u or exceeds u. On the first set bound the new factor by u^s and retain $|K_i|^2$ without expanding it at another stage. Removing the previous restrictions increases this nonnegative contribution. Its bound, including the initial weight, is $R^{29n(i-j)}u^sE_{j,i-1}(g,x)$. On the second set keep the new weight and continue. The restrictions on earlier areas are preserved under subsequent left multiplications as above. If all the areas encountered up to ν exceed u, they still do so after the final θ_n-multiplication. The full left multiplier is a product of elements of the nested groups $G_{j+1},\dots,G_n$, hence lies in G_{j+1}. The hypothesis then makes K_n zero by Lemma 148.2. This also proves the claim directly when $j=\nu$. Summing the stopped contributions and bounding $i-j$ by $\nu-j$ gives the result. □

Corollary 151.3. *For the empty head let $c_B(x)=\int\langle V(y),B(yx)\rangle\,\mathrm{d}\bar{y}$, with $B=V,F$, or $V-F$. At sufficiently large scales, for every $x\in G_1$,*

$$\lambda^2|c_B(x)|^2 \le R^{29n\nu+5k}Z_\nu(B,1;x).$$

If $T\ge\max(4,r,R)$ and $\|x-I\|_{\mathrm{op}}>T^{16n}$, then

$$\lambda^2|c_V(x)|^2 \le R^{29n\nu}(\pi/T^2)^s\sum_{l=0}^{\nu-1}Z_l(V,1;x).$$

Proof. In the empty head $P_0=\mu_{\mathrm{up}}$ and $d_*=1$. The two factors of the inner product have the same unitary transformation on S_0. Its quotient integral is absolutely convergent by the two quotient norm bounds in Theorem 124.2 and right invariance in Lemma 123.2. Applying Theorem 136.2 to the absolute value and then to the scalar gives $K_0(1,x)=\lambda c_B(x)$. Use $|K_0(1,x)|\le U_0(1,x)$ and the first inequality of Proposition 151.2. For $B=V$, a nonzero raw pairing at h,hx implies $S(h),S(hx)\le R$ by Theorem 124.2. At $j=0$ both near conditions

are automatic since $Q_0 = G_1$. Thus Theorem 145.2 gives exactly the hypothesis of the second stopping inequality with $u = \pi/T^2$. □

152. The full real energy of the cut coefficient

Lemma 152.1. *At sufficiently large scales set* $U = (2R)^{16n}$ *and*

$$\mathcal{A} = n(1 + 2^{70n^2+1})R^{29n\nu+5k}U^{70n^2}, \qquad C_{\mathrm{B}} = \frac{u_k}{u_r}\pi^{(2\nu+1)k}.$$

Let B *satisfy Theorem 150.3 for the empty head. Define* c_B *by the same quotient integral as in Corollary 151.3. Then*

$$\int_{G_1} |c_V(x)c_B(x)|\,\mathrm{d}x \le D^2 e\, C_{\mathrm{B}}\mathcal{A}.$$

Proof. The integral for c_B is absolutely convergent by the quotient Cauchy–Schwarz argument in Corollary 151.3. Its first pointwise bound extends to this B: that proof uses only the common equivariance and the finite squared quotient norm. Write $m_+ = k - 1$ and $C = R^{29n\nu+5k}$. On the ball of radius U about the identity, apply that pointwise bound to both coefficients, followed by Cauchy–Schwarz in x and Theorem 150.3 for each function. Since $e = 1$ is valid for V, the integral on this ball is at most

$$D^2 e\,\lambda^{-1} n\pi^{2m_+\nu} C U^{70n^2}.$$

The factor λ^{-1} comes from using $\lambda\,\mathrm{d}x$ in the squared norm estimate and from the λ^2 on each left side of Corollary 151.3.

On the shell $2^aU < \|x - I\|_{\mathrm{op}} \le 2^{a+1}U$, $a = 0, 1, \ldots$, use $T_a = (2^aU)^{1/(16n)}$. Here $T_a \ge 2R \ge \max(4, r, R)$ eventually by Proposition 10.5 and Theorem 124.2. Apply the second pointwise bound of Corollary 151.3 to c_V and the first to c_B. Enlarge the integrals of the nonnegative Z_l's to the ball of radius $2^{a+1}U$. The preceding estimate for the inner ball, multiplied by

$$2^{(a+1)70n^2}(\pi/T_a^2)^{s/2} \le 2^{70n^2}2^{-a(s/(16n)-70n^2)},$$

bounds the mixed integral on this shell, since $\sqrt{\pi}/(2R) \le 1$. Now $s = k - 3n - 2$ by Theorem 144.3, $r = k + 2n - 1$, $r \ge x_{\mathrm{ar}}^5$ and $n \le \sqrt{x_{\mathrm{ar}}}$, where x_{ar} denotes the arithmetic scale of Proposition 10.5. Hence $s/(16n) - 70n^2 \ge 1$ eventually. The geometric sum is at most two. Finally $\lambda^{-1}\pi^{2m_+\nu} \le C_{\mathrm{B}}$ by Theorem 136.2. Adding the inner ball and all shells gives the claim by Lemma A.2. □

Theorem 152.2. *Let* $d = \mathcal{D}_\tau$ *be the formal degree of the actual row at a real place* τ *in Theorem 123.3. For each fixed* $0 < \eta < 1$ *and at all sufficiently large arithmetic scales, uniformly in the place and the permitted window parameter,*

$$\frac{1}{D^2}\int_{G_1} |c_V(x)|^2\,\mathrm{d}x \le \frac{1 + b^{-(5-\eta)r^2}}{d} \le \frac{1 + b^{-10}}{d}.$$

Proof. Take $B = V - F$. Its admissible norm parameter is $e = b^{-5r^2}$ by Theorem 124.2. Expanding $|c_F|^2 = |c_V - c_B|^2$ gives $|c_V|^2 \le |c_F|^2 + 2|c_Vc_B|$. Use Theorem 137.2 on the first term and Lemma 152.1 on the second. This gives

$$\frac{d}{D^2}\int |c_V|^2 \le 1 + 2b^{-5r^2} d C_{\mathrm{B}}\mathcal{A}.$$

We estimate the constants using X for the arithmetic scale. Since $R = b^{6n+2}$, $\log_b U = 16n\log_b(2R) = O(n^2)$. Consequently $\log_b(2\mathcal{A}) = O(n^4 + nk + n^3) = o(r^2)$ by Proposition 10.5, since $n \le \sqrt{X}$ and $k \le r$, $r \ge X^5$. All constants here are absolute. The constant C_{B} equals the one in Definition 56.1, whose parameter m is $2n$. By Proposition 56.2,

$$\log_b(dC_{\mathrm{B}}) \le 2n(k-1)\log_b w(\tau) + O\left(\frac{nk/X + n^2\log X}{\log b}\right).$$

Proposition 122.1 and its proof give $1 \le w(\tau) \le W_\tau \le 2b^2$. Thus the last upper bound is $o(r^2)$ uniformly in the indicated data. For the fixed η this gives $2dC_{\mathrm{B}}\mathcal{A} \le b^{\eta r^2}$ eventually. The second asserted inequality follows from $(5-\eta)r^2 \ge 10$ at the same scales. □

153. Integration over the preceding hyperbolic pairs

Definition 153.1. For $0 \le j \le \nu$ put

$$\mathfrak{m}_j = (\eta_j * \theta_j) * \cdots * (\eta_1 * \theta_1),$$
$$\mathfrak{h}_j = \kappa * \theta_n * (\eta_\nu * \theta_\nu) * \cdots * (\eta_{j+1} * \theta_{j+1}).$$

An empty product of measures is unit mass at 1. For a kernel of initial length $j_0 \le j$, write $S_j(g,x) = \sum_{l=j}^{\nu} E_{j,l}(g,x)$ and $Z_{\ge j}(x) = \sum_{l=j}^{\nu} Z_l(B, d_*; x)$, using Definition 151.1. Write $W_j(g,x) = \mathcal{W}_j(g, gx)$.

Lemma 153.2. *For every j in Definition 153.1, $\mathfrak{m}_j = \mathrm{d}u_{N^j} * \tau_j$ and $\mathfrak{h}_j * \mathfrak{m}_j = \lambda\,\mathrm{d}g$. The measure $\mathfrak{h}_j$ is carried by G_{j+1} and is right invariant under that group. For a nonnegative Borel function A invariant on the left by G_{j+1} and any w commuting with G_{j+1}, one has*

$$\int A(wg)\,\mathrm{d}\mathfrak{m}_j(g) = \int A(g)\,\mathrm{d}\mathfrak{m}_j(g).$$

The particular choices $w = w_p^{\pm 1}$ are permitted for a head of length j.

Proof. In the defining convolution for $\mathfrak{m}_j$ move the first $j-1$ radicals to the left of $\eta_j * \theta_j$ by induction: both factors are supported on G_j, which preserves measure on N^{j-1} by Lemma 146.2. The leftmost factors now give the product on N^j of that lemma, and the rest give τ_j. The full convolution is exactly $\mu_{\mathrm{all}} = \lambda\,\mathrm{d}g$ of Theorem 136.2.

At $j = \nu$, right invariance of $\mathfrak{h}_\nu$ is Lemma 141.1. Descend from $j = i$ to $i-1$. The measure $\mathfrak{h}_i * \eta_i$ is right invariant under the stabilizer of e_i in G_i. Indeed the stabilizer is G_{i+1} times the i-th layer by Lemma 146.2 on V_i: an isometry fixing e_i acts by an isometry on its perpendicular modulo that line, and the kernel consists of exactly the matrices $n_i(t,\beta)$. Right translation by this kernel preserves η_i. Right translation by G_{i+1} can be moved past η_i by its determinant-one conjugation and then preserves $\mathfrak{h}_i$ by induction. Averaging a nonnegative Borel test on the left by $\mathfrak{h}_i * \eta_i$ therefore gives a function of the orbit vector $g^{-1}e_i$. This function is Borel on the full-measure chart by the section in Proposition 138.3. Right translation by an element of G_i preserves the law of this vector under θ_i, including the null complement of the chart. Thus $\mathfrak{h}_{i-1} = \mathfrak{h}_i * \eta_i * \theta_i$ has the required right invariance. Support follows from the nested groups.

For the last identity choose a strictly positive Haar integrable Borel function ϕ on G_1. For example sum the indicators of the norm balls of positive integral radii $a = 1, 2, \ldots$, with coefficients $2^{-a}/(1 + |B_G(a)|)$; the balls are compact and have finite Haar measure by Lemma 95.2. Put $H(g) = \int \phi(hg)\,\mathrm{d}\mathfrak{h}_j(h)$, which is positive since the defining product has positive measure. It is left G_{j+1}-invariant by the right invariance just proved. Both $H(g)$ and $H(wg)$ are finite for $\mathfrak{m}_j$-almost every g. Indeed their integrals equal $\lambda \int \phi\,\mathrm{d}g$, using the full convolution, commutation with w, and left Haar invariance. Define $a(g) = \phi(g)/H(g)$ when $H(g) < \infty$ and zero elsewhere. The averages of $a(hg)$ and $a(whg)$ over $\mathfrak{h}_j$ are consequently one for almost every such g. Inserting the respective averages and using Lemma 123.1 identifies the two integrals in the statement with $\lambda \int A(wy)a(wy)\,\mathrm{d}y$ and $\lambda \int A(y)a(y)\,\mathrm{d}y$. Left invariance identifies them. All $w_p^{\pm 1}$ act as the identity on V_{j+1} and preserve its complementary pairs, proving the commutation. □

Lemma 153.3. *For fixed x, $S_j(\,\cdot\,, x)$ is nonnegative Borel, is left Q_j-invariant and vanishes outside $\mathcal{N}_j$. Its integral against τ_j equals $Z_{\ge j}(x)$.*

Proof. In the definition of $E_{j,l}$, the function at its last stage is $W_l(y,x)|K_{l+1}(y, yx)|^2$. Its invariance under Q_{l+1} follows from Lemmas 146.2 and 148.2, since $l + 1 > j_0$. Both factors

are Borel and the pairing requires $y \in \mathcal{N}_{l+1}$. Applying Lemma 144.2 to each successive integration gives invariance under Q_j at the end. The support condition descends because $Q_i, G_i \subset Q_{i-1}$ and θ_i is carried by G_i. Finally the product with τ_j is precisely τ_{l+1}. Integrate and sum using Lemma 123.1. □

Proposition 153.4. *Fix heads p, q of length j. In Definition 153.1, let S, Z denote $S_j, Z_{\geq j}$ for $V, 1$, and S', Z' the corresponding functions for B, d_* with $d_* = w_q^{-1} w_p$. Let $L \geq \max(4, r, R)$ bound every $|p_1 \cdots p_a|^{-1}$, $1 \leq a \leq j$, at sufficiently large scales. Fix $x \in G_1$. Suppose f is nonnegative Borel, vanishes unless both $g, w_p^{-1} g \in \mathcal{N}_j$, and on its support obeys*

$$f(g)^2 \leq c^2 \, \frac{S(g,x)}{W_j(g,x)} \frac{S'(w_p^{-1} g, x)}{W_j(w_p^{-1} g, x)}, \qquad c > 0.$$

Whenever $Z(x), Z'(x) < \infty$,

$$\int f \, \mathrm{d}\mathfrak{m}_j \leq c L^{125 n^2 j} \prod_{a=1}^{j} |p_a|^{-2d_a - 1} \sqrt{Z(x) Z'(x)}.$$

Proof. In Cauchy–Schwarz pair $S(g,x)$ with the reciprocal weight and the near indicator at $w_p^{-1} g$. Pair $S'(w_p^{-1} g, x)$ with the reciprocal weight and the near indicator at g. For the first integral use Lemma 153.2 and Corollary 147.2 with $\rho' = \tau_j$ and the head $\bar{p}$, since $w_p^{-1} = w_{\bar{p}}$. This gives $L^{125 n^2 j} \prod |p_a|^{-2d_a - 1} Z(x)$ by Lemma 153.3. In the second integral the integrand is $A(w_p^{-1} g)$ with

$$A(y) = S'(y, x) \, \mathbf{1}_{\mathcal{N}_j}(w_p y) / W_j(w_p y, x).$$

This function is left G_{j+1}-invariant since this group commutes with w_p, preserves the weights and near sets, and fixes S' by the same lemma. Apply Lemma 153.2 to replace its argument by g, then apply Corollary 147.2 with head p. The second squared integral has the identical constant times $Z'(x)$. The measure τ_j is sigma-finite here: $\mathfrak{m}_j$ integrates the positive H in the preceding proof to a finite number, and $\mathfrak{m}_j = \mathrm{d}u_{N^j} * \tau_j$. Integrating $H(ug)$ in u gives a positive function with finite τ_j-integral. Both applications of the corollary are therefore valid, including when a squared integral is zero. Taking the two square roots proves the claim. □

154. Absolute convergence for arbitrary real heads

Definition 154.1. Fix $0 \leq j \leq \nu$, $p, q \in (\mathbb{C}^\times)^j$, and put $i = j + 1$. In Definition 148.1 use $B = V$ throughout. Use K_a for the empty head, and K'_a for the twist $d_* = w_q^{-1} w_p$. Set $P_j(p) = \prod_{a \leq j} |p_a|^{d_a}$, $\kappa_j = P_j(p) P_j(q)$, and $C_j = \kappa_j / (\lambda D)^2$. For functions of $\sigma, t \in G_i$ write

$$H(\sigma) = K_i(\sigma g, \sigma g x), \qquad H'(t) = K'_i(t w_p^{-1} g, t w_p^{-1} g x),$$

with g, x, p, q understood. Define the following functions of g, x, using $f_i = e_i$ if $i < n$, $f_n = z$:

$$B_j(g,x) = \iint |H(\sigma) H'(t)| \, \mathrm{d}\theta_i(t) \, \mathrm{d}\theta_i(\sigma),$$

$$B_j^\xi(g,x) = \iint \psi(\xi[\sigma^{-1} f_i, t^{-1} f_i]) H(\sigma) \overline{H'(t)} \, \mathrm{d}\theta_i(t) \, \mathrm{d}\theta_i(\sigma).$$

Here $\xi \in \mathbb{C}$. These integrals are finite at every g, x by Lemma 148.2 and Theorem 144.3. Write $\mathcal{I}_j^\xi(p,q) = C_j \iint B_j^\xi(g,x) \, \mathrm{d}\mathfrak{m}_j(g) \, \mathrm{d}x$ whenever absolutely convergent. Use $J_j^*(p,q) = \mathcal{I}_j^0(p,q)$.

Lemma 154.2. *At sufficiently large scales suppose*

$$L \geq \max(4, r, R, |D_{p,a}|^{-1}, |D_{q,a}|^{-1} : 0 \leq a \leq j), \qquad L' \geq \max(4, r, R),$$

where $D_{p,a} = p_1 \cdots p_a$, $D_{p,0} = 1$, and similarly for q. For any Borel $X \subset \{x : \|x - I\|_{\mathrm{op}} \leq L'\}$,

$$C_j \int_X \int B_j(g,x) \, \mathrm{d}\mathfrak{m}_j(g) \, \mathrm{d}x \leq n C_{\mathrm{B}} L^{125 n^2 j + 1} (L')^{70 n^2} R^{29 n \nu + 5k}.$$

If $T \ge \max(4, r, R, L^{4j+1})$ and every $x \in X$ satisfies $\|x - I\|_{\mathrm{op}} > T^{16n}$, the same bound holds multiplied by $(\pi/T^2)^{s/2}$. If either head has $|D_{p,a}| > R^{2a}$ or $|D_{q,a}| > R^{2a}$ for some $a \le j$, then B_j vanishes identically.

Proof. Write S, Z, S', Z' as in Proposition 153.4. The two factors of B_j are exactly the U_j's of Definition 151.1 at g and $w_p^{-1}g$. Positivity of the first yields some $h \in G_i$ with both $V(hg), V(hgx) \neq 0$. To see this, dominate the averaged pairing by the integral of the absolute raw pairing and use that the upper multipliers lie in G_i. Thus $g, gx \in \mathcal{N}_j$. Positivity of the second yields $h' \in G_i$ with $V(h'w_p^{-1}g), V(h'w_q^{-1}gx) \neq 0$: the twist and both Weyl matrices commute with G_i, and $d_* w_p^{-1} = w_q^{-1}$. This gives $w_p^{-1}g, w_q^{-1}gx \in \mathcal{N}_j$. Theorem 143.2 at g, p and at gx, q now proves the assertion about large minors (vacuous at $j = 0$).

Proposition 151.2 gives the pointwise hypothesis of Proposition 153.4 for B_j with $c = R^{29n\nu+5k}$. For the assertion with T, consider the first factor on the set where $B_j > 0$. If the raw pairing at hg, hgx is nonzero for $h \in G_i$, both sizes are at most R. The two required near conditions still hold after this translation because $h \in Q_j$ commutes with w_p, w_q. Apply Theorem 145.2 at hg. The second stopping estimate improves c by the claimed factor. When $j = \nu$ its hypothesis forces this product to be zero.

Apply Proposition 153.4 in g and Cauchy–Schwarz in x. Theorem 150.3 shows the two sums of squared norms are finite almost everywhere on the ball and gives

$$\int_X \sqrt{Z(x)Z'(x)}\,\mathrm{d}x \le n\pi^{2(k-1)\nu}(L')^{70n^2}\lambda D^2\,\delta(d_*)^{1/2}.$$

The empty-head sum starting at j is bounded by its sum starting at zero. The twisted sum starts at precisely j; use $e = 1$ for both functions. On multiplying by C_j, its remaining pivot factor is

$$\kappa_j \prod_{a=1}^{j} |p_a|^{-2d_a-1} \prod_{a=1}^{j} |p_a/q_a|^{d_a+1} = |D_{q,j}|^{-1} \le L$$

by Lemma 150.2. Finally $\lambda^{-1}\pi^{2(k-1)\nu} \le C_{\mathrm{B}}$ by Lemma 152.1. This proves both bounds. □

Theorem 154.3. *The forms $\mathcal{I}_j^{\xi}(p,q)$ of Definition 154.1 are absolutely convergent at every fixed pair of heads and every $\xi \in \mathbb{C}$, at sufficiently large arithmetic scales. More precisely, with L as in Lemma 154.2 and $U = (2L^{4j+1})^{16n}$, one has*

$$C_j \iint B_j(g,x)\,\mathrm{d}\mathfrak{m}_j(g)\,\mathrm{d}x \le nC_{\mathrm{B}}(1 + 2^{70n^2+1})L^{125n^2j+1}U^{70n^2}R^{29n\nu+5k}.$$

Proof. Use Lemma 154.2 on the inner ball of radius U. On the shell of inner radius 2^aU and outer radius $2^{a+1}U$, $a \ge 0$, take $T = (2^aU)^{1/(16n)}$. It exceeds L^{4j+1}. Relative to the inner ball bound, the shell contributes at most $2^{70n^2}2^{-a(s/(16n)-70n^2)}$, since $\sqrt{\pi}/(2L^{4j+1}) \le 1$. Sum this series as in Lemma 152.1. The modulus of each oscillatory integrand, before either floor integration, equals the integrand of B_j. Lemma 123.1 therefore gives all asserted absolute integrals. □

155. Descent from the real Gram kernel to its total mass

Lemma 155.1. *At sufficiently large scales take $0 \le j < \nu$ and arbitrary heads p, q of length j. For $i = j + 1$, the function $c \mapsto J_i^*((p,c),(q,c))$ is absolutely integrable on $\mathbb{C}^\times$. For every $\xi \in \mathbb{C}$,*

$$\int_{\mathbb{C}^\times} \psi(\xi c) J_i^*((p,c),(q,c))\,\mathrm{d}c = \mathcal{I}_j^{\xi}(p,q).$$

Proof. Use H, H' of Definition 154.1. For fixed σ, g, x, the integrand in t on the right depends on t only through $t^{-1}f_i$. Indeed K_i' is invariant under simultaneous multiplication by the stabilizer of f_i in G_i, since it is contained in $Q_i \cap G_i$; apply Lemma 148.2 with initial length $i - 1$. Right multiplying t by σ preserves integrals of such functions by Proposition 138.3.

Replacing t with $t\sigma$ changes the phase to $\psi(\xi[f_i, t^{-1}f_i])$ and changes the second pairing to $H'(t\sigma)$.

Disintegrate $t = w_i(c)^{-1}u$, with $|c|^{2d_i}\,\mathrm{d}c$ on $\mathbb{C}^\times$ and $\mathrm{d}\eta_i(u)$ on its layer, by Definition 136.1. As $uf_i = f_i$, one has $[f_i, t^{-1}f_i] = [f_i, w_i(c)f_i] = c$. The untwisted pairing $H(\sigma)$ equals $K_i(u\sigma g, u\sigma g x)$ by Lemma 148.2 with empty head, since $u \in Q_i$. The matrices $u, \sigma, w_i(c)$ commute with w_p, w_q, since they act only on V_i. Set $z' = u\sigma g$. The resulting product of the two pairings is

$$K_i(z', z'x)\,\overline{K_i'((w_p w_i(c))^{-1}z', (w_p w_i(c))^{-1}z'x)}.$$

The new Weyl matrices are $w_p w_i(c), w_q w_i(c)$; their ratio is still d_*. Thus K_i' is exactly the pairing needed for the extended heads. By Definition 153.1 the law of z' is $\eta_i * \theta_i * \mathfrak{m}_j = \mathfrak{m}_i$. Also $C_j|c|^{2d_i}$ is the normalization constant C_i for these two heads. At zero frequency B_i^0 is the product of K_i and the conjugate translated K_i', by Lemma 148.2. This identifies the integral in (z', x) with $J_i^*((p, c), (q, c))$ after normalization.

Each change applies first to the modulus, whose original total integral is finite by Theorem 154.3. Consequently the transformed product is absolutely integrable jointly in c, z', x. The kernels are Borel in their head variables: their defining integrands are compositions of Borel functions with continuous matrix maps, and the fixed parameter measures give measurability under successive integrations by Lemma 123.1. The same lemma gives absolute integrability in c after integrating the pairings and permits the preceding changes of order for the complex integral. □

Lemma 155.2. *For heads p, q of length ν, at the same scales and every $\xi \in \mathbb{C}$, one has*

$$\int_{\mathbb{C}^\times} \psi(\xi c) J_n((p, c), (q, c))\,\mathrm{d}c = \mathcal{I}_\nu^\xi(p, q),$$

where J_n is the Gram kernel of Definition 134.1. Its integrand is absolutely integrable.

Proof. Use A_p, X, M of Proposition 141.2. At $i = n$, Lemma 148.2 gives exactly the raw pairings $U_{g,x}, W_{g,x}$ of that proposition. Proposition 138.3 converts the two θ_n-integrals in B_ν^ξ to its two quadric integrals. For fixed x, B_ν^ξ is invariant under $g \mapsto hg$ for $h \in M$. Indeed replace the two orbit vectors by their simultaneous images under h^{-1}, using the invariant orbit law and signed pairing and commutation of h with both Weyl matrices. The same holds for the nonnegative majorant B_ν.

If $B_\nu(g, x) > 0$, choose $\sigma, t \in M$ at which both pairings are nonzero. Put $y = \sigma g,\ y' = yx,\ h = \sigma t^{-1}$. Then $V(y), V(h^{-1}w_p^{-1}y), V(y'), V(h^{-1}w_q^{-1}y')$ are all nonzero. The compact-rotation reduction in Proposition 141.2 gives $\|y\|_{\mathrm{op}} \le A_p,\ \|y'\|_{\mathrm{op}} \le A_q$. In particular $g \in MB_G(A_p)$ and $x \in B_G(X)$. Insert the cutoff a_{A_p} of Lemma 141.1; its integral along M is one on this support. Since $\mathfrak{h}_\nu = \kappa * \theta_n$ is exactly the measure used in that lemma, Lemma 153.2 gives

$$\int B_\nu^\xi(g, x)\,\mathrm{d}\mathfrak{m}_\nu(g) = \lambda \int a_{A_p}(y) B_\nu^\xi(y, x)\,\mathrm{d}y$$

for almost every x. This follows first for the majorant and then for the complex integrand by absolute integrability in Theorem 154.3. The cutoff on the right vanishes outside $B_G(2A_p)$. Finally $C_\nu\lambda = u_k P_\nu(p)P_\nu(q)/(u_r D^2)$. The asserted formula now follows from Proposition 141.2, which also gives absolute integrability at the last pivot. □

Theorem 155.3. *Define $J_j = J_j^*$ for $0 \le j \le \nu$ and use J_n of Definition 134.1. At every fixed pair of heads of length $j < n$, integration of J_{j+1} in one common appended complex coordinate gives J_j absolutely. The Fourier integral there at any ξ equals $\mathcal{I}_j^\xi$. All J_j are positive semidefinite. For $0 < \eta < 1$ fixed, at sufficiently large scales one has*

$$\int_{(\mathbb{C}^\times)^n} \|\Xi(p)\|_2^2\,\mathrm{d}p = J_0 = D^{-2}\int_{G_1} |c_V(x)|^2\,\mathrm{d}x \le (1 + b^{-(5-\eta)r^2})/\mathcal{D}_\tau.$$

Proof. Lemmas 155.1 and 155.2 give the Fourier identities. Set $\xi = 0$ for ordinary integration. In particular the definition at ν agrees with Corollary 142.2. The top kernel is positive semidefinite by Theorem 134.2. For a finite list of heads and complex coefficients, multiply the descent identity by the corresponding coefficient products and sum. The quadratic form is the integral of a nonnegative quadratic form at the next stage. This proves positive semidefiniteness by downward induction; all finite sums can be integrated termwise by the absolute convergence just proved. On the diagonal the integrands are nonnegative, so Lemma 123.1 identifies the iterated integral with the product integral of $\|\Xi\|_2^2$. Finally $\mathfrak{m}_0$ is unit mass at 1, $C_0 = (\lambda D)^{-2}$, and $B_0^0(1,x) = |K_0(1,x)|^2 = \lambda^2 |c_V(x)|^2$ by Corollary 151.3 and the pairing recurrence. Apply Theorem 152.2. □

156. Fourier integration at an arbitrary real pivot

Lemma 156.1. *Use j, i, p, q, L, H, H' of Definition 154.1 and Lemma 154.2. For fixed g, x the two amplitudes $H(\sigma), H'(t)$ are bounded Borel functions of $v = \sigma^{-1} f_i$, $v' = t^{-1} f_i$ on their orbit, up to orbit null sets. If $B_j(g,x) > 0$, there is a single $h_0 \in G_i$ such that their respective supports, up to null sets, satisfy*

$$\|h_0 v\| \le L^{6j+4}, \qquad \|h_0 v'\| \le L^{6j+4}.$$

Proof. For both amplitudes the stabilizer of f_i in G_i belongs to the subgroup of simultaneous invariance of K_i or K_i' in Lemma 148.2. The Borel sections and orbit laws in Proposition 138.3 give the factorization and that lemma on the pairings gives boundedness. If $B_j > 0$, then $g, w_p^{-1} g \in \mathcal{N}_j$ by Lemma 154.2 and its proof. For $j > 0$, Theorem 143.2 gives $h_0 \in G_i$ for which both $\|h_0 g\|_{\mathrm{op}}$ and $\|h_0 w_p^{-1} g\|_{\mathrm{op}}$ are at most L^{6j+3}. For $j = 0$ take $h_0 = g^{-1}$.

Where $H(\sigma) \neq 0$, choose $a \in Q_i$ with $\|a\sigma g\|_{\mathrm{op}} \le R$ using Lemma 148.2. The projection to V_i of $h_0 \sigma^{-1} a^{-1} f_i$ is $h_0 \sigma^{-1} f_i$: $a^{-1} f_i - f_i$ is in the span of the preceding frame entries, fixed pointwise by G_i. The unprojected norm is at most $R\|h_0 g\|_{\mathrm{op}}$, by writing the matrix as $(h_0 g)(a\sigma g)^{-1}$. Use the same proof with $w_p^{-1} g$ at a nonzero value of H'. Since $R \le L$, this gives both bounds with the same h_0. □

Proposition 156.2. *For $\xi \neq 0$, and L, L', X as in Lemma 154.2, one has*

$$C_j \int_X \int |B_j^{\xi}(g,x)| \, \mathrm{d}\mathfrak{m}_j(g) \, \mathrm{d}x \le n C_{\mathrm{B}} L^{125n^2 j + 50n^2 + 1} (L')^{70n^2} |\xi|^{-(k-1)}.$$

Proof. For $B_j(g,x) > 0$, apply the common isometry of Lemma 156.1 to the two orbit vectors. The measure on each is $\delta_\gamma^{(i)}$ by Proposition 138.3; both the measures and the phase pairing are preserved by this change. The two amplitudes are supported on the flat box of radius $T = L^{6j+4}$. These boxes have finite measure by Theorem 139.2, so the bounded amplitudes are square integrable. Cauchy–Schwarz against the first amplitude and that Fourier theorem applied to the conjugate second amplitude give

$$|B_j^{\xi}(g,x)|^2 \le M_{\mathrm{quad}}(T)^2 (\pi/|\xi|)^{2(k-1)} \int |H|^2 \, \mathrm{d}\theta_i \int |H'|^2 \, \mathrm{d}\theta_i.$$

Here $M_{\mathrm{quad}}(T) = 4\pi T^2 (\pi T^2)^{2(n-i)}$. Since $L \ge 4$, $j \le n-1$, one has $\log_L M_{\mathrm{quad}}(T) \le 2 + 12n + 2n(1+12n) \le 50n^2$. The first squared amplitude integral is $E_{j,j}(g,x)/W_j(g,x)$, and the second is its twisted counterpart at $w_p^{-1} g$, as $G_i \subset Q_j$ preserves W_j by Lemma 146.2. Each numerator is one summand of the corresponding S_j. If $B_j = 0$, its oscillatory integral vanishes.

Apply Proposition 153.4 with $c = L^{50n^2} (\pi/|\xi|)^{k-1}$. Integrate over X by the squared-norm and determinant calculation of Lemma 154.2. Before its final replacement by C_{B}, the factor involving π, λ in that calculation is $\lambda^{-1} \pi^{2(k-1)\nu}$. Including the extra factor from the Fourier integration gives $\lambda^{-1} \pi^{(2\nu+1)(k-1)} \le C_{\mathrm{B}}$. The remaining factors give the asserted bound. □

157. Uniform real Fourier bounds in the initial minors

Definition 157.1. For $t > 0,\ h \geq 0$ put $d_h(t) = t^h \min(1,t)^6$. For a head $p \in (\mathbb{C}^\times)^j$ let

$$\mathcal{H}_j(p) = \prod_{a=1}^{j} d_{n^6}(|D_{p,a}|^{-1}), \qquad K = k - n^7.$$

Here $K \geq 0$ at the scales under consideration by Proposition 10.5.

Lemma 157.2. *Fix $\delta > 0$. At all sufficiently large arithmetic scales take heads p, q of length $j \leq \nu$, $\xi \neq 0$, and set*

$$L = \max(\{R, 4, r\} \cup \{|D_{p,a}|^{-1}, |D_{q,a}|^{-1} : 1 \leq a \leq j\}).$$

Let $0 \leq f(g,x) \leq B_j(g,x)$ be Borel. Suppose that for every $L' \geq \max(4,r,R)$ the integral of $C_j f$ on $\|x - I\|_{\mathrm{op}} \leq L'$, with measures $\mathrm{d}\mathfrak{m}_j(g)\,\mathrm{d}x$, is at most $nC_{\mathrm{B}} L^{n^5} (L')^{70n^2} |\xi|^{-(k-1)}$. Then

$$C_j \iint f(g,x)\,\mathrm{d}\mathfrak{m}_j(g)\,\mathrm{d}x \leq C_{\mathrm{B}} b^{(12+\delta)n^9} d_K(|\xi|^{-1})\,\mathcal{H}_j(p)\mathcal{H}_j(q).$$

Proof. By Lemma 154.2 we need only treat $|D_{p,a}|, |D_{q,a}| \leq R^{2a}$. Write $P = 70n^2,\ A = 16nP,\ t = \max(1, R^{10}|\xi|)$, and $U = (2L^{4j+1}t)^{16n}$. On the ball of radius U, take the smaller of the hypothesized estimate and the unconditional estimate of Lemma 154.2. On each shell of inner radius $2^a U$ and outer radius $2^{a+1}U$, use that lemma with $T = (2^a U)^{1/(16n)}$. Since $\sqrt{\pi}/(2L^{4j+1}) \leq 1$, the additional factor satisfies $(\pi/T^2)^{s/2} \leq t^{-s} 2^{-as/(16n)}$. As in Lemma 152.1, the sum over a costs at most $2^{P+1} t^{-s}$ relative to the unconditional inner ball estimate. Let $E = n^5 + (4j+1)A$. Substituting U^P and using $125n^2 j + 1 \leq n^5$ eventually gives the total bound

$$nC_{\mathrm{B}} 2^{P+A+1} L^E R^{29n\nu} t^A \left(\min(R^{5k}, |\xi|^{-(k-1)}) + R^{5k} t^{-s}\right).$$

Set $y = |\xi|^{-1}$. Eventually $5k \leq 10K,\ 5k \leq 10s$ and $s - A \geq K + 6$, as $k \geq X^5 - 2\sqrt{X},\ n \leq \sqrt{X},\ s = k - 3n - 2,\ A = 1120n^3$, for the scale X in Proposition 10.5. Also $k - 1 \geq s$. If $t = 1$, then $y \geq R^{10}$ and the parenthesized expression is at most $2R^{5k} \leq 2d_K(y)$. If $t > 1$, then $R^{5k} t^{-s} \leq y^s$ and $t^A = R^{10A} y^{-A}$. When $y \leq 1$, the resulting sum is at most $2R^{10A} y^{K+6}$. When $1 < y < R^{10}$, use $y^{-A} \leq 1$ and $y^{k-1} \leq y^K R^{10n^7}$. In every case the factor beginning with t^A costs at most $2R^{10A+10n^7} d_K(y)$.

It remains to translate the single scale L into the two head weights. For each occurrence of $z = |D_{p,a}|^{-1}$ or $|D_{q,a}|^{-1}$ below one, its factor in the product of head weights is $z^{n^6+6} \geq R^{-2a(n^6+6)}$. The factors for $z \geq 1$ are at least one, and any such factor attaining L is L^{n^6}. Since $R \geq 4, r$ eventually and $E \leq n^6$ eventually, whether $L = R$ or some $z = L$ this gives

$$L^E \leq R^{n^6 + 2j(j+1)(n^6+6)} \mathcal{H}_j(p)\mathcal{H}_j(q).$$

The remaining factor apart from C_{B} and the three prescribed weights has logarithm to base b at most

$$\log_b(n 2^{P+A+2}) + (6n+2)(29n\nu + 10A + 10n^7 + n^6 + 2j(j+1)(n^6+6)).$$

Divide by n^9. Since $j \leq n-1,\ n \to \infty,\ \log b \to \infty$, the upper limit is at most 12, uniformly in both heads and the frequency. This proves the assertion for each fixed δ. □

Theorem 157.3. *For each fixed $\delta > 0$ at all sufficiently large arithmetic scales, for every $0 \leq j \leq \nu$, every $p, q \in (\mathbb{C}^\times)^j$, and every $\xi \in \mathbb{C}^\times$, one has*

$$\left| \int_{\mathbb{C}^\times} \psi(\xi c) J_{j+1}((p,c),(q,c))\,\mathrm{d}c \right| \leq C_{\mathrm{B}} b^{(12+\delta)n^9} d_{k-n^7}(|\xi|^{-1})\,\mathcal{H}_j(p)\mathcal{H}_j(q).$$

Proof. The integral is $\mathcal{I}_j^{\xi}(p,q)$ by Theorem 155.3. In Lemma 157.2 take $f = |B_j^{\xi}|$, which is bounded by B_j. Proposition 156.2 implies its ball hypothesis since $125n^2j + 50n^2 + 1 \le n^5$ eventually. The triangle inequality for the defining integral gives the result. In particular $\delta = 1$ supplies the frequency bound with the larger exponent $2000n^9$. □

158. The real unipotent integral for unequal last entries

Lemma 158.1. *Take $1 \le j \le \nu$ and heads p, q of length j with $p_j \ne q_j$. Set $\xi = p_j^{-1} - q_j^{-1}$. Use the twisted K'_a of Definition 154.1. For $g, x \in G_1$, write $F_g(\sigma^{-1} f_{j+1}) = K'_{j+1}(\sigma g, \sigma g x)$ for $\sigma \in G_{j+1}$ on the orbit of f_{j+1}. With $\delta' = \delta_{\gamma}^{(j+1)}$, where $\gamma = 0$ for $j+1 < n$ and $\gamma = -1$ for $j+1 = n$, one has*

$$K'_j(n_j(t,\beta)g, n_j(t,\beta)gx) = \int \psi(p_j \xi[t,v]) F_g(v)\, \mathrm{d}\delta'(v).$$

Proof. The stabilizer of f_{j+1} in G_{j+1} fixes the pairing, and its Borel orbit sections give F_g as in Lemma 156.1. Unfold the recurrence for K'_j in Lemma 148.2. Conjugation of $n_j(t,\beta)$ by $\sigma \in G_{j+1}$ gives $n_j(\sigma t, \beta) \in N^j$, as in Proposition 148.3. Its only possibly nonzero simple-root entry is $-[\sigma t, f_{j+1}]$. In the second argument conjugation by d_* multiplies that entry by $r_j = p_j/q_j$, since $r_{j+1} = 1$. The two separate transformation laws therefore give the phase of $(1 - r_j)[t, \sigma^{-1} f_{j+1}] = p_j \xi[t, \sigma^{-1} f_{j+1}]$. Proposition 138.3 yields the integral. The recurrence is absolutely convergent by Lemma 148.2. □

Proposition 158.2. *Let L be as in Lemma 154.2, with the heads of Lemma 158.1. For every g, x at sufficiently large scales one has*

$$\int \mathbf{1}_{\mathcal{N}_j}(w_p u g) |K'_j(ug, ugx)|^2 \, \mathrm{d}\eta_j(u) \\ \le L^{100n^2} |p_j|^{-2d_j - 1} (\pi/|\xi|)^{2(k-1)} \int |K'_{j+1}(\sigma g, \sigma g x)|^2 \, \mathrm{d}\theta_{j+1}(\sigma).$$

Proof. If the integral on the left is positive there is a layer element u_0 with $g' = u_0 g \in \mathcal{N}_j$ and $w_p g' \in \mathcal{N}_j$, by Lemma 148.2. Replace g by g' there, using right invariance of the layer measure. The squared amplitude integral on the right is unchanged: $\sigma u_0 \sigma^{-1} \in N^j$ contributes only a phase to K'_{j+1} by the same lemma. Thus take $g, w_p g \in \mathcal{N}_j$. Choose $a \in Q_j$ with $\|ag\|_{\mathrm{op}} \le R$, and write $h_0 \in G_{j+1}$ for its action on the middle quotient. For the amplitude in Lemma 158.1, $\|h_0 v\| \le R^2$ on its nonzero set by the first assertion of Lemma 149.2: its proof only uses $\sigma g \in \mathcal{N}_{j+1}$.

For every $u = n_j(t,\beta)$ with $w_p u g \in \mathcal{N}_j$, apply Theorem 143.2 to ug, the Weyl matrix $w_p^{-1} = w_{\bar{p}}$, and the near multiplier au^{-1}. It gives $\|au^{-1} e_j^*\| \le L^{4j+1}$. Use it also at the identity layer element to obtain $\|ae_j^*\| \le L^{4j+1}$. By Definition 146.1, $u^{-1} e_j^* - e_j^* = -t + (-i\beta - [t,t]/2) e_j$. Projecting its image under a to V_{j+1} gives $-h_0 t$; in particular $\|h_0 t\| \le 2L^{4j+1}$. For fixed t, the difference of the two images for admissible β, β' is $-i(\beta - \beta') a e_j$. Its norm is at most $2L^{4j+1}$, and $\|ae_j\| \ge 1$ by the unit diagonal of the flag action. The admissible central coordinates have length at most $2L^{4j+1}$, in the sense of their Lebesgue measure.

Apply the phase formula and integrate first in the center. Change t, v by the same h_0; this preserves ambient and orbit measures by Proposition 138.3 and $|\det h_0| = 1$. Use ambient output of radius $T = 2L^{4j+1}$, quadric input of radius R^2, and frequency $p_j \xi$ in Theorem 139.2. The squared amplitude integral on the orbit is exactly the one in the statement. The other factors are bounded by

$$T\, M_{\mathrm{amb}}(T) M_{\mathrm{quad}}(R^2) (\pi/|p_j \xi|)^{2(k-1)}.$$

Here the two box factors have $h = n - j - 1$. Since $T \le L^{4j+2}$ and $L \ge 4$, their product with T has logarithm to base L at most $(4j+2) + (2h+2)(1+8j+4) + 2 + 4 + 10h \le 70n^2$. Also $|D_{p,j}| \le$

R^{2j} by the head-size theorem at g; hence $|p_j| \le L^{2j+1}$. As $2d_j + 1 - 2(k-1) = 4(n-j)+1$, replacing $|p_j|^{-2(k-1)}$ by $|p_j|^{-2d_j-1}$ incurs a factor at most $L^{(2j+1)(4(n-j)+1)} \le L^{30n^2}$. These estimates give the result, including the zero case. □

159. The real family of kernels in the orbital sieve

Proposition 159.1. *For the heads and nonzero $\xi = p_j^{-1} - q_j^{-1}$ of Lemma 158.1, and L, L', X of Lemma 154.2, one has*

$$C_j \int_X \int |B_j^0(g,x)| \,\mathrm{d}\mathfrak{m}_j(g)\,\mathrm{d}x \le nC_{\mathrm{B}} L^{125n^2j+50n^2+1}(L')^{70n^2}|\xi|^{-(k-1)}.$$

Proof. Fix x, writing $W_a(g) = W_a(g,x)$, $I_a(g) = \mathbf{1}_{\mathcal{N}_a}(g)/W_a(g)$, and put $w = w_p$. Let w_0 flip the first $j-1$ pairs with their entries from p. Write $Z = Z_{j-1}(V,1;x)$ and $Z' = Z_{\ge j}(x)$ for the twisted kernel. Both are finite for almost every x in the ball by Theorem 150.3. The integrand on the left is $|K_j(g,gx)K_j'(w^{-1}g, w^{-1}gx)|$, by Lemma 148.2. Its nonzero set requires both $g, w^{-1}g \in \mathcal{N}_j$. In Cauchy–Schwarz in g, use the squared factors

$$A(g)I_j(w^{-1}g), \qquad A(g) = W_{j-1}(g)|K_j(g,gx)|^2,$$
$$\Phi(w^{-1}g), \qquad \Phi(y) = \mathbf{1}_{\mathcal{N}_j}(wy)W_j(y)I_{j-1}(w_0y)|K_j'(y,yx)|^2.$$

Their product is the squared integrand wherever the latter is nonzero. Indeed $w_0w^{-1} = w_j(p_j)^{-1} \in G_j \subset Q_{j-1}$ preserves the earlier weights and near condition by Lemma 146.2 and Definition 144.1. All weights are positive.

Put $H_p = \prod_{a \le j} |p_a|^{-2d_a-1}$. For the first squared factor, A is left Q_j-invariant and supported in $\mathcal{N}_j$ by Lemma 148.2 for the empty head. Its integral against τ_j equals Z by definition. Write $\mathfrak{m}_j = \mathrm{d}u_{N^j} * \tau_j$ by Lemma 153.2 and apply Corollary 147.2 with head $\bar{p}$. This bounds the first integral by $L^{125n^2j}H_pZ$.

For the second squared factor, Φ is left G_{j+1}-invariant: the two Weyl matrices commute with that group, the weights and near sets are invariant, and K_j' is simultaneously invariant there by Lemma 148.2. Hence Lemma 153.2 replaces $\Phi(w^{-1}g)$ by $\Phi(g)$ in the integral. In $\mathfrak{m}_j = \eta_j * \mathrm{d}u_{N^{j-1}} * \tau_j$, permitted by Lemma 146.2, integrate first over $g = uy$, u in layer j. The factors $W_j(uy)$ and $I_{j-1}(w_0uy)$ are constant in u, since $u \in Q_j \subset Q_{j-1}$ commutes with w_0. Proposition 158.2 bounds the remaining inner integral by

$$L^{100n^2}|p_j|^{-2d_j-1}(\pi/|\xi|)^{2(k-1)} \int |K_{j+1}'(\sigma y, \sigma yx)|^2 \,\mathrm{d}\theta_{j+1}(\sigma).$$

After multiplying by $W_j(y)$, the squared integral is $E_{j,j}(y,x)$ for the twisted kernel. Enlarge this to $S_j(y,x)$. It is left Q_j-invariant and supported in $\mathcal{N}_j$ by Lemma 153.3. It is thus left N^{j-1}-invariant and supported in $\mathcal{N}_{j-1}$. Apply Corollary 147.2 to this function, the head of length $j-1$ giving w_0, and $\rho' = \tau_j$. The remaining integral against τ_j equals Z'. The second squared factor consequently has integral at most

$$L^{100n^2+125n^2(j-1)}H_p(\pi/|\xi|)^{2(k-1)}Z'.$$

Cauchy–Schwarz gives the product of the two square roots. Integration of $\sqrt{ZZ'}$ over X is bounded by Theorem 150.3 as in Lemma 154.2; Z is one summand of the empty-head sum. The pivot powers cancel to $|D_{q,j}|^{-1} \le L$ as in that lemma. Finally the total power of π is absorbed by C_{B} exactly as in Proposition 156.2. □

Theorem 159.2. *At every sufficiently large scale and every actual real place, use the functions $J_0, \dots, J_n$ of Theorem 155.3. For each fixed $\delta > 0$, these satisfy all the kernel conditions of Definition 54.2 over the single factor $C = \mathbb{C}$, with length parameter $\nu = n-1$, phase $\psi(z) = \exp(z - \bar{z})$, $I(v) = v^{-1}$, and bounds*

$$E_0 = (1 + b^{-10})/\mathcal{D}_\tau, \qquad \Lambda_j(p) = \mathcal{H}_j(p),$$
$$\beta(\xi) = C_{\mathrm{B}} b^{(12+\delta)n^9} d_{k-n^7}(|\xi|^{-1}) \qquad (\xi \ne 0).$$

The scale threshold may depend on δ and the fixed arithmetic parameters. At the top the kernel is the Gram kernel of the actual vector Ξ in Definition 134.1, including its identity with the matrix integral for any middle isometry with the specified last entry. One may in particular use $2000n^9$ as the exponent of b in β.

Proof. Theorem 155.3 gives the bottom bound, absolute common-coordinate integration and its value at zero frequency. Assigning one to an appended zero coordinate changes none of these Lebesgue integrals. Theorem 157.3 gives the bound at nonzero common-coordinate frequencies. Theorem 134.2 proves top continuity, positive semidefiniteness and the claimed identity at every admissible middle isometry.

It remains to bound $J_j(p,q)$ when $1 \le j \le \nu$ and $\xi = p_j^{-1} - q_j^{-1} \neq 0$. Use $f(g,x) = |B_j^0(g,x)| \le B_j(g,x)$. The ball hypothesis of Lemma 157.2 follows from Proposition 159.1, since $125n^2j + 50n^2 + 1 \le n^5$ eventually. Apply that lemma, then the triangle inequality in the definition of J_j^*. This gives precisely $\beta(\xi)\Lambda_j(p)\Lambda_j(q)$, as required. Taking $\delta = 1$ proves the last statement. □

160. Finite-place phase functions and congruence cosets

Definition 160.1. Let T be a finite set of finite places of the fixed field F; over E take all factors above T, a set stable under conjugation. Write $A_T = \prod_{v\in T} A_v$ and $B_T = \prod_{v\in T} B_v$ for the integer rings of Lemma 57.1. Their fraction algebras are F_T, E_T, allowing the two factors at a split place. Let G_T be the E_T-linear isometries of the row form in Definition 15.1. Denote the specified integral row lattice over B_T by $\mathcal{L}$, its stabilizer by K_T, and the kernel of the action on $\mathcal{L}/g\mathcal{L}$ by K_g, where $g = g_0 g_1$ as in Proposition 12.2. Let U_T be the flag unipotent group of Definition 20.3 over F_T, and set

$$\chi_g(u) = \psi_T^0(\alpha(u)/g).$$

Here ψ_T^0 is the restriction of ψ_f^0 of Proposition 18.5, and α is from Proposition 21.2. Write $|\pi_v|_v = Q_v^{-1}$ for a uniformizer of F_v.

Lemma 160.2. *The subgroup K_T is compact open, K_g is open and normal in it, and K_T/K_g is finite. The character χ_g is continuous and trivial on $U_T \cap K_g$. There is a bi-invariant measure $\mathrm{d}u$ on U_T assigning $U_T \cap K_T$ volume one, for which*

$$v_g := |U_T \cap K_g| = \prod_{v\in T} |g|_v^{\nu(2k+2\nu+1)}.$$

If T contains every place above g_0, g_1, this equals $g^{-f\nu(2k+2\nu+1)}$.

Proof. In the integral basis, the Gram matrix and its inverse are integral by Proposition 15.2. Thus an integral matrix satisfying the isometry equation has integral inverse. The stabilizer is the closed solution set of that equation among integral matrices, hence compact, since the rings are inverse limits of finite residue rings. It is open by the open condition that the entries be integral. The congruence subgroup is the kernel of a homomorphism to a finite group and is open by the entry conditions. The simple-root sum is additive on the unipotent group by the matrix computation of Proposition 21.2. For an element of $U_T \cap K_g$ its value is in gB_T, since the frame vectors are integral and their outer coordinates are integral coordinate functionals. Proposition 18.5 gives $\chi_g = 1$ there and continuity everywhere.

At layer $a = 1, \dots, \nu$, write l for the row on the intervening space, of rank $d_a = k - 1 + 2(n-a)$. Use the coordinates (l, s) with corner entry $-\omega l J^{-1} l^* + s$, where $s + \bar{s} = 0$ and $\omega + \bar{\omega} = 1$ by Theorem 18.2. By Lemma 20.4, integrality in this layer is exactly integrality of the row on the intervening lattice and of s; at principal level these are multiplied by g. Indeed the quadratic term is then in $g^2 B_T$. The free row lattice has $2d_a$ coordinates over A_T, and the skew lattice is iA_T by Lemma 20.1 after completion. Give each of these additive lattices volume one. Layer multiplication translates l, then translates s at each fixed l, on both the left and the right. The inner group acts on l by an isometry whose

determinant has norm one, and leaves its scalar quadratic term invariant. By the diagonal index computation in Lemma 20.2, this preserves the measure. The semidirect products of the layers therefore give a bi-invariant measure with the required normalization. The measure ratio at layer a is $\prod_v |g|_v^{2d_a+1}$; summing the exponents gives $\nu(2k+2\nu+1)$. Each g_i splits into f places of F with $Q_v = g_i$ and exponent one by Lemma 13.2, proving the last formula. □

Definition 160.3. For $y \in G_T$ put $\phi_g(y) = \chi_g(u)$ if $y = ua$ with $u \in U_T,\ a \in K_g$, and put it equal to zero otherwise. For $K_g \subseteq Z \subseteq K_T$ a subgroup, let $D_Z = K_g\backslash Z$ be the finite set of cosets $K_g a$. In the Hilbert space $\mathbb{C}^{D_Z}$ with the sum inner product define

$$W_Z(y)_{K_g a} = \phi_g(ya^{-1}).$$

Proposition 160.4. *The functions of Definition 160.3 are well defined, locally constant, and bounded. One has $\phi_g(uyh) = \chi_g(u)\phi_g(y)$ for $u \in U_T,\ h \in K_g$. Left multiplication by u acts on W_Z by $\chi_g(u)$; right multiplication by $z \in Z$ permutes its coordinates via $K_g a \mapsto K_g a z^{-1}$. For every $y, x \in G_T$,*

$$v_g\langle W_Z(y), W_Z(yx)\rangle = \int_{U_T} \mathbf{1}_Z(uy)\phi_g\big((uy)x(uy)^{-1}\big)\,\mathrm{d}u.$$

In particular,

$$\|W_Z(y)\|^2 = [U_T \cap Z : U_T \cap K_g]\,\mathbf{1}_{U_T Z}(y).$$

Proof. Two decompositions $y = ua = u'a'$ have $(u')^{-1}u \in U_T \cap K_g$, where Lemma 160.2 gives trivial character. The support is unchanged by the indicated left and right multiplications and the formula on it follows from the character law. In particular ϕ_g is locally constant by right K_g-invariance, and its modulus is at most one. Changing a to ha, $h \in K_g$, multiplies its inverse on the right by h^{-1}, so the coordinate formula is independent of the representative. Right translation by z uses $za^{-1} = (az^{-1})^{-1}$. Normality in Z implies that K_g fixes every coordinate. This proves local constancy and the stated equivariances.

Partition the integration set by $uy \in K_g a$ for $K_g a \in D_Z$. This set in u is nonempty exactly when $W_Z(y)_{K_g a} \neq 0$; in that case it is a right translate of $U_T \cap K_g$ and has measure v_g. On it write $uy = ha$ with $h \in K_g$. Then $\phi_g(ya^{-1}) = \chi_g(u)^{-1}$, and by the left and right transformation laws

$$\phi_g\big((uy)x(uy)^{-1}\big) = \chi_g(u)\phi_g(yxa^{-1}) = \overline{W_Z(y)_{K_g a}}\, W_Z(yx)_{K_g a}.$$

Integration on the finitely many cosets gives the identity. For $x = 1$ the integral is the measure of $\{u : uy \in Z\}$, either empty or a translate of $U_T \cap Z$. This compact subgroup is a finite union of cosets of $U_T \cap K_g$ since it maps into the finite group Z/K_g. Dividing its measure by v_g proves the squared-norm formula. □

161. The finite coefficient and its congruence contribution

Lemma 161.1. *For any $c > 0$ the group G_T of Definition 160.1 has a left and right invariant Borel measure μ, finite on compact sets, with $\mu(K_T) = c$. Its normalized restriction to K_T is the compatible uniform probability on the finite congruence images.*

Proof. First take one place v, and put $A = A_v,\ B = B_v$, with uniformizer π of A. In the integral basis write J for the Gram matrix and $K(\pi^h)$ for isometries congruent to the identity modulo π^h. The compact group $K = K_T$ in this one-place notation is the inverse limit of its finite images on the lattices modulo π^h: a compatible tuple of matrices lifts entrywise by completeness, and the closed isometry equation passes to the limit. Uniform probabilities on these images are consistent because all fibres of a surjective group homomorphism have the same size. To construct the probability on K, choose representatives for each successive kernel and lifts for the previous image. One may sample each successive digit from length measure on $[0,1)$ by successively subdividing each interval into the same number of equal

half-open intervals as its digit choices. The resulting compatible residues give a Borel map to K and its cylinder probabilities are the indicated ones. They determine the measure, since the congruence cylinders form a countable basis; the generating-family argument of Lemma 123.1 gives uniqueness. Both left and right multiplication preserve these probabilities at each depth.

Extend this probability by left translation to each left coset of K and sum. The cosets are countable because the finite-dimensional ambient local field has a countable basis of residue balls, obtained from the finite residue digits and powers of π. Changing the representative of a coset preserves the measure there. Thus the result is left invariant and finite on compact sets, which meet only finitely many open cosets. Any left invariant Borel measure finite on compact sets is determined by its value on K: in each congruence quotient all cosets have equal measure.

For right invariance put $X^{\dagger} = J^{-1}\bar{X}^{\mathsf{T}}J$ and $\mathfrak{l} = \{X \in M_r(B) : X^{\dagger} = -X\}$. On $\pi^h\mathfrak{l}$, $h \geq 1$, the Cayley map $C(X) = (1+X)(1-X)^{-1}$ is a bijection to $K(2\pi^h)$, where the argument of K denotes the congruence ideal. Indeed its inverse is $(y-1)(y+1)^{-1}$; when $y = 1 + 2\pi^h b$ the matrix $(y+1)/2 = 1 + \pi^h b$ has integral inverse by its geometric series. Taking adjoints proves the two defining equations. The difference

$$C(X) - C(Y) = 2(1-X)^{-1}(X-Y)(1-Y)^{-1}$$

shows that residue cylinders on this lattice correspond exactly to congruence cosets on its image, with an extra factor 2 in the congruence ideal. Hence on this chart the group measure is a constant multiple of additive measure on the skew space: all cylinders of the same depth have equal mass on both sides, and their partitions determine both measures.

For any isometry y, conjugation $X \mapsto yXy^{-1}$ has determinant one on the skew space over F_v. After extension to $E_v = F_v \otimes_F E$ it is conjugation on the full matrix algebra: every matrix decomposes uniquely over F_v as a skew matrix plus i times a skew matrix by taking its adjoint symmetric and antisymmetric parts. The scalar extension map is thus an isomorphism, also for a split pair. On all matrices left multiplication has determinant $(\det y)^r$ by columns, and right multiplication by y^{-1} has its reciprocal by rows. Scalar extension is injective on F_v, giving the assertion there. Conjugate a sufficiently deep chart so that it and its image lie in the chart at $h = 1$; this is possible by clearing the finitely many denominators of y, y^{-1}. The Cayley map commutes with conjugation and the additive determinant-one map preserves volume, by the diagonal lattice calculation of Lemma 20.2 in coordinates over $\mathbb{Q}_p$. Conjugation therefore preserves the mass of a single open congruence subgroup. Its image measure is left invariant; the uniqueness above and finite index in K show that the ratio is one everywhere. Right invariance follows by composing conjugation and left translation. For arbitrary T take the finite product and multiply by c; when T is empty use mass c at the identity. □

Definition 161.2. With the two measures of Lemmas 160.2 and 161.1, put $v_Z = |U_T \cap Z|$. For a left U_T-invariant function h supported on $U_T Z$ use the quotient integral $v_Z^{-1}\int_Z h(y)\,\mathrm{d}\mu(y)$. Define the coefficient

$$c_Z(x) = v_Z^{-1}\int_Z \langle W_Z(y), W_Z(yx)\rangle\,\mathrm{d}\mu(y), \qquad m_Z = \mu(Z)\mu(K_g)/v_g^2.$$

The integral is the quotient integral of the displayed inner product by Proposition 160.4.

Proposition 161.3. *Let $C_Z(x)$ be the mean of $\phi_g(yxy^{-1})$ for $y \in Z$ under the probability $\mathrm{d}\mu(y)/\mu(Z)$. Then $v_g c_Z = \mu(Z)C_Z$, with $C_Z = 1$ on K_g. The function C_Z is invariant on*

both sides by K_g and under conjugation by Z. One has the exact identities

$$v_Z^{-1}\int_Z \|W_Z(y)\|^2\,\mathrm{d}\mu(y)=\mu(Z)/v_g,$$

$$\frac{1}{m_Z}\int_{G_T}|c_Z(x)|^2\,\mathrm{d}\mu(x)=\mu(Z)\left(1+\sum_{\substack{xK_g\in G_T/K_g\\ xK_g\neq K_g}}|C_Z(x)|^2\right).$$

The latter equality holds in the extended nonnegative reals.

Proof. The first identity is immediate from the norm formula in Proposition 160.4, since v_Z/v_g is the index there. For $y\in Z$ the condition $uy\in Z$ in its unfolding is $u\in U_T\cap Z$. Integrating in y over Z, left translation by such u removes it from the conjugation argument. Absolute integrability follows from boundedness and compactness of the two groups. Lemma 123.1 then yields the asserted formula for c_Z. Right translation of the integration variable in Z gives conjugacy invariance there. Since Z normalizes K_g, right translation of x by K_g only introduces a right K_g multiplier into the phase function and preserves it. Left invariance by K_g follows since hx is conjugate by $h\in K_g$ to xh. On K_g every conjugate belongs to K_g and has phase one. Finally all cosets xK_g have the same positive mass $\mu(K_g)$; there are countably many by the basis argument of Lemma 161.1. Summing the constant squared modulus on these cosets and dividing by m_Z gives the second identity. □

162. A diagonal average at and away from the level

Definition 162.1. In G_T consider the diagonal subgroup $\mathcal{T}_g$ with entries d_a on e_a and $1/\bar{d}_a$ on e_a^* for $1\le a\le\nu$, fixing the middle space, where $d_a\in H_g=\{v\in B_T^\times : v-1\in gB_T\}$. This subgroup lies in K_g by Definition 160.1. Give it product probability and set $d_{\nu+1}=1$. For $u\in U_T$ denote by $b_a(u)$ the e_a-coordinate of ue_{a+1} when $a<\nu$, and of uz when $a=\nu$. Their sum is $\alpha(u)$ from Proposition 21.2. Write $w\mid T$ for each field component of E_T; let O_w be its integer ring, π_w a uniformizer, q_w its residue cardinality and ψ_w^0 its factor of ψ_T^0.

Proposition 162.2. *Define $\Phi_g(y)=\int_{\mathcal{T}_g}\phi_g(tyt^{-1})\,\mathrm{d}t$. It vanishes outside U_TK_g. For $y=uh$ with $u\in U_T$ and $h\in K_g$ it is*

$$\Phi_g(uh)=\prod_{a=1}^{\nu}\prod_{w\mid T}A_w(b_a(u)_w),$$

where for $g\in\pi_wO_w$ one has

$$A_w(b)=\mathbf{1}_{O_w}(b)\,\psi_w^0(b/g),$$

and for $g\in O_w^\times$ one has

$$A_w(b)=\begin{cases}1, & b\in O_w,\\ -1/(q_w-1), & b\in\pi_w^{-1}O_w\setminus O_w,\\ 0, & b\notin\pi_w^{-1}O_w.\end{cases}$$

For every intermediate Z of Definition 160.3, $C_Z(x)$ of Proposition 161.3 equals the mean of $\Phi_g(yxy^{-1})$ for $y\in Z$.

Proof. The group $\mathcal{T}_g$ normalizes U_T by the flag equations and K_g by Lemma 160.2, so it preserves the support under conjugation. For $y=uh$, the last factor remains in K_g, whereas the simple-root entries of tut^{-1} are $(d_a/d_{a+1})b_a(u)$ by diagonal multiplication. The map taking (d_a) to (d_a/d_{a+1}) is an automorphism of H_g^ν: its inverse takes successive products from the last entry. At every finite congruence quotient it permutes the points and preserves uniform probability, hence it preserves the compact probability by the cylinder argument of

Lemma 161.1. The components over $w \mid T$ are independent as B_T is their direct product. The desired integral reduces to the average of $\psi_w^0(sb/g)$ over $s \in O_w^\times,\ s-1 \in gO_w$.

Proposition 18.5 applied in one component shows that the additive annihilator of O_w for multiplication and ψ_w^0 is O_w. Indeed the involution permutes the component rings, and the test elements in its statement can be varied independently in each component. A nontrivial character on a compact additive subgroup has mean zero, since translation by a point with character different from one multiplies the mean by that value. The trivial character has mean one. If g is a nonunit, write $s = 1 + gz$ with $z \in O_w$. The uniform additive probability on z gives Haar probability on these units: multiplication by $1 + gz_0$ induces the measure-preserving affine map $z \mapsto z_0 + (1 + gz_0)z$. The mean is thus $\psi_w^0(b/g)$ times the additive mean at b. If g is a unit, s ranges over all of $O_w^\times$. Its probability is additive measure restricted there and divided by $1 - q_w^{-1}$, since multiplication by a unit preserves every residue count. Subtracting the integral on $\pi_w O_w$ from that on O_w gives the numerator $\mathbf{1}_{O_w}(b) - q_w^{-1}\mathbf{1}_{\pi_w^{-1}O_w}(b)$. This evaluates all factors. Finally left translation by $t \in \mathcal{T}_g \subset Z$ preserves probability on Z. Integrate first over t in the conjugation argument using Lemma 123.1, which applies to these bounded functions on compact groups. □

Corollary 162.3. *Put $\bar K = K_T/K_g,\ \bar Z = Z/K_g$, and let $\bar U$ be the image of $U_T \cap K_T$. Its character induced by χ_g is denoted by $\bar\chi$. Then the part of the normalized energy of Proposition 161.3 with $x \in K_T$ is the explicit finite sum*

$$\mu(Z) \sum_{a \in \bar K} \left| |\bar Z|^{-1} \sum_{\substack{z \in \bar Z \\ zaz^{-1} \in \bar U}} \bar\chi(zaz^{-1}) \right|^2 .$$

The complementary part is supported only on those cosets xK_g where some $y \in Z$ has $yxy^{-1} = uh$ with $h \in K_g$ and with every $b_a(u)_w \in O_w$ for $w \mid g$ and $b_a(u)_w \in \pi_w^{-1}O_w$ for the other $w \mid T$.

Proof. If $x = uh \in K_T,\ h \in K_g$, then $u \in U_T \cap K_T$. Conversely membership of the residue of x in $\bar U$ gives such a decomposition, with $\phi_g(x)$ equal to its residue character by Lemma 160.2. Uniform probability on Z pushes to uniform probability on $\bar Z$ because all its cosets have equal measure. Proposition 161.3 gives the sum. For $C_Z(x) \neq 0$, the last average in Proposition 162.2 has a nonzero integrand at some y. Its product formula gives exactly the stated coordinate requirements. □

163. Integral hyperbolic coordinates on the finite quadrics

Lemma 163.1. *Let $a_* = 3\nu + 6,\ m_* = k - 1 - 2a_*$, and $1 \le i \le n$. At all sufficiently large scales $m_* > 0$. After removing the first $i-1$ outer pairs, the remaining finite lattice over B_T is an orthogonal sum of $\ell_i = n - i + a_* + 1$ integral hyperbolic planes and a standard lattice of rank m_*. In these coordinates the norm and product additive measure are*

$$Q(a, b, w) = \mathrm{Tr}_{E_T/F_T}(a^*b) + w^*w, \qquad \mathrm{d}a\ \mathrm{d}b\ \mathrm{d}w.$$

Here $a, b \in E_T^{\ell_i},\ w \in E_T^{m_*}$, every integer coordinate ring has volume one, and $a^*b = \sum_h \bar a_h b_h$. The plane on v, v' in Lemma 70.1 can be retained among the displayed planes.

Proof. That lemma gives the plane on v, v' and $k-1$ mutually orthogonal standard lines. At each $v_0 \in T$ choose $\epsilon \in B_{v_0}^\times$ with $\epsilon\bar\epsilon = -1$. For a split pair take $(-1, 1)$. For an inert place choose a residue solution by the cyclic norm calculation in Proposition 57.3. A unit solution modulo π^h lifts modulo π^{h+1}: varying by $\pi^h d$ changes the next digit by the trace of $\bar\epsilon d$ on the residue field, which is onto by the differential calculation in that proposition. Completeness gives the desired unit. On two standard lines x_1, x_2 put $e = x_1 + \epsilon x_2,\ e' = x_1 - \eta e$, with $\eta + \bar\eta = 1$ from Lemma 70.1. Then e, e' have zero norm and pairing one; their determinant

in the given basis is $-\epsilon$, a unit. Apply this to a_* disjoint pairs and retain all outer planes. There are enough lines by Proposition 10.5. These integral basis changes and their inverses preserve additive measure by Lemma 20.2. Taking the product over the places gives the assertion. □

Definition 163.2. At a place $v \in T$, write $A = A_v,\ B = B_v,\ Q_v = |A/\pi A|$. For $a \in E_v^\ell$, $\ell \geq 1$, $a \neq 0$, put $d_v(a) = \max\{d \in \mathbb{Z} : a \in \pi^d B^\ell\}$. Write $L_a(b) = \mathrm{Tr}_{E_v/F_v}(a^*b)$. Choose an integral A-basis of B as in Lemma 20.1. Among the 2ℓ coefficients of this linear form over F_v, let $c(a)$ be the first of largest absolute value. On the hyperplane $L_a(b) = h$ define $\lambda_{a,h}$ by solving for that coordinate and taking $|c(a)|_v^{-1}$ times product additive measure in the other coordinates. For T take the product of these measures, omitting the null set where $a_v = 0$ for some v.

Lemma 163.3. *One has $|c(a)|_v = Q_v^{-d_v(a)}$. For $\theta \in F_v^\times$ and all $h \in F_v$,*

$$\lambda_{a,h}(\theta B_v^\ell) = |\theta|_v^{2\ell-1} Q_v^{d_v(a)} \mathbf{1}_{\theta \pi^{d_v(a)} A_v}(h).$$

Write $\Delta_T(\theta) = \prod_{v \in T} |\theta_v|_v$ for $\theta \in F_T^\times$, and define

$$M_T(\ell) = \prod_{v \in T} \frac{1 - Q_v^{-2\ell}}{1 - Q_v^{1-2\ell}}.$$

For the product measures in Definition 163.2,

$$\int_{a \in \theta B_T^\ell} \sup_{h \in F_T} \lambda_{a,h}(\theta B_T^\ell)\, \mathrm{d}a = \Delta_T(\theta)^{4\ell-2} M_T(\ell).$$

Proof. The image $L_a(B_v^\ell)$ is $\pi^{d_v(a)} A_v$. After division of a by $\pi^{d_v(a)}$, at least one coordinate has nonzero residue. Its trace pairing with a freely chosen coordinate is onto on the residue field: in the field case use the surjective trace in Proposition 57.3, and in the split case the two coordinates on the other side are independent. All coefficients are integral and one is therefore a unit, proving the asserted image and the value of $|c(a)|_v$. For the largest coefficient the solved coordinate lies in θA_v precisely when $h \in c(a)\theta A_v$, provided the other coordinates lie in θA_v. This proves the first formula.

On B_v^ℓ the measure of $d_v(a) \geq h$ is $Q_v^{-2\ell h}$ for integers $h \geq 0$, since B_v^ℓ is free of rank 2ℓ over A_v. The point $a = 0$ has zero measure by letting h increase. Thus the integral of $Q_v^{d_v(a)}$ on this lattice is $(1 - Q_v^{-2\ell}) \sum_{h \geq 0} Q_v^{(1-2\ell)h}$. On dilating a by θ, the factor $Q_v^{d_v(a)}$ is multiplied by $|\theta|_v^{-1}$ and its integration measure by $|\theta|_v^{2\ell}$. The supremum in the statement is attained simultaneously at $h = 0$. Multiplying the single-place integrals and using Lemma 123.1 proves the formula. □

Proposition 163.4. *On the finite space in Lemma 163.1, let $\delta_\gamma^{[i]}$, $\gamma \in F_T$, be the measure on $Q = \gamma$ given in the variables (a, b, w) by*

$$\mathrm{d}a\ \mathrm{d}w\ \mathrm{d}\lambda_{a,\gamma - w^*w}(b).$$

It is finite on compact sets, integrates continuously in γ on compact locally constant tests, and is invariant under every E_T-linear isometry of this space. For every nonnegative Borel test $H(\gamma, z)$,

$$\int H(Q(z), z)\, \mathrm{d}z = \int_{F_T} \int H(\gamma, z)\, \mathrm{d}\delta_\gamma^{[i]}(z)\, \mathrm{d}\gamma.$$

For $i = n$ the measure is the middle norm measure in Theorem 70.2.

Proof. All chosen coordinate maps are Borel, since the ordering of the finitely many coefficients partitions the nonzero a's into Borel sets. At fixed a, w and all but one coordinate of b_v, changing its last coordinate to $L_a(b_v) + (w^*w)_v$ gives exactly the factor $|c(a)|_v^{-1}$. The excluded zero sets have zero additive measure. Applying Lemma 123.1 gives disintegration.

For a test bounded by C and supported in a product box of radius θB_T on every entry, the solved-coordinate integral is dominated, independently of γ, by C times the upper fibre volume of Lemma 163.3. The w and a range over the displayed finite boxes, and that lemma proves integrability of this majorant. More explicitly, before integrating the free b-coordinates one can take $C\prod_v |c(a_v)|_v^{-1}$ times the indicators of their boxes and the a, w boxes. For fixed a, w and free coordinates, the solved coordinates are continuous in γ. Dominated convergence in Lemma A.2 thus gives continuity for compact locally constant tests and local finiteness for their indicator majorants. An isometry preserves Q and ambient additive measure: its determinant has norm one from the hermitian matrix identity, and one uses the determinant computation of Lemma 20.2. For a compact open indicator its two disintegrations before and after the change have equal integrals over every compact open set of γ's. Their continuous difference must vanish at every point, since every open ball has positive measure. The countable-basis uniqueness argument of Lemma 123.1, on increasing compact boxes, extends equality to all Borel sets. The same argument compares at $i = n$ with the continuous disintegration of Theorem 70.2 and Proposition 67.2. □

164. Fourier integration on the finite norm fibres

Lemma 164.1. *For $\xi \in E_T^\times$ put $m_T(\xi) = \prod_{v\in T} |\mathrm{N}_{E_v/F_v}(\xi_v)|_v$. Give E_T^b the additive measure for which B_T^b has volume one, for any integer $b \geq 0$. If $H \in L^1 \cap L^2(E_T^b)$, then*

$$\int_{E_T^b} \left| \int_{E_T^b} \psi_T^0(\xi z^* w) H(w)\,\mathrm{d}w \right|^2 \mathrm{d}z \leq m_T(\xi)^{-b} \int |H|^2.$$

Proof. For a box $D = \alpha B_T^b$, $\alpha \in F_T^\times$, the integral over $z \in D$ of $\psi_T^0(\xi z^*(w - w'))$ is $|D|\mathbf{1}_{\xi^{-1}\alpha^{-1}B_T^b}(w - w')$. Indeed conjugation preserves the integer rings and Proposition 18.5 gives the annihilator coordinatewise; translation gives zero for any nontrivial character as in Proposition 162.2. The box $\xi^{-1}\alpha^{-1}B_T^b$ has volume $m_T(\xi)^{-b}/|D|$ by the determinant and diagonal lattice calculation of Lemma 20.2. Expand the square on D and integrate in z first, using $|D|\|H\|_1^2 < \infty$. For each difference $d = w - w'$, Cauchy–Schwarz and additive invariance give $\int |H(w)H(w - d)|\,\mathrm{d}w \leq \|H\|_2^2$. The product of the two box volumes now gives the assertion on D. Enlarge D to exhaust the space and use Lemma A.2. For $b = 0$ the same calculation has both volumes equal to one. □

Theorem 164.2. *Use the space of Lemma 163.1 at floor i, put $\ell = \ell_i$, and let μ, μ' be either ambient additive measure or any of the respective quadric measures $\delta_\gamma^{[i]}$. For $\theta \in F_T^\times$ write D_θ for the box in which every coordinate is in θB_T, and set*

$$A_{\mathrm{amb}}(\theta) = \Delta_T(\theta)^{4\ell}, \qquad A_{\mathrm{quad}}(\theta) = \Delta_T(\theta)^{4\ell-2} M_T(\ell).$$

Use A_μ and $A_{\mu'}$ with these meanings. If $h \in L^2(\mu)$ vanishes outside D_θ, $\theta' \in F_T^\times$, $\xi \in E_T^\times$, and z_0 is any fixed vector, then

$$\int_{D_{\theta'}} \left| \int \psi_T^0(\xi[z + z_0, y]) h(y)\,\mathrm{d}\mu(y) \right|^2 \mathrm{d}\mu'(z) \leq m_T(\xi)^{-m_*} A_{\mu'}(\theta') A_\mu(\theta) \int |h|^2\,\mathrm{d}\mu.$$

Here $[\,,\,]$ is the hermitian pairing with norm Q.

Proof. Keep a, w as independent variables in both measures. For ambient measure the volume of the b-fibre restricted to D_θ is at most $M(a) = \mathbf{1}_{\theta B_T^\ell}(a)\Delta_T(\theta)^{2\ell}$. For a quadric use the same indicator times the supremum of the fibre volume in Lemma 163.3. In both cases $\int M(a)\,\mathrm{d}a = A_\mu(\theta)$. The identical construction in the output gives $M'(a')$ and $A_{\mu'}(\theta')$. The w boxes have finite measure, so h is also integrable.

At fixed a_z, a_y, w_y, integrate $h(y)$ on its b_y-fibre with the phase corresponding to $a_z^* b_y + [z_0, y]$, including the factor ξ. Denote the result as a function of w_y by $H(w_y)$, suppressing the two fixed a's. Its squared modulus is bounded by $M(a_y)\int_{\mathrm{fibre}} |h|^2$ by Cauchy–Schwarz.

The other terms of $[z,y]$ are $w_z^* w_y + b_z^* a_y$. The latter term contributes a phase of modulus one independent of w_y. Therefore the modulus of the original transform is bounded by the integral in a_y of the modulus of the transform of H in Lemma 164.1. This bound is independent of b_z. Weighted Cauchy–Schwarz in a_y, with weight $M(a_y)$, bounds its square by $A_\mu(\theta) \int |\widehat{H}(w_z)|^2 M(a_y)^{-1} \, \mathrm{d}a_y$. Omit the zero set of M, where the corresponding fibres make no contribution. For almost every a_y, H has finite squared integral by its fibre bound and is integrable since its w_y box has finite measure. At each a_z, integrate w_z over the full space and apply Lemma 164.1 with $b = m_*$. The result is at most $A_\mu(\theta) m_T(\xi)^{-m_*} \int |h|^2$ by cancellation of the fibre weight. Integration of the output fibre upper bound $M'(a_z)$ gives the other factor. The changes of order follow from Proposition 163.4 and Lemma 123.1. □

Corollary 164.3. *At all sufficiently large arithmetic scales, uniformly over finite sets T, one has*

$$1 \le \prod_{i=1}^{n} M_T(\ell_i) \le 1 + 2^{-5n}.$$

For every γ, the mass under $\delta_\gamma^{[i]}$ of the integral lattice box D_1 is at most $M_T(\ell_i)$.

Proof. The second assertion follows by integrating the upper fibre volume of Lemma 163.3 at $\theta = 1$ over the w lattice, which has mass one. For the first, put $h = 2\ell_i - 1 \ge 6n + 7$ by Lemma 163.1. Each local factor is at least one and its logarithm is at most $2Q_v^{-h}$, since $-\log(1-t) = \int_0^t (1-u)^{-1} \, \mathrm{d}u \le 2t$ for $0 \le t \le 1/2$. By Lemma 57.1 at most f places lie over each rational prime and each Q_v is at least that prime. Also $\sum_{m \ge 2} m^{-h} \le 2^{-h} + \int_2^\infty t^{-h} \, \mathrm{d}t \le 2 \cdot 2^{-h}$. Hence the logarithm of the product is at most $4nf \, 2^{-(6n+7)}$. Propositions 10.5 and 12.2 give $f \le X^{12}$, $n \ge \sqrt{X}/2$ eventually at scale X. Thus $4nf \, 2^{-(6n+7)} \le 2^{-5n-1}$ eventually. Using the exponential series, $\exp(t) \le 1 + 2t$ for $0 \le t \le 1/2$, which proves the bound. □

165. Sections of the finite unitary orbits

Definition 165.1. Number the outer pairs of Definition 15.1 from 1 through $\nu = n - 1$. Let V_i be the orthogonal complement of the first $i - 1$ pairs over E_T, and put $\mathcal{L}_i = \mathcal{L} \cap V_i$. Let G_i consist of the isometries of V_i, extended by the identity on the removed pairs, and let K_i be the stabilizer of $\mathcal{L}_i$ in G_i. Set $f_i = e_i$ for $i < n$ and $f_n = z$. For $i < n$ let N_i be the group of matrices $H(l,t)$ of Lemma 20.4 on the i-th pair with intervening space V_{i+1} and Gram matrix J. With $t = -\omega l J^{-1} l^* + s$, give N_i the measure $\eta_i = \mathrm{d}l \, \mathrm{d}s$ of Lemma 160.2. For $c \in E_T^\times$, let $w_i(c)$ exchange the two basis vectors by $e_i \mapsto c e_i^*$, $e_i^* \mapsto \bar{c}^{-1} e_i$, and fix V_{i+1}. These assignments preserve the pairing. Use $d_i = \dim_{E_T} V_{i+1} = k - 1 + 2(n - i)$ for the common rank in the field factors. Define the measure on G_i

$$\theta_i = (c \mapsto w_i(c)^{-1})_* (m_T(c)^{d_i} \, \mathrm{d}c) * \eta_i.$$

Here m_T is from Lemma 164.1. The convolution is the pushforward by ordered multiplication. Write $u_T = |B_T^\times|$ for the additive volume of the integral units. Empty products of measures have mass one.

Lemma 165.2. *For $1 \le i < n$ the image of θ_i under $\sigma \mapsto \sigma^{-1} f_i$ is $\delta_0^{[i]}$ of Proposition 163.4. In the coordinates $d e_i + y + c e_i^*$, this map on the parameters of θ_i is a Borel bijection onto the open part of the quadric with $c \in E_T^\times$. Its complement is null for $\delta_0^{[i]}$, and $\theta_i(K_i) \ge u_T$.*

Proof. Take $\sigma = w_i(c)^{-1} H(l,t)$. The inverse of $H(l,t)$ is $H(-l, \bar{t})$ by the trace equation of Lemma 20.4. Acting on $c e_i^*$ gives $d = c\bar{t}$, $y = c J^{-1} l^*$. Conjugation preserves additive measure since it preserves each integral lattice and its residue balls. The integral invertible matrix J preserves it as well, by Lemma 20.2. Hence at fixed c the change from l to y gives $\mathrm{d}y = m_T(c)^{d_i} \, \mathrm{d}l$. Writing $Q(y) = [y,y]$, the missing coordinate is

$$d = -\bar{c}^{-1} \bar{\omega} Q(y) - cs.$$

For general $\gamma \in F_T$ replace this by $d = \bar{c}^{-1}\bar{\omega}(\gamma - Q(y)) - cs$. Then $\mathrm{Tr}(\bar{c}d) + Q(y) = \gamma$. The lattice splitting $B_T = \bar{\omega}A_T \oplus iA_T$ follows by subtracting $\bar{\omega}$ times the trace, as in Lemma 70.1. In this splitting $\bar{c}d = \bar{\omega}(\gamma - Q(y)) - c\bar{c}s$. On the skew line the multiplier $-c\bar{c}$ changes volume by $m_T(c)$, the same factor as multiplication by $\bar{c}$ on E_T. This proves that $(\gamma, s) \mapsto d$ is an affine bijection carrying product measure to $\mathrm{d}d$. All inverse formulas on the open part are continuous in the field coordinates.

To identify the slice at every γ, first take a compact open test on d, y, c supported where c, c^{-1} are bounded in every field component. For γ in a compact neighborhood, the inverse formula bounds s on the support by a fixed compact box. Dominated convergence, Lemma A.2, gives continuity of its slice integral in γ. Integrating over any compact open set of γ's gives its ambient integral there by the preceding Jacobian and Lemma 123.1. Proposition 163.4 gives the same integral and continuity, hence equality at every γ. In the coordinates of Lemma 163.1 we may take the e_i^*-coefficient as the first a coordinate, exchanging the two sides of this hyperbolic plane if necessary. The set where any field component of c vanishes has measure zero in $\mathrm{d}a$, hence in that disintegration even on $Q = 0$. Exhaust the complement by compact open boxes with c, c^{-1} bounded and use the uniqueness on generating boxes from Lemma 123.1. This proves the measure assertion. For integral unit c the Weyl element is integral and $m_T(c) = 1$. Restricting also $H(l, t)$ to its integral subgroup of volume one gives the lower bound. □

Proposition 165.3. *On $Q = -N$ in V_n, a Borel section $\sigma(e)$ satisfies $\sigma(e)^{-1}z = e$ for $\delta_{-N}^{[n]}$-almost every e. Its pushforward measure θ_n on G_n satisfies*

$$(\sigma \mapsto \sigma^{-1}z)_*\theta_n = \delta_{-N}^{[n]}, \qquad \theta_n(K_n) \geq u_T.$$

Proof. In the notation of Lemma 70.1 write $e = dv + y + cv'$, with y in the $k-1$ standard lines. By Theorem 70.2 and Proposition 163.4 we may take

$$c \in E_T^\times, \qquad d = \bar{c}^{-1}\eta(-N - Q(y)) + c\alpha, \qquad \mathrm{d}\delta_{-N}^{[n]} = \mathrm{d}c\ \mathrm{d}y\ \mathrm{d}\alpha,$$

where $\alpha \in iF_T$. Let D act diagonally by $-\bar{c}^{-1}, -c$ on v, v', fixing the standard lines. It is an isometry and $Dz = d_0 v + cv'$ with $d_0 = -N\eta/\bar{c}$, since $z = N\eta v - v'$. Put $u = y/c,\ h = (d - d_0)/c$. Equality of the two norms $Q(e) = Q(Dz)$ gives $h + \bar{h} = -Q(u)$. The map A fixing v, sending v' to $v' + u + hv$, and sending a standard-space vector y' to $y' - [u, y']v$ is triangular with unit diagonal. The added term in the pairing of Av' and Ay' cancels, and $Q(Av') = Q(u) + h + \bar{h} = 0$. All remaining basis pairings are unchanged. Thus A is an isometry and $ADz = e$. Take $\sigma(e) = (AD)^{-1}$. These are continuous matrix formulas on the open chart, whose complement is null. For $c \in B_T^\times$, integral y and $\alpha \in iA_T$, each displayed matrix and its inverse is integral, since N, η are integral. The product volume of this subset of the chart is u_T. This proves both assertions. □

166. Group measure obtained from the finite quadrics

Definition 166.1. In the groups of Definition 165.1, let S_i be the stabilizer of f_i in G_i and put $H = S_n$. Take on H its bi-invariant measure μ_H with $\mu_H(H \cap K_n) = 1$. Indeed z has nonzero norm in every field factor, so an isometry fixing it acts freely as an arbitrary isometry of its perpendicular, the k original standard lines. Conversely every such isometry extends by fixing z. Lemma 161.1, applied in that standard basis, supplies bi-invariant measure. The subgroup $H \cap K_n$ is compact since H is closed, and open in H since K_n is open. It has positive finite mass by the congruence cylinder construction in that lemma, so the specified scaling exists. Use θ_i from Lemma 165.2 and Proposition 165.3 and define

$$\mathcal{M}_n = \mu_H * \theta_n, \qquad \mathcal{M}_i = \mathcal{M}_{i+1} * \eta_i * \theta_i \quad (i < n), \qquad \mathcal{M}_{\mathrm{up}} = \mu_H * \theta_n * \cdots * \theta_1.$$

Theorem 166.2. *Each $\mathcal{M}_i$ is a bi-invariant measure on G_i, positive and finite on K_i. In particular the measure $\mu = \mathcal{M}_1$ on G_T satisfies*

$$u_T^n \;\le\; \mu(K_T) \;\le\; \prod_{i=1}^{n} M_T(\ell_i) \;\le\; 1 + 2^{-5n}$$

at sufficiently large arithmetic scales, uniformly over T.

Proof. For $i < n$, the equation of an isometry fixing e_i forces the bottom row in this three-block basis to be $(0, 0, 1)$, by pairing with e_i. The induced matrix on $e_i^{\perp}/E_T e_i$ is an isometry of V_{i+1}. Dividing out its block diagonal lift leaves exactly N_i by the computation of Lemma 20.4. Thus $S_i = G_{i+1} N_i$ with a unique factorization in these two subgroups and with N_i normal. Conjugation by $h \in G_{i+1}$ changes the row l to lh^{-1}, fixing s: the norm $lJ^{-1}l^*$ is preserved. The determinant of h has norm one, so η_i is invariant under this action by Lemma 160.2. It is also invariant on both sides in N_i. If $\mathcal{M}_{i+1}$ is bi-invariant, the product $\nu_i = \mathcal{M}_{i+1} * \eta_i$ is consequently bi-invariant on S_i. Indeed multiplication by a member of either subgroup either translates that factor alone or also conjugates the other integration variable with its measure preserved. Its mass on $S_i \cap K_i$ is $\mathcal{M}_{i+1}(K_{i+1})$: integrality of an integral inverse pair forces integrality of the inner diagonal block and its inverse, after which the remaining layer must be integral, and the converse holds by multiplication. The integral layer has volume one. At $i = n$ use $\nu_n = \mu_H$, with corresponding mass one.

To prove right invariance in both cases, take a nonnegative Borel test B on G_i. The function $g \mapsto \int_{S_i} B(sg)\,\mathrm{d}\nu_i(s)$ depends only on $e = g^{-1} f_i$, since two representatives differ on the left by S_i and ν_i is right invariant. On the full-measure open chart of Lemma 165.2 or Proposition 165.3 it has a Borel representative $b(e)$ obtained by inserting the explicit section. Extend b by zero off that chart. The pushforward laws in those two results give $\int B\,\mathrm{d}\mathcal{M}_i = \int b\,\mathrm{d}\delta^{[i]}_{Q(f_i)}$. For each $y \in G_i$, replacing g by gy replaces e by $y^{-1}e$. Both e and $y^{-1}e$ lie in the open chart almost everywhere because the quadric measure is invariant by Proposition 163.4. Integration of $b(y^{-1}e)$ therefore gives the same value. The measurability and iterated integrations here follow from the Borel parameter formulas and Lemma 123.1.

For $B = \mathbf{1}_{K_i}$, the set of $s \in S_i$ with $sg \in K_i$ is empty or a right coset of $S_i \cap K_i$: if $s_0 g \in K_i$, the condition is $s s_0^{-1} \in S_i \cap K_i$. In the latter case $e = (s_0 g)^{-1} f_i \in \mathcal{L}_i$. The integral is thus at most $\nu_i(S_i \cap K_i)\delta^{[i]}_{Q(f_i)}(\mathcal{L}_i)$. It is at least $\nu_i(S_i \cap K_i)\theta_i(K_i)$, by restricting g to K_i before pushing forward. The first bound is at most $\nu_i(S_i \cap K_i) M_T(\ell_i)$ by Corollary 164.3, and the second bound is at least $\nu_i(S_i \cap K_i) u_T$ by the two section results. The number u_T is positive because each of the finitely many field factors contains an open ball of integral units. Right invariance now gives local finiteness since every compact set meets only finitely many open right cosets of K_i. Inversion carries this measure to a left invariant measure of the same positive finite mass on K_i. Uniqueness by residue cylinders as in Lemma 161.1 identifies the inverse measure with the bi-invariant measure of that mass in the integral basis of V_i. The latter is preserved by inversion, by the same uniqueness. This proves bi-invariance at each induction step. Multiply the successive bounds, starting at $i = n$, and apply Corollary 164.3. □

Corollary 166.3. *For the measure $\mathrm{d}u$ of Lemma 160.2, one has $\mu = \mathrm{d}u * \mathcal{M}_{\mathrm{up}}$. If A is a nonnegative Borel left U_T-invariant function supported on $U_T Z$ for $K_g \subseteq Z \subseteq K_T$, then*

$$\int A(y)\,\mathrm{d}\mathcal{M}_{\mathrm{up}}(y) = v_Z^{-1} \int_Z A(y)\,\mathrm{d}\mu(y).$$

In particular the integrals of $\|W_Z(y)\|^2$ and of $\langle W_Z(y), W_Z(yx)\rangle$ against $\mathcal{M}_{\mathrm{up}}$ equal $\mu(Z)/v_g$ and $c_Z(x)$, respectively, with $\mu = \mathcal{M}_1$ in Definition 161.2.

Proof. In the product defining $\mathcal{M}_1$, first move the independent N_1 factor immediately preceding θ_1 to the front. Every preceding factor lies in G_2 and preserves η_1 by the conjugation calculation of Theorem 166.2. Repeat inside the remaining product with $N_2, \ldots, N_\nu$.

The resulting leading product has measure $\mathrm{d}u$ by the successive layer decomposition in Lemma 160.2. The other factors are exactly $\mathcal{M}_{\mathrm{up}}$. Lemma 123.1 proves the convolution identity on nonnegative tests. At $y \in U_T Z$ the set of u with $uy \in Z$ is a right coset of $U_T \cap Z$, of volume v_Z. Elsewhere it is empty. Apply the identity to $A\mathbf{1}_Z$ to obtain the quotient formula. Proposition 160.4 gives the required support and invariance for both displayed tests. For the complex test apply the formula first to its absolute value, which is bounded on Z as both factors are bounded. Taking the real and imaginary parts and using Proposition 161.3 gives the two values. □

167. Products of coefficient integrals at vector stabilizers

Definition 167.1. Use $\mu = \mathcal{M}_1$ from Theorem 166.2. For $K_g \subseteq Z \subseteq K_T$, define

$$\mathcal{E}_T(Z) = \frac{1}{\mu(Z)\mu(K_g)} \int_{G_T} \left| \int_Z \phi_g(yxy^{-1}) \, \mathrm{d}\mu(y) \right|^2 \mathrm{d}\mu(x),$$
$$\mathcal{R}_T(Z) = \mu(Z) \sum_{\substack{xK_g \in G_T/K_g \\ xK_g \neq K_g}} |C_Z(x)|^2.$$

The sum runs over the other left cosets xK_g of the level subgroup, and C_Z is the mean in Proposition 161.3.

Proposition 167.2. *One has $\mathcal{E}_T(Z) = \mu(Z) + \mathcal{R}_T(Z) = m_Z^{-1} \int |c_Z|^2 \, \mathrm{d}\mu$. For a partition $T = T_1 \sqcup \cdots \sqcup T_a$, write μ_j for the same group measure constructed over T_j. Then $\mu = \bigotimes_j \mu_j$. If $Z = \prod_j Z_j$ is a product of intermediate groups, then $\mathcal{E}_T(Z) = \prod_j \mathcal{E}_{T_j}(Z_j)$, also in the extended nonnegative reals.*

Proof. Proposition 161.3 identifies the inner integral with $v_g c_Z(x)$ and gives the coset sum. The normalizing denominator is $v_g^2 m_Z$, proving the first identities. The coordinate sections of Lemma 165.2 and Proposition 165.3 act componentwise. The chart variables, additive measures and Weyl densities are products, as are the layer measures. The groups $H, H \cap K_n$ in Definition 166.1 split as products, and the product of the respective $\mu_{H,j}$ is bi-invariant with the prescribed mass one on $H \cap K_n$. Uniqueness on compact open cylinders in Lemma 161.1 gives exactly μ_H. Products and group multiplications now commute in all the convolutions defining $\mathcal{M}_1$ by Lemma 123.1. This proves the measure identity, independent of the integral bases used to describe the quadric measures, by Proposition 163.4.

The group $U_T K_g$ is the product of the corresponding sets. On it the phase χ_g is the product of its local phases by Proposition 18.5 and the componentwise simple-root sum in Definition 160.1. Off that product at least one factor is zero. Thus ϕ_g itself factors. The compact inner integral factors by iterated integration and the normalizing masses factor by the measure identity. The outer integrand is nonnegative, so the outer integrals factor as well. Each factor is at least $\mu_j(Z_j) > 0$. For the empty set of places all masses and the identity coefficient are one. □

Theorem 167.3. *Let T contain the places above g_0 and g_1, and let S be any subset of the split pairs above $p = g_0$ with one prescribed condition per pair as in Definition 19.3. Identify each pair with its underlying place of F. There is a product Z between K_g and K_T whose members are exactly the integral isometries over T satisfying these conditions. Its contribution from K_g is*

$$\mu(Z) = \mu(K_T)(p^r - 1)^{-|S|}.$$

Put $T_0 = T \setminus S$, and write Z_v for its factor at a selected $v \in S$. The following estimates suffice at every sufficiently large scale for $\mathcal{E}_T(Z) \leq (1 + b^{-6}) b^{-6k|S|}$:

$$\mathcal{R}_{T_0}(K_{T_0}) \leq 2^{-n}, \qquad \mathcal{R}_{\{v\}}(Z_v) \leq p^{-3k/4} \quad (v \in S).$$

Proof. At $v \in S$ take the inverse image of the indicated vector or covector stabilizer in the split reduction $\mathrm{GL}_r(\mathbf{F}_p)$. The reduction is onto by Proposition 15.2. At other places take the entire integral subgroup. The product contains K_g by its defining congruence, and is exactly the prescribed inverse image. Its index follows from Proposition 19.4, since the conditions have independent factors and each prescribed vector is nonzero. Partition into cosets of equal group measure to obtain the identity.

Use Proposition 167.2 with one block T_0 and one for each selected pair. On T_0 the assumed remainder and Theorem 166.2 give

$$\mathcal{E}_{T_0}(K_{T_0}) \le 1 + 2^{-5n} + 2^{-n} \le 1 + b^{-6}.$$

The last inequality holds eventually since $n \ge \sqrt{X}/2$ and $\log b = X^{\epsilon}$, $0 < \epsilon < 1/100$, by Proposition 10.5, where X denotes the arithmetic scale. At one selected place $\mu_v(K_v) \le 2$ at those scales. Thus the contribution from the level subgroup is at most $2/(p^r - 1) \le 4p^{-k}$, since $r \ge k$. Our other hypothesis gives $\mathcal{E}_{\{v\}}(Z_v) \le 4p^{-k} + p^{-3k/4}$. Proposition 12.2 gives $p \ge b^{10}$, so the last bound is at most b^{-6k} for sufficiently large X, independently of v and of its prescribed vector. Multiplying proves the conclusion, including $S = \varnothing$. □

168. Denominators in a relative position of two finite flags

Definition 168.1. Use the groups of Definition 165.1. For $0 \le j \le \nu$ put $U^{(j)} = N_1 \cdots N_j$, and $P_j = U^{(j)} G_{j+1}$. The empty unipotent product is the identity. Each inner group normalizes the outer layers by the proof of Theorem 166.2, so these are subgroups. For $p = (p_1, \ldots, p_j) \in (E_T^{\times})^j$ write

$$w_p = \prod_{a=1}^{j} w_a(p_a), \quad D_a(p) = \prod_{h=1}^{a} p_h, \quad B(p) = \prod_{a=1}^{j} D_a(p), \quad d(p) = B(p)\overline{B(p)}.$$

All empty scalar products are one. For a self-conjugate $d \in B_T \cap E_T^{\times}$ write $\mathcal{B}(d) = \{x \in G_T : dx\mathcal{L} \subseteq \mathcal{L}\}$. This is a compact open set. Indeed in the integral basis it is the intersection of the isometry equations with the product of the compact open entry boxes $d^{-1}B_T$; the equations force invertibility since the Gram matrix is invertible in every field factor. The identity $x^{-1} = J_0^{-1} x^* J_0$ with J_0, J_0^{-1} integral by Proposition 15.2 also shows that $\mathcal{B}(d)$ is stable under inversion.

Theorem 168.2. *Let $u_1, u_2 \in U^{(j)}$, $l \in G_{j+1}$. If $k_0 = u_1 w_p l u_2$ belongs to K_T, then*

$$D_a(p) \in B_T \quad (1 \le a \le j), \qquad u_1, u_2, l \in \mathcal{B}(d(p)).$$

More precisely, $D_j(p)$ clears the entries of the block of l on V_{j+1} when $j > 0$.

Proof. When $j = 0$ the assertion is simply that $l = k_0$ is integral. Order the column basis in blocks as $(e_1, \ldots, e_j, V_{j+1}, e_1^*, \ldots, e_j^*)$, and order the row basis as $(e_1^*, \ldots, e_j^*, V_{j+1}, e_1, \ldots, e_j)$. In both occurrences of V_{j+1} use its integral basis. In these coordinates the matrix is

$$L \operatorname{diag}(p_1, \ldots, p_j, l|_{V_{j+1}}, \bar{p}_1^{-1}, \ldots, \bar{p}_j^{-1})\, R,$$

where L is u_1 in the row basis and R is u_2 in the column basis. Relative to the three blocks, L is lower and R upper triangular with identity middle blocks. The first diagonal block of L is lower unitriangular and that of R is upper unitriangular. Indeed the flag action in Definition 20.3 is upper unitriangular on the first j isotropic vectors and inverse conjugate transpose on their partners, by the pairing. This also follows by multiplying the layer matrices of Lemma 20.4.

The determinant of the first a rows and columns of the displayed matrix is $D_a(p)$: the two triangular corners have determinant one, and terms using an intermediate index beyond a vanish. Since k_0 is integral, each $D_a(p)$ is integral in every component. For $s > a$, take rows $1, \ldots, a-1, s$ and columns $1, \ldots, a$. The determinant is $L_{sa} D_a(p)$: in the left minor the first $a - 1$ rows have zero last column and determinant one on the other columns. Using

instead columns $1,\dots,a-1,s$ and the first a rows gives $D_a(p)R_{as}$. Thus each entry of the first j columns of L, respectively rows of R, has denominator dividing $D_a(p)$ in column or row a. Bordering the first j rows and columns by one middle row and one middle column gives $D_j(p)$ times the corresponding entry of $l|_{V_{j+1}}$. Indeed in that bordered matrix one may subtract combinations of the first j rows and columns to remove the off-diagonal triangular blocks. The surviving middle entry is the stated one because both middle blocks of L, R are the identity. This argument is over E_T; the bordered determinant itself belongs to B_T.

It remains to control entries outside the indicated rows and columns. In the column basis write

$$R=\begin{pmatrix} A & C & E\\ 0 & I & F\\ 0 & 0 & A^{-*}\end{pmatrix},\qquad F=-J^{-1}(A^{-1}C)^*,$$

where J is the integral unimodular Gram matrix on V_{j+1} and $A^{-*}=(A^{-1})^*$. These equations follow by pairing the first and last block columns, then the middle and last block columns. Write $A=I+U$ with U strictly upper triangular. In $A^{-1}=\sum_{h=0}^{j-1}(-U)^h$, every entry of U^h is a sum indexed by strictly increasing chains. A term uses one entry per distinct row, with denominator the corresponding $D_a(p)$. In U^hC the additional C-factor uses the last index of the chain, still distinct from the preceding rows. Thus $B(p)$ clears both A^{-1} and $A^{-1}C$, as well as every entry already known in the first j rows. Since $B(p)$ and its conjugate are integral, $d(p)$ clears every block of R. For L take its conjugate transpose in the row basis. This upper block matrix preserves the inverse Gram matrix in that basis, which again has identity cross blocks and an integral unimodular middle Gram matrix, by the isometry equation. Its first j rows have the conjugate denominators. The same chain argument clears all its entries by the same self-conjugate $d(p)$. Finally $d(p)/D_j(p)$ is integral, so the middle bound also gives $l\in\mathcal{B}(d(p))$. □

Corollary 168.3. *If $y, w_p^{-1}y\in P_jK_T$, then every $D_a(p)$ is integral. There is $s\in G_{j+1}$ with $sy\in\mathcal{B}(d(p))$ and $sw_p^{-1}y\in\mathcal{B}(d(p)^2)$. If $j=\nu,\ l\in G_n$ and both $W_Z(y)$ and $W_Z(lw_p^{-1}y)$ are nonzero, then $y, l, lw_p^{-1}y$ all belong to $\mathcal{B}(d(p))$.*

Proof. Write $y=uhk,\ w_p^{-1}y=u'h'k'$, where $u,u'\in U^{(j)},\ h,h'\in G_{j+1}$ and $k,k'\in K_T$. Then $k(k')^{-1}=u_1w_pl_0u_2$ with

$$u_1=h^{-1}u^{-1}h,\qquad l_0=h^{-1}h',\qquad u_2=(h')^{-1}u'h'.$$

Here w_p commutes with G_{j+1}, since their nonidentity blocks act on disjoint summands. Apply Theorem 168.2. With $s=h^{-1}$ the two representatives are $u_1^{-1}k$ and l_0u_2k'. The bounds follow from inversion stability, integrality of the compact factors and multiplication of denominators. For the last assertion, the support formula of Proposition 160.4 gives $y=uk,\ lw_p^{-1}y=vk'$, where $u,v\in U_T=U^{(\nu)}$ and $k,k'\in Z$. Now $k(k')^{-1}=u^{-1}w_pl^{-1}v$. Apply the same theorem and inversion stability to u,v,l, and multiply by k,k' to obtain the claim. □

169. The last finite pivot as a Gram kernel

Definition 169.1. Use d_a and m_T of Definition 165.1 and $\mu=\mathcal{M}_1$ of Theorem 166.2. For $p\in(E_T^\times)^\nu,\ l\in G_n$ put

$$H(p)=\prod_{a=1}^{\nu}m_T(p_a)^{d_a/2},\qquad \mathcal{J}(p,l)=H(p)\int_{G_T}W_Z(y)W_Z(lw_p^{-1}y)^*\,\mathrm{d}\mu(y).$$

Vectors are columns indexed by D_Z from Definition 160.3; the product is a matrix, integrated entry by entry. Put $\tau(l)=[z,l^{-1}z]$. For a compact open additive subgroup $L\subseteq E_T$ of

volume $|L| > 0$ and $t \in E_T$, set

$$V(p,t;l) = \frac{\mathbf{1}_{t+L}(\tau(l))}{\sqrt{m_Z|L|}}\,\mathcal{J}(p,l).$$

Here m_Z is from Definition 161.2. Use the Hilbert–Schmidt inner product $\langle A,B\rangle = \operatorname{tr}(A^*B)$.

Lemma 169.2. *The integral defining $\mathcal{J}(p,l)$ is absolutely convergent for all p,l. It vanishes unless every $D_a(p)$ is integral, and otherwise is supported in $l \in G_n \cap \mathcal{B}(d(p))$. As a function of p,l it is locally constant. For each p there is a neighborhood on which $\mathcal{J}(p',l) = \mathcal{J}(p,l)$ for all $l \in G_n$.*

Proof. By Corollary 168.3, a nonzero integrand requires the asserted integrality and, on that domain, $l, y \in \mathcal{B}(d(p))$. Multiplication by K_T preserves the integral lattice, and the ball in Definition 168.1 is compact. The integrand is bounded by Proposition 160.4 and Haar measure is finite on compact sets by Theorem 166.2. Each entry is therefore absolutely integrable. Give p a compact open neighborhood on which the valuations of all p_a in every field factor are fixed. There the integrality conditions are constant, $H(p')$ is constant, and the common bounding ball is fixed because its defining principal fractional ideal $d(p')^{-1}B_T$ is fixed. If some minor fails integrality the whole function vanishes there. Otherwise put $D = \mathcal{B}(d(p))$. The matrix-valued integrand is locally constant jointly in p, l, y by the explicit formula for w_p, continuity of multiplication, and local constancy in Proposition 160.4. On the compact sets $y \in D,\ l \in G_n \cap D$, cover the slice at fixed p by finitely many product neighborhoods of constancy. Intersect their neighborhoods of p; then the integrand agrees with its value at p for every y, l in these compact sets. Outside them it vanishes whenever the product defining it could be nonzero. Integrating proves the claimed uniformity. At fixed p, l, the same argument with $y \in D$ gives constancy in a neighborhood of l. This proves joint local constancy. □

Theorem 169.3. *For heads p, q of length ν and last entries $t, t' \in E_T$ define*

$$J_n((p,t),(q,t')) = \int_{G_n} \langle V(p,t;l), V(q,t';l)\rangle \,\mathrm{d}\mathcal{M}_n(l).$$

This is a jointly locally constant function, and its matrices on all finite lists of heads and last entries are positive semidefinite. For common last entry $t = t'$ it is

$$\frac{1}{m_Z|L|}\int_{\tau(l)\in t+L} \langle \mathcal{J}(p,l), \mathcal{J}(q,l)\rangle \,\mathrm{d}\mathcal{M}_n(l).$$

For fixed p, q the common-entry function is compactly supported and integrable on E_T. Write $L^\perp = \{\xi \in E_T : \psi_T^0(\xi a) = 1 \text{ for all } a \in L\}$. For every $\xi \in E_T$,

$$\int_{E_T} \psi_T^0(\xi t) J_n((p,t),(q,t))\,\mathrm{d}t = \frac{\mathbf{1}_{L^\perp}(\xi)}{m_Z}\int_{G_n} \psi_T^0(\xi\tau(l))\langle \mathcal{J}(p,l), \mathcal{J}(q,l)\rangle \,\mathrm{d}\mathcal{M}_n(l).$$

These formulas apply to the terminal pivot of Definition 54.2 after restricting the entries to $E_T^\times$.

Proof. The preceding lemma bounds the supports in l by compact sets of finite $\mathcal{M}_n$-mass. On them the locally constant functions $\mathcal{J}$ are bounded by compactness. The varying heads are locally constant as functions with values in this space of bounded functions, by that lemma. The indicators are constant when t, t' vary in their respective cosets of L. Consequently the defining integral exists, is jointly locally constant, and its quadratic form on coefficients (a_j) is the integral of $\|\sum_j a_j V(p_j,t_j;l)\|_{\mathrm{HS}}^2$. The common-entry formula follows by squaring the indicator. Its support in t is contained in the sum of the compact image under τ of a common support in l and the compact group L. Thus it is integrable, and Lemma 123.1 interchanges the two integrals after multiplication by the bounded character. At fixed l,

change t to $\tau(l)+a$ with $a\in L$. The integral of $\psi_T^0(\xi a)$ on L is its volume if $\xi\in L^\perp$, and otherwise zero by translating by an element with character different from one. This proves the Fourier identity. The complement of $E_T^\times$ is a finite union of coordinate zero sets of additive measure zero. Replacing individual zero components by one under the integration therefore preserves the common-entry formulas. Both occurrences of the terminal kernel in Definition 54.2 use a common last entry, for integration or for a matrix on a common slice. Multiplying the two indicators in the definition gives exactly the single indicated restriction on every such slice. □

Corollary 169.4. *Fix $p_1,\dots,p_a$, coefficients $b_1,\dots,b_a\in\mathbb{C}$ and a common t. With $\sigma(e)$ from Proposition 165.3, one has*

$$m_Z|L|\sum_{i,j=1}^{a}\bar b_i b_j J_n((p_i,t),(p_j,t))$$
$$=\int_{[z,e]\in t+L}\int_H\left\|\sum_{i=1}^{a}b_i\mathcal{J}(p_i,h\sigma(e))\right\|_{\mathrm{HS}}^2\,\mathrm{d}\mu_H(h)\,\mathrm{d}\delta_{-N}^{[n]}(e).$$

The inner integral is independent of the chosen orbit representative.

Proof. Apply the common-entry formula of Theorem 169.3 to the quadratic combination, giving a squared norm. Substitute $\mathcal{M}_n=\mu_H*\theta_n$ from Definition 166.1. The vector $(h\sigma)^{-1}z$ is $\sigma^{-1}z$ because $h\in H$. A second representative for the same vector differs on the left by a member of H, which may be absorbed using right invariance of μ_H. Push forward θ_n by the orbit map using Proposition 165.3 and Lemma 123.1 to obtain the identity. □

170. A common integral subspace for two finite positions

Definition 170.1. Call two vectors $z_1,z_2\in V_1$ close if $z_1\in\mathcal{L}$ and $z_1-z_2\in g\mathcal{L}$, with the level $g\in A_T\cap F_T^\times$ of Definition 160.1. Let $I:E_T^{m_*}\to V_1$ include the m_* standard lines of Lemma 163.1. Set

$$\mathcal{P}_T=\{t^{-1}ut:t\in G_T,\ u\in U_T\}K_g.$$

For $x\in K_T,\ y\in\mathcal{P}_T$ and a matrix C over E_T with m_* columns and at most ν rows, define the module

$$W=\{w:Cw=0,\ (x^{-1}Iw,(xy)^{-1}Iw)\text{ is close}\}.$$

For $v\in E_T^{m_*}$ write $\mathfrak{a}_v=\{v^*w:w\in W\}$. Throughout this section the arithmetic scale is large enough that $m_*>5n$, as follows from Proposition 10.5.

Lemma 170.2. *The set W is a compact B_T-submodule contained in $B_T^{m_*}$, and it contains YB_T^d, where $d=m_*-3\nu$ and Y consists of the last d columns of some $V\in\mathrm{GL}_{m_*}(B_T)$. For every v, $\mathfrak{a}_v$ is a product of compact fractional ideals, allowing a zero ideal in some factors.*

Proof. The closeness conditions are closed and B_T-linear, as is $Cw=0$. The first integrality condition implies $Iw\in\mathcal{L}$, because x stabilizes $\mathcal{L}$. The integral direct decomposition of Lemma 163.1 then gives $w\in B_T^{m_*}$. This proves compactness. Write $y=t^{-1}uth$ with $u\in U_T,\ h\in K_g$. In each field component $\operatorname{rank}(u^{-1}-1)\le 2\nu$: in the three blocks consisting of all outer isotropic vectors, the middle space and the dual vectors, the only potentially nonzero entries lie in the first ν rows or the last ν columns. This block form follows from the successive matrices in Lemma 20.4, also for u^{-1}. Stack C with the matrix $(u^{-1}-1)tx^{-1}I$. Their combined rank is at most 3ν componentwise.

Over a single discrete valuation ring, multiply this stacked matrix by a common nonzero denominator. Move an entry of least valuation to the upper left by row and column permutations. Its quotients into the entries of its column and row are integral, so elementary row and column subtractions clear that column and row apart from the pivot. Repeating on the smaller block gives a diagonal matrix up to integral invertible row and column operations, stopping when the remaining block is zero. The number of nonzero pivots is the

rank over the field. Thus an invertible integral change of columns puts at least the last d columns in the kernel of the original stacked matrix. Choose these changes independently in all field components and combine them to give V, Y. If $w \in YB_T^d$, then $Cw = 0$, $x^{-1}Iw \in \mathcal{L}$, and $(xy)^{-1}Iw = h^{-1}x^{-1}Iw$. Since $h^{-1} \in K_g$, the two vectors are close. This proves the inclusion.

The linear image $\mathfrak{a}_v$ is a compact B_T-module in E_T. Its components split independently by multiplication by the coordinate idempotents of B_T. A nonzero compact submodule of a local field is bounded, so the integer valuations of its nonzero elements have a minimum. Any member of minimum valuation generates the whole submodule over the integer ring, by division into every other member. This proves the ideal description. □

Theorem 170.3. *Use W, Y, d from Lemma 170.2, and define*

$$\phi(v) = |\{c \in E_T : cY^*v \in B_T^d\}|.$$

Use the value $+\infty$ when this set has infinite measure. Then $1/|\mathfrak{a}_v| \le \phi(v)$, with $1/0 = +\infty$. Moreover the set of $\alpha \in E_T$ such that for every $w \in W$ there are close pairs $(z_1, z_2), (z_1', z_2')$ satisfying

$$(\alpha v)^* w = g^{-1}([z_1, z_1'] - [z_2, z_2'])$$

has volume exactly $1/|\mathfrak{a}_v|$. For $\beta \in E_T^\times$ and real $0 < s < d$ one has

$$\int_{\beta^{-1}B_T^{m_*}} \phi(v)^s \, \mathrm{d}v = m_T(\beta)^{-(m_*-s)} \prod_{w|T} \frac{1 - q_w^{-d}}{1 - q_w^{-(d-s)}}.$$

Here w ranges over the field components of E_T, of residue cardinality q_w. In particular one may take $s = d - 2n$. Then the n-th power of the product on the right is at most $1 + 2^{-n}$ at sufficiently large scales, uniformly in T.

Proof. For close pairs all four vectors lie in $\mathcal{L}$. Subtract the two pairings by changing one argument at a time. The differences of the arguments belong to $g\mathcal{L}$, and $\bar{g} = g$, so the divided difference is integral by Proposition 15.2. Conversely take $z_1 = z_2 = e_1$, $z_1' = gbe_1^*$, $z_2' = 0$, for any $b \in B_T$. Both pairs are close and the divided difference is b. Thus the asserted scalar set is exactly $\{\alpha : \bar{\alpha}\mathfrak{a}_v \subseteq B_T\}$. For a nonzero ideal $\pi_w^a O_w$ in a field component, the scalars multiplying it into O_w form $\pi_w^{-a} O_w$, of reciprocal additive volume. A zero component gives the full field of infinite volume. All nonzero components give open sets of positive volume. Taking products, and using the lattice-preserving conjugation, proves the asserted equality. Since $YB_T^d \subseteq W$, the subset $v^*YB_T^d \subseteq \mathfrak{a}_v$ has the corresponding scalar set of volume $\phi(v)$, giving the inequality.

The change of variables $v \mapsto V^*v$ preserves every box $\beta^{-1}B_T^{m_*}$ and additive measure, as its matrix and inverse are integral and the map is E_T-linear. This follows by partition into residue boxes as in Lemma 20.2. Its last d coordinates are Y^*v. On the unit box, at component w let h be the minimum of their valuations. For $j \ge 0$, the fraction with $h \ge j$ is q_w^{-dj}, since all d coordinates must lie in $\pi_w^j O_w$. The set with all zero has volume zero. The scalar bound there is q_w^h, so the integral of its s-th power is $(1 - q_w^{-d}) \sum_{j \ge 0} q_w^{-(d-s)j}$. These formulas also give measurability, including on the exceptional zero sets. Under $v = \beta^{-1}v_0$ the scalar volume becomes $m_T(\beta)\phi(v_0)$, and the vector measure changes by $m_T(\beta)^{-m_*}$. Lemma 123.1 and the geometric sum give the formula.

For $d - s = 2n$, bound the logarithm of each factor by $2q_w^{-2n}$ as in Corollary 164.3. Each rational prime lies under at most $2f$ field components by Lemma 57.1, and q_w is at least that prime. Thus the logarithm of the n-th power is at most $8nf\,2^{-2n}$, using the series comparison in that corollary. Since $f \le X^{12}$ and $n \ge \sqrt{X}/2$ eventually, this is at most 2^{-n-1} eventually. Exponentiation as there proves the stated bound. □

171. Congruence displacement detected by flag ideals

Definition 171.1. For the remainder of this section take a single finite place v of F, of residue cardinality Q, with $A = A_v,\ B = B_v$ and a uniformizer $\pi \in A$. Use the measures and groups above with $T = \{v\}$. For $a, b' \in G_T$, let

$$\mathcal{L}(a,b') = \{w : (a^{-1}w, (b')^{-1}w) \text{ is close}\},$$
$$I_i(a,b') = \{[e_i, w] : w \in \mathcal{L}(a,b'),\ [e_j, w] = 0 \text{ for } j < i\},$$
$$A_i(a,b') = |I_i(a,b')| \qquad (1 \le i \le \nu).$$

Let $h(x)$ be the least integer $h \ge 0$ with $(x-1)\mathcal{L} \subset g\pi^{-h}\mathcal{L}$. Thus $h(x) = 0$ exactly when $x \in K_g$: if $x - 1$ has entries in gB, then x is integral and its inverse is integral by the isometry equation and Proposition 15.2.

Lemma 171.2. *Each $I_i(a,b')$ is a principal fractional B-ideal with invertible generator in E_v. Its positive finite volume is a Borel function, invariant under simultaneous left multiplication by P_i of Definition 168.1. If $a \in P_iK_T$, then $I_i(a,b') \subseteq B$. If $\langle W_Z(a), W_Z(ax)\rangle \neq 0$, there exist $u, v_1 \in U_T$ with $k_1 = ua,\ k_2 = v_1ax \in Z$ and $k_1^{-1}k_2 \in K_g$. In particular $x \in \mathcal{P}_T$ and every $I_i(a, ax) \subseteq B$.*

Proof. The close-vector module is a compact open full lattice: it is the intersection of $a\mathcal{L}$ with the inverse image of $g\mathcal{L}$ under $a^{-1} - (b')^{-1}$. It is open, closed and contained in a compact set. Its intersection with the stated orthogonal subspace therefore contains sufficiently small multiples of the entire lattice in that subspace, in particular of e_i^*. Projection gives a compact module containing a nonzero open ideal. The last argument of Lemma 170.2 makes its ideals principal, with no zero field components.

Membership of a scalar c in this image is open in (a, b', c): perturb a witness by $(c' - c)e_i^*$ at a nearby scalar c'; the earlier orthogonalities persist, and the two lattice conditions are open also in a, b'. Integration of the resulting indicator in c gives Borel measurability by Lemma 123.1. For $q \in P_i$, the vector $q^{-1}e_j$ differs from e_j by a combination of earlier frame vectors for $j \le i$. The change $w \mapsto qw$ therefore preserves the scalar and earlier orthogonalities, and preserves closeness after left multiplication of a, b' by q. This proves invariance. At $a \in K_T$ the first vector condition forces $w \in \mathcal{L}$, giving an integral scalar. The same follows at $a \in P_iK_T$.

A nonzero inner product has a coordinate with both entries nonzero, say at $K_gc,\ c \in Z$. By Definition 160.3, the points a, ax then belong respectively to U_TK_gc and U_TK_gc. Choose the indicated unipotent multipliers taking them to $k_1, k_2 \in K_gc$. Normality of K_g in Z gives $k_1^{-1}k_2 \in K_g$, and $x = k_1^{-1}uv_1^{-1}k_1(k_1^{-1}k_2) \in \mathcal{P}_T$ by Definition 170.1. Finally $U_T \subset P_i$: after removing i outer layers the remaining factors belong to G_{i+1}. Thus $a \in P_iK_T$ for all i. □

Theorem 171.3. *If $\langle W_Z(a), W_Z(ax)\rangle \neq 0$, then*

$$h(x) \le 2\sum_{i=1}^{\nu}\bigl(-\log_Q A_i(a, ax)\bigr).$$

In particular for $x \notin K_g$ some i satisfies

$$A_i(a, ax) \le Q^{-\max\{1, h(x)/(2\nu)\}}.$$

Proof. Take u, v_1, k_1, k_2 from Lemma 171.2, and put $S = v_1u^{-1} \in U_T$. For each i choose a generator t_i of $I_i(a, ax)$ and a witness w_i with $[e_i, w_i] = t_i$ and the earlier coordinates zero. Put $\Delta = \prod_i t_i$. Use the matrices of columns $Y = (uw_i)_i,\ Y' = (v_1w_i)_i$. They are integral and $Y' - Y$ has entries in gB. Indeed, writing $z = a^{-1}w_i,\ z' = (ax)^{-1}w_i$, the two columns are k_1z, k_2z', with z, z' integral and $z - z' \in g\mathcal{L}$. Also $k_1^{-1}k_2 \in K_g$. In the decomposition into all outer vectors, V_n, and all dual vectors (the duals in the same order), the bottom blocks D, D' of Y, Y' are lower triangular with diagonal $t_1, \dots, t_\nu$. This follows from the flag action of u, v_1 as in Lemma 171.2. Both determinants are Δ; the adjugate formula makes

$\Delta D^{-1}, \Delta (D')^{-1}$ integral. In particular Δ is an invertible scalar in the field algebra and is integral.

Write $YD^{-1} = (X, Y_0, 1)^{\mathsf{T}}$, meaning three stacked blocks with identity bottom block. Then $\Delta X, \Delta Y_0$ are integral, and $(S-1)YD^{-1} = (Y'-Y)D^{-1}$ has entries in $g\Delta^{-1}B$. In the same blocks put

$$S = \begin{pmatrix} A_0 & B_0 & C_0 \\ 0 & 1 & F \\ 0 & 0 & A_0^{-*} \end{pmatrix}.$$

Its bottom equation gives $A_0^{-*} = D'D^{-1}$; hence $A_0^{*} = D(D')^{-1}$ and $A_0^{*} - 1 = (D - D')(D')^{-1} \in g\Delta^{-1} M_\nu(B)$. In particular A_0 is bounded entrywise by the fractional ideal $\bar{\Delta}^{-1}B$, and $A_0 - 1$ by $g\bar{\Delta}^{-1}B$. The middle difference is F, so its entries belong to $g\Delta^{-1}B$. The isometry equation gives $B_0 = -A_0 F^{*} J$, for the integral unimodular middle matrix J, exactly as in Theorem 168.2. Its entries belong to $g\bar{\Delta}^{-2}B$. The top difference is $(A_0 - 1)X + B_0 Y_0 + C_0$; this puts the entries of C_0 in $g(\Delta\bar{\Delta}^2)^{-1}B$. The bottom difference $A_0^{-*} - 1 = (D' - D)D^{-1}$ also has the required bound. All entries of $S - 1$ consequently lie in $g(\Delta\bar{\Delta})^{-2}B$. This last ideal is conjugation invariant, so $S^{-1} - 1 = J_0^{-1}(S-1)^{*}J_0$ has entries in it as well, with J_0 the full Gram matrix. Since $x = k_1^{-1}S^{-1}k_2$ and $k_1^{-1}k_2 - 1$ has entries in gB, the same bound holds for $x - 1$.

The valuation of $\Delta\bar{\Delta}$ in F_v is $-\log_Q m_T(\Delta)$ by Lemma 164.1. Each $|t_i B| = m_T(t_i) = A_i(a, ax)$, by the additive determinant calculation of Lemma 70.1. This proves the first inequality. The individual exponents $-\log_Q A_i$ are nonnegative integers by integrality of the ideals. When $h(x) > 0$ their maximum is positive and at least $h(x)/(2\nu)$, which proves the second assertion. □

172. Integrable pairings on a finite flag

Definition 172.1. At one place as in Definition 171.1, extend $P_0, \dots, P_\nu$ by $P_n = U_T H$ and put $\mathcal{N}_i = P_i K_T$. Write $M_i = M_T(\ell_i)$, $s = m_* - 3\nu - 2n$, and

$$C_v = \prod_{w|v} \frac{1 - q_w^{-(m_* - 3\nu)}}{1 - q_w^{-2n}}.$$

For two lists p, q of the same length $j_0 \le \nu$, put $d_* = w_q^{-1} w_p$ and $\lambda_a = p_a / q_a$ for $a \le j_0$, $\lambda_a = 1$ for $j_0 < a \le n$. This diagonal isometry acts by λ_a on e_a, by $\bar{\lambda}_a^{-1}$ on e_a^{*}, and fixes V_n. Set

$$T_i = \mu_H * \theta_n * \cdots * \theta_{i+1}, \quad T_n = \mu_H, \qquad K_i(a, b') = \int \langle W_Z(\sigma a), W_Z(d_* \sigma b') \rangle \, \mathrm{d}T_i(\sigma).$$

In T_0 the last factor is θ_1, so $T_0 = \mathcal{M}_{\mathrm{up}}$. All these definitions allow empty lists. The notation $K_i(a, b')$ denotes a complex pairing; K_i without arguments in Definition 165.1 denotes a compact group.

Proposition 172.2. *For all $a \in G_T$ and $1 \le i \le n$,*

$$\theta_i\{\sigma : \sigma a \in \mathcal{N}_i\} \le M_i.$$

For $i < n$ and $x \in \mathcal{P}_T$ one also has

$$\int_{\sigma a \in \mathcal{N}_i} A_i(\sigma a, \sigma a x)^{-s} \, \mathrm{d}\theta_i(\sigma) \le M_i C_v.$$

Finally $\mu_H\{h : ha \in U_T K_T\} \le 1$ for every a.

Proof. For $i \le n$, the quotient map taking the middle block of $P_{i-1} = U^{(i-1)} G_i$ has kernel $U^{(i-1)}$ and sends P_i onto $S_i = \mathrm{Stab}_{G_i}(f_i)$. For $i < n$ this follows from the factorization $S_i = G_{i+1} N_i$ in Theorem 166.2; at $i = n$ it is the definition of P_n. These groups contain the outer unipotents and are nested. Their sets $\mathcal{N}_i$ are Borel since they are unions of open

cosets of K_T. If $\sigma a \in \mathcal{N}_i$, then $a \in P_{i-1}K_T$, as $\sigma \in G_i$. For a P_i-invariant nonnegative Borel function F, the integral $\int F(\sigma a)\,\mathrm{d}\theta_i$ is left P_{i-1}-invariant as a function of a. Indeed a member of $U^{(i-1)}$ conjugates through σ into the same subgroup of P_i. For $h_0 \in G_i$, $\sigma \mapsto F(\sigma a)$ depends only on $\sigma^{-1}f_i$, since $S_i \subset P_i$; on the full-measure orbit chart its representative is Borel by Lemma 165.2 and Proposition 165.3. Replacing σ by σh_0 preserves that integral by quadric invariance in Proposition 163.4. Thus in both asserted θ_i-integrals one may assume $a \in K_T$, since the weights have the stated invariance by Lemma 171.2. If $\sigma a \in \mathcal{N}_i$, there is $q' \in P_i$ with $q'\sigma a \in K_T$, hence $q'\sigma \in K_T \cap P_{i-1}$. Its middle block lies in K_i, with inverse integral by the inverse block equation, and is $s'\sigma$ for $s' \in S_i$. Consequently $\sigma^{-1}f_i = (s'\sigma)^{-1}f_i \in \mathcal{L}_i$. The orbit laws and Corollary 164.3 give the first estimate.

For $i < n$ use W of Definition 170.1 with integral first position a, translation x, and no row equations. For $w \in W$, the vector $\sigma I w$ is orthogonal to all earlier frame vectors and is admissible in $\mathcal{L}(\sigma a, \sigma a x)$. Its scalar is $[f_i, \sigma I w] = v_+^* w$, where v_+ is the projection of $\sigma^{-1}f_i$ onto the m_* standard lines. Hence the reciprocal of $A_i(\sigma a, \sigma a x)$ is at most $\phi(v_+)$ by Theorem 170.3. Integrate on the integral quadric box just obtained. In the disintegration of Proposition 163.4, integrate the b-fibre and then the hyperbolic a-coordinates using Lemma 163.3. Their upper bound is M_i independently of v_+. The moment integral on the remaining integral box is C_v by Theorem 170.3 with $\beta = 1$.

For the final bound, suppose $h_0 a = u_0 k_0,\ ha = uk$ with $h_0, h \in H,\ u, u_0 \in U_T,\ k, k_0 \in K_T$. Then $u^{-1}hh_0^{-1}u_0 = kk_0^{-1}$ is integral, and its middle diagonal block is the action of hh_0^{-1} on V_n, since the two unipotents have identity middle diagonal block. The inverse matrix gives integrality of the inverse block. Thus $h \in (H \cap K_n)h_0$, of mass one by Definition 166.1. An empty set has measure zero, proving the general assertion. □

Theorem 172.3. *The integrals defining K_i are absolutely convergent and Borel for all arguments. Each vanishes unless $a \in \mathcal{N}_i$ and has absolute value at most $(v_Z/v_g)\prod_{b=i+1}^{n} M_b$. One has*

$$K_{i-1}(a, b') = \int K_i(\sigma a, \sigma b')\,\mathrm{d}\theta_i(\sigma), \qquad K_0(1, x) = c_Z(x) \quad (j_0 = 0)$$

for the recurrence $1 \le i \le n$ and the indicated empty head, respectively. For $u, u' \in U^{(i-1)}$, $1 \le i \le n$,

$$K_i(ua, u'b') = \overline{\chi_g(u)}\chi_g(d_* u' d_*^{-1}) K_i(a, b').$$

For $i \ge j_0$, K_i is invariant under simultaneous left multiplication of both arguments by $P_i \cap G_{j_0+1}$. For $i > j_0$ its modulus is invariant under simultaneous left multiplication by all of P_i.

Proof. The pointwise norm formula of Proposition 160.4 bounds the raw pairing by v_Z/v_g and restricts its first argument to $U_T Z$. At H the last bound of Proposition 172.2 applies, and positivity of the support measure implies $a \in U_T H K_T = \mathcal{N}_n$ because H normalizes U_T. Apply its first bound successively to the integral of the absolute value. The support at each step lies in $\mathcal{N}_{i-1}$ by the block observation in its proof. This gives the asserted estimates, Borel measurability and the recurrence by Lemma 123.1. Corollary 166.3 identifies the empty-head value.

For the separate translation formula, T_i is carried by G_{i+1} for $i < n$, and by H at $i = n$. These multipliers normalize $U^{(i-1)}$ and leave the simple-root coordinates of its elements unchanged. Indeed a multiplier in G_{i+1} fixes both frame vectors entering each of the first $i-1$ coordinates. All later simple entries of an element of that subgroup are zero by the block equations in Lemma 20.4. At the last step H also fixes the successor z. Conjugation by d_* multiplies the a-th simple entry by λ_a/λ_{a+1}; here $\lambda_n = 1$. Thus the two characters from translating the arguments in Proposition 160.4 are constant in the integral, giving the formula.

For simultaneous invariance at $i = n$, the H-factor is absorbed by right invariance of μ_H; d_* commutes with H. A simultaneous unipotent multiplier in $U_T \cap G_{j_0+1}$ has equal first

and second character. For the induction from i to $i-1 \geq j_0$, factor the given multiplier as a member of G_i times one of $U^{(i-1)} \cap G_{j_0+1}$ using the middle block. Conjugating the latter by a member of G_i stays in the same subgroup and it can be removed by the induction hypothesis. As a function of $\sigma \in G_i$ the resulting simultaneous integrand is left invariant by S_i, which lies in $P_i \cap G_{j_0+1}$. The orbit-section argument of Proposition 172.2 therefore absorbs the G_i-factor. This proves the first simultaneous invariance.

For $j_0 < i < n$ a member of $U^{(i)}$ conjugated by the upper multiplier in G_{i+1} can change its i-th simple entry, but that entry has character multiplier $\lambda_i/\lambda_{i+1} = 1$. All entries of index at most j_0 are preserved, as both frame vectors there are fixed. The ratio of the two characters for a simultaneous translation is thus constant in the integral and has absolute value one. The G_{i+1}-factor belongs to the simultaneous invariance already proved. At $i = n$ use the separate formula for $U^{(\nu)}$ and the H-invariance. This proves the modulus assertion. □

173. The weighted unipotent integral on a congruence ball

Definition 173.1. Continue at a single finite place and put $\mathcal{W}_j(a,b') = \prod_{i=1}^{j} A_i(a,b')^s$ with s from Definition 172.1. For an integer $H_0 \geq 0$ let $\Omega_{H_0} = \{x \in \mathcal{P}_T : h(x) \leq H_0\}$. It is open, since $\mathcal{P}_T$ is a union of open cosets of K_g and the entry conditions are open. Let $\chi_j(a,b')$ indicate that $a^{-1}ub' \in \Omega_{H_0}$ for some $u \in U^{(j)}$. This is Borel since the defining condition is a union of open sets. For $i \leq \nu$ define

$$v_{g,i} = m_T(g)^{d_i} |g|_v = |U_T \cap K_g \cap N_i|_{\eta_i}.$$

The equality and $\prod_i v_{g,i} = v_g$ are the layer calculation of Lemma 160.2. Use $n_i(t,c) \in N_i$ for the element with row $-t^*J$ and corner $-\omega Q(t)+c$ on V_i; it sends e_i^* to $e_i^* + t + (-\omega Q(t)+c)e_i$. Its measure is $\mathrm{d}t\ \mathrm{d}c$ on $V_{i+1} \times iF_v$, since the row change preserves the integral lattices. Here the i in iF_v denotes the skew unit $\sqrt{-1}$.

Lemma 173.2. *If $h(x), h(y) \leq H_0$, then the entries of $xy^{-1} - 1$ belong to $g\pi^{-2H_0}B$. Conjugation by K_T preserves this matrix condition. In $n_i(t,c)$ it implies $t \in g\pi^{-2H_0}\mathcal{L}_{i+1}$ and restricts c, at each fixed t, to a set of volume at most $|g|_v Q^{2H_0}$.*

Proof. The relation $y^{-1} = J_0^{-1} y^* J_0$ and integrality of the Gram matrices give the same entry bound for $y^{-1} - 1$ as for $y - 1$. Now $xy^{-1} - 1 = (x-y)y^{-1}$; the entries of $x - y$ lie in $g\pi^{-H_0}B$, those of y^{-1} in $\pi^{-H_0}B$ because g is integral. Multiplication on both sides by integral invertible matrices preserves the ideal condition. Read the last column of $n_i(t,c) - 1$. Besides the claimed condition on t, the corner belongs to $g\pi^{-2H_0}B$. Two admissible c's at fixed t differ by an element of the intersection with the skew line, $g\pi^{-2H_0}iA$ by Lemma 20.1. This group has exactly the stated additive volume. □

Theorem 173.3. *For $0 \leq j \leq \nu$, $a \in \mathcal{N}_j$, $b' \in G_T$, one has*

$$\int_{U^{(j)}} \mathbf{1}_{\Omega_{H_0}}(a^{-1}ub')\mathcal{W}_j(a,ub')\,\mathrm{d}u \ \leq\ \chi_j(a,b')\, C_v^j Q^{60njH_0} \prod_{i=1}^{j} v_{g,i}.$$

The measure on the partial unipotent group assigns integral unipotents mass one. The threshold on the arithmetic scale is independent of the residue cardinality Q.

Proof. If there is no admissible u, the integral is zero. Simultaneous left multiplication of a, b' by a member of P_j, and conjugation of u by that member, preserve the integral: P_j normalizes $U^{(j)}$, its inner isometry factors preserve the row and skew measures by Theorem 166.2, and the unipotent measure is bi-invariant as in Lemma 160.2. The weight is unchanged by Lemma 171.2 and the nesting of the subgroups. We may thus take $a \in K_T$. Replace b' by $u_* b'$ for one admissible u_*, using right invariance in the integral. We may now also take $a^{-1}b' \in \Omega_{H_0}$. For any contributing u, conjugate the quotient of $a^{-1}ub'$ and $a^{-1}b'$ by a. By Lemma 173.2, $u - 1$ has entries in $g\pi^{-2H_0}B$.

Induct on j, the initial assertion being the bound for one indicator. For $j > 0$ write $u = u_0 n_j(t,c)$, with the product of the $U^{(j-1)}$-measure and $\mathrm{d}t\,\mathrm{d}c$ by the split projection in Lemma 20.4. The induced middle block after removing $j-1$ pairs is $n_j(t,c)$, so the bounds on t, c in Lemma 173.2 apply. Let t_+ be the component of t on the m_* standard lines of Lemma 163.1. Take the module W in Definition 170.1 at $x = a,\ y = a^{-1}b'$, with no row equations. The choice of b' guarantees $y \in \mathcal{P}_T$. We claim $A_j(a, ub') \le \phi(t_+/g)$, with ϕ given by Theorem 170.3. For $\lambda = [e_j, w] \in I_j(a, ub')$ choose a witness w, and for $v \in W$ put $z = Iv$. Then

$$g^{-1}([a^{-1}w, a^{-1}z] - [(ub')^{-1}w, (b')^{-1}z]) = g^{-1}[w, z - uz] = (\lambda t_+/g)^* v.$$

Indeed $n_j(t,c)z = z - [t,z]e_j$ and u_0 acts trivially on this vector modulo the earlier frame vectors, to which w is orthogonal. Both argument pairs on the left are close by the definitions of I_j, W. The scalar-volume assertion of that theorem implies the claim.

At fixed t, c, use this bound for the last weight factor and the induction hypothesis at $a, n_j(t,c)b'$ for the other $j-1$ factors, dropping its indicator χ_{j-1} from the upper bound. This use is valid since $\mathcal{N}_j \subseteq \mathcal{N}_{j-1}$. It remains to integrate $\phi(t_+/g)^s$ over the indicated range of t, c. Integrating the central variable costs $|g|_v Q^{2H_0}$. Put $t = gt_0$, giving the factor $m_T(g)^{d_j}$. Now $t_0 \in \pi^{-2H_0}\mathcal{L}_{j+1}$. The $d_j - m_*$ other coordinates have volume $Q^{4H_0(d_j - m_*)}$, and the moment on the standard coordinates is $C_v Q^{4H_0(m_* - s)}$ by Theorem 170.3. All changes of order are for nonnegative integrands by Lemma 123.1. The cost of this step is thus at most

$$v_{g,j} C_v Q^{(2+4(d_j - s))H_0}.$$

From $d_j = k - 1 + 2(n-j)$, $a_* = 3\nu + 6$, $m_* = k - 1 - 2a_*$ and $s = m_* - 3\nu - 2n$, one gets $d_j - s = 13n + 3 - 2j$. Hence $2 + 4(d_j - s) \le 60n$ at the indicated scales. Multiplication over the steps completes the induction. □

174. A Fourier integral centered on a finite flag layer

Lemma 174.1. *In Definition 172.1 let $j_0 < i \le \nu$ and $a, b' \in G_T$. Put $f = f_{i+1}$, $\delta = \delta_{Q(f)}^{[i+1]}$, where $Q(f)$ here denotes the hermitian norm. There is a bounded Borel amplitude F on the quadric such that $F(\sigma^{-1}f) = K_{i+1}(\sigma a, \sigma b')$ for θ_{i+1}-almost every σ. It satisfies*

$$K_i(a, n_i(t,c)b') = \int \psi_T^0(-[t,e]/g) F(e)\, \mathrm{d}\delta(e).$$

If $a \in \mathcal{N}_i$ and $qa \in K_T$, $q \in P_i$ has middle block $h \in G_{i+1}$ on the quotient after removing i pairs, then $F(e) = 0$ almost everywhere outside $h^{-1}\mathcal{L}_{i+1}$. In particular $F \in L^1 \cap L^2(\delta)$. For $a \notin \mathcal{N}_i$ one may take $F = 0$ almost everywhere.

Proof. The stabilizer S_{i+1} is contained in $P_{i+1} \cap G_{j_0+1}$. By Theorem 172.3 the indicated function on G_{i+1} therefore factors through $\sigma^{-1}f$. Use the Borel sections of Lemma 165.2 and Proposition 165.3 and extend by zero off their charts. In the recurrence for K_i, conjugation of $n_i(t,c)$ by $\sigma \in G_{i+1}$ gives $n_i(\sigma t, c)$: the corner is unchanged and the middle column transforms by σ. This unipotent commutes with d_* since $i > j_0$. Its only possibly nonzero simple-root coordinate is that at i, namely the e_i-entry of the image of f, equal to $-[\sigma t, f] = -[t, \sigma^{-1}f]$. Use the separate translation law of Theorem 172.3 at $i+1$ and push forward by the orbit map. This gives the displayed formula.

If the amplitude is nonzero at $\sigma^{-1}f$, then $\sigma a \in \mathcal{N}_{i+1}$ by that theorem, forcing $a \in \mathcal{N}_i$ as in Proposition 172.2. Choose $p' \in P_{i+1}$ with $p'\sigma a \in K_T$. Since $qa \in K_T$, the matrix $p'\sigma q^{-1}$ lies in $P_i \cap K_T$. Its middle block after removing i pairs lies in K_{i+1} and has the form $s'\sigma h^{-1}$ with $s' \in S_{i+1}$. Taking its inverse on f gives $h\sigma^{-1}f \in \mathcal{L}_{i+1}$. The support assertion follows. The transformed lattice is compact, and δ is locally finite by Proposition 163.4. Boundedness from Theorem 172.3 proves integrability of the amplitude and its square. □

Theorem 174.2. *For $j_0 < i \le \nu$ and every $a, b' \in G_T$, with the indicators of Definition 173.1, one has*

$$\int_{N_i} \chi_{i-1}(a, nb')|K_i(a, nb')|^2 \,\mathrm{d}\eta_i(n)$$
$$\le \chi_i(a, b')\, v_{g,i} M_{i+1} Q^{40nH_0} \int_{G_{i+1}} |K_{i+1}(\sigma a, \sigma b')|^2 \,\mathrm{d}\theta_{i+1}(\sigma).$$

Proof. The inner square integral is finite by Lemma 174.1 and the orbit law. The left side is zero if $a \notin \mathcal{N}_i$, or if no $n_i(t_0, c_0)$ has $\chi_{i-1}(a, n_i(t_0, c_0)b') = 1$. Otherwise fix such a pair, choose $qa = k \in K_T$ with $q \in P_i$, and use its middle block h from that lemma. Necessarily $\chi_i(a, b') = 1$ by the layer factorization $U^{(i)} = U^{(i-1)}N_i$. For every admissible t, c there are $u, u_0 \in U^{(i-1)}$ so that $a^{-1}un_i(t, c)b'$ and $a^{-1}u_0 n_i(t_0, c_0)b'$ belong to Ω_{H_0}. Their quotient, conjugated by k, is

$$qun_i(t, c)n_i(t_0, c_0)^{-1}u_0^{-1}q^{-1}.$$

Its difference from the identity lies in the matrix ideal $g\pi^{-2H_0}B$ by Lemma 173.2. Take its block on the quotient after the first $i-1$ pairs. There u, u_0 disappear and q acts as an element of N_iG_{i+1} with G_{i+1}-block h. In the product of two layer matrices their vector parameters add and the corner entries add with one extra term determined by their vector parameters, by multiplying the three-block matrices of Lemma 20.4. It follows that the quotient block has vector parameter $h(t - t_0)$, and its corner depends on c additively with coefficient one at fixed t. Thus

$$h(t - t_0) \in g\pi^{-2H_0}\mathcal{L}_{i+1}, \qquad |\{c : \chi_{i-1}(a, n_i(t, c)b') = 1\}| \le |g|_v Q^{2H_0}.$$

For the second bound use that differences of the admissible skew parameters lie in $g\pi^{-2H_0}iA$. This reasoning uses only a fixed translate in estimating the domain.

The Fourier formula in Lemma 174.1 is independent of c. Integrate that parameter first. Change t to $t' = h(t - t_0)/g$, giving a factor $m_T(g)^{d_i}$; the isometry has determinant of norm one. In the amplitude integral send e to $e' = he$, preserving δ by Proposition 163.4. The phase from t_0 has modulus one and may be absorbed into the amplitude. Now the input amplitude is supported on the integral quadric box and the output t' ranges in $\pi^{-2H_0}\mathcal{L}_{i+1}$. Apply Theorem 164.2 there with frequency -1, ambient output and quadric input. Its factor is at most $Q^{8\ell_{i+1}H_0}M_{i+1}$. Its input squared norm is the square integral in the statement by the orbit law. Multiplying by the central and vector Jacobians gives $v_{g,i}M_{i+1}Q^{(8\ell_{i+1}+2)H_0}$. Finally $\ell_{i+1} = 4n + 3 - i$ by Lemma 163.1, so $8\ell_{i+1} + 2 \le 40n$ at sufficiently large scales. All integrations of squared absolute values use Lemma 123.1. □

175. Squared inner products on the finite unipotent quotient

Definition 175.1. In the notation of Definition 172.1, put $\tau_i = \theta_i * \cdots * \theta_1$ for $1 \le i \le n$ and let τ_0 be unit mass at the identity. Put $\rho_l = \eta_{l+1} * \cdots * \eta_\nu * \mathcal{M}_{\mathrm{up}}$ for $0 \le l \le \nu$, so $\rho_\nu = \mathcal{M}_{\mathrm{up}}$. Write $\mathrm{d}u_l$ for the measure on $U^{(l)}$ of Theorem 173.3. For the diagonal d_* of that definition of the pairings, set

$$D_* = \prod_{a=1}^{j_0} m_T(p_a/q_a)^{d_a+1}.$$

Lemma 175.2. *One has $\mathrm{d}u_l * \rho_l = \mu$. Suppose $h_1 \in G_T$ normalizes $U^{(l)}$ and conjugation by h_1 multiplies volumes there by $\Delta(h_1) > 0$. For every nonnegative Borel function F invariant under left $U^{(l)}$-multiplication,*

$$\int F(h_1 y) \,\mathrm{d}\rho_l(y) = \Delta(h_1) \int F(y) \,\mathrm{d}\rho_l(y).$$

For $h_1 \in G_{l+1}$ one has $\Delta(h_1) = 1$. At $l = \nu$, $h_1 = d_*$ one has $\Delta(h_1) = D_*$.

Proof. The successive semidirect group products from Lemma 160.2 give $\mathrm{d}u_l * \eta_{l+1} * \cdots * \eta_\nu = \mathrm{d}u$. Corollary 166.3 gives the first identity. A compact set C has finite ρ_l-measure: restrict u to the compact mass-one subgroup $U_0 = U^{(l)} \cap K_T$ and bound $\rho_l(C)$ by $\mu(U_0 C)$ using that identity. Measures of the other products can throughout be integrated in their defining coordinates by Lemma 123.1.

Choose one y_b in each double coset $U^{(l)} y_b K_T$; there are at most countably many since G_T/K_T is countable by Lemma 161.1. On $y_b K_T$ put $a_l = 1/|U^{(l)} \cap y_b K_T y_b^{-1}|$, and put it equal to zero outside their union. The denominator is a positive finite volume since the intersection is compact open. For $y \in U^{(l)} y_b K_T$, the set of u taking y into $y_b K_T$ is a right coset of that intersection. Hence $\int a_l(uy)\,\mathrm{d}u_l = 1$ everywhere and a_l is Borel. Applied to $F a_l$, the convolution identity yields $\int F\,\mathrm{d}\rho_l = \int F a_l\,\mathrm{d}\mu$. Apply this also to $F(h_1 \cdot)$, which is left invariant under $U^{(l)}$, and use left Haar invariance to get $\int F(z) a_l(h_1^{-1} z)\,\mathrm{d}\mu(z)$. Expanding the convolution once more, the inner cutoff integral is $\int a_l(h_1^{-1} u z)\,\mathrm{d}u_l$; substituting $u = h_1 v h_1^{-1}$ gives $\Delta(h_1)$. This proves the asserted formula.

The determinant-one action for $h_1 \in G_{l+1}$ follows on each earlier layer from Theorem 166.2. For $h_1 = d_*$, conjugation on layer a sends its vector and skew coordinate to $(\bar\lambda_a d_*|_{V_{a+1}} t, \lambda_a \bar\lambda_a c)$, by applying the matrix to e_a^* and using the corner parametrization in Definition 173.1. The middle isometry preserves vector volume; the scalar factors give $m_T(\lambda_a)^{d_a}$ on the row and $m_T(\lambda_a)$ on the skew line. Multiplying gives D_*. □

Proposition 175.3. *For the same group and level,*

$$\iint |\langle W_Z(a), W_Z(b')\rangle|^2\,\mathrm{d}\mathcal{M}_{\mathrm{up}}(a)\,\mathrm{d}\mathcal{M}_{\mathrm{up}}(b') = m_Z\, v_Z/v_g.$$

Consequently the last averaged pairing of Theorem 172.3 satisfies

$$\iint |K_n(a,b')|^2\,\mathrm{d}\tau_n(a)\,\mathrm{d}\rho_\nu(b') \le D_* m_Z\, v_Z/v_g.$$

Proof. The first integrand is left U_T-invariant independently in each variable, and vanishes off $(U_T Z)^2$. By Corollary 166.3 it is enough to integrate on Z^2 with measure $\mathrm{d}\mu(a)\,\mathrm{d}\mu(b')/v_Z^2$. Let $\bar Z = Z/K_g$ and let B_0 be the image of $U_T \cap Z$ in $\bar Z$. It has size $b_0 = v_Z/v_g$. The character on it induced by χ_g is well defined by Lemma 160.2. At $a \in Z$ the support of the vector in $\bar Z$ is the set $B_0 \bar a$: indeed $\phi_g(a c^{-1}) \neq 0$ exactly when $\bar a \bar c^{-1} \in B_0$. On this support the entries have modulus one and are the values of that character at $\bar a \bar c^{-1}$. Two such supports are disjoint or equal. If they are equal, the two vectors differ by a scalar of modulus one, because $\bar b' \bar a^{-1} \in B_0$ multiplies every character argument on the left. Thus the squared inner product is b_0^2 on the coincident supports and zero otherwise. For independent uniform elements the probability of coincidence is $b_0/|\bar Z| = b_0 \mu(K_g)/\mu(Z)$. Uniformity of the residues follows by equality of the measures of their cosets. The resulting quantity is $(\mu(Z)/v_Z)^2 b_0^3 \mu(K_g)/\mu(Z) = m_Z v_Z/v_g$, as claimed.

For $K_n(a,b')$, the H-integration has support in $\{h : ha \in U_T K_T\}$ of measure at most one by Proposition 172.2. Cauchy–Schwarz bounds its square by the H-integral of $|\langle W_Z(ha), W_Z(d_* h b')\rangle|^2$. At fixed h, a the corresponding function of hb' is left U_T-invariant, since d_* normalizes U_T and its character has modulus one. Lemma 175.2 with $h \in H \subset G_n$ permits the replacement of hb' by b' under ρ_ν. The remaining measure in ha is $\mu_H * \tau_n = \mathcal{M}_{\mathrm{up}}$. Now apply the same lemma with $h_1 = d_*$ in the second variable. The first equality, with the resulting factor D_*, gives the assertion. All these changes of order use Lemma 123.1 on nonnegative functions. □

176. Second moments on finite congruence balls

Definition 176.1. With Definitions 173.1 and 175.1, for $j_0 \le l \le \nu$ set

$$B_l = \iint \chi_l(a,b')\,|K_{l+1}(a,b')|^2\,\mathrm{d}\rho_l(b')\,\mathrm{d}\tau_{l+1}(a),$$
$$Z_l(x) = \int \mathcal{W}_l(a,ax)|K_{l+1}(a,ax)|^2\,\mathrm{d}\tau_{l+1}(a).$$

Here B_l uses the fixed integer H_0 in the indicator, and Z_l is independent of it. Both are nonnegative integrals.

Lemma 176.2. *For $j_0 < i \le \nu$ one has*

$$B_{i-1} \le v_{g,i} M_{i+1} Q^{40nH_0} B_i, \qquad B_\nu \le D_* m_Z v_Z / v_g.$$

For every $j_0 \le l \le \nu$,

$$\int_{\Omega_{H_0}} Z_l(x)\,\mathrm{d}\mu(x) \le C_v^l Q^{60nlH_0} \left(\prod_{i=1}^{l} v_{g,i}\right) B_l.$$

Proof. The terminal estimate follows from Proposition 175.3 after dropping χ_ν. Write $\rho_{i-1} = \eta_i * \rho_i$ in B_{i-1} and apply Theorem 174.2. For $\sigma \in G_{i+1}$, normalizing $U^{(i)}$ gives $\chi_i(\sigma a, \sigma b') = \chi_i(a,b')$. At fixed σ, a, the function

$$b'' \longmapsto \chi_i(\sigma a, b'')|K_{i+1}(\sigma a, b'')|^2$$

is invariant under left $U^{(i)}$ because its indicator absorbs the unipotent in the witness and the separate translation law in Theorem 172.3 has unit modulus. Use Lemma 175.2 with this σ to replace $\sigma b'$ by b'' under ρ_i. The measure on σa is $\theta_{i+1} * \tau_i = \tau_{i+1}$. This gives B_i times the asserted factor.

For the last inequality change x to $y = ax$ at fixed a. Write $\mathrm{d}\mu(y) = \mathrm{d}u_l * \rho_l$ by Lemma 175.2. The modulus of $K_{l+1}(a, ub')$ is independent of $u \in U^{(l)}$ by the separate translation formula, also when $l+1 = n$. A nonzero value forces $a \in \mathcal{N}_{l+1} \subseteq \mathcal{N}_l$. The integral over u of the remaining indicator and weight is bounded by Theorem 173.3, including its factor $\chi_l(a,b')$. This gives the claimed multiple of B_l. All integrals interchanged are nonnegative by Lemma 123.1. □

Theorem 176.3. *At all sufficiently large arithmetic scales, uniformly in the finite place, intermediate group Z, heads of fixed common length $j_0 \le \nu$, and integer radius parameter $H_0 \ge 0$,*

$$\int_{\Omega_{H_0}} \sum_{l=j_0}^{\nu} Z_l(x)\,\mathrm{d}\mu(x) \le 2n\, m_Z\, v_Z\, D_*\, Q^{60n^2 H_0}.$$

Proof. For a fixed l, iterate Lemma 176.2 from ν down to l and insert the result into its last inequality. The product of the level factors is exactly v_g by Definition 173.1; it cancels the denominator v_g in B_ν. The remaining local density factor is

$$C_v^l \prod_{i=l+1}^{\nu} M_{i+1} \le (1+2^{-n})(1+2^{-5n}) \le 2.$$

Here Theorem 170.3 and Corollary 164.3 apply to this single place and all factors are at least one. The exponent of Q^{H_0} is at most $60nl + 40n(\nu - l) \le 60n^2$. There are at most n indices. Summing the bounds proves the theorem. □

177. The finite coefficient away from the principal level

Lemma 177.1. *Use the empty heads of Definition 172.1 and $Z_l(x)$ of Definition 176.1. For $x \in \mathcal{P}_T \setminus K_g$ let $e_x = Q^{-\max\{1,h(x)/(2\nu)\}}$. At sufficiently large scales,*

$$|c_Z(x)|^2 \le 2e_x^s \sum_{l=0}^{\nu-1} Z_l(x).$$

For $x \notin \mathcal{P}_T$, $c_Z(x) = 0$.

Proof. The integral for $K_0(1,x)$ equals $c_Z(x)$ by Theorem 172.3. If any raw inner product $\langle W_Z(a), W_Z(ax)\rangle$ is nonzero, Lemma 171.2 gives $x \in \mathcal{P}_T$, proving the last assertion. For $x \in \mathcal{P}_T \setminus K_g$, apply Cauchy–Schwarz to the recurrence at the i-th step, $1 \le i \le \nu$. The integrand vanishes unless $\sigma a \in \mathcal{N}_i$. Using the reciprocal s-th power bound of Proposition 172.2 on that set gives

$$|K_{i-1}(a,ax)|^2 \le M_i C_v \int A_i(\sigma a, \sigma ax)^s |K_i(\sigma a, \sigma ax)|^2 \,\mathrm{d}\theta_i(\sigma).$$

Start this inequality at $a = 1,\ i = 1$. At each step split the right integral where $A_i \le e_x$ and where $A_i > e_x$. On the former subset bound A_i^s by e_x^s, keeping the squared pairing there. On the latter keep the weight and expand the pairing by the inequality at the next step. All earlier weights and comparisons with e_x are unchanged by that next left multiplier, because $G_{i+1} \subset P_b$ for $b \le i$ and Lemma 171.2 applies. The resulting contribution from the first small ideal occurring at i is at most

$$\left(\prod_{b=1}^{i} M_b C_v\right) e_x^s Z_{i-1}(x) :$$

indeed the preceding weights give $\mathcal{W}_{i-1}$, the product of the first i measures gives τ_i, and the restrictions may be dropped from this nonnegative integral. This procedure uses the recurrence only finitely many times.

On the remaining set at $i = \nu$, all ν ideal volumes exceed e_x. This still holds after every left translation by an element of G_n. However $T_\nu = \mu_H * \theta_n$ is supported there, and Theorem 171.3 makes every raw pairing in $K_\nu(a, ax)$ zero on this set. Thus only the stopped contributions remain. Their products of density factors are at most two by Theorem 170.3 and Corollary 164.3. Summing proves the claim, including cases of infinite individual upper integrals. □

Theorem 177.2. *At every sufficiently large arithmetic scale, for each finite place v of F of residue cardinality Q and every intermediate group $K_g \subseteq Z \subseteq K_{\{v\}}$,*

$$\mathcal{R}_{\{v\}}(Z) \le Q^{-k+120n^3} \le Q^{-9k/10}.$$

In particular the full integral in Definition 167.1 is finite.

Proof. By Proposition 161.3 the remainder equals $m_Z^{-1} \int_{x \notin K_g} |c_Z(x)|^2 \,\mathrm{d}\mu(x)$. At height $h(x) = h \ge 1$ use Lemma 177.1, then enlarge the domain for the nonnegative Z_l's to Ω_h. Theorem 176.3 applies with $D_* = 1$ and $v_Z \le 1$ by Lemma 160.2. The bound becomes

$$4n \sum_{h=1}^{\infty} Q^{60n^2 h - s\max\{1,h/(2\nu)\}}.$$

Here $s = k - 11n - 4$, by Lemma 163.1 and Definition 172.1. For $h \le 2\nu$ every term is at most $Q^{120n^2\nu - s}$. Beyond 2ν the terms form a geometric progression with ratio $Q^{60n^2 - s/(2\nu)} \le 1/2$: indeed $Q \ge 2$, $n \le \sqrt{X}$ and $k \ge X^5/2$ eventually by Proposition 10.5. Their sum is bounded by one further copy of $Q^{120n^2\nu - s}$. Countable additivity therefore bounds the remainder by $4n(2\nu+1)Q^{120n^2\nu - s}$. Since $\nu = n - 1$ and $11n + 4 + \log_2(4n(2\nu+1)) \le 120n^2$ eventually,

this is at most $Q^{-k+120n^3}$. Also $120n^3 \le k/10$ eventually by the same scale bounds, proving the two estimates uniformly. □

Theorem 177.3. *For every finite set T of places of F, $\mathcal{E}_T(K_T) \le \mu(K_T) + 2^{-n}$ at all sufficiently large scales. For T, S and the actual vector stabilizer Z of Theorem 167.3,*

$$\mathcal{E}_T(Z) \le (1 + b^{-6}) b^{-6k|S|}.$$

Proof. At v write K_v for its integral isometry group. The lower bound in Theorem 166.2 gives $\mu_v(K_v) \ge (1 - Q_v^{-1})^{2n} \ge 2^{-2n}$. Indeed the split integer algebra has unit volume $(1 - Q_v^{-1})^2$, and the inert one has unit volume $1 - Q_v^{-2}$, by deleting the maximal ideals. Consequently Theorem 177.2 gives

$$\sum_{v \in T} \frac{\mathcal{R}_{\{v\}}(K_v)}{\mu_v(K_v)} \le 2^{2n} f \sum_{h=2}^{\infty} h^{-9k/10} \le 2^{2n+1} f \, 2^{-9k/10}.$$

Here at most f places lie above each rational prime by Lemma 57.1, and we used the decreasing integral estimate of Corollary 164.3. The last bound is at most 2^{-n-2} eventually because $f \le X^{12}$, $n \le \sqrt{X}$ and $k \ge X^5/2$. By Proposition 167.2 the ratio $\mathcal{E}_T(K_T)/\mu(K_T)$ is the product of one plus each local ratio in this sum. Using $1+t \le \exp(t)$ and $\exp(t)-1 \le 2t$ for $0 \le t \le 1/2$, its excess over one is at most 2^{-n-1}. Multiplying by $\mu(K_T) \le 1+2^{-5n} \le 2$ from Theorem 166.2 gives the first assertion. In particular the unselected block in Theorem 167.3 has the remainder bound required there. At each selected place $Q_v = g_0$, Theorem 177.2 gives $g_0^{-9k/10} \le g_0^{-3k/4}$ for its intermediate group. That reduction proves the second assertion. □

178. A simultaneous second moment over the finite places

Definition 178.1. Let T be an arbitrary finite set of places of F and $Z = \prod_{v \in T} Z_v$ a product of intermediate groups, in particular the group in Theorem 177.3. Use all groups, orbit and layer measures, partial quotients, averaged pairings K_i, and flag ideals as in Definitions 171.1, 172.1 and 175.1, now over the product algebras E_T, F_T. Use $\mathcal{P}_T = \prod_v \mathcal{P}_{\{v\}}$ and $\mathcal{N}_i = P_i K_T$, also at $i = n$. For an integer tuple $\mathbf{H} = (H_v \ge 0)_{v \in T}$, put

$$R(\mathbf{H}) = \prod_{v \in T} Q_v^{H_v}, \qquad \Omega_{\mathbf{H}} = \prod_{v \in T} \Omega_{H_v}, \qquad R(x) = \prod_{v \in T} Q_v^{h(x_v)}.$$

Here and below $\mathcal{W}_j(a, b') = \prod_{i=1}^{j} A_i(a, b')^s$ uses $s = m_* - 3\nu - 2n$, and $A_i(a, b') = \prod_v A_{i,v}(a_v, b'_v)$ is the additive volume of the product ideal. Put $C_T = \prod_v C_v$ and use $D_* = \prod_{a=1}^{j_0} m_T(p_a/q_a)^{d_a+1}$. Write $Z_l(x)$ as in Definition 176.1 with product measures. For $j_0 \le j \le \nu$ put

$$U_j(a, x) = \int |K_{j+1}(ha, hax)| \, \mathrm{d}\theta_{j+1}(h),$$

$$S_j(a, x) = \sum_{l=j}^{\nu} \int \mathcal{W}_l(ha, hax) |K_{l+1}(ha, hax)|^2 \, \mathrm{d}(\theta_{l+1} * \cdots * \theta_{j+1})(h).$$

An empty tuple of places carries the singleton probability on every group and scalar set.

Proposition 178.2. *The orbit laws, recurrence, separate and simultaneous translation laws, and support conclusions of Theorem 172.3 hold with these product measures. One has*

$$\int_{\Omega_{\mathbf{H}}} \sum_{l=j_0}^{\nu} Z_l(x) \, \mathrm{d}\mu(x) \le 2n \, m_Z v_Z D_* \, R(\mathbf{H})^{60n^2}.$$

In addition $S_j(\,\cdot\,, x)$ is left P_j-invariant, vanishes off $\mathcal{N}_j$, and $\int S_j(a, x) \, \mathrm{d}\tau_j(a) = \sum_{l=j}^{\nu} Z_l(x)$.

Proof. The orbit and unipotent measures are products by the coordinate calculations of Proposition 167.2. All the iterated group measures are products by ordered multiplication in each component. Also $D_Z = \prod_v D_{Z_v}$, and the vector entry is the product of the respective

entries by the factorization of ϕ_g in that proposition. Thus the inner product, then its absolutely convergent successive averages, factor. The local translation and support laws give their stated product versions. Each Z_l factors separately by Lemma 123.1 on nonnegative functions. Apply Lemma 176.2 at each place for one fixed l, iterating from ν. Its level factors cancel its terminal $v_{g,v}^{-1}$. Before taking any sum over layers the resulting bound is

$$m_{Z_v} v_{Z_v} D_{*,v}\ C_v^l \prod_{i=l+1}^{\nu} M_{\{v\}}(\ell_{i+1})\ Q_v^{60n^2 H_v}.$$

The normalizations m_Z, v_Z, D_* are the products of their one-place quantities by their definitions and Proposition 167.2. After multiplication the two density factors have product at most $C_T^n \prod_i M_T(\ell_i) \le 2$ at all sufficiently large scales by Theorem 170.3 and Corollary 164.3. Summing over at most n indices proves the bound.

For the last assertions fix l. The innermost test $\mathcal{W}_l(y, yx)|K_{l+1}(y, yx)|^2$ is P_{l+1}-invariant since $l + 1 > j_0$, by the modulus law and Lemma 171.2. Its support lies in $\mathcal{N}_{l+1}$. The orbit integration in the proof of Proposition 172.2 descends the invariance and support through one layer; its proof uses the orbit law and applies verbatim to a product via its explicit product sections. In detail a preceding unipotent conjugates into the invariance subgroup and the remaining isometry acts on the invariant orbit measure. A nonzero contribution at a layer forces membership in the preceding near set as both the integrating group and the next subgroup lie in the preceding subgroup. Iterating proves the claims concerning S_j, also for infinite values. Finally $(\theta_{l+1} * \cdots * \theta_{j+1}) * \tau_j = \tau_{l+1}$. Nonnegative integration gives the identity for its integral. □

Lemma 178.3. *For $a \in G_T,\ x \in \mathcal{P}_T,\ j_0 \le j \le \nu$, one has*

$$\mathcal{W}_j(a, ax) U_j(a, x)^2 \le 2 S_j(a, x).$$

If $e > 0$ and for every $h \in G_{j+1}$ with $\langle W_Z(ha), W_Z(d_* hax)\rangle \ne 0$ some $j < i \le \nu$ has $A_i(ha, hax) \le e$, then the right side may be replaced by $2e^s S_j(a, x)$.

Proof. For $i \le \nu$, Proposition 172.2 gives

$$\int_{\sigma a \in \mathcal{N}_i} A_i(\sigma a, \sigma a x)^{-s}\, \mathrm{d}\theta_i(\sigma) \le M_T(\ell_i) C_T$$

by product integration. At $i = n$ the integral of one there is at most $M_T(\ell_n)$. In U_{i-1}, the integrand vanishes off the indicated set by Proposition 178.2. Weighted Cauchy–Schwarz, using $A_i^{s/2}$ and its reciprocal inside the integral of the absolute value, gives

$$U_{i-1}(a, x)^2 \le M_T(\ell_i) C_T \int A_i(\sigma a, \sigma a x)^s |K_i(\sigma a, \sigma a x)|^2\, \mathrm{d}\theta_i(\sigma) \qquad (i < n).$$

At $i = n$ use the same inequality with weight one and the stated terminal mass. The recurrence gives $|K_i(a, ax)| \le U_i(a, x)$. Iterate the inequalities starting at $i = j + 1$, multiplying by $\mathcal{W}_j$. Later multipliers preserve the earlier ideals because they belong to the corresponding P_b. The terminal contribution is at most the $l = \nu$ term of S_j times $C_T^\nu \prod_i M_T(\ell_i) \le 2$, proving the first estimate. For the second, stop as soon as a new A_i is at most e, bounding A_i^s by e^s and retaining $|K_i|^2$. Dropping the earlier restrictions bounds that stopped term by the $l = i - 1$ summand of $2e^s S_j$ just as in Lemma 177.1. If no such index is reached, the raw upper factors, including H, are all in G_{j+1}, and all previous comparisons persist under them. The hypothesis makes each remaining raw integrand zero. This argument also covers $j = \nu$, when the hypothesis requires all such integrands to vanish. □

179. Reciprocal ideal volumes on the opposite finite flag

Definition 179.1. For a list p of length $j \le \nu$ with all $D_a(p) \in B_T$, use $B(p), d(p)$ of Definition 168.1 and put

$$L(p) = m_T(B(p))^{-1}, \qquad L_v(p) = Q_v^{h_v}, \quad h_v = \operatorname{ord}_v d(p)_v.$$

Then $L(p) = \prod_v L_v(p) \ge 1$ by the norm determinant formula in Lemma 70.1. Put $\mathfrak{m}_0$ equal to unit mass at one and $\mathfrak{m}_j = (\eta_j * \theta_j) * \cdots * (\eta_1 * \theta_1)$.

Theorem 179.2. *For $a \in \mathcal{N}_j$, $x \in \mathcal{P}_T$ and any p as in Definition 179.1, one has*

$$\int_{U^{(j)}} \frac{\mathbf{1}_{\mathcal{N}_j}(w_p u a)}{\mathcal{W}_j(w_p u a, w_p u a x)} \, \mathrm{d}u_j \le C_T^j L(p)^{60nj} \prod_{b=1}^{j} m_T(p_b)^{-d_b - 1/2}.$$

If some $D_b(p)$ fails integrality, the integration domain in the numerator is empty.

Proof. It suffices to prove the result at one place: all domains, weights and measures factor before integration by Proposition 178.2. Replacing a by an element of $U^{(j)}a$ leaves the integral unchanged by right invariance in u. Replacing it by ha, $h \in G_{j+1}$, conjugates u and commutes h past w_p, which preserves the integral by Lemma 171.2 and the determinant-one conjugation in Theorem 166.2. Thus take $a \in K_T$ locally. On the domain there is $q' \in P_j$ with $q' w_p u a \in K_T$. Factoring $q' = u_1 l$, $l \in G_{j+1}$, gives $u_1 w_p l u \in K_T$; thus Theorem 168.2 proves all the leading minors integral and bounds $u \in \mathcal{B}(d(p))$. This proves also the empty-domain assertion.

Induct on j, the empty case having bound one. Write $u = n_j(t,c)u_0$, $u_0 \in U^{(j-1)}$, in this order. It gives the product of their measures since conjugation by N_j preserves the earlier layers by Theorem 166.2. The middle block after removing $j-1$ pairs is $n_j(t,c)$. Writing $h_v = \operatorname{ord}_v d(p)$, we therefore have $t \in \pi^{-h_v}\mathcal{L}_{j+1}$, and at fixed t the admissible c's have volume at most Q^{h_v} by the corner calculation of Lemma 173.2. The valuations of $p_j = D_j(p)/D_{j-1}(p)$ in each field component are between $-h_v$ and h_v: all the initial minors are integral and their product and its conjugate give $d(p)$. Consequently $m_T(p_j)^{1/2} \le Q^{h_v}$.

At fixed u_0, its action on Iv_0, for $v_0 \in E_v^{m_*}$, adds a combination of $e_1, \dots, e_{j-1}$. Take the coefficients to be the rows of Cv_0 in Definition 170.1, with first position a and translation x. For v_0 in its module W we have $u_0 I v_0 = I v_0$. The vector $w = w_p u I v_0$ has earlier dual coordinates zero and j-th dual coordinate $-p_j t_+^* v_0$: the layer inserts $-[t, Iv_0]e_j$ and the j-th flip sends e_j to $p_j e_j^*$. The two inverse images of w under $w_p u a, w_p u a x$ are close by the choice of W. Hence

$$A_j(w_p u a, w_p u a x)^{-1} \le \phi(-\bar{p}_j t_+)$$

by Theorem 170.3. Use $t' = -\bar{p}_j t$, which gives volume factor $m_T(p_j)^{-d_j}$ and puts t' in $\pi^{-2h_v}\mathcal{L}_{j+1}$. The moment formula of that theorem on the standard coordinates, and the box volumes for the other $d_j - m_*$ coordinates, bound the integral of $\phi(t'_+)^s$ by $C_v Q^{4(d_j - s)h_v}$. For the center use $Q^{h_v} \le m_T(p_j)^{-1/2} Q^{2h_v}$. Since $2 + 4(d_j - s) \le 60n$ as computed in Theorem 173.3, the cost of this step is at most $m_T(p_j)^{-d_j - 1/2} C_v Q^{60nh_v}$. The bound is uniform in u_0; all original integrands are Borel by Lemma 171.2.

If p' is the prefix of length $j-1$, then $w_p u a = w_j(p_j) n_j(t,c) w_{p'} u_0 a$. The first two factors belong to G_j, hence preserve the earlier weights and may be removed from the necessary condition $w_{p'} u_0 a \in \mathcal{N}_{j-1}$. Apply the induction hypothesis to that prefix and use $h_v(p') \le h_v(p)$, which follows since $D_j(p)$ is integral. Nonnegative integration now gives the asserted one-place bound, and multiplication gives the claimed product bound. □

Lemma 179.3. *For $0 \le j \le \nu$, $\mathfrak{m}_j = \mathrm{d}u_j * \tau_j$ and $\mu = \mathcal{M}_{j+1} * \mathfrak{m}_j$. Consequently if a nonnegative Borel function A is left G_{j+1}-invariant and w commutes with G_{j+1}, then*

$$\int A(wy) \, \mathrm{d}\mathfrak{m}_j(y) = \int A(y) \, \mathrm{d}\mathfrak{m}_j(y).$$

Proof. The second convolution identity follows by expanding Definition 166.1 down to layer j. For the first identity use induction on j: in $(\eta_j * \theta_j) * \mathrm{d}u_{j-1} * \tau_{j-1}$, the first two factors lie in G_j and their conjugations preserve $\mathrm{d}u_{j-1}$ by Theorem 166.2. Move it to the front; $\mathrm{d}u_{j-1} * \eta_j = \mathrm{d}u_j$, and $\theta_j * \tau_{j-1} = \tau_j$. To obtain the last assertion use a cutoff $a(y)$ with $\int_{G_{j+1}} a(hy)\,\mathrm{d}\mathcal{M}_{j+1}(h) = 1$ for every y. Construct it on one $y_b K_T$ per double coset $G_{j+1} y_b K_T$, with value the reciprocal volume of $G_{j+1} \cap y_b K_T y_b^{-1}$, and set it to zero elsewhere. The proof of Lemma 175.2 applies since $\mathcal{M}_{j+1}$ is bi-invariant, finite and positive on compact open subgroups by Theorem 166.2. Insert a using the second convolution identity and the left invariance of A. Replacing y by wy uses left invariance of μ; the inner cutoff integral is still one by commutation with w. This proves the formula also for infinite integrals. □

180. Two finite flags with prescribed initial pivots

Definition 180.1. For p, q of length $j \le \nu$ put $i = j+1$, and extend $H(p) = \prod_{a=1}^{j} m_T(p_a)^{d_a/2}$ to these lengths. Use K_b, S_j, Z_l of Definition 178.1 with empty initial lists, and write K'_b, S'_j, Z'_l for the pairings and sums with $d_* = w_q^{-1} w_p$. At fixed a, x let

$$F(\sigma) = K_i(\sigma a, \sigma a x), \qquad F'(t) = K'_i(t w_p^{-1} a, t w_p^{-1} a x).$$

Define

$$B_j(a,x) = \iint |F(\sigma) F'(t)|\,\mathrm{d}\theta_i(\sigma)\,\mathrm{d}\theta_i(t),$$

$$B_j^{\xi}(a,x) = \iint \psi_T^0(\xi[\sigma^{-1} f_i, t^{-1} f_i])\, F(\sigma)\overline{F'(t)}\,\mathrm{d}\theta_i(\sigma)\,\mathrm{d}\theta_i(t),$$

$$\mathcal{I}_j^{\xi}(p,q) = \frac{H(p)H(q)}{m_Z} \iint B_j^{\xi}(a,x)\,\mathrm{d}\mathfrak{m}_j(a)\,\mathrm{d}\mu(x)$$

for $\xi \in E_T$, the last expression whenever absolutely convergent. The two inner integrals exist at every a, x by the finite support-mass bounds in Proposition 172.2 and the product pairing laws of Proposition 178.2.

Lemma 180.2. *If $B_j(a,x) > 0$ then $x \in \mathcal{P}_T$,*

$$a,\ ax,\ w_p^{-1}a,\ w_q^{-1}ax \in \mathcal{N}_j, \qquad D_b(p), D_b(q) \in B_T \quad (1 \le b \le j).$$

Let $L(p), L(q)$ be as in Definition 179.1 and put $A = m_T(g)^{-1} L(p)^2 L(q)^2$. If $R(x) > A^{4n}$, then for every $h \in G_{j+1}$ with $\langle W_Z(ha), W_Z(hax)\rangle \neq 0$ there is some $j < i' \le \nu$ such that $A_{i'}(ha, hax) \le R(x)^{-1/(4n)}$.

Proof. Positivity in Definition 180.1 yields a product of upper multipliers in G_{j+1} making the raw untwisted pairing nonzero. Lemma 171.2 at each place gives $x \in \mathcal{P}_T$, and the two points then lie in $U_T Z \subseteq \mathcal{N}_j$. The upper multipliers can be removed since $G_{j+1} \subseteq P_j$. The same support argument for the twisted factor gives $w_p^{-1}a, w_q^{-1}ax \in \mathcal{N}_j$, since d_* commutes with G_{j+1} and $d_* w_p^{-1} = w_q^{-1}$. Corollary 168.3 gives the integrality assertion.

Replace a by ha with its raw pairing nonzero. All four near conditions persist by the commutation and subgroup inclusion just used. Choose $u, v_1 \in U_T$ with $ua = k_1,\ v_1 ax = k_2 \in Z,\ k_1^{-1} k_2 \in K_g$, now at these replaced arguments, by Lemma 171.2 at each place. Write $u = u_{\mathrm{out}} u_{\mathrm{in}}$ with $u_{\mathrm{out}} \in U^{(j)},\ u_{\mathrm{in}} \in G_{j+1}$, by splitting the first j layers. Since $w_p^{-1} a \in P_j K_T$, write $a = w_p l u' k'$ with $l \in G_{j+1}, u' \in U^{(j)}, k' \in K_T$. Thus $u_{\mathrm{out}} w_p (u_{\mathrm{in}} l) u' = k_1 (k')^{-1}$ is integral, so $u_{\mathrm{out}} \in \mathcal{B}(d(p))$ by Theorem 168.2. Since u_{in} fixes the first j dual vectors, $d(p) u e_b^* \in \mathcal{L}$ for $b \le j$. Using ax, q, v_1 in the identical factorization gives $d(q) v_1 e_b^* \in \mathcal{L}$.

Put $\alpha = g d(p) d(q)$, so $m_T(\alpha)^{-1} = A$. In the determinant argument of Theorem 171.3 choose $w_b = \alpha e_b^*$ for $b \le j$. Their two images $u w_b, v_1 w_b$ both lie in $g\mathcal{L}$, their dual entries before position b vanish, and their b-th entries equal α. For $b > j$ choose witnesses for generators t_b of $I_b(a, ax)$ at every component as in that theorem. The two matrices of columns are thus integral, congruent modulo g, and their lower triangular blocks have the

same determinant $\Delta = \alpha^{j}\prod_{b>j} t_b$. The adjugate and block calculation of that theorem, which used precisely these three facts about the columns, gives at each v $h(x_v) \le 2\,\mathrm{ord}_v(\Delta\bar{\Delta})$. Multiply $Q_v^{h(x_v)}$ over v to obtain

$$R(x) \le A^{2j}\left(\prod_{b>j} A_b(a, ax)\right)^{-2}.$$

In the stated range $A^{2j} \le R(x)^{1/2}$, whence the product of the remaining volumes is at most $R(x)^{-1/4}$. For $j < \nu$, at least one factor is at most $R(x)^{-1/(4(\nu-j))} \le R(x)^{-1/(4n)}$. For $j = \nu$ the empty product makes such a raw pairing impossible. This proves the assertion at the originally replaced arguments. □

Proposition 180.3. *Fix $x \in \mathcal{P}_T$ and two integral-minor heads p, q as in Lemma 180.2. Let $f(a)$ be a nonnegative Borel function supported where $a, w_p^{-1}a \in \mathcal{N}_j$. Suppose on its support*

$$f(a)^2 \le c^2\,\frac{S_j(a,x)}{\mathcal{W}_j(a,ax)}\,\frac{S_j'(w_p^{-1}a,x)}{\mathcal{W}_j(w_p^{-1}a, w_p^{-1}ax)}$$

with $c > 0$. Write $Z = \sum_{l=j}^{\nu} Z_l(x)$ and $Z' = \sum_{l=j}^{\nu} Z_l'(x)$ in this statement only. If these are finite, then

$$\int f\,\mathrm{d}\mathfrak{m}_j \le cC_T^{j}L(p)^{60nj}\prod_{b=1}^{j} m_T(p_b)^{-d_b-1/2}\sqrt{ZZ'}.$$

Proof. Use Cauchy–Schwarz, pairing $S_j(a,x)$ with the reciprocal weight and near indicator at $w_p^{-1}a$, and $S_j'(w_p^{-1}a,x)$ with those at a. In the first integral use $\mathfrak{m}_j = \mathrm{d}u_j * \tau_j$ of Lemma 179.3. The S_j-factor is left P_j-invariant and supported on $\mathcal{N}_j$ by Proposition 178.2. Apply Theorem 179.2 in u_j with head $\bar{p}$, since $w_p^{-1} = w_{\bar{p}}$ by the pair matrices. Its factor is the displayed product times $C_T^{j}L(p)^{60nj}$; the integral of S_j against τ_j is Z. The other integral has the form $\int A_1(w_p^{-1}a)\,\mathrm{d}\mathfrak{m}_j(a)$ with

$$A_1(y) = S_j'(y,x)\frac{\mathbf{1}_{\mathcal{N}_j}(w_p y)}{\mathcal{W}_j(w_p y, w_p yx)}.$$

This function is left G_{j+1}-invariant because that group commutes with w_p and preserves each factor under simultaneous translation. Lemma 179.3 removes w_p^{-1} from its argument. The same unipotent integration now applies with head p, giving the same factor times Z'. Taking square roots of the product proves the assertion. □

181. Absolute convergence on finite height shells

Definition 181.1. For a nonnegative integer tuple $\mathbf{h} = (h_v)_{v\in T}$ put $X_{\mathbf{h}} = \{x \in \mathcal{P}_T : h(x_v) = h_v\ (v \in T)\}$, $R = R(\mathbf{h})$ and $P = 60n^2$. For heads having integral initial minors write $A = m_T(g)^{-1}L(p)^2L(q)^2$, and put

$$\kappa(R) = \begin{cases} 1, & R \le A^{4n}, \\ R^{-s/(8n)}, & R > A^{4n}. \end{cases} \qquad C_j = \frac{H(p)H(q)}{m_Z}.$$

All measures and pairings in this section are those of Definition 180.1. For $T = \varnothing$ there is just one shell and $R = 1$.

Proposition 181.2. *For integral-minor heads of length $0 \le j \le \nu$ one has*

$$C_j\int_{X_{\mathbf{h}}}\int B_j(a,x)\,\mathrm{d}\mathfrak{m}_j(a)\,\mathrm{d}\mu(x) \le 8nL(p)^{60nj}L(q)^{1/2}R^{P}\kappa(R).$$

At all sufficiently large arithmetic scales the integral defining $\mathcal{I}_j^{\xi}$ is absolutely convergent for all p, q, ξ. If either list has a nonintegral initial minor this integral is zero.

Proof. For fixed x, B_j is the product of $U_j(a,x)$ for the empty head and $U_j'(w_p^{-1}a,x)$ for the prescribed heads. By Lemma 180.2 it vanishes off the indicated near sets, and is identically zero for a nonintegral initial minor. In the remaining case apply Lemma 178.3 to both factors. For $R > A^{4n}$ the displacement lemma allows the stronger bound for the first squared factor with $e = R^{-1/(4n)}$. Taking square roots gives the hypothesis of Proposition 180.3 with $c = 2\kappa(R)$. Its Z, Z' are finite almost everywhere on $X_{\mathbf{h}} \subseteq \Omega_{\mathbf{h}}$, by Proposition 178.2. That proposition and Cauchy–Schwarz in x bound the integral of $\sqrt{ZZ'}$ on this shell by $2n\, m_Z v_Z D_*^{1/2} R^P$. The head factors simplify exactly:

$$H(p)H(q)D_*^{1/2}\prod_{b\le j} m_T(p_b)^{-d_b-1/2} = m_T(D_j(q))^{-1/2} \le L(q)^{1/2}.$$

The final inequality uses integrality of every initial minor, with $D_0 = 1$. The facts $C_T^j \le 2$ and $v_Z \le 1$ follow from Theorem 170.3 and the integral unipotent normalization in Proposition 160.4. This proves the estimate.

By the place count of Lemma 57.1,

$$\sum_{\mathbf{h}} R(\mathbf{h})^{-2} = \prod_{v\in T}(1-Q_v^{-2})^{-1} \le \exp(2f).$$

Indeed $-\log(1-y) \le 2y$ for $0 \le y \le 1/2$ by integration, and $\sum_{m\ge 2} m^{-2} \le \int_1^\infty t^{-2}\,\mathrm{d}t = 1$. Since $k = r+1-2n$, Proposition 10.5 gives $s/(8n) \ge P+2$ eventually. For $R \le A^{4n}$ one has $R^{P+2} \le A^{4n(P+2)}$; for $R > A^{4n}$, $R^P\kappa(R) \le R^{-2}$. Summing the shell estimate therefore bounds the integral of the full absolute majorant. Lemma 180.2 covers its complement and $|B_j^\xi| \le B_j$, as required. □

Lemma 181.3. *Fix a, x with $B_j(a,x) > 0$ and write $i = j+1$, $d = d(p)$. There is $h \in G_i$ such that the two functions $F(\sigma)$ and $F'(t)$ in Definition 180.1 factor through $v = h\sigma^{-1}f_i$, $v' = ht^{-1}f_i$, respectively. As functions on the two quadrics they are bounded and vanish almost everywhere outside $d^{-1}\mathcal{L}_i$ and $d^{-2}\mathcal{L}_i$, respectively. On the product of these boxes, $[v,v'] \in d^{-3}B_T$.*

Proof. The near conditions of Lemma 180.2 allow Corollary 168.3 to give one $h \in G_i$ with $ha \in \mathcal{B}(d)$ and $hw_p^{-1}a \in \mathcal{B}(d^2)$. As in Lemma 174.1, the stabilizer of f_i in G_i is contained in $P_i \cap G_{j+1}$. The translation laws of Proposition 178.2 give bounded Borel amplitudes through the orbit sections for both lists. The simultaneous factor h preserves the quadric measure by Proposition 163.4.

If $F(\sigma) \ne 0$, choose $u' \in P_i$ with $u'\sigma a = k' \in K_T$ by the same support law. Thus $ha(k')^{-1}f_i = h\sigma^{-1}(u')^{-1}f_i$ lies in $d^{-1}\mathcal{L}$. The difference $(u')^{-1}f_i - f_i$ belongs to the span of $e_1, \dots, e_j$: on the quotient by these preceding frame vectors P_i fixes f_i, by its block definition, also for $f_n = z$, $P_n = U_T H$. Since $h, \sigma \in G_i$ preserve that span and fix its vectors, projection onto the middle summand gives $v \in d^{-1}\mathcal{L}_i$. Apply this argument with first position $w_p^{-1}a$ to obtain the other bound. The integral basis in Lemma 163.1 has integral hermitian pairings, and $d = \bar{d}$. This proves the last claim. □

Proposition 181.4. *For the heads in Proposition 181.2 set $d = d(p)$. If $\xi \in E_T$, $\eta \in E_T^\times$ and $\xi - \eta \in d^3 B_T$, then*

$$C_j \int_{X_{\mathbf{h}}} \int |B_j^\xi|\,\mathrm{d}\mathfrak{m}_j\,\mathrm{d}\mu \le 8nL(p)^{60nj+30n}L(q)^{1/2}R^P \cdot \min\{\kappa(R), m_T(\eta)^{-m_*/2}\}.$$

Proof. Where $B_j > 0$, Lemma 181.3 and Proposition 18.5 allow replacement of ξ by η in the phase since the argument changes by an integer at each component. Isometry preserves the pairing in the two new orbit coordinates. Apply Cauchy–Schwarz against the first amplitude on its support and apply Theorem 164.2 with the second amplitude conjugated, input $d^{-2}\mathcal{L}_i$

and output $d^{-1}\mathcal{L}_i$, both with quadric measure. The bound for $|B_j^\xi|$ is the product of the two amplitude norms in $L^2(\delta_i)$ times

$$m_T(\eta)^{-m_*/2} M_T(\ell_i) L(p)^{3(4\ell_i-2)/2}.$$

Here $\Delta_T(d^{-1}) = L(p)$; $M_T(\ell_i) \le 2$ by Corollary 164.3 and $3(4\ell_i - 2)/2 \le 30n$ eventually by Lemma 163.1. The square of each amplitude norm times its corresponding $\mathcal{W}_j$ is the $l = j$ summand of S_j or S_j', using the orbit measure identity and Definition 178.1. The hypotheses of Proposition 180.3 therefore hold with $c = 2L(p)^{30n} m_T(\eta)^{-m_*/2}$. Integrate this bound on the shell using the same second moments and exact head cancellation as in Proposition 181.2. That proposition also bounds $|B_j^\xi|$ using $\kappa(R)$. Since $L(p) \ge 1$, taking the smaller estimate gives the result. □

182. Denominators in the common-pivot frequency bound

Definition 182.1. Write $K = k - n^7$ and let $w \mid T$ run over the field components of E_T. For $\xi \in E_T^\times$ and $p \in (E_T^\times)^j$ put

$$D_T(\xi) = \prod_{w|T} q_w^{\max(0, -\operatorname{ord}_w \xi_w)},$$

$$\beta_T(\xi) = b^{n^9 f}\, m_T(\xi)^{-K/2} D_T(\xi)^{-6},$$

$$\Lambda_{T,j}(p) = \prod_{a=1}^{j} \mathbf{1}_{B_T}(D_a(p))\, m_T(D_a(p))^{-n^6/2}.$$

Here the empty head has weight one, and D_T agrees with the finite denominator in Theorem 52.3. The identity $m_T(\xi) = \prod_{w|T} q_w^{-\operatorname{ord}_w \xi_w}$ follows from the determinant of multiplication on the local integer rings in Lemma 70.1.

Lemma 182.2. *At all sufficiently large scales, for any $u > 0$,*

$$\sum_{\mathbf{h}} R^P \min\{\kappa(R), u^{-m_*/2}\} \le e^{2f} A^{4n(P+2)} \min(1, u^{-1/2})^{K+12}.$$

Moreover, for any integral-minor heads, $0 \le e \le 160n^2$, $0 \le e' \le 1$, one has

$$8ne^{2f} L(p)^{e+36} L(q)^{e'} A^{4n(P+2)+6} \;\le\; b^{11000n^3 f} \big(L(p)L(q)\big)^{500n^3}.$$

Proof. For $m_* = k - 6n - 7$, $s = k - 11n - 4$, put $\theta = 1 - (K+12)/m_*$. Proposition 10.5 implies $K > 0$, $m_* > 0$, $s/m_* \ge 1/2$ eventually, since $r \ge X^5$, $n \le \sqrt{X}$, $k = r + 1 - 2n$ at scale X. Also $m_* - (K+12) = n^7 - 6n - 19 \ge n^7/2$ eventually. Hence $0 < \theta < 1$ and $\theta s/(8n) \ge n^6/32 \ge P + 2$. For positive a, b' their minimum is at most $a^\theta (b')^{1-\theta}$, by comparison with each factor. Since $\kappa \le 1$, use $a = \kappa$, $b' = \min(1, u^{-m_*/2})$. On $R > A^{4n}$ one has $R^P \kappa^\theta \le R^{-2}$; on the remaining shells it is at most $A^{4n(P+2)} R^{-2}$. The sum of R^{-2} was bounded in the proof of Proposition 181.2. This proves the first claim.

By Theorem 11.4 one has $g = g_0 g_1 \le b^{22}$. The local lattice indices give $m_T(g)^{-1} \le g^{2f}$: the complete product of the local indices is g^{2f}, the determinant of multiplication by g on the rank $2f$ integer lattice, by Lemma 20.2. All omitted factors are at least one. Thus $A \le b^{44f} L(p)^2 L(q)^2$. With $P = 60n^2$, both $160n^2 + 36 + 2(4n(P+2)+6)$ and $1 + 2(4n(P+2)+6)$ are at most $500n^3$ eventually. Also $44(4n(P+2)+6) + n \le 11000n^3$ eventually. Finally $8ne^{2f} \le b^{nf}$ for large n, b, as $f \ge 1$. These comparisons prove the second bound. □

Theorem 182.3. *For $0 \le j \le \nu$, $\xi \in E_T^\times$ and every pair of heads of length j, one has, eventually in the arithmetic scale,*

$$C_j \iint |B_j^\xi(a,x)|\, \mathrm{d}\mathfrak{m}_j(a)\, \mathrm{d}\mu(x) \le \beta_T(\xi) \Lambda_{T,j}(p) \Lambda_{T,j}(q).$$

For heads with integral initial minors the right side can be replaced by the following quantity:

$$b^{11000n^3 f}\big(L(p)L(q)\big)^{500n^3} m_T(\xi)^{-K/2} D_T(\xi)^{-6}.$$

Proof. Proposition 181.2 gives zero outside the asserted integral-minor case. With $d = d(p)$ integral and nonzero on every component, choose

$$\eta_w = \begin{cases} \xi_w, & \operatorname{ord}_w \xi_w < \operatorname{ord}_w d^3, \\ d^3, & \operatorname{ord}_w \xi_w \geq \operatorname{ord}_w d^3. \end{cases}$$

Then η is a unit of E_T and $\eta - \xi \in d^3 B_T$. The minimum of the two valuations gives $m_T(\eta) \geq m_T(\xi)$. For each component of negative valuation of ξ, its factor in $m_T(\eta)$ is exactly the denominator factor of ξ there. At every other component it is at least $q_w^{-\operatorname{ord}_w d^3}$, and these latter local lower bounds are at most one at every component. Hence

$$m_T(\eta) \geq D_T(\xi) m_T(d^3) = D_T(\xi) L(p)^{-6},$$

where $d = B(p)\overline{B(p)}$. In particular

$$\min(1, m_T(\eta)^{-1/2})^{K+12} \leq m_T(\xi)^{-K/2} D_T(\xi)^{-6} L(p)^{36}.$$

Proposition 181.4 and Lemma 182.2 now yield the improved bound: the exponent $60nj + 30n$ is at most $160n^2$, and $A \geq 1$. Finally $11000n^3 \leq n^9$ and $500n^3 \leq n^6/2$ for all sufficiently large n. Definition 182.1 and $\Lambda_{T,j}(p) = L(p)^{n^6/2}$ on the surviving heads give the stated inequality. □

183. The finite unipotent integral at unequal last entries

Lemma 183.1. *Let $1 \leq j \leq \nu$, let p, q have length j, and suppose $\xi = g^{-1}(p_j^{-1} - q_j^{-1}) \in E_T^{\times}$. For all $a, x \in G_T$ there is a bounded Borel amplitude F_a on the quadric of f_{j+1}, with measure δ, given by $F_a(\sigma^{-1} f_{j+1}) = K'_{j+1}(\sigma a, \sigma a x)$ for θ_{j+1}-almost every σ. One has*

$$K'_j(n_j(t,c)a, n_j(t,c)ax) = \int \psi_T^0(p_j \xi[t,v]) F_a(v)\, \mathrm{d}\delta(v).$$

If $a \in \mathcal{N}_j$ and $A_0 a \in K_T$, $A_0 \in P_j$ has middle quotient $h \in G_{j+1}$, the amplitude is supported almost everywhere on $h^{-1}\mathcal{L}_{j+1}$. If $a \notin \mathcal{N}_j$ it may be taken identically zero.

Proof. The stabilizer of f_{j+1} in G_{j+1} preserves the twisted pairing by Proposition 178.2. The orbit sections and their measure formulas in Lemma 165.2 and Proposition 165.3 give the stated amplitude. If it is nonzero at a representative σ, choose $A_1 \in P_{j+1}$ with $A_1 \sigma a \in K_T$ by the pairing support law. Then $a \in \mathcal{N}_j$. On the middle quotient, the integral matrix $A_1 \sigma A_0^{-1} \in P_j \cap K_T$ is $s_1 \sigma h^{-1}$ with s_1 fixing f_{j+1}. Its inverse preserves $\mathcal{L}_{j+1}$, so $h\sigma^{-1} f_{j+1} \in \mathcal{L}_{j+1}$. Local finiteness of δ and boundedness of the pairing give integrability and square integrability.

Use the pairing recurrence at level $j+1$. Conjugation of $n_j(t,c)$ by σ gives $u = n_j(\sigma t, c)$, as in Lemma 174.1. Its only possible simple-root entry is $-[\sigma t, f_{j+1}]$. Conjugation of u by $d_* = w_q^{-1} w_p$ multiplies that entry by p_j/q_j, since d_* is the identity on the middle quotient. The two translation characters at level $j+1$ therefore give the phase with argument

$$g^{-1}(1 - p_j/q_j)[\sigma t, f_{j+1}] = p_j \xi[t, \sigma^{-1} f_{j+1}].$$

The assertion follows by the orbit measure formula, also for the last layer with $f_{j+1} = z$. □

Proposition 183.2. *Under the hypotheses of Lemma 183.1, assume p has integral initial minors and put $L_p = L(p)$. For all a, x one has*

$$\int_{N_j} \mathbf{1}_{\mathcal{N}_j}(w_p u a) |K'_j(ua, uax)|^2 \,\mathrm{d}\eta_j(u)$$
$$\leq 2 L_p^{50n} m_T(p_j)^{-d_j - 1/2} m_T(\xi)^{-m_*} \int |K'_{j+1}(\sigma a, \sigma a x)|^2 \,\mathrm{d}\theta_{j+1}(\sigma).$$

Proof. If the left integral is positive, there exists $u_0 \in N_j$ with $w_p u_0 a, u_0 a \in \mathcal{N}_j$. Replace a by $u_0 a$. The left integral is unchanged by right invariance of η_j from Theorem 166.2. On the right, $\sigma u_0 \sigma^{-1} \in N_j \subseteq U^{(j)}$ contributes only factors of modulus one by Proposition 178.2. Thus assume $w_p a, a \in \mathcal{N}_j$. Choose A_0, h as in Lemma 183.1 and put $d = d(p)$.

For $u = n_j(t, c)$ with $w_p u a \in \mathcal{N}_j$, choose $A_1 \in P_j$ with $A_1 w_p u a \in K_T$. Multiplication gives

$$(A_0 u^{-1}) w_p^{-1} A_1^{-1} = (A_0 a)(A_1 w_p u a)^{-1} \in K_T.$$

Write $A_0 u^{-1} = u_1 h$, $A_1^{-1} = h_1 u_2$ with the unipotents in $U^{(j)}$ and $h_1 \in G_{j+1}$. Both middle factors commute with $w_p^{-1} = w_{\bar{p}}$. By Theorem 168.2, $u_1 \in \mathcal{B}(d)$. Hence $A_0 u^{-1} e_j^* = u_1 e_j^* \in d^{-1}\mathcal{L}$. This holds also at $u = 1$. Their difference projects onto $-ht$ in V_{j+1}, since $u^{-1} e_j^* - e_j^*$ equals $-t$ plus a multiple of e_j by Lemma 20.4. Thus $ht \in d^{-1}\mathcal{L}_{j+1}$. At fixed t, the difference between the images for c, c' is $-(c - c')A_0 e_j$, whose e_j coordinate is $-(c - c')$. Consequently every nonempty admissible central fibre lies in one translate of the integral skew lattice scaled by d^{-1}. Its measure is at most $\Delta_T(d^{-1}) = L_p$, by the skew coordinate normalization of Lemma 173.2 in each field place.

The phase formula is independent of c. Integrate the center and send t, v to ht, hv. This preserves ambient measure because the determinant of an isometry has norm of absolute value one. It preserves δ by Proposition 163.4. The input is supported on the integral quadric box and the output lies in the ambient box $d^{-1}\mathcal{L}_{j+1}$. Theorem 164.2 gives a squared-norm factor at most

$$M_T(\ell_{j+1}) L_p^{1+4\ell_{j+1}}\, m_T(p_j)^{-m_*} m_T(\xi)^{-m_*}$$

including the central measure. Its squared input norm is the last integral of the assertion. At each v, every field valuation of p_j has absolute value at most $\operatorname{ord}_v d_v$ by the minor calculation in Theorem 179.2. There are two degrees over F_v, giving $m_T(p_j) \le L_p^2$. Also $d_j + 1/2 - m_* > 0$, and $1 + 4\ell_{j+1} + 2(d_j + 1/2 - m_*) \le 50n$ eventually by Lemma 163.1 and Definition 165.1. Use this inequality to extract $m_T(p_j)^{-d_j - 1/2}$ and apply Corollary 164.3 to bound $M_T(\ell_{j+1})$ by two. This proves the result. □

184. Uniform decay for unequal finite pivots

Proposition 184.1. *Use the heads and frequency of Lemma 183.1 and assume the initial minors of both heads are integral. For $1 \le j \le \nu$ and each shell of Definition 181.1,*

$$C_j \int_{X_{\mathbf{h}}} \int |B_j^0(a, x)|\, \mathrm{d}\mathfrak{m}_j(a)\, \mathrm{d}\mu(x) \le 8nL(p)^{60nj+25n} L(q)^{1/2} R^P \cdot \min\{\kappa(R), m_T(\xi)^{-m_*/2}\}.$$

Proof. Use $w = w_p$, and let w_0 be the product of the flips for the first $j - 1$ entries of p. At fixed $x \in X_{\mathbf{h}}$ abbreviate

$$W_i(y) = \mathcal{W}_i(y, yx), \quad I_i(y) = \mathbf{1}_{\mathcal{N}_i}(y)/W_i(y), \qquad H_p = \prod_{b=1}^{j} m_T(p_b)^{-d_b - 1/2}.$$

The weights are finite and positive by Lemma 171.2. By the pairing recurrence, $B_j^0(a, x)$ equals $K_j(a, ax)\overline{K_j'(w^{-1}a, w^{-1}ax)}$. Its nonzero set requires $a, w^{-1}a \in \mathcal{N}_j$. Use Cauchy–Schwarz in a with squared factors $A(a) I_j(w^{-1}a)$ and $\Phi(w^{-1}a)$, where

$$A(y) = W_{j-1}(y) |K_j(y, yx)|^2,$$
$$\Phi(y) = \mathbf{1}_{\mathcal{N}_j}(wy) W_j(y) I_{j-1}(w_0 y) |K_j'(y, yx)|^2.$$

Indeed their product equals $|B_j^0|^2$ on that set: $w_0 w^{-1} = w_j(p_j)^{-1} \in G_j$ preserves W_{j-1} and $\mathcal{N}_{j-1}$ by Lemma 171.2.

For the first squared factor, A is P_j-invariant and supported on $\mathcal{N}_j$ by the empty-head pairing laws of Proposition 178.2. Also $\int A\, \mathrm{d}\tau_j = Z_{j-1}(x)$ for the empty head, by

Definition 176.1. Factor $\mathfrak{m}_j = \mathrm{d}u_j * \tau_j$ using Lemma 179.3. Then Theorem 179.2 for $\bar{p}$ bounds the first squared integral by $C_T^j L(p)^{60nj} H_p Z_{j-1}(x)$.

The function Φ is G_{j+1}-invariant. Indeed this group commutes with both Weyl matrices, preserves the earlier ideals and the near sets, and preserves the simultaneous twisted pairing at j by Proposition 178.2. Lemma 179.3 therefore replaces $\Phi(w^{-1}a)$ by $\Phi(a)$ in its integral. In the resulting integral disintegrate $\mathfrak{m}_j = \eta_j * \mathrm{d}u_{j-1} * \tau_j$: this follows from the same lemma since conjugation by N_j preserves the preceding radical measure, as in Theorem 166.2. At $a = uy,\ u \in N_j$, both $W_j(a)$ and $I_{j-1}(w_0 a)$ are constant in u; here u commutes with w_0. Apply Proposition 183.2 to the other two factors. After multiplication by $W_j(y)$, the squared amplitude integral is the $l = j$ summand of $S'_j(y,x)$. Enlarge it to $S'_j(y,x)$, which vanishes off $\mathcal{N}_j$ and is invariant under the preceding radical. Integrate $I_{j-1}(w_0 y)$ over that radical by Theorem 179.2 with the prefix of p. Its L is at most $L(p)$. Proposition 178.2 evaluates the integral of S'_j against τ_j as $Z'(x) = \sum_{l=j}^{\nu} Z'_l(x)$. The second squared integral is thus at most

$$2C_T^{j-1} L(p)^{50n+60n(j-1)} H_p\, m_T(\xi)^{-m_*} Z'(x).$$

Both moments are finite almost everywhere on this shell by the same proposition. Take square roots and integrate in x. The integral of $\sqrt{Z_{j-1} Z'}$ is at most $2n\, m_Z v_Z D_*^{1/2} R^P$, by that proposition and Cauchy–Schwarz. The exact head cancellation in Proposition 181.2, $C_T^j \le 2$ and $v_Z \le 1$ give the bound involving $m_T(\xi)$. The other bound follows from $|B_j^0| \le B_j$ and that shell majorant. Taking the minimum proves the assertion. □

Theorem 184.2. *For every $1 \le j \le \nu$, heads p, q of length j with $\xi = g^{-1}(p_j^{-1} - q_j^{-1}) \in E_T^{\times}$, and all sufficiently large arithmetic scales, one has*

$$C_j \iint |B_j^0(a,x)|\, \mathrm{d}\mathfrak{m}_j(a)\, \mathrm{d}\mu(x) \le \beta_T(\xi) \Lambda_{T,j}(p) \Lambda_{T,j}(q).$$

On heads with integral initial minors one may use the improved bound of Theorem 182.3.

Proof. By Proposition 181.2 only the integral-minor case contributes. The valuation estimate for p_j in Theorem 179.2 implies that $d(p)/p_j \in B_T$, and likewise $d(q)/q_j \in B_T$. Hence $\alpha\xi \in B_T$ for $\alpha = gd(p)d(q) \in B_T$. At each field component the denominator exponent of ξ is therefore at most $\mathrm{ord}_w\, \alpha$. Thus $D_T(\xi) \le m_T(\alpha)^{-1} = A$. Summing Proposition 184.1 with Lemma 182.2, use

$$\min(1, m_T(\xi)^{-1/2})^{K+12} \le m_T(\xi)^{-K/2} A^6 D_T(\xi)^{-6}.$$

Here $K \ge 0,\ A/D_T(\xi) \ge 1$; the same inequality holds whether $m_T(\xi)$ exceeds one or lies below it. Now $60nj + 25n \le 160n^2$, so the second assertion of Lemma 182.2 proves the improved bound. Its comparison to $\beta_T \Lambda_{T,j}(p) \Lambda_{T,j}(q)$ is that of Theorem 182.3. □

185. The finite-place kernels in the orbital sieve

Lemma 185.1. *For $0 \le j \le \nu$ set $J_j(p,q) = \mathcal{I}_j^0(p,q)$. For $0 \le j < \nu$, and heads p, q of length j, the function $c \mapsto J_{j+1}((p,c),(q,c))$ is integrable on $E_T^{\times}$ for additive measure. For every $\xi \in E_T$,*

$$\int_{E_T^{\times}} \psi_T^0(\xi c) J_{j+1}((p,c),(q,c))\, \mathrm{d}c = \mathcal{I}_j^{\xi}(p,q).$$

Proof. Put $i = j + 1$. In the inner integral in t of $B_j^{\xi}(a,x)$, both the phase and the second pairing depend on t only through $t^{-1} f_i$, by the stabilizer argument of Lemma 181.3, also before applying the isometry used there. The orbit law of Lemma 165.2 is invariant under G_i by Proposition 163.4. Thus for fixed σ one may replace t by $t\sigma$. The phase argument becomes $\xi[f_i, t^{-1} f_i]$, by isometry.

Disintegrate $t = w_i(c)^{-1}u$ according to Definition 165.1, with measure $m_T(c)^{d_i}\,\mathrm{d}c\,\mathrm{d}\eta_i(u)$. Then $[f_i, t^{-1}f_i] = [uf_i, w_i(c)f_i] = c$. The first pairing $K_i(\sigma a, \sigma a x)$ is unchanged by simultaneously multiplying its arguments by $u \in P_i$, by Proposition 178.2 for the empty head. Put $y = u\sigma a$. The product of pairings is now

$$K_i(y, yx)\,\overline{K_i'((w_p w_i(c))^{-1}y, (w_p w_i(c))^{-1}yx)}.$$

Indeed w_p, w_q commute with G_i and the ratio of the extended Weyl matrices for $(p,c),(q,c)$ is still d_*. This product equals the zero-frequency B_i^0 for the extended lists by the pairing recurrence. The measure in y is $\eta_i * \theta_i * \mathfrak{m}_j = \mathfrak{m}_i$, and the normalization is $C_j m_T(c)^{d_i} = H(p,c)H(q,c)/m_Z$.

Every change just made also applies to the modulus before either orbit integration. Its original integral is finite by Proposition 181.2. The transformed product is therefore absolutely integrable jointly in c, y, x. The parameterized functions are Borel since their defining matrix maps are continuous and their integrands and fixed measures are Borel, by Lemma 123.1. That lemma permits integration first in x, y, giving the claimed identity and integrability in c. □

Lemma 185.2. *For $j = \nu$ use J_n of Theorem 169.3 with any compact open additive subgroup $L \subset E_T$. For $\xi \in E_T$ and all heads p, q of length ν,*

$$\int_{E_T} \psi_T^0(\xi c) J_n((p,c),(q,c))\,\mathrm{d}c = \mathbf{1}_{L^\perp}(\xi)\mathcal{I}_\nu^\xi(p,q).$$

Proof. Start with the Fourier identity of Theorem 169.3. Expand $\langle \mathcal{J}(p,l), \mathcal{J}(q,l)\rangle$ by Definition 169.1, using integration variables $y, y' \in G_T$. The trace of the product of the two rank-one matrices, with the first adjointed, is

$$\langle W_Z(y), W_Z(y')\rangle\,\overline{\langle W_Z(l w_p^{-1} y), W_Z(l w_q^{-1} y')\rangle}.$$

In this expansion, a nonzero term forces y, l into the compact box for p, and y', l into the compact box for q, by Corollary 168.3. A failing minor makes every term zero. In the remaining case the integrand is bounded by Proposition 160.4 and the product of the compact boxes has finite measure. All changes below therefore apply to the absolute integral as well.

Set $y' = yx$ using left invariance of μ, and write $y = ba$ with $\mathrm{d}\mu = \mathcal{M}_n * \mathfrak{m}_\nu$ from Lemma 179.3. Change l to $t = lb$ by right invariance of $\mathcal{M}_n$ in Theorem 166.2. Both b and t lie in G_n, commuting with w_p, w_q. The phase is $\psi_T^0(\xi[b^{-1}z, t^{-1}z])$ since $l = tb^{-1}$. The two pairings now have arguments

$$(ba, bax), \qquad (t w_p^{-1} a, t w_q^{-1} a x).$$

Independently in b, t, use $\mathcal{M}_n = \mu_H * \theta_n$ of Definition 166.1. The left H-factors fix z in the phase, and the respective averages of the two pairings are $K_n(\sigma a, \sigma a x)$ and $K_n'(t' w_p^{-1} a, t' w_p^{-1} a x)$, with σ, t' now distributed by θ_n. Indeed $d_* w_p^{-1} = w_q^{-1}$ and d_* commutes with G_n. Thus the normalized integral is exactly $\mathbf{1}_{L^\perp}(\xi)\mathcal{I}_\nu^\xi(p,q)$ by Definition 180.1. Integrability in c was proved in Theorem 169.3. □

Theorem 185.3. *At all sufficiently large arithmetic scales take T, S, Z of Theorem 167.3. For every compact open additive subgroup $L \subset E_T$, use $J_0, \ldots, J_\nu$ of Lemma 185.1 and the kernel J_n of Theorem 169.3, $n = \nu + 1$. They satisfy Definition 54.2 over $C = E_T$ with length parameter ν, additive character ψ_T^0, $I(z) = g^{-1}z^{-1}$, bounds $\beta_T, \Lambda_{T,j}$ of Definition 182.1, and*

$$E_0 = (1 + b^{-6})b^{-6k|S|}, \qquad J_0 = \frac{1}{m_Z}\int_{G_T} |c_Z(x)|^2\,\mathrm{d}\mu(x).$$

The top common-entry quadratic form is the middle-orbit integral of Corollary 169.4.

Proof. By the recurrence and the empty lists, $B_0^0(1,x) = |K_0(1,x)|^2 = |c_Z(x)|^2$, using Proposition 178.2. The preceding measure is unit mass at one. Hence the formula for J_0 follows from Definition 180.1. It equals $\mathcal{E}_T(Z)$ by Proposition 161.3, giving the bottom bound by Theorem 177.3. Lemmas 185.1 and 185.2 give absolute integrability and the required values at zero frequency, since $0 \in L^\perp$. Replacing zero components by one changes the integrand only on an additive null set by Theorem 169.3. At invertible frequencies Theorem 182.3 bounds the Fourier integrals at every level, including the last, since the factor $\mathbf{1}_{L^\perp}$ has modulus at most one. Theorem 184.2 bounds $J_j = \mathcal{I}_j^0$ at each required inverse difference. Theorem 169.3 supplies the continuity and positive semidefiniteness of the top kernel. Its common-entry formula gives the asserted orbit integral. □

186. The rational pivot sum for the constructed kernels

Lemma 186.1. *Let $C = \prod_{\alpha\in I} C_\alpha$ be a finite product of the local scalar algebras in Definition 54.2, with product additive measure and character, and componentwise maps $u, I(\,\cdot\,)$ as in that definition. At each factor let $J_{j,\alpha}$ satisfy the three kernel conditions with length ν and bounds $E_{0,\alpha}, \beta_\alpha, \Lambda_{\alpha,j}$. Then the kernels*

$$J_j(p,q) = \prod_{\alpha\in I} J_{j,\alpha}(p_\alpha, q_\alpha)$$

satisfy those conditions on C, with the products of the corresponding bounds.

Proof. At fixed heads the absolute value of each common-entry function factors, and every factor is integrable. Integration of the nonnegative product gives the product of its finite integrals by Lemma 123.1. The same lemma evaluates the complex Fourier integral as the product of its local Fourier integrals. At zero this is the preceding product kernel. At a unit frequency every local component is a unit. The same observation applies to the difference of the two inverses in the last condition, giving both product bounds. Multiplication gives the bottom inequality and preserves continuity at the top.

For positivity, represent each finite matrix on a common slice as a Gram matrix. Explicitly give the formal span of its column indices the sesquilinear pairing with that matrix. Quotient by the vectors of zero square length. They pair to zero with every vector by positivity of the quadratic polynomial obtained by adding any complex multiple of a second vector, so the quotient has a positive inner product. In an orthonormal basis there, obtained by successive orthogonal subtraction and normalization, the images of the index vectors have the prescribed pairings. For each index take the array of products of its vector coordinates over all factors. Summing products of its conjugate coordinates against the array for a second index factors into the product of the prescribed pairings. The resulting product matrix is therefore itself a Gram matrix, proving positivity. □

Theorem 186.2. *Take the arithmetic scales and $l_{\tau i}$ of Lemma 55.1, write $n = \nu + 1$, and let T contain all places above g, taking both conjugate components over split places. For S a subset of the marked places and a prescribed vector stabilizer as in Theorem 167.3, choose any compact open additive subgroup $L \subset E_T$. Use as J_i the product of the f real kernels of Theorem 159.2 and the finite kernel of Theorem 185.3. For a finite set A of rational T-units in $E^\times$ and a subset $A' \subset A$, form $D_i(A')$ and C_{i+1} from these kernels by Theorem 55.2. Uniformly in these choices and throughout the arithmetic window, for $0 \le i \le \nu$ and all sufficiently large x,*

$$0 \le \frac{D_i(A')}{C_{i+1}} \le \frac{1 + b^{-6} + b^{-8}}{\mathcal{D}\, b^{6k|S|}}.$$

Proof. The product character is exactly that of Corollary 53.4, by Definition 160.1; the inverse map at the finite component is $v \mapsto g^{-1}v^{-1}$. Apply Lemma 186.1. At unit frequencies use Theorem 159.2 with $\delta = 1$ and Definition 182.1. The product of all the scalar powers of

b in the frequency bound is $b^{14n^9 f}$, since each real factor contributes $13n^9$ to the exponent and the finite product contributes $n^9 f$. The other factors are exactly

$$m_T(\xi_T)^{-(k-n^7)/2} D_T(\xi_T)^{-6} \prod_{\tau} C_{\mathrm{B}}\, d_{k-n^7,6}(|\xi_\tau|^{-1}).$$

For a rational prefix product z_j in Theorem 55.2, the finite head weight is zero if any z_j is nonintegral at T. Otherwise $D_T(z_j) = 1$ for every prefix, and its finite weight factor is $G_T(z_j)^{n^6/2}$ by Definition 182.1. The real factors are exactly $d_{n^6,6}(|L_{\tau j}\tau z_j|^{-1})$ from Definition 157.1. Thus both hypotheses of Theorem 55.2 hold with its $h = n$, $B_\tau = C_{\mathrm{B}}$. Its sums are nonnegative since each common-entry sum is a nonnegative quadratic form at the top and successive common-entry integrals preserve nonnegativity, as proved in Theorem 54.3.

Use Theorem 56.3 with $\eta = 1/2$. It bounds the error by $\mathcal{D}^{-1} b^{-12kf}$. The product bottom bound is $\mathcal{D}^{-1} b^{-6k|S|}(1 + b^{-6})(1 + b^{-10})^f$, by the two local theorems and Definition 56.1. Here $|S| \le f$ by the split-place description in Proposition 66.1. Also $f \le x^{12} \le b$ eventually by Proposition 12.2. Thus $(1 + b^{-10})^f \le \exp(fb^{-10}) \le 1 + 2b^{-9}$ eventually, using the exponential series. Relative to $\mathcal{D}^{-1} b^{-6k|S|}$, the error is at most b^{-6kf}. For sufficiently large x, $2b^{-9}(1 + b^{-6}) + b^{-6kf} \le b^{-8}$. Adding proves the bound. □

187. The weighted Siegel inequality for finite correlation entries

Definition 187.1. Fix a finite index set $\mathcal{A}$, heads $p_i \in (E_T^\times)^\nu$ for $i \in \mathcal{A}$, and a positive semidefinite matrix $R = (R_{ij})_{i,j\in\mathcal{A}}$. For a coordinate $d \in D_Z \times D_Z$ of the correlation matrix of Definition 169.1, let

$$v_d(l) = (\mathcal{J}(p_i, l)_d)_{i\in\mathcal{A}}, \qquad Q_R(l) = \sum_d v_d(l)^* R v_d(l) \quad (l \in G_n), \qquad H_0 = H \cap K_g.$$

For $e = l^{-1}z$ define $E_R(e) = \int_H Q_R(hl)\,\mathrm{d}\mu_H(h)$; set $E_R(e) = 0$ elsewhere on the quadric of norm $-N$.

Lemma 187.2. *The function E_R is well-defined, nonnegative, compactly supported and locally constant on the entire quadric. It is invariant under H_0. For every compact open additive subgroup $L \subset E_T$ and $t \in E_T$ one has*

$$\int_{[z,e]\in t+L} E_R(e)\,\mathrm{d}\delta_{-N}^{[n]}(e) = m_Z|L| \sum_{i,j\in\mathcal{A}} R_{ij} J_n((p_i, t), (p_j, t)).$$

Proof. By Lemma 169.2, Q_R is locally constant and supported on a compact subset of G_n; positivity of R gives $Q_R \ge 0$. Two representatives of e differ by left multiplication by H. Right invariance of μ_H and its finiteness on compact sets show that E_R is independent of the representative and finite. There is a single open subgroup V of G_n such that $Q_R(lv) = Q_R(l)$ for all $v \in V$. Indeed its nonzero set is compact and open. Cover it by finitely many right neighborhoods $l_i V_i$ with V_i open subgroups and Q_R constant there. Such subgroups form a basis at one by the congruence neighborhoods in Lemma 161.1. Their intersection works on the union and preserves its complement. It follows that $l \mapsto \int_H Q_R(hl)\,\mathrm{d}\mu_H(h)$ is locally constant as well.

The chart in Proposition 165.3 contains $z = N\eta v - v'$, whose last coefficient is -1. Its formula gives $\sigma(z) = 1$. Around $e_0 = l_0^{-1}z$ in the quadric, $l(e) = \sigma(l_0 e) l_0$ is therefore a continuous section, with $l(e_0) = l_0$. This proves local constancy at each represented point. All nonzero values are on the compact image of $\mathrm{supp}(Q_R)$ under $l \mapsto l^{-1}z$. This image is closed and consists of represented points. Its complement has zero value, proving the support and continuity assertions everywhere.

For $h_0 \in H_0$, the Bessel vector satisfies $W_Z(h_0 y) = W_Z(y)$. Indeed h_0 normalizes U_T, fixes the frame pairs and z, and preserves the simple-root sum there. In each coordinate $\phi_g(h_0 y a^{-1})$, the factor $h_0 \in K_g$ can be moved past the unipotent in the support $U_T K_g$; the

phase is unchanged. Its inverse gives the same support in the opposite direction, proving the identity by Definition 160.3. It follows that $\mathcal{J}(p,h_0l)=\mathcal{J}(p,l)$. Substituting $y=h_0y'$ in the matrix integral and using commutation with w_p also gives $\mathcal{J}(p,lh_0^{-1})=\mathcal{J}(p,l)$. The second identity shows $E_R(h_0e)=E_R(e)$. Finally expansion of Q_R gives $\sum_{i,j}R_{ij}\langle\mathcal{J}(p_i,l),\mathcal{J}(p_j,l)\rangle$. The group disintegration $\mathcal{M}_n=\mu_H*\theta_n$ and the pushforward in Proposition 165.3 give the displayed formula by Theorem 169.3. □

Proposition 187.3. *For a represented point e fix a finite list of pairs (l_θ,d_θ) with $l_\theta^{-1}z=e$ and $d_\theta\in D_Z^2$. Suppose all pairs (H_0l_θ,d_θ) are distinct. Then*

$$\mu_H(H_0)\sum_\theta v_{d_\theta}(l_\theta)^*Rv_{d_\theta}(l_\theta)\le E_R(e).$$

The same lower bound holds on the H_0-translates of a neighborhood of e in the norm quadric.

Proof. Fix l over e and put $h_\theta=l_\theta l^{-1}\in H$. For each fixed coordinate d, those right cosets H_0h_θ having $d_\theta=d$ are pairwise disjoint. On each, the d-term of $Q_R(hl)$ equals the term at l_θ by the left H_0-invariance of the matrix correlations proved in Lemma 187.2. Its integral over that coset is the term times $\mu_H(H_0)$, by right invariance. Use nonnegativity to retain these cosets for each d, then sum over coordinates. This proves the first assertion. Local constancy and H_0-invariance in the same lemma prove the second. □

Theorem 187.4. *Suppose T contains all places above gN. Take $\mathcal{S},t$ and β as in Theorem 67.3. At the point $e_{c,y}=P(t,(h_c^{-1}y)_T)$ of each $(c,y)\in\mathcal{S}$ choose a list as in Proposition 187.3, allowing an empty list. Use one R and one list of heads for all pairs. Write $G(t)=\prod_\tau(|\tau t|^2-N^2)^{k-1}$. For all sufficiently small compact open additive subgroups L of E_T,*

$$\mu_H(H_0)\sum_{(c,y)\in\mathcal{S}}\sum_\theta v_{d_\theta}(l_\theta)^*Rv_{d_\theta}(l_\theta)\ \le|\mathcal{C}(K_c)|c_*P_T(\beta)G(t)\,m_Z\cdot\sum_{i,j\in\mathcal{A}}R_{ij}J_n((p_i,t),(p_j,t)).$$

The double sum on the right is real and nonnegative.

Proof. Use the individual bounds from Proposition 187.3 as weights, with weight zero for an empty list. The image of the local column level in the middle group is H_0 by Proposition 66.1. To apply the ambient neighborhood condition in Theorem 67.3, extend E_R from the closed quadric to the rest of the middle space by the largest of the finitely many weights, taking zero for an empty collection. This extension is Borel and invariant under H_0. Each relative neighborhood on which the weight is attained or exceeded is the intersection of an open ambient set with the quadric. Thus that ambient inequality holds there and after translation by any element of H_0. Theorem 67.3 and the integral identity in Lemma 187.2 give the assertion after cancellation of $|L|>0$. The same identity proves the last statement. □

188. The unipotent expansion for the cut Poincaré columns

Definition 188.1. Use $G_\infty,G_f,\Gamma,K_Y^0,K_Y,h_c$ of Definition 21.3; the rational unipotent group is $U(F)$, $F=E^{\mathfrak{c}}$. For $\gamma\in\Gamma$ write γ_∞,γ_f for the archimedean matrix in the scaled basis and the finite adelic matrix. Set

$$V_\infty(y)=\bigotimes_\tau V_\tau(y_\tau),\qquad A_c(u)=h_c^{-1}u_fh_c,\qquad\Delta_{c,c'}(\gamma)=h_c^{-1}\gamma_fh_{c'}.$$

Here V_τ is the bounded continuous cut vector of Theorem 124.2. The tensor space is given the product orthonormal basis. A finite matrix $x\in G_f$ is integral off T if it preserves the specified row lattice at every place outside T. For $k_0\in K_Y^0$ with image $s\in\mathcal{G}$, let $\mathcal{T}(c,s,c')=\{\gamma\in\Gamma:\Delta_{c,c'}(\gamma)\in k_0K_Y\}$.

Proposition 188.2. *For every column class c there is a nonnegative Borel function $\beta_c \le 1$ on G_∞ such that $\phi_c = \beta_c V_\infty$ is continuous and compactly supported and $\sum_{\delta\in\Gamma_c} \phi_c(\delta_\infty y) = V_\infty(y)$. Periodizing the coordinates of ϕ_c at $h_c K_Y$ as in Theorem 21.5 gives continuous square-integrable automorphic functions. For two classes put*

$$M_{c,c'}(\gamma) = \int_{G_\infty} \phi_c(y)\phi_{c'}(\gamma_\infty^{-1} y)^* \, \mathrm{d}y.$$

For $s \in \mathcal{G}$ their matrix of translated inner products in that theorem is

$$\sum_{\gamma\in\mathcal{T}(c,s,c')} M_{c,c'}(\gamma).$$

Entry (a, a') pairs the translate of the a'-th function for c' against the a-th function for c. Only finitely many rational matrices in this sum can have anywhere nonzero integrand.

Proof. Under $u \in U_\infty$ the transformation law in Theorems 122.2 and 124.2 gives $V_\infty(uy) = \chi_\infty(u)V_\infty(y)$. Its support lies in $U_\infty C$ for a compact C by the last assertion of the truncation theorem, taking the finite product of the compact representatives in its proof. The cutoffs in Theorem 21.5 therefore apply simultaneously to all its coordinates. In that construction $\beta_c = \xi_c / \sum_{\delta\in\Gamma_c} \xi_c(\delta_\infty \cdot\,)$ where the denominator is positive, and zero elsewhere. The identity term gives $\beta_c \le 1$. A change of index shows $\sum_{\delta\in\Gamma_c} \beta_c(\delta_\infty y) = 1$ throughout the support of V_∞. The character is one on Γ_c by Proposition 21.2. These facts prove the periodization identity and all the asserted regularity properties by Theorem 21.5. Continuous functions of compact support on the quotient are square-integrable by the local fundamental-domain measure of Lemma 21.4.

For the finiteness assertion put $B_c = \operatorname{supp}\phi_c$. If the integrand is nonzero at y, then $\gamma_\infty \in B_c B_{c'}^{-1}$, which is compact. The finite matrix belongs to $h_c k_0 K_Y h_{c'}^{-1}$, also compact. The bounded denominator and bounded conjugate argument of Lemma 21.4 gives finitely many rational matrices in their product. The inner product formula in Theorem 21.5 now gives the stated matrix entry by entry. □

Lemma 188.3. *Let T contain every place above g. For every $x \in G_f$ integral off T and $y \in G_\infty$,*

$$\sum_{\substack{u\in U(F)\\ A_c(u)x\in K_Y}} \phi_c(u_\infty y) = \phi_g(x_T)V_\infty(y).$$

The sum has only finitely many nonzero terms at each y.

Proof. If $A_c(u)x \in K_Y$, then at T its component lies in K_g, so $\phi_g(x_T) \ne 0$. Conversely, if $\phi_g(x_T) \ne 0$, some $v \in U_T$ has $vx_T \in K_g$ by Definition 160.3. Extend v by the identity off T. Then $vx \in K_Y$, since x was integral there. The open condition $v'x \in K_Y$ on the finite adelic unipotent group therefore holds on a nonempty open set. Conjugation by h_c is a homeomorphism of this group because it acts on the middle space fixing the distinguished vector and all the outer pairs. Lemma 21.1 gives $u_0 \in U(F)$ with $A_c(u_0)x \in K_Y$. All such elements form $\Gamma_c u_0$: their pairwise quotients satisfy $A_c(u)A_c(u_0)^{-1} \in K_Y$, and conversely multiplication on the left by any element of Γ_c preserves the level condition by Definition 21.3.

In that case $A_c(u_0)$ is integral off T, since both $A_c(u_0)x$ and x^{-1} are integral there. The simple-root sum is unchanged by conjugation by h_c, which fixes each frame vector and its dual and fixes z. Its character is trivial on the integral unipotent group outside T since g is a unit there. By Proposition 21.2 and the phase identity of Proposition 160.4,

$$\chi_\infty((u_0)_\infty) = \chi_f((u_0)_f)^{-1} = \chi_g(A_c(u_0)_T)^{-1} = \phi_g(x_T).$$

Summing the left side of the assertion over $\Gamma_c u_0$ now gives $V_\infty((u_0)_\infty y)$ by Proposition 188.2, the desired right side. The sum is locally finite in y because ϕ_c has compact support and Γ_c preserves the finite coset $h_c K_Y$, by Lemma 21.4. In the remaining case both sides vanish by the support criterion already proved. □

189. Unfolding one rational unipotent double coset

Definition 189.1. Let $G_{\mathrm{mid}}(F)$ be the group of rational isometries fixing each e_j, e_j^*, acting on the middle space of Definition 15.1. For a rational head $p = (p_1, \dots, p_\nu) \in (E^\times)^\nu$ use the rational flip w_p with the same pair matrices as in Definition 165.1. For $\gamma = w_p\sigma$, $\sigma \in G_{\mathrm{mid}}(F)$, define

$$B^T_{c,s,c'}(\gamma) = \sum M_{c,c'}(u\gamma v^{-1}),$$
$$\mathsf{T}_\infty(\gamma) = \int_{G_\infty} V_\infty(y) V_\infty(\gamma_\infty^{-1} y)^* \, \mathrm{d}y.$$

The sum ranges over $u, v \in U(F)$ such that $A_c(u)$ is integral off T and $u\gamma v^{-1} \in \mathcal{T}(c, s, c')$. Write $B_{c,s,c'}(\gamma)$ for the same sum with the exterior integrality condition omitted.

Lemma 189.2. *For fixed $\gamma = w_p\sigma$ the map $(u, v) \mapsto u\gamma v^{-1}$ on $U(F)^2$ is injective. Both sums in Definition 189.1 have only finitely many terms with anywhere nonzero integrand. Given finitely many choices of c, s, c', γ, one has $B^T_{c,s,c'}(\gamma) = B_{c,s,c'}(\gamma)$ for all of them as soon as T contains a sufficiently large fixed finite set. The integral defining $\mathsf{T}_\infty(\gamma)$ converges absolutely, and equals the tensor product of the single-place matrix correlations.*

Proof. The middle group normalizes U, by the flag equations of Definition 20.3. An element of $U \cap w_p U w_p^{-1}$ preserves both the flag of the e_j and the flag of the e_j^*, with diagonal entries one on each. It preserves the two spans and their common orthogonal complement, the middle space. The matrices on the two spans are upper triangular in their listed orders, and each is the inverse conjugate transpose of the other by the pairing. They are therefore diagonal and equal to the identity. The induced middle action is already the identity for U. It follows that the whole intersection is trivial. Equality of two products $u\gamma v^{-1}$ puts the quotient of their left unipotents in this intersection, proving injectivity. Proposition 188.2 gives the asserted finiteness of the rational matrices that can contribute. For their finitely many corresponding u's each $A_c(u)$ is integral off finitely many places, by the definition of the finite adeles. Taking the union proves stabilization.

At τ, conjugation by the diagonal basis change of Proposition 20.5 sends the j-th flip to $w_j(s^2_{j-1,\tau}\tau p_j)$: apply it to the scaled pair. It sends σ to a middle isometry. The integral of the norm product of the two finite-dimensional vectors is finite at each place by the compact support-intersection argument in Theorem 134.2. Finite product integration using Lemma 123.1 gives absolute convergence and the tensor product formula. □

Theorem 189.3. *Let T contain the places above g, and suppose $\Delta = \Delta_{c,c'}(w_p\sigma)$ is integral off T. Choose $k_0 \in K_Y^0$ of class s with $(k_0)_T \in Z$. Let $a, b' \in D_Z = K_g\backslash Z$ satisfy $a(b')^{-1} = (k_0)_T \bmod K_g$, using the quotient group, and let p_T be the head evaluated at T. Then with $l = (h_c^{-1}\sigma_f h_{c'})_T^{-1} \in G_n$,*

$$\mathcal{J}(p_T, l)_{a,b'} \, \mathsf{T}_\infty(w_p\sigma) = H(p_T)\mu(K_g) \, B^T_{c,s,c'}(w_p\sigma).$$

Proof. Write $\gamma = w_p\sigma$. Since $h_c, h_{c'}$ act on the middle space, $\Delta_T^{-1} = l w_{p_T}^{-1}$. In the defining integral for the (a, b') entry of $\mathcal{J}/H(p_T)$, let the group variable be $x_T a$, using any representative of the coset and right invariance of μ. Right K_g-invariance of ϕ_g in Proposition 160.4 gives the integrand $\phi_g(x_T)\overline{\phi_g(\Delta_T^{-1} x_T (k_0)_T)}$. Let x_f be x_T at T and one elsewhere, and put $x'_f = \Delta^{-1} x_f k_0$. Both are integral off T. Multiply the last integrand by $V_\infty(y)V_\infty(\gamma_\infty^{-1}y)^*$. Lemma 188.3 in the two arguments identifies this product with the sum over $u, v \in U(F)$ of

$$\mathbf{1}_{K_Y}(A_c(u)x_f) \, \mathbf{1}_{K_Y}(A_{c'}(v)x'_f)\phi_c(u_\infty y)\phi_{c'}(v_\infty \gamma_\infty^{-1} y)^*.$$

Put $\delta = u\gamma v^{-1}$. Its finite position is $A_c(u)\Delta A_{c'}(v)^{-1}$. When $J = A_c(u)x_f \in K_Y$, the other finite condition is $\Delta_{c,c'}(\delta)^{-1} J k_0 \in K_Y$. Since k_0 normalizes K_Y, this is equivalent to $\Delta_{c,c'}(\delta) \in k_0 K_Y$, independent of J. The first condition requires $A_c(u)$ integral off T; if this holds, its allowed x_T's form the single left translate $A_c(u)_T^{-1} K_g$ of mass $\mu(K_g)$. Change y to

$u_\infty y$ by left invariance. The archimedean matrix integral becomes $M_{c,c'}(\delta)$. By Lemma 189.2, there are finitely many pairs for which this integrand can take a nonzero value with both finite conditions. For each, its finite set of integration is a compact coset and its archimedean integrand is bounded with compact support. Thus one may integrate the expansion term by term. The separated left side is absolutely integrable by that lemma and by the compact estimate in Lemma 169.2. Integrating proves the claimed identity. □

190. Separation and multiplicity of the rational middle terms

Lemma 190.1. *Let T contain all places above gN, and fix a rational t with $|\tau t| > N$ for all τ. Take a finite family of distinct tuples $(c_i, s_i, c_i', \sigma_i)$, with column classes c_i, c_i', lifts $k_i \in K_Y^0$ of s_i, $(k_i)_T \in Z$, and $\sigma_i \in G_{\mathrm{mid}}(F)$ satisfying $[z, \sigma_i z] = t$. Assume $\Delta_i = h_{c_i}^{-1}(\sigma_i)_f h_{c_i'}$ integral off T. Write*

$$\sigma_i z = \sum_{a=1}^{k} y_{i,a} x_a - \frac{t}{N} z, \qquad y_i = (y_{i,a})_a, \qquad l_i = (\Delta_i)_T^{-1}.$$

For each $a \in D_Z$, let $d_{i,a} = (a, \bar{k}_i^{-1} a)$ where $\bar{k}_i$ denotes $(k_i)_T \bmod K_g$. Then (c_i, y_i) satisfies the conditions of Theorem 67.3, and $l_i^{-1} z = P(t, (h_{c_i}^{-1} y_i)_T)$. In each fibre of $i \mapsto (c_i, y_i)$ the pairs $(H_0 l_i, d_{i,a})$, indexed by i, a, are distinct.

Proof. The coefficient of z follows from its norm $-N$. Taking norms gives $H(y_i) = (t\bar{t} - N^2)/N$. The representatives h_c fix z and act on its perpendicular in the middle space by the definite matrix. This gives the formula for $l_i^{-1} z$ using Definition 67.1. Off T, the integral middle basis differs from the rational one by a matrix integral in both directions since N is a unit and $\pi_N \bar{\pi}_N = N$. The vector $\Delta_i z$ is integral, since Δ_i preserves the lattice. Taking its definite coordinates therefore gives precisely the exterior integrality required of $h_{c_i}^{-1} y_i$.

Suppose two pairs agree in a fibre. Their first coordinate in $d_{i,a}$ gives $a = a'$ and the second gives $\bar{k}_i = \bar{k}_{i'}$. Since all places of the global level lie in T, $s_i = s_{i'}$ by Proposition 15.2. Also $c_i = c_{i'}$ and $\sigma_i z = \sigma_{i'} z$. Consequently $\delta = \sigma_i^{-1} \sigma_{i'}$ is a rational definite isometry extended by the identity on z and all outer pairs. At T,

$$(h_{c_i'}^{-1} \delta_f h_{c_{i'}'})_T = l_i l_{i'}^{-1} \in H_0.$$

Outside T this position equals $\Delta_i^{-1} \Delta_{i'}$ and preserves the row lattice. It therefore belongs globally to the extended K_c, by Proposition 66.1. The two right classes coincide, $c_i' = c_{i'}'$, and the representatives for that class are the same. Its rational stabilizer at K_c is trivial by Proposition 13.5. This gives $\delta = 1$ and $\sigma_i = \sigma_{i'}$. Distinctness of the original tuples yields $i = i'$, as claimed. □

Lemma 190.2. *In the preceding setting write u, S_0 as in Definition 68.1. One has the exact identity*

$$\frac{m_Z}{\mu_H(H_0)\,|D_Z|\,\mu(K_g)^2} = S_0 u.$$

Proof. By Definition 161.2, $m_Z = \mu(Z)\mu(K_g)/v_g^2$. Partitioning Z into its congruence cosets gives $\mu(Z) = |D_Z|\mu(K_g)$. Since T contains all places above g, the explicit volume in Lemma 160.2 is $v_g = u^{-1}$. The measure on H gives $H \cap K_n$ volume one by Definition 166.1. Reduction on $H \cap K_n$ has kernel H_0 and image of order $|\bar{H}|$, as in Proposition 66.1: all reductions at the marked pairs lift independently, and the other components may be set to the identity. Thus $\mu_H(H_0) = |\bar{H}|^{-1}$. Substitute and use $S_0 = |\bar{H}| u$ from the same proposition. □

Theorem 190.3. *With the tuples of Lemma 190.1, fix a finite family $\mathcal{A}$ of rational heads p of length ν. Assume the positions $\Delta_{c_i,c_i'}(w_p \sigma_i)$ are integral off T. Suppose $\rho_p, r_0 > 0$ satisfy $\rho_p H(p_T) = r_0 \sqrt{G(t)}$ for every p, and that the Gram matrices of*

$$\left(\rho_p \, \mathsf{T}_\infty(w_p \sigma_i)\right)_{p \in \mathcal{A}}$$

in the Hilbert–Schmidt inner product are the same matrix R for every i. Then for all sufficiently small compact open additive subgroups $L \subset E_T$,

$$r_0^2 \sum_i \left\| \sum_{p \in \mathcal{A}} B^T_{c_i, s_i, c_i'}(w_p \sigma_i) \right\|_{\mathrm{HS}}^2 \le P_{\mathrm{pkt}}\, P_T(\beta) \sum_{p,q \in \mathcal{A}} R_{pq} J_n((p_T, t), (q_T, t)),$$

where P_{pkt} is the constant of Definition 68.1. Here t in the finite kernel denotes its image at T.

Proof. Take the distinct pairs (c_i, y_i) in Theorem 187.4. Use in each fibre the representatives l_i with every coordinate $d_{i,a}$ supplied by Lemma 190.1. The matrix R is positive semidefinite as a Gram matrix. The scalar in that theorem at i, a is the squared norm of

$$\sum_{p \in \mathcal{A}} \mathcal{J}(p_T, l_i)_{d_{i,a}}\, \rho_p \mathsf{T}_\infty(w_p \sigma_i).$$

The classes $a, b' = \bar{k}_i^{-1} a$ satisfy $a (b')^{-1} = \bar{k}_i$ in the finite quotient. By Theorem 189.3 the display is

$$\mu(K_g)\, r_0 \sqrt{G(t)} \sum_{p \in \mathcal{A}} B^T_{c_i, s_i, c_i'}(w_p \sigma_i).$$

This quantity is independent of a, giving the multiplicity $|D_Z|$. Cancel $G(t) > 0$ from the Siegel inequality and use Lemma 190.2. The remaining constant is $|\mathcal{C}(K_c)| c_* S_0 u = P_{\mathrm{pkt}}$, as asserted. □

191. The squared mass of the rational open-cell blocks

Lemma 191.1. *For every fixed $D > 0$, the bound of Theorem 186.2 holds with $1 + b^{-D}$ in place of $1 + b^{-6} + b^{-8}$, at all sufficiently large scales depending also on D.*

Proof. On the unselected finite places, Theorems 177.3 and 166.2 give bottom value at most $1 + 2^{-5n} + 2^{-n}$. At a selected place the proof of Theorem 167.3 bounds the principal contribution by $4 g_0^{-k}$; the remainder is at most $g_0^{-9k/10}$ by Theorem 177.2. Their sum is at most b^{-6k} for large scales since $g_0 \ge b^{10}$. Multiply using Proposition 167.2. The factor $2^{-5n} + 2^{-n}$ is $o(b^{-D})$, since $n \ge \sqrt{x}/2$ and $\log b = x^{\epsilon}$, $\epsilon < 1/100$ by Proposition 10.5. At the real places Theorem 155.3, with $\eta = 1/2$, gives an additional factor bounded by $(1 + b^{-9r^2/2})^f = 1 + o(b^{-D})$, using $f \le b$, $r \to \infty$ and the exponential series. Thus relative to $\mathcal{D}^{-1} b^{-6k|S|}$ the product bottom is at most $1 + o(b^{-D})$. The frequency error in Theorem 56.3 with $\eta = 1/2$ is at most b^{-6kf} relative to the same quantity, as in Theorem 186.2. This too is $o(b^{-D})$, proving the claim uniformly in all the choices. □

Lemma 191.2. *For $l_{\tau a} = s_{a,\tau}^2$ and $C_n = \prod_{a=0}^{\nu} (V_E \prod_\tau l_{\tau a}^2)^{-1}$ put*

$$r_0 = \frac{C_{\mathrm{op}}^{f/2} Z_{\mathrm{sc}}}{J_c \prod_{\tau, a=0}^{\nu} l_{\tau a}}, \qquad \rho_p(t) = \prod_\tau \omega_\tau((l_{\tau, j-1} \tau p_j)_{1 \le j \le \nu}, l_{\tau,\nu} \tau t).$$

The product in the denominator of r_0 ranges over all τ and $0 \le a \le \nu$; J_c, Z_{sc} are from Definition 68.1. If every p_j is a rational T-unit and $|\tau t| > N$ for all τ, then

$$\rho_p(t) H(p_T) = r_0 \sqrt{G(t)}, \qquad \frac{P_{\mathrm{pkt}} C_n}{r_0^2\, \mathfrak{v}^2\, \mathcal{D}} = \frac{P_{\mathrm{pkt}}}{Z_{\mathrm{pkt}}} E_2.$$

For any $\sigma \in G_{\mathrm{mid}}(F)$ with $[z, \sigma z] = t$, the Gram matrix of $(\rho_p(t) \mathsf{T}_\infty(w_p \sigma))_p$ is exactly the product of the real top kernels at these scaled heads with common last entry $l_{\tau,\nu} \tau t$.

Proof. In the scaled basis the last frame pairing is $s_{\nu,\tau}^2 \tau t = \tau t / N$, and the other flips acquire the scales in Lemma 189.2. The top Gram identity in Theorem 134.2 and the tensor product formula of that lemma give the final assertion since the trace of a tensor product is the product of the traces, by diagonal entries in a product basis.

In Definition 134.1 the factor at τ is $\sqrt{C_{\mathrm{op}}}/\dim \pi_\tau$ times

$$N^{-(k-1)}(|\tau t|^2 - N^2)^{(k-1)/2} \prod_{j=1}^{\nu} (l_{\tau,j-1}|\tau p_j|)^{d_j}.$$

By Proposition 122.1, π_τ is the symmetric power of degree a at the selected embeddings and degree zero elsewhere, so $\prod_\tau \dim \pi_\tau = J_c$ by the monomial count of Proposition 13.5. By the product formula in Lemma 52.1, $m_T((p_j)_T) \prod_\tau |\tau p_j|^2 = 1$; all finite components off T are units. Thus $H(p_T)$ cancels the powers of the rational pivots. For $a < \nu$, $d_{a+1} = k+2(\nu-a)-1$, and $N^{-(k-1)} = l_{\tau,\nu}^{k-1}$. The remaining scale product is $Z_{\mathrm{sc}}/\prod_{\tau,a} l_{\tau a}$, giving the first identity. Finally $C_n = V_E^{-n}(\prod_{\tau,a} l_{\tau a})^{-2}$. Insert the formulas for Z_{pkt} and E_2 from Definitions 68.1 and 69.1 to obtain the second identity. □

Theorem 191.3. *Fix S and its vector conditions as in Theorem 167.3. Let $\mathcal{I}$ be any finite set of distinct $(c_i, s_i, c_i', \sigma_i)$ with $s_i \in \mathcal{G}$ satisfying those conditions and $\sigma_i \in G_{\mathrm{mid}}(F)$ such that, for $t_i = [z, \sigma_i z]$, every $|\tau t_i| > N$. For every fixed $D > 0$, uniformly in $\mathcal{I}, S$, at all sufficiently large scales,*

$$\frac{1}{\mathfrak{v}^2} \sum_{i \in \mathcal{I}} \Big\| \sum_{p \in (E^\times)^\nu} B_{c_i, s_i, c_i'}(w_p \sigma_i) \Big\|_{\mathrm{HS}}^2 \le (1 + b^{-D}) E_2 \, b^{-6k|S|}.$$

The sum over p is finite after deleting its zero terms. The bound also holds for the sum over all such tuples, defined as the supremum of finite subsums.

Proof. Fix one tuple. If a rational matrix δ occurs for a head p, its frame matrix $([e_a, \delta e_b])_{a,b \le \nu}$ has initial minor of order j equal to $\prod_{a \le j} p_a$. Indeed $\delta = u w_p \sigma v^{-1}$, and in the two ordered bases of Theorem 168.2 its first j rows and columns have the two unitriangular factors and diagonal $p_1, \dots, p_j$. This computation is over E and uses no integrality. Consecutive minor ratios thus recover the entire head. By Proposition 188.2 only finitely many δ can have nonzero integrand, giving the assertion about the pivot sum.

Choose a finite set $A \subset E^\times$ containing all entries of these contributing heads for the finite family, as well as all t_i. The empty family already satisfies the assertion. Sum over the full A^ν for each tuple. Choose T containing all places above gN, the zeros and poles of A, and the finitely many places required in Lemma 189.2 for stabilization for every tuple and every head in A^ν. Enlarge it also to ensure the exterior integrality of all their finite middle and flipped positions, possible for a finite list of finite adelic matrices. Take all conjugate components at each place. Lifts of the s_i have T-components in the subgroup Z of Theorem 167.3.

Group the finite family by the exact rational value $t_i = t$. Lemma 191.2 supplies the weights $\rho_p(t), r_0$ and the common matrix R for Theorem 190.3. Take a single compact open L sufficiently small for all these finitely many inequalities, using a product of sufficiently small ideal balls. For any fixed $D' > D$, Theorem 66.4 bounds every exterior factor $P_T(\beta)$ by $1 + b^{-D'}$. The right double sum in each inequality is nonnegative by Theorem 187.4. After substituting the real kernels for R, the sum of these double sums over the distinct t's is $D_\nu(A')$ of Theorem 186.2 with $A' = \{t_i\}$. There is no multiplicity in this set of last entries; all multiplicities on the left were retained within their respective Siegel sums. Lemma 191.1 bounds $D_\nu(A')/C_n$ by $(1 + b^{-D'})/(\mathcal{D} b^{6k|S|})$. Use the second identity of Lemma 191.2 and $P_{\mathrm{pkt}}/Z_{\mathrm{pkt}} \le 1 + b^{-D'}$ from Theorem 68.2. The relative factor is at most $(1+b^{-D'})^3 \le 1 + b^{-D}$ eventually by expansion. Every bound is uniform in the finite family. Taking the supremum gives the final assertion. □

192. Rational frame intersections and open double cosets

Definition 192.1. Set $f_a = e_a$ for $1 \le a \le \nu$, $f_n = z$, $n = \nu + 1$, and let $L_f = \sum_{a=1}^n E f_a$. For $\gamma \in \Gamma$ put $Z_{ab}(\gamma) = [f_a, \gamma f_b]$, and let $D_j(\gamma)$ be the leading minor of size j of $Z(\gamma)$, with

$D_0 = 1$. Write

$$\mathfrak{d}(\gamma) = D_n(\gamma)\overline{D_n(\gamma)} - N^2 D_\nu(\gamma)\overline{D_\nu(\gamma)} \in F.$$

Call the condition $D_1 \cdots D_\nu \neq 0$ the open flag condition. These symbols apply to rational matrices. The initial minors $D_j(p)$ of a pivot list are the scalar products of Definition 168.1.

Theorem 192.2. *For $1 \le b \le n$ let $c_b = (-1)^{n+b} \det Z_{\widehat{n,b}}$, deleting the indicated row and column. Let $Y \in E^k$ be the definite coordinates of $w = \sum_b c_b \gamma f_b$. Then*

$$\mathfrak{d}(\gamma) = NH(Y), \qquad \mathfrak{d}(\gamma) = 0 \quad \Longleftrightarrow \quad L_f \cap \gamma L_f \neq \{0\}.$$

In particular a nonzero discriminant is totally positive. All $D_j, \mathfrak{d}$ are invariant under left and right multiplication by $U(F)$. If $\gamma \in \mathcal{T}(c, s, c')$, its frame entries and all D_j are algebraic integers. Their reductions at the marked places depend only on s.

Proof. Expansion along the last row gives $[z, w] = D_n$. Using a repeated row in that expansion gives $[e_a, w] = 0$ for $a \le \nu$, so w has no dual coordinates. Its coefficient at z is $-D_n/N$ by Definition 15.1. Thus $[w, w] = H(Y) - D_n \bar{D}_n/N$. On the other hand, the original frame has Gram matrix $\operatorname{diag}(0, \dots, 0, -N)$, so $[w, w] = -N c_n \bar{c}_n = -N D_\nu \bar{D}_\nu$. This proves the identity. Since H is the sum of coordinate norms, it is positive at every real embedding for nonzero Y.

Suppose the first ν rows of Z are independent. Then $c = (c_b)_b$ spans their common nullspace and is nonzero. Indeed one of their minors of order ν is nonzero and the nullspace has dimension one by elimination. The equalities just proved show $w \in L_f$ exactly when $Y = 0$; and $w \neq 0$ since the γf_b are independent. Conversely any vector of $L_f \cap \gamma L_f$ is orthogonal to the first ν frame vectors, so its coordinates in (γf_b) are a multiple of c. This proves the equivalence in this case. If these rows are dependent, all $c_b = 0$ and the discriminant vanishes. Choose a nonzero v in the span of the e_a orthogonal to all γf_b using a nonzero left relation among the rows, conjugating its coefficients. Then $\gamma^{-1} v$ has no dual coordinates and no z coordinate. Its norm is zero, so definiteness forces all its x_a coordinates to be zero. Hence v lies in the span of the γe_a as well, proving the equivalence in this case.

On L_f a unipotent in U is upper unitriangular in the listed basis, by Definition 20.3; in particular the image of z differs from z by earlier frame vectors. Left and right unipotents therefore transform Z on the left by a lower unitriangular matrix and on the right by an upper unitriangular matrix. Each initial determinant is invariant. For the arithmetic assertion, all h_c fix the frame, so the finite entries for γ are precisely those of the frame matrix for $\Delta_{c,c'}(\gamma)$. That matrix preserves the row lattice and its prescribed marked reduction is s. The frame vectors and their pairings are integral in that lattice, by Proposition 15.2. Integrality at all finite places gives membership in O_E by Lemma 52.1. Reduction and determinants commute by their polynomial formulas. □

Theorem 192.3. *A rational isometry γ satisfies the open flag condition if and only if it can be written $\gamma = u w_p \sigma v^{-1}$, with $u, v \in U(F)$, $p \in (E^\times)^\nu$, $\sigma \in G_{\mathrm{mid}}(F)$. All four factors are unique, and then*

$$p_j = D_j(\gamma)/D_{j-1}(\gamma), \qquad t = [z, \sigma z] = D_n(\gamma)/D_\nu(\gamma), \qquad \mathfrak{d}(\gamma) = D_\nu(\gamma)\overline{D_\nu(\gamma)}(t\bar{t} - N^2).$$

Thus in this open flag condition a nonzero discriminant is equivalent to $|\tau t| > N$ at every infinite place.

Proof. In the asserted decomposition the matrix $Z(w_p \sigma)$ is diagonal with first ν entries $p_1, \dots, p_\nu$ and last entry t. Theorem 192.2 therefore gives all the displayed identities and necessity of the open flag condition.

For existence remove one outer pair at a time. In the first pair write $\gamma e_1 = d e_1 + y + c e_1^*$, where $c = D_1 \neq 0$ and y belongs to the intervening space. In its first Heisenberg layer send e_1^* to $e_1^* + y/c + (d/c)e_1$. This is an allowed element u_1: isotropy of γe_1 gives $d/c + \overline{d/c} = -[y/c, y/c]$, the necessary and sufficient trace condition of Lemma 20.4. The

isometry $w_1(c)^{-1}u_1^{-1}\gamma$ fixes e_1. Its induced isometry on $e_1^{\perp}/Ee_1$ is a middle matrix g'. After removal of its diagonal extension, the matrix has first column e_1, last row the last coordinate functional by isometry, and identity intervening block. The same Heisenberg calculation puts the remainder in the first layer. Thus $\gamma = u_1 w_1(c) g' v_1^{-1}$ for some v_1 there. The invariance proved above shows that each remaining leading minor in g' is the next one in γ divided by c, by the diagonal first block. Repeat this construction on g'. All factors from deeper layers commute with the earlier flips. Collect their unipotent factors on the two sides to get the assertion. In the case of zero outer pairs the middle group is already the whole group.

For uniqueness the formulas recover p. If two decompositions with this p agree, conjugating the quotient of their left unipotents by w_p^{-1} gives an element simultaneously in $w_p^{-1}Uw_p$ and $G_{\mathrm{mid}}U$. The former preserves the flag of dual vectors with diagonal one; the latter preserves the original flag with diagonal one. The proof of Lemma 189.2 forces identity on both spans. Their common orthogonal complement is preserved, and the first of these groups has identity induced middle action, giving identity everywhere. In the second group the middle projection now gives equality of the two middle factors, after which the right unipotents agree also. Finally the discriminant is totally positive whenever it is nonzero by Theorem 192.2. The last displayed identity gives the claimed equivalence at infinity. □

Corollary 192.4. *For fixed c, s, c', the cut Poincaré coefficient coming from terms $\delta \in \mathcal{T}(c, s, c')$ with the open flag condition and $\mathfrak{d}(\delta) \neq 0$ is the sum, over all $\sigma \in G_{\mathrm{mid}}(F)$ with $|\tau[z, \sigma z]| > N$ for every τ, of the matrices $\sum_p B_{c,s,c'}(w_p\sigma)$. Only finitely many terms can contribute. If at each $j \leq \nu$ the marked reductions of D_j prescribed by s include a nonzero component, the open flag condition holds for every $\delta \in \mathcal{T}(c, s, c')$.*

Proof. By Theorem 192.3 each indicated rational term determines exactly one quadruple u, p, σ, v in the definition of $B_{c,s,c'}$, and conversely every summand on the right meets the two stated conditions. The finiteness assertion of Proposition 188.2 therefore permits this regrouping of its coefficient formula. The last assertion follows from the integrality and reduction in Theorem 192.2, since a zero algebraic integer reduces to zero in every component. □

193. Cancellation by the character on a common unipotent subgroup

Definition 193.1. For $\delta \in \Gamma$, put

$$U_\delta = U(F) \cap \delta U(F) \delta^{-1}, \qquad \epsilon_\delta(v) = \chi_\infty(v_\infty)\overline{\chi_\infty((\delta^{-1}v\delta)_\infty)} \quad (v \in U_\delta).$$

Say the rational characters agree at δ when $\epsilon_\delta = 1$ on U_δ. By Proposition 21.2 this is equivalent to agreement of the two finite adelic characters on the same rational elements. For $\delta \in \mathcal{T}(c, s, c')$, let $\Gamma_c \delta \Gamma_{c'}$ be its arithmetic unipotent double orbit, and put $\Lambda = \Gamma_c \cap \delta \Gamma_{c'} \delta^{-1}$.

Lemma 193.2. *With the preceding notation put $F_\delta(y) = V_\infty(y) V_\infty(\delta_\infty^{-1} y)^*$. It is left Λ-invariant and integrable on $\Lambda \backslash G_\infty$, where integration means integration on any measurable domain of representatives for left Λ. One has*

$$\int_{\Lambda \backslash G_\infty} F_\delta(y)\, \mathrm{d}y = \sum_{\delta' \in \Gamma_c \delta \Gamma_{c'}} M_{c,c'}(\delta').$$

Proof. For $v \in U_\delta$ equivariance in Proposition 188.2 gives $F_\delta(v_\infty y) = \epsilon_\delta(v) F_\delta(y)$. The character is trivial on Λ since the two unipotents in its definition belong to $\Gamma_c, \Gamma_{c'}$, respectively. The left action of Λ on G_∞ is free and properly discontinuous, since $\Lambda \subset \Gamma_c$ and Γ_c preserves $h_c K_Y$ at finite places. The construction of Lemma 21.4 gives a measurable domain D tiling G_∞ under its left translates, and independence of the domain for nonnegative or integrable tests.

Choose one pair $(u, v) \in \Gamma_c \times \Gamma_{c'}$ for each distinct matrix $\delta' = u\delta v^{-1}$ in the orbit. All pairs giving that matrix are $(u\lambda, v\delta^{-1}\lambda\delta)$ for $\lambda \in \Lambda$. The entire orbit stays in $\mathcal{T}(c, s, c')$,

since multiplication by the finite positions of $\Gamma_c, \Gamma_{c'}$ preserves the level coset by normality. In $M_{c,c'}(\delta')$, replace the variable by $u_\infty y$. Its integrand becomes

$$\beta_c(u_\infty y)\beta_{c'}(v_\infty \delta_\infty^{-1} y)F_\delta(y).$$

Tile the integral over $\lambda_\infty D$ and use invariance of F_δ. Summing over λ and the representative pairs gives every pair in $\Gamma_c \times \Gamma_{c'}$ once. At a point where $F_\delta \neq 0$ both sums over cutoffs are one by the proof of Proposition 188.2. The same calculation applies first to the nonnegative norm of the integrand by Lemma 123.1. Its original sum is finite by the last assertion of that proposition, since the associated integrands have compact support and only finitely many matrices can contribute. Thus $\int_D \|F_\delta\| < \infty$. Applying the calculation entry by entry gives the identity. □

Theorem 193.3. *If the rational characters fail to agree at $\delta \in \mathcal{T}(c, s, c')$, the sum of the cut coefficients on $\Gamma_c \delta \Gamma_{c'}$ vanishes. The same conclusion holds for a union of such orbits at fixed c, s, c'.*

Proof. In G_f put

$$K' = h_c K_Y h_c^{-1} \cap \delta_f h_{c'} K_Y h_{c'}^{-1} \delta_f^{-1}.$$

This group is compact open and $\Lambda = \{a \in U_\delta : a_f \in K'\}$ by Definition 21.3. Choose $v \in U_\delta$ with $\epsilon_\delta(v) \neq 1$. Conjugation by v preserves U_δ, since it belongs to that group. Hence $\Lambda' = \Lambda \cap v^{-1}\Lambda v$ is characterized there by the finite condition $K' \cap v_f^{-1} K' v_f$. It has finite index m in Λ and finite index m' in $v^{-1}\Lambda v$: the two maps on left coset sets inject into the respective coset sets of compact groups by an open subgroup. Those coset sets are finite since the open cosets cover a compact set.

Write $I = \int_{\Lambda \backslash G_\infty} F_\delta$, absolutely convergent by Lemma 193.2. The function is invariant also under $v^{-1}\Lambda v$, since it transforms on U_δ by a scalar character trivial on Λ. A domain for Λ' is a disjoint union of m translates of a domain for Λ, using representatives of the right cosets $\Lambda' \backslash \Lambda$. Thus its integral is mI. If D is a domain for Λ, then $v_\infty^{-1} D$ is a domain for $v^{-1}\Lambda v$ and its integral is $\epsilon_\delta(v)^{-1} I$, by left invariance of group measure. Tiling a domain for Λ' using that subgroup instead gives $m' \epsilon_\delta(v)^{-1} I$. Independence of the integral from the chosen domain gives equality of the two expressions. If $I \neq 0$, a nonzero matrix coordinate shows $\epsilon_\delta(v) = m'/m$, a positive real of modulus one. This contradicts the choice of v. Thus $I = 0$, proving the first claim. At fixed c, s, c' there are only finitely many contributing matrices by Proposition 188.2, so sums over unions of these orbits also vanish. □

194. The shared flag of an agreeing rational term

Definition 194.1. Write $L_a = \sum_{b=1}^{a} E f_b$ for $0 \leq a \leq n$, where $L_0 = 0$, $L_n = L_f$. A vector shared at index a with coefficients c, c' is a vector $\sum_{b \leq a} x_b f_b = \sum_{b \leq a} y_b \gamma f_b$ with $x_a = c$, $y_a = c'$. The height of a nonzero vector in L_f is the largest index with nonzero coefficient, and the height in γL_f uses the basis (γf_b).

Lemma 194.2. *Suppose the rational characters agree at γ. If there is a vector shared at $a < n$ with coefficients c, c', then there is a vector shared at $a + 1$ with coefficients c, c'. At any index a, existence of such a vector implies $c\bar{c} = c'\bar{c}'$.*

Proof. Let w be a shared vector at $a < n$ and let m be any E-linear functional vanishing on $L_a + \gamma L_a$. Nondegeneracy of the row matrix in Proposition 15.2 gives v with $m = [v, \cdot]$ by inverting that matrix and conjugating coefficients. For $r \in E$ put $X = rw$ and choose $\eta \in E$ with $\eta + \bar{\eta} = 1$, for instance $1/2$. We have $[X, X] = [v, X] = 0$, since $a < n$. Define

$$U_r y = y + X[v, y] - v[X, y] - \eta[v, v]X[X, y].$$

This is a rational isometry. In expanding $[U_r y, U_r y']$, the two terms involving $\overline{[v, y]}[X, y']$ cancel, as do those involving $\overline{[X, y]}[v, y']$. The remaining excess is $(1 - \eta - \bar{\eta})[v, v]\overline{[X, y]}[X, y'] =$

0. Preservation of a nondegenerate form implies injectivity and hence invertibility in finite dimension.

For every frame vector f_b the last two added terms vanish. When $b \le a$, the first one does too; when $b > a$, it belongs to L_a. On $L_\nu^\perp$ the action is the identity modulo L_ν, since $[X, y] = 0$ there. These facts give $U_r \in U(F)$ by Definition 20.3. The same proof in the frame γf_b, using $X \in \gamma L_a$ and $m(\gamma L_a) = 0$, shows that $U_r \in \gamma U(F)\gamma^{-1}$. For $w = \sum_{b \le a} x_b f_b$, the simple-root sum is $r \sum_{b \le a} x_b m(f_{b+1}) = r c\, m(f_{a+1})$. In the other frame it is $r\bar{c}'\, m(\gamma f_{a+1})$. By Definition 193.1, the finite characters of these two values agree for every r. Writing $d = c\, m(f_{a+1}) - c'\, m(\gamma f_{a+1})$, this says $\psi_f(-rd/(g\varpi)) = 1$ for every $r \in E$. If $d \ne 0$, take $r = g\varpi/(4fd)$. Proposition 18.5 evaluates the character by the rational trace; the trace here is $-1/2$ since $[E : \mathbb{Q}] = 2f$. Its value is -1, impossible.

Thus every functional annihilating $L_a + \gamma L_a$ annihilates $c f_{a+1} - c' \gamma f_{a+1}$. Extending a basis of that subspace to a full basis and testing the remaining coordinate functionals proves that the difference belongs to $L_a + \gamma L_a$. Moving the earlier terms to the two sides gives a shared vector at $a + 1$ with the asserted coefficients. Iterate up to n. There the norms of the two descriptions are $-Nc\bar{c}$ and $-Nc'\bar{c}'$; their equality proves the last assertion, also when $a = n$ initially. □

Theorem 194.3. *Assume the rational characters agree at γ, and set $M = L_f \cap \gamma L_f$. If $M \ne 0$, then there is a unique $0 \le j \le \nu$ and a unique $\lambda \in E^\times$ with $\lambda\bar{\lambda} = 1$ such that M has a basis $(w_a)_{j<a\le n}$ of shared vectors at a with coefficients $\lambda, 1$. The heights of every nonzero vector of M in the two flags coincide. At this level $D_j(\gamma) \ne 0$ and $\mathfrak{d}(\gamma) = 0$. If $M = 0$, then $\mathfrak{d}(\gamma) \ne 0$ and $D_n(\gamma) \ne 0$.*

Proof. Given a nonzero vector of M, take the larger of its heights in the two flags. At that index at least one of its leading coefficients is nonzero. The equality of their norms in Lemma 194.2 forces both to be nonzero, proving equality of heights. At a fixed occurring height two nonzero leading coefficient pairs are proportional: subtracting a suitable multiple eliminates the leading coefficient on one side, and the same lemma forces the other to vanish. Hence their ratio is determined at that height and has norm one.

Let $j+1$ be the minimum occurring height, normalize the second leading coefficient to one and call the first λ. Lemma 194.2 supplies vectors with these same leading coefficients up to height n. They are independent by their distinct heights and span M: subtract suitable multiples to eliminate successive leading coefficients in the second flag, and a nonzero remainder of height at most j is excluded by minimality. This also proves uniqueness of j, λ.

If $j > 0$ and $D_j(\gamma) = 0$, then a nonzero vector $y \in \gamma L_j$ is orthogonal to L_j by the singularity of the pairing matrix. It is orthogonal also to the w_a, since $j \le \nu$, $y \in \gamma L_\nu$ and $w_a \in \gamma L_f$. The first leading coefficients show $L_f = L_j + \sum_{a>j} E w_a$, so $y \perp L_f$. In row coordinates this eliminates its dual and z entries. It is isotropic, being in γL_j, so definiteness eliminates its k middle coordinates as in Theorem 192.2. This puts $y \in M$, although its height in the second flag is at most j, a contradiction. For $j = 0$ the minor is one. The assertion about $\mathfrak{d}$ in both cases follows from Theorem 192.2. In the case $M = 0$ it is positive at every real embedding, so $D_n \bar{D}_n = \mathfrak{d} + N^2 D_\nu \bar{D}_\nu$ gives $D_n \ne 0$. □

195. The middle relation of two rational flags

Definition 195.1. In the space of Definition 15.1 set $P = L_\nu$, $M_0 = X_0 \oplus Ez$, $X_0 = \sum_{a=1}^{k} E x_a$. Write h_0 for the restriction of the hermitian form to M_0 and $\pi : P^\perp = P \oplus M_0 \longrightarrow M_0$ for the projection. For a rational isometry γ, put

$$W_\gamma = (P + \gamma P)^\perp, \qquad \mathcal{L}_\gamma = \{\, (\pi v, \pi\gamma^{-1} v) : v \in W_\gamma \,\} \subset M_0 \oplus M_0.$$

Equip this last direct sum with the hermitian form $B = h_0 \oplus (-h_0)$. A subspace is called totally isotropic if the form vanishes on every pair of its elements.

Proposition 195.2. *Let $c = \dim_E(P \cap \gamma P)$ and let a be the rank of the ν-by-ν matrix $([e_i, \gamma e_j])_{i,j \le \nu}$. Then $\mathcal{L}_\gamma = \mathcal{L}_\gamma^{\perp_B}$ has dimension $k+1$, and each coordinate projection on $\mathcal{L}_\gamma$ has kernel of dimension $\nu - a - c \in \{0,1\}$. It is the graph of an isometry of M_0 exactly when $a + c = \nu$. Moreover, for $u, v \in U(F)$,*

$$\mathcal{L}_{u\gamma v^{-1}} = \mathcal{L}_\gamma, \qquad \mathcal{L}_{\gamma^{-1}} = \{(y,x) : (x,y) \in \mathcal{L}_\gamma\}.$$

Proof. The form is nondegenerate by Proposition 15.2. For any subspace its independent basis vectors give independent orthogonality conditions, by the invertible matrix of the form. It follows that $\dim_E W_\gamma = r - (2\nu - c) = k + 1 + c$. Both coordinates of its map to the relation vanish precisely on $P \cap \gamma P$, all of which is indeed in W_γ by isotropy of the two outer spaces. Eliminating that kernel gives dimension $k+1$. For $y, y' \in P^\perp$ one has $[y, y'] = h_0(\pi y, \pi y')$ because P is the radical there. Applying this identity to elements of W_γ and to their inverse images under γ gives total isotropy of the relation for B. Its orthogonal complement has dimension $2(k+1) - (k+1) = k+1$, proving equality.

The vectors of W_γ with first coordinate zero are $P \cap (\gamma P)^\perp$, of dimension $\nu - a$ by the pairing matrix. Quotienting by the simultaneous kernel gives $\nu - a - c$. Interchanging the flags conjugate-transposes the pairing matrix, so the second kernel has the same dimension. In either kernel the other component is injective and has totally isotropic image in M_0. Projection of that image on Ez is injective, since in X_0 a nonzero vector has strictly positive norm at each complex embedding. This bounds its dimension by one. When the kernels vanish both projections are isomorphisms by the dimension calculation. Their composite, with the first inverted, preserves h_0 by total isotropy. Conversely every graph of an isometry has zero kernels. For the last identities, $W_{u\gamma v^{-1}} = uW_\gamma$. The action of U on $P^\perp/P$ is the identity by Definition 20.3, so the two projected coordinates of uy agree with the old ones of y. For inversion use $y \mapsto \gamma^{-1} y$. □

Theorem 195.3. *On an open flag orbit $uw_p\sigma v^{-1}$ the relation $\mathcal{L}_\gamma$ is the graph of $\sigma^{-1}|_{M_0}$. Two such relations are equal exactly when the two rational middle factors are equal. If the characters agree and $\mathfrak{d}(\gamma) = 0$, let j, λ be as in Theorem 194.3. Then $\mathcal{L}_\gamma$ is also a graph. Its isometry A satisfies $Az = \lambda^{-1} z$ and restricts to an isometry of X_0.*

Proof. By Proposition 195.2 remove the two outer unipotents. For $w_p\sigma$ the second outer space is the span of the dual vectors, so $W_{w_p\sigma} = M_0$. The Weyl representative acts identically on this middle space and σ fixes the hyperbolic pairs, as in Theorem 192.3. This proves the first two assertions.

In the singular case Theorems 192.2 and 194.3 give a basis $w_{j+1}, \dots, w_n$ for $L_f \cap \gamma L_f$. The vectors with both heights at most ν form $P \cap \gamma P$ and are exactly the span of the first $\nu - j$ of these vectors. In fact the coefficient at z of w_n is $\lambda \neq 0$ and all earlier ones have zero coefficient there. Thus $c = \nu - j$. The nonzero leading minor D_j gives $a \ge j$ with the empty minor included. The kernel computation in Proposition 195.2 gives $a \le \nu - c = j$, so that proposition gives a graph. The vector w_n belongs to both perpendicular spaces, since $L_f \subset P^\perp$ and $\gamma L_f \subset (\gamma P)^\perp$. Its two projected coordinates are $\lambda z, z$. This proves the claimed action on the line and, by preservation of pairings, on its orthogonal complement in M_0. □

Proposition 195.4. *Under the singular hypotheses of Theorem 195.3 put $D = D_j$, $Z = Z(\gamma)$, and write*

$$P_j = \begin{pmatrix} Z_{1:j,\,1:j} & Z_{1:j,\,j+1:n} \\ 0 & I_{n-j} \end{pmatrix}.$$

For each $b > j$ there is a unique shared vector v_b whose last $n - j$ coordinates in the transported frame are those of the b-th coordinate vector ε_b of E^n. Its entire transported coordinate column is $P_j^{-1}\varepsilon_b$. It has height b in both flags and first-frame coefficient λ at b. If $\gamma \in \mathcal{T}(c, s, c')$, the coordinates of Dv_b in both frames belong to O_E.

Proof. The shared basis from Theorem 194.3 has upper unitriangular matrix on the final $n-j$ coordinates in the transported frame. Successively removing its last nonzero coordinate proves unique solvability for any such tail. For the indicated tail this procedure uses w_b with coefficient one and only earlier vectors besides it. This proves the heights and leading coefficient assertion. If y denotes the full transported coordinate column, the equality $v_b = \sum y_i \gamma f_i \in L_f$ gives $(Zy)_i = 0$ for $1 \le i \le j$ by orthogonality to those e_i. Together with the tail, this is $P_j y = \varepsilon_b$. Expansion along the last rows gives $\det P_j = D \neq 0$.

For an arithmetic term, the entries of P_j are integral by Theorem 192.2. Thus $Dy = \operatorname{adj}(P_j)\varepsilon_b$ is integral by the cofactor formula. At every finite place the vector Dv_b belongs to $\gamma\Lambda_{c'} = \Lambda_c$, since the transported frame vectors are in the transported lattice. In Λ_c the f_i are part of a local integral basis by Definitions 15.1 and 20.3: apply the local class matrix to the displayed integral basis, which fixes each e_i and z. The first-frame coordinates are therefore integral at every finite place. Apply Lemma 52.1 to each rational entry. □

196. The determinant of the pairing between two middle relations

Definition 196.1. Put $t(x, x') = ([z, x], [z, x'])$ on $M_0 \oplus M_0$. On that space let

$$Q = \tfrac{1}{2}(h_0 \oplus h_0) + N^{-1} t^* t,$$

where t^*t denotes the pullback of the usual hermitian product on E^2. In the coordinates of Definition 15.1 its diagonal is $\frac{1}{2}(I_k, N, I_k, N)$, so it is positive definite at every infinite place. Write $\widehat{O} = \prod_{\mathfrak{p}} O_{E,\mathfrak{p}}$. For a family $\alpha = (\alpha_{\mathfrak{p}})$ of nonzero finite local scalars, units at almost all primes, put

$$I(\alpha) = \prod_{\mathfrak{p}} (\#\, O_E/\mathfrak{p})^{\operatorname{ord}_{\mathfrak{p}} \alpha_{\mathfrak{p}}}.$$

It is multiplicative, and $I(\bar{\alpha}) = I(\alpha)$, since conjugation permutes the places with their residue degrees. For $a \in E^{\times}$ one has $I(a) = |\operatorname{Nm}_{E/\mathbb{Q}} a|$: for a integral both sides give the index of multiplication by a in O_E, by the integral-basis determinant and Lemma 20.2; division gives the general case.

A finite lattice of a rational space is a family of full local integral lattices equal to the span of a fixed rational basis at almost all places. For such a lattice A and a totally positive definite hermitian form Q' on its rational span, put $\Delta_{Q'}(A) = I(\det(A^* Q' A))^{1/2}$. In this formula A denotes any tuple of local basis matrices in the ambient rational coordinates. The conjugate transpose uses both factors at a split pair. A basis change multiplies the determinant by units, so the value is independent of the basis. Empty Gram determinants are one. For a rational $\gamma \in \mathcal{T}(c, s, c')$ let A_γ be the image of $W_\gamma \cap \widehat{\Lambda}_c$ under the map in Definition 195.1, extended at each finite place; $\widehat{\Lambda}_c$ is the finite completion of the row lattice of Definition 20.3. This is a finite lattice on $\mathcal{L}_\gamma$. Indeed kernels, images and rational spans can be obtained by Gaussian elimination, and, off the finite set of denominators of the bases and nonzero pivots and their inverses, the same operations give local integral bases. At the other places elimination over the discrete valuation rings gives full lattices by Lemma 25.1, whose pivot argument also applies to rectangular matrices.

Lemma 196.2. *Let $C = \mathcal{L}_\gamma \cap \mathcal{L}_\delta$, $i = k + 1 - \dim_E C$, $l = \operatorname{rank}(t|_C)$, and let T be the first projection of $C \cap \ker t$. Then $T \subset X_0$ and*

$$k - \dim_E T = i + l - 1, \qquad l \le 2.$$

On this subspace of pairs the first projection is an isometry from Q to the ordinary positive hermitian form on T; the second projection has the same property. On the whole $\mathcal{L}_\gamma$, t has rank two if $\mathfrak{d}(\gamma) \neq 0$, and rank one if the characters agree and $\mathfrak{d}(\gamma) = 0$. In the latter case $k - \dim_E T \le i$.

Proof. In $\ker t = X_0 \oplus X_0$, the two ordinary inner products on pairs of members of C are equal because B vanishes by Proposition 195.2. Their half-sum is Q; positivity gives injectivity of each projection. Thus $\dim T = \dim C - l$, proving the identity.

If t on $\mathcal{L}_\gamma$ has rank below two, the two functionals $y \mapsto [z, y]$, $y \mapsto [\gamma z, y]$ are linearly dependent on W_γ. The double orthogonal of a subspace for a nondegenerate form equals that subspace by the dimension computation in Proposition 195.2. Thus $\alpha z + \beta\gamma z \in P + \gamma P$ for some $(\alpha, \beta) \neq (0, 0)$, conjugating the coefficients of the functional relation. Write this as $p + \alpha z = \gamma(p' - \beta z)$ with $p, p' \in P$. Norms give $\alpha\bar{\alpha} = \beta\bar{\beta}$, so both coefficients are nonzero. This is a nonzero common frame vector and Theorem 192.2 gives $\mathfrak{d}(\gamma) = 0$. For the agreeing singular case use the graph of Theorem 195.3; the two z functionals are then proportional and nonzero. Their rank on a subspace C is at most one. □

Proposition 196.3. *If $\gamma, \delta \in \mathcal{T}(c, s, c')$, then*

$$B(A_\gamma, A_\delta) \ \subset \ g\widehat{O}.$$

The induced pairing on $(\mathcal{L}_\gamma / C) \times (\mathcal{L}_\delta / C)$ is nondegenerate.

Proof. For lifts $y \in W_\gamma,\ y' \in W_\delta$, the pairing is

$$[y, y'] - [\gamma^{-1} y, \delta^{-1} y'] = [y, (1 - \gamma\delta^{-1}) y'].$$

The first equality uses $P^\perp = P \oplus M_0$ on both sides, as in Proposition 195.2. For finite integral lifts, $\gamma\delta^{-1}$ preserves the class-c lattice and acts as the identity modulo g because the two maps from the class-c' lattice have the same reduction. At the conjugate split components equality of the inverse transposes follows from Proposition 15.2. The pairing on the lattice is integral, giving the asserted divisibility. A member of $\mathcal{L}_\gamma$ perpendicular to all of $\mathcal{L}_\delta$ belongs to $\mathcal{L}_\delta$, by Proposition 195.2. The converse holds for every member of C, and the same argument applies to the other variable. These are exactly the two radicals. □

Theorem 196.4. *In Proposition 196.3 let $A_\gamma^\perp, A_\delta^\perp$ denote the images of the two lift lattices under the Q-orthogonal projections away from C, within their respective relations. Then*

$$\Delta_Q(A_\gamma^\perp)\Delta_Q(A_\delta^\perp) \ \geq \ (g/2)^{2fi}.$$

For either of the two lattices one has the exact identity $\Delta_Q(A) = \Delta_Q(A \cap C)\Delta_Q(A^\perp)$, with intersections and projections taken locally.

Proof. The projection is rational because solving the orthogonality equations uses the inverse of the rational Gram matrix on C, which is invertible by positivity even at one infinite place. The local intersection is saturated: an element of the lattice whose nonzero scalar multiple lies in the subspace lies there itself. Diagonal elimination as in Lemma 25.1 extends a basis of this intersection to a basis of the full local lattice. The added columns project to a basis of the image. Subtract their C-components over the local fields. This is a triangular basis transformation of determinant one, and now the Gram determinant factors, proving the identity.

Choose rational bases Y, Y' for the two orthogonal complements of C, with local basis matrices $YH, Y'H'$ for the projected lattices. Their dimension is i, and $D = \det(Y^* B Y')$ is nonzero by Proposition 196.3. Subtracting a C-component leaves B on each cross pair unchanged. Thus all entries of $H^* Y^* B Y' H'$ belong to $g\widehat{O}$. Taking determinants gives

$$g^{2fi} \leq I(\det H)\, |\operatorname{Nm}_{E/\mathbb{Q}} D|\, I(\det H').$$

Here an integral finite scalar has $I \geq 1$, and $I(g) = g^{2f}$ by Definition 196.1.

The rational operator $J = Q^{-1}B$ is diagonal with entries 2 or -2; thus $J^* Q J = 4Q$. At an infinite place apply the mixed Gram determinant bound in the proof of Lemma 17.1 to Y, JY' and the positive diagonal form Q. This gives

$$|\tau D|^2 \leq 4^i \det(\tau(Y^* Q Y)) \det(\tau(Y'^* Q Y')).$$

Multiplication over the f chosen embeddings, using each conjugate pair once, gives

$$|\operatorname{Nm}_{E/\mathbb{Q}} D| \le 2^{2fi} |\operatorname{Nm}_{E/\mathbb{Q}} \det(Y^* Q Y)|^{1/2} |\operatorname{Nm}_{E/\mathbb{Q}} \det(Y'^* Q Y')|^{1/2}.$$

After restoring the two finite basis determinants, the right side is $2^{2fi}\Delta_Q(A_\gamma^\perp)\Delta_Q(A_\delta^\perp)$. Combining the inequalities proves the theorem, including $i = 0$ by the empty determinant convention. □

197. An exact discriminant formula for integral lifts

Definition 197.1. Fix $\gamma \in \mathcal{T}(c, s, c')$. If $\mathfrak{d}(\gamma) \neq 0$, set $j = n$. Otherwise assume character agreement and take the $j \le \nu$ of Theorem 194.3. Set $d = \min(j, \nu)$, $h = \nu - d$, $V = P + \gamma P$, and $P_O = \sum_{a \le \nu} \widehat{O} e_a$, with the sum starting at $a = 1$. For lattices $L' \subset L$ of the same rank, let $[L : L']$ denote the cardinality of the finite quotient. Write $R_d(\gamma)$ for the $\widehat{O}$-span of the first d rows of $Z(\gamma)$ and define

$$\begin{aligned} s_\gamma &= [\, V_f \cap \widehat{\Lambda}_c : P_O + \gamma P_O \,], \\ i_\gamma &= [\, R_\nu(\gamma) : R_d(\gamma) \,], \\ m_\gamma &= \begin{cases} |\operatorname{Nm}_{E/\mathbb{Q}} D_j|, & j \le \nu, \\ |\operatorname{Nm}_{E/\mathbb{Q}} D_n| / N^{2f}, & j = n. \end{cases} \end{aligned}$$

Here V_f means scalar extension to the finite adeles. Use the same j with the two frames exchanged to define $i_{\gamma^{-1}}$; its matrix is $Z(\gamma)^*$. All these indices are finite by the following lemma.

Lemma 197.2. *Let $C_0 = P \cap \gamma P$ and $F = (e_1, \dots, e_\nu, \gamma e_1, \dots, \gamma e_d)$. The columns of F are a basis of V. The projections of C_0 to the last h outer coordinates in the original and transported frames are both isomorphisms onto E^h. The images of the integral intersection $C_{0,f} \cap \widehat{\Lambda}_c$ have indices i_γ and $i_{\gamma^{-1}}$ in the two respective copies of $\widehat{O}^h$. In particular*

$$[\, V_f \cap \widehat{\Lambda}_c : F \widehat{O}^{\,\nu+d} \,] = s_\gamma \, i_{\gamma^{-1}}.$$

Proof. If $j = n$, Theorem 192.2 gives $C_0 = 0$. Both row indices are then one since $d = \nu$. If $j \le \nu$, the first $\nu - j$ shared vectors of Theorem 194.3 form a basis of C_0 by the proof of Theorem 195.3. Their two tail matrices are upper triangular with nonzero diagonal. In either case γL_d meets P in zero because a member of this intersection has zero transported tail. The remaining transported columns are combinations of the displayed columns because any transported tail can be removed with a vector of C_0. This proves the basis and projection assertions.

For $h > 0$ the first d rows of Z are independent by $D_d \neq 0$ (for $d = 0$ the list is empty). The coordinates of every $v \in C_0$ give a conjugate-linear relation on its first ν rows since $v \perp \gamma L_f$. Prescribing a tail determines such a relation uniquely: existence uses the tail isomorphism and uniqueness uses the independence just proved. For $t = (t_b)_{d<b\le\nu} \in \widehat{O}^h$, the sum of $\bar{t}_b$ times the corresponding rows gives a surjection of additive groups onto R_ν / R_d. The kernel consists exactly of the tails whose unique relation has integral head coefficients. This is exactly the image of $C_{0,f} \cap \widehat{\Lambda}_c$, because the outer vectors are part of a basis of every class lattice by Proposition 195.4 and its basis argument. This is a full lattice, giving finiteness and the asserted index. On exchanging the frames the same proof computes the other index: $\gamma \widehat{\Lambda}_{c'} = \widehat{\Lambda}_c$ and $Z(\gamma^{-1}) = Z(\gamma)^*$ by preservation of the form. When $h = 0$ the tail assertions hold with trivial quotients.

The quotient of $P_O + \gamma P_O$ by the span of F is generated by the last h transported outer vectors. An integral combination with tail t is zero in this quotient exactly when one can adjust it by an integral transported head to lie in P_O. This says precisely that t is the transported tail of an integral vector in C_0. The index of the span of F in $P_O + \gamma P_O$ is

therefore $i_{\gamma^{-1}}$. Both are full lattices in V, so multiplication of the two finite indices proves the formula. □

Lemma 197.3. *In the notation of Lemma 197.2 let $B_1 = (e^*_{d+1}, \dots, e^*_\nu, z, \gamma z)$, with $h+2$ columns, and let J be the rational row form. Write $H = [F\ B_1]^* J [F\ B_1] + \operatorname{diag}(0_{\nu+d}, N I_{h+2})$. Then*

$$|\operatorname{Nm}_{E/\mathbb{Q}} \det H| = (N^{2f} m_\gamma)^2.$$

Proof. Reorder the columns to put $(e_1, \dots, e_\nu, z)$ first and $(\gamma e_1, \dots, \gamma e_d, e^*_{d+1}, \dots, e^*_\nu, \gamma z)$ last. Apply the same permutation to the rows. The first diagonal block vanishes: the one nonzero norm $-N$ of z is canceled by the added N. Write K for the upper right block. Interchanging the two column blocks then gives a block triangular matrix, so $\det H = (-1)^n \det K \overline{\det K}$. The intermediate dual columns of K are the corresponding identity columns. When $j = n$ there are none and $K = Z$, giving the formula. In the singular case expansion along these columns gives the minor of Z on rows and columns $1, \dots, j, n$. Subtract from its last column the unique combination of the first j columns that clears its first j entries. The resulting vector of column coefficients, extended by zeros at $j < b < n$, is $P_j^{-1}\varepsilon_n$ of Proposition 195.4. Hence the last entry is $[z, v_n] = -N\lambda$, and the remaining determinant is $-N\lambda D_j$. Since $\lambda\bar\lambda = 1$, its absolute field norm is $N^{2f}|\operatorname{Nm}_{E/\mathbb{Q}} D_j|$. This also proves the assertion when $j = 0$ or $j = \nu$. □

Theorem 197.4. *For the positive metric Q and the entire lattice A_γ of integral lifts from Definition 196.1,*

$$\Delta_Q(A_\gamma)\, s_\gamma\, i_\gamma\, i_{\gamma^{-1}} = m_\gamma.$$

Proof. Work in finite local bases of $\widehat{\Lambda}_c$, grouping conjugate places. Its Gram matrix J_0 is integral unimodular by Proposition 15.2. Use F, B_1 also for the coordinates of the preceding columns in these bases. Take a basis F_0 of the saturation of F in the local lattice, and write $F = F_0 R$. Such bases and their extensions to full lattice bases are obtained by diagonal elimination over each valuation ring, as in Lemma 25.1; saturation replaces the diagonal elementary divisors by ones. Thus $I(\det R) = s_\gamma i_{\gamma^{-1}}$ by Lemma 197.2. The map $F_0^* J_0$ from the lattice onto the coordinate module of rank $\nu + d$ is surjective: F_0 extends to an invertible integral matrix and so does its adjoint, and J_0 is unimodular. Choose integral lifts D of the standard basis, and a basis W of its kernel. Then $[D\ W]$ is an integral basis and W spans $W_{\gamma,f} \cap \widehat{\Lambda}_c$. All choices can be made integral in fixed rational coordinates off finitely many places by the pivot argument of Definition 196.1.

Compute the determinant of

$$\begin{pmatrix} J_0 & J_0F & J_0B_1 \\ F^*J_0 & 0 & 0 \\ B_1^*J_0 & 0 & -NI_{h+2} \end{pmatrix}.$$

Eliminating its first diagonal block gives $\det J_0$ times $\det(-H)$. Eliminating its last diagonal block first instead replaces J_0 by $J' = J_0 + N^{-1} J_0 B_1 B_1^* J_0$ and produces a factor $(-N)^{h+2}$. These eliminations subtract row multiples to clear the entries below or above the pivot block. In the remaining matrix use the basis $[D\ W]$ on the original coordinates. Since $F^* J_0 [D\ W] = [R^*\ 0]$, expansion along the last rows, and then along the matching columns, leaves $\det(R^*) \det R \det(W^* J' W)$ up to sign. All integral basis changes and $\det J_0$ have $I = 1$. Hence

$$I(\det H) = I(N)^{h+2} I(\det R)^2 I(\det(W^* J' W)).$$

For $y \in W_{\gamma,f}$, write $a(y)$ for the last h coefficients along $e_{d+1}, \dots, e_\nu$, and $b(y)$ for its image in $\mathcal{L}_\gamma$. The map $y \mapsto (a(y), b(y))$ is an isomorphism onto $E_f^h \oplus \mathcal{L}_{\gamma,f}$. Indeed the kernel of b is $C_{0,f}$, which maps isomorphically to the first summand by Lemma 197.2, and b is surjective by definition. In these coordinates $J'|_{W_\gamma}$ is $N^{-1} I_h \oplus Q|_{\mathcal{L}_\gamma}$. To see this, the entries of $B_1^* J_0 y$ are $a(y)$ and $t(b(y))$. The original pairing of two lifts equals the middle pairing on

either of their two coordinates, hence their half-sum. Adding the displayed correction gives exactly the claimed form.

The image of the integral kernel of b in the first summand has index i_γ in $\widehat{O}^h$. Extend its basis by lifts of a basis of A_γ. Their first components can be removed over the fields by subtracting kernel columns, a shear of determinant one. The Gram determinant in the two orthogonal summands therefore gives

$$I(\det(W^* J' W)) = I(N)^{-h} i_\gamma^2 \Delta_Q(A_\gamma)^2.$$

Using $I(N) = N^{2f}$ in the preceding determinant identity and substituting Lemma 197.3 leaves the claimed equality after taking positive square roots. This calculation uses $I(\overline{\det R}) = I(\det R)$ on the product of all finite places; it includes the split factors. □

198. A fixed frame vector at a marked prime

Definition 198.1. Call $s \in \mathcal{G}$ agreeing if, for every $u, u' \in U_f \cap K_Y^0$ whose reductions satisfy $\bar{u}s = s\bar{u}'$, one has $\chi_f(u) = \chi_f(u')$, with χ_f as in Proposition 21.2. A set of rational terms is stable if it is invariant under left and right multiplication by $U(F)$. Write $T_f(x, x') = ([z, x], [z, x'])$ for the finite extension of t of Definition 196.1.

Proposition 198.2. *At a nonagreeing s, for each c, c' and each stable set $\mathcal{B}$ of rational terms, the sum $\sum_{\delta \in \mathcal{T}(c,s,c') \cap \mathcal{B}} M_{c,c'}(\delta)$ is zero.*

Proof. Choose u, u' as in Definition 198.1 with unequal characters. Lemma 21.1 gives $v, v' \in U(F)$ whose finite positions $A_c(v), A_{c'}(v')$ lie in $uK_Y, u'K_Y$, respectively. Indeed the two inverse images of these cosets are nonempty open sets in the finite unipotent group: the respective conjugates by $h_c, h_{c'}$ supply points in them. The class matrices normalize the unipotent by preserving the flag and acting invertibly on its middle quotient. They fix every frame vector and leave unchanged the coefficients along its outer lines. Conjugation therefore preserves the simple-root character, as in Lemma 188.3. On a principal integral unipotent the character is one by the same integrality argument as in Proposition 21.2. Consequently $\chi_\infty(v_\infty) \neq \chi_\infty(v'_\infty)$ by that proposition.

The map $\delta \mapsto v \delta v'^{-1}$ permutes the terms in the stated sum, since $\bar{u}s = s\bar{u}'$. It permutes their arithmetic double orbits as well: v, v' normalize the principal arithmetic unipotents by normality of K_Y in K_Y^0. Apply Lemma 193.2 on each orbit. Its integral is multiplied by $\chi_\infty(v_\infty)\overline{\chi_\infty(v'_\infty)}$ under this permutation, by the substitution $y \mapsto v_\infty y$. This carries the old stabilizer domain to a domain for the conjugate stabilizer. Only finitely many orbit integrals can be nonzero by Proposition 188.2. Thus the total equals itself times a scalar different from one. □

Lemma 198.3. *Let s be agreeing. Choose either component $\mathfrak{p}$ of a marked split pair at $p \in \{g_0, g_1\}$. Work over its residue field $\mathbf{F}_p$ in the fixed integral basis, write S for the action of s there, and let L_a denote the span of the first a reduced frame vectors. If $1 \le a < n$ and*

$$\sum_{b \le a} x_b f_b = \sum_{b \le a} y_b S f_b \qquad (x_a = c_1,\ y_a = c_2),$$

there is such a shared vector at $a+1$ with the same two leading coefficients.

Proof. Let w be the displayed vector and m any linear functional annihilating $L_a + SL_a$. For $r \in \mathbf{F}_p$ the matrix $I + rwm$ in the $\mathfrak{p}$-component has conjugate by S^{-1} equal to $I + r(S^{-1}w)(mS)$. These matrices lift on the two sides to integral finite unipotents, with all other pairs trivial. For the first lift choose W in the integral span of the first a outer basis vectors with the prescribed coefficients, and a lift M of m annihilating those basis vectors exactly. Use $I + \tilde{r}WM$ over $O_{E,\mathfrak{p}}$ with $\tilde{r}$ lifting r. Its inverse is $I - \tilde{r}WM$ since $M(W) = 0$. On the other component take the forced inverse adjoint, which is integral by unimodularity in Proposition 15.2. In the chosen component the addition has image in L_a and is zero on

L_a. Thus it preserves each outer flag, acts identically on its successive quotients and on $P^\perp/P$. In the partner component the addition annihilates all of $P^\perp$, since it pairs with $W \in P$ in the first component. These are precisely the requirements of Definition 20.3. In the second lift use the coefficients of $S^{-1}w$ in the outer frame and lift mS, which also annihilates L_a. The resulting reductions intertwine by s at both components.

In the first component the simple-root sum of the first lift reduces to $rc_1 m(f_{a+1})$: the terms with indices less than a vanish because $m(L_a) = 0$, and those above a because $w \in L_a$. In the second lift it reduces to $rc_2 m(Sf_{a+1})$. The partner simple-root sums vanish because both partner actions fix the entire frame. By Proposition 18.5 the character $x \mapsto \psi_f(-x/(g\varpi))$ on the chosen integral component induces a nontrivial additive character of $\mathbf{F}_p$: the self-dual integral lattice there has annihilator $O_{E,\mathfrak{p}}$, and g is a uniformizer times a unit. More explicitly the component is $\mathbb{Q}_p$; dividing an integral variable by $g\varpi$ at this split unramified prime gives conductor p, since ϖ is a unit there by Proposition 18.4. All other components of each simple-root sum are zero. Equality for every r implies $c_1 m(f_{a+1}) = c_2 m(Sf_{a+1})$, because multiplication by a nonzero residue permutes the field. Thus the vector $c_1 f_{a+1} - c_2 S f_{a+1}$ lies in $L_a + SL_a$, by extending a basis and testing all annihilating functionals. Rearrangement gives the required shared vector. □

Theorem 198.4. *Suppose s is agreeing. Let $\mathcal{P}$ be a set of distinct primes of E in the marked split pairs, possibly including both partners, such that s fixes a nonzero vector in the reduced frame span at each $\mathfrak{p} \in \mathcal{P}$. Put $X = \prod_{\mathfrak{p}\in\mathcal{P}} \#(O_E/\mathfrak{p})$, with empty product one. If $\gamma \in \mathcal{T}(c, s, c')$, then*

$$T_f(A_\gamma) \subset \widehat{O}^2, \qquad a_1 \equiv a_2 \pmod{\bar{\mathfrak{p}}} \quad ((a_1, a_2) \in T_f(A_\gamma),\ \mathfrak{p} \in \mathcal{P}).$$

In particular for any two columns of $T_f(A_\gamma)$ whose determinant α is a finite idele, $I(\alpha) \geq X$.

Proof. At a prime in $\mathcal{P}$ take a fixed nonzero frame vector $\sum_{b\leq a} x_b f_b = S\sum_{b\leq a} x_b f_b$ of height a. Its two leading coefficients agree and are nonzero. Iteration of Lemma 198.3, or this vector itself if $a = n$, gives a shared vector at n with the same nonzero leading coefficient in the two frames. Dividing by that coefficient gives $z - Sz \in P + SP$ in the residue space. Applying the finite class frames, which fix all the f_b, this says $z - \gamma z$ modulo $\mathfrak{p}\widehat{\Lambda}_c$ is in the reduction of $P_O + \gamma P_O$.

For an integral lift $y \in W_{\gamma,f} \cap \widehat{\Lambda}_c$ the two entries of the evaluation are $[z, y]$ and $[\gamma z, y]$. Both are integral by Proposition 15.2. The lift is orthogonal to $P_O + \gamma P_O$. Pairing the congruence in the first argument therefore gives $[z, y] - [\gamma z, y] \equiv 0 \bmod \bar{\mathfrak{p}}$; the conjugate prime occurs because the first variable is conjugate-linear. This proves the inclusions. The two rows of any evaluation matrix of two columns then agree modulo each $\bar{\mathfrak{p}}$. Its determinant is integral everywhere and has valuation at least one at each of these distinct primes. Multiplication of residue cardinalities, which are unchanged by conjugation, gives the asserted bound by Definition 196.1. □

199. Discriminant bounds for pairs of congruent terms

Definition 199.1. For a term at a level of Definition 197.1 put

$$b_\gamma = \prod_{\tau\in\Phi} |\tau\mathfrak{d}(\gamma)|, \qquad G_\gamma = \frac{\max(b_\gamma, N^f m_\gamma)}{i_\gamma i_{\gamma^{-1}}}.$$

For a rational subspace $T \subset X_0$ let $L_c(T) = T_f \cap \widehat{\Lambda}_c$, $d_c(T) = \Delta_{h_0}(L_c(T))$. For $a = 1$ or g write

$$F_\gamma(v) = ([\gamma f_b, v])_{1\leq b\leq n},$$
$$o_{\gamma,a}(T) = [\, F_\gamma(L_c(T)) : F_\gamma(L_c(T)) \cap aF_\gamma(P_O)\,], \qquad P_\gamma(T) = d_c(T) o_{\gamma,1}(T) s_\gamma.$$

The index is used below only when finite. All local coordinates in these definitions are those in the rational row space. The positive form on X_0 is the ordinary one of Definition 15.1.

Lemma 199.2. *On the relation $\mathcal{L}_\gamma$ the two coordinates of t have representing vectors r_1, r_2 for Q, meaning $t(v) = (Q(r_1, v), Q(r_2, v))$. There is $u \in E$ such that their Gram matrix is*

$$N \begin{pmatrix} 1 & -\bar{u} \\ -u & 1 \end{pmatrix}, \qquad 1 - u\bar{u} = \frac{\mathfrak{d}(\gamma)}{D_j \bar{D}_j}.$$

When $j = n$, $u = ND_\nu / D_n$.

Proof. First construct $r_1 = (a - z, uz)$ in the relation, with $a \in X_0$. For the regular level take the vector w from the cofactor construction in Theorem 192.2. Its inverse image under γ is in L_f, and $w \perp P$. Thus $w \in W_\gamma$. Multiply it by N/D_n. The first middle coordinate now has z-coefficient -1, while the second is $ND_\nu z / D_n$. The discriminant identity gives the indicated $1 - u\bar{u}$. In the singular case use the vector w_n of Theorem 194.3 multiplied by $-\lambda^{-1}$; then $a = 0,\ u = -\lambda^{-1}$ and both sides of the discriminant identity are zero. Interchanging the two frames and using Proposition 195.2 gives a member $r_2 = (u'z, a' - z)$ for some $a' \in X_0$. The hypotheses for the inverse follow because $Z(\gamma^{-1}) = Z(\gamma)^*$, and rational agreement is symmetric by Definition 193.1. From $B(r_1, r_2) = 0$ we get $N(u' - \bar{u}) = 0$.

For any $v = (x, x')$ in the relation the two terms $h_0(a - z, x), h_0(uz, x')$ coincide because $B(r_1, v) = 0$. Their half-sum in $Q(r_1, v)$ therefore equals $\bar{u}[z, x']$. The contribution of $N^{-1} t^* t$ is $[z, x] - \bar{u}[z, x']$, giving the first representing property. Exchanging the slots gives the second one. Evaluating on r_1, r_2 gives the displayed matrix. □

Proposition 199.3. *Take two terms in $\mathcal{T}(c, s, c')$ at levels as in Definition 197.1, and denote them by γ, δ. With the notation of Lemma 196.2 set $C_1 = C \cap \ker t$, and let D be the Q-orthogonal projection of $A_\gamma \cap C$ away from C_1 inside C. Assume $s, \mathcal{P}, X$ satisfy Theorem 198.4, and put $X_l = X$ for $l = 2$ and $X_l = 1$ otherwise. Then*

$$\Delta_Q(A_\gamma \cap C_1) \geq d_c(T) o_{\gamma,1}(T) > 0,$$
$$\Delta_Q(D) \geq \frac{X_l \, m_\gamma}{\max(b_\gamma, N^f m_\gamma)}.$$

In particular both $o_{\gamma,1}(T)$ and $o_{\gamma,g}(T)$ are finite.

Proof. For any member of $A_\gamma \cap C_1$, an integral lift y has the form $p + x$ with $p \in P_O,\ x \in L_c(T)$, since the outer planes split off integrally in Definition 20.3. Its second middle entry is also orthogonal to z, so $y \perp \gamma L_f$ and $F_\gamma(x) = -F_\gamma(p)$. The first projection is a Q-isometry on C_1 by Lemma 196.2. Its image lattice L' spans T and lies inside the preimage of $F_\gamma(P_O)$ in $L_c(T)$. Thus $[L_c(T) : L'] \geq o_{\gamma,1}(T)$ by the induced quotient map. This full lattice has finite index by local elimination and Lemma 20.2. Its discriminant equals $d_c(T)[L_c(T) : L']$, by taking the determinant of the basis change in Definition 196.1. This proves the first assertion. For $a = g$ the preimage contains gL', again of finite index.

A projection away from C_1 preserves t, so $t(D)$ is integral at every place and satisfies the congruences of Theorem 198.4. For $l = 0$ its source has discriminant one, which suffices since $N \geq 1$. For $l = 1$ choose a nonzero evaluation coordinate on the rational span of D. On local basis vectors it gives a finite idele η with $I(\eta) \geq 1$. By Lemma 199.2 its representing vector has Q-norm squared N. Applying the mixed Gram bound of Lemma 17.1 over all embeddings, exactly as in Theorem 196.4, gives $I(\eta) \leq N^f \Delta_Q(D)$. This yields the stated inequality.

For $l = 2$ both evaluations are independent on the rational span of D. The determinant of their values on local basis vectors is an idele η and $I(\eta) \geq X$ by Theorem 198.4. This case requires $j = n$ by Lemma 196.2. By Lemma 199.2, the discriminant of the lattice generated by the two rational representing vectors in their own span is

$$\left| \mathrm{Nm}_{E/\mathbb{Q}} \frac{N^2 \mathfrak{d}(\gamma)}{D_n \bar{D}_n} \right|^{1/2} = \frac{b_\gamma}{m_\gamma}.$$

The same mixed Gram bound gives $I(\eta) \le (b_\gamma/m_\gamma)\Delta_Q(D)$. Indeed in a rational basis Y for the span of D the evaluation determinant is $\det((r_1, r_2)^* QY)$ times the determinant of the finite basis change; this latter factor occurs on both sides after taking I. Combining the inequalities proves the claim. □

Theorem 199.4. *For the two terms in Proposition 199.3 one has*

$$\Delta_Q(A_\gamma^\perp)\, P_\gamma(T) X_l \le G_\gamma,$$
$$P_\gamma(T) P_\delta(T) X_l^2 \le (2/g)^{2fi} G_\gamma G_\delta.$$

The two flag levels may be different. The number i is zero precisely when the relations are equal. For a regular term with $i = 0$ one has $l = 2$; for each singular term one has $k - \dim T \le i$, and in all cases $k - \dim T \le i + 1$.

Proof. Factor the discriminant of A_γ along C, then factor the discriminant of $A_\gamma \cap C$ along C_1. The lattice factorization in Theorem 196.4 applies to both orthogonal projections. Use both lower bounds of Proposition 199.3 and the exact value of $\Delta_Q(A_\gamma)$ from Theorem 197.4. Cancelling $m_\gamma > 0$ gives the first inequality. Exchange γ and δ. The image T of C_1, its codimension, and X_l are unchanged by this interchange. Multiply the two bounds and apply the lower bound for the two complementary discriminants in Theorem 196.4. This proves the second inequality. Both relations have the same dimension $k + 1$ by Proposition 195.2, and the remaining rank assertions follow from Lemma 196.2. □

200. Counting definite subspaces by their discriminants

Definition 200.1. For $u \in G^u$ of Definition 13.1 let $L_u = u\widehat{O}^k$ be the finite lattice in X_0, and write $L_u^{\mathrm{rat}} = L_u \cap E^k$. For a rational subspace $T \subset X_0$ put $d_u^0(T) = \Delta_{h_0}(L_u \cap T_f)$. Set $\|v\|_\infty^2 = \sum_{\tau\in\Phi} \sum_{a=1}^k |\tau v_a|^2$ for $v \in E^k$; this is the Euclidean norm on the product of the complex spaces at infinity.

Lemma 200.2. *For every T one has $d_u^0(T) = d_u^0(T^{\perp_{h_0}}) > 0$. If A is a finite lattice on an E-subspace $V \subset X_0$ of dimension s, the Euclidean covolume of $V \cap A$ in $\prod_{\tau\in\Phi} \tau V$ equals $V_E^s \Delta_{h_0}(A)$, where $V_E = 2^{-f} \mathrm{rd}_E^f$.*

Proof. In an arbitrary rational basis Y of V, write A as $YH\widehat{O}^s$. The coefficient lattice has covolume $V_E^s I(\det H)$ by Lemma 20.2 and Definition 196.1. At each infinite place the Jacobian of $v \mapsto (\tau Y)v$ between the underlying real Euclidean spaces is $\det(\tau(Y^*Y))$: in an orthonormal basis for the image the complex matrix is square, and its real determinant is its absolute determinant squared, by replacing each complex scalar by its real two-dimensional matrix. Multiplying over τ gives exactly the stated discriminant by its definition. The value of V_E follows from the integral embedding determinant in Lemma 20.1.

Locally L_u is self-dual for h_0 since its local defining maps are isometries of the standard form and the standard lattice is self-dual. The orthogonal projection of L_u onto T_f is the h_0-dual of $L_u \cap T_f$. Indeed the projected lattice pairs integrally with this intersection. Conversely an integral functional on the intersection extends to the full lattice by completing an integral basis as in Theorem 196.4. Self-duality represents that functional by a member of the full lattice; its orthogonal projection is the required vector. At a split pair the functional on one component is represented in the partner, by the same nondegenerate pairing. In a local basis with Gram matrix H_1 the dual basis is obtained by multiplication by H_1^{-1} and its Gram matrix is H_1^{-1}; the two discriminants therefore multiply to one. Factor $\Delta_{h_0}(L_u) = 1$ along $T^{\perp_{h_0}}$ using the basis and projection argument in Theorem 196.4. The projected factor is $1/d_u^0(T)$, giving the equality. All these determinants are positive after taking I, since each restricted rational form is positive definite at infinity. □

Lemma 200.3. *Let $0 < s \le k$ and let T have codimension s. There are E-independent $v_1, \dots, v_s \in L_u^{\mathrm{rat}}$ whose common orthogonal space is T, such that*

$$\left(\prod_{\alpha=1}^{s} \|v_\alpha\|_\infty \right)^{2f} \le (fs \operatorname{rd}_E)^{fs} \, d_u^0(T).$$

Proof. Take the real Euclidean lattice $A = L_u^{\mathrm{rat}} \cap T^{\perp_{h_0}}$, of dimension $D = 2fs$ in its real span, by Lemma 200.2. Choose real independent lattice vectors successively of least length outside the previous span, possible by discreteness and full rank. Write λ_j for their lengths, which are nondecreasing. In their successive real flag take the orthogonal lines, and scale the j-th line by $1/\lambda_j$. A nonzero lattice vector first belonging to the j-th flag space had length at least λ_j; hence after scaling its length is at least one. In the scaled lattice the open balls of radius $1/2$ centered at lattice points are disjoint. The volume of one ball is at most the covolume, by integrating its periodization on a fundamental parallelotope. It contains an open cube of side $1/\sqrt{D}$. Thus $\prod_{j=1}^{D} \lambda_j \le D^{D/2} \operatorname{covol}(A)$.

Now choose the v_α successively of least length outside the E-span of their predecessors. At step α that span at infinity has real dimension $2f(\alpha-1)$, so one of the first $2f(\alpha-1)+1$ real independent lattice vectors lies outside it. Consequently $\|v_\alpha\|_\infty \le \lambda_{2f(\alpha-1)+1}$. Grouping the nondecreasing λ_j into consecutive groups of $2f$ gives the asserted product bound since $D^{D/2} V_E^s = (fs \operatorname{rd}_E)^{fs}$. By construction the vectors span the s-dimensional orthogonal complement over E, so their common orthogonal space is T. □

Theorem 200.4. *For every $0 \le s \le k$, with $k \ge 1$,*

$$\sum_{\substack{T \subset E^k \\ \operatorname{codim}_E T = s}} d_u^0(T)^{-(k+1)} \le (25fk \operatorname{rd}_E)^{f(k+1)s}.$$

The bound is independent of the finite unitary isometry u.

Proof. The summation set is countable since rational bases are finite lists in a number field. For nonzero $v \in L_u^{\mathrm{rat}}$, $h_0(v,v)$ is a totally positive algebraic integer by the local isometries and Lemma 52.1. The product of its f real conjugates is a positive integer by the integral multiplication determinant in the fixed field. Hence at least one conjugate is at least one, and $\|v\|_\infty \ge 1$. The same gives unit separation of distinct lattice points. In real dimension $D_0 = 2fk$, disjoint open balls of radius $1/2$ around the points of length at most R fit inside the ball of radius $R + 1/2$. Thus their count is at most $(1+2R)^{D_0}$. Put $p = 2f(k+1) > D_0$. Using $y^{-p} = p \int_y^\infty R^{-p-1} \, \mathrm{d}R$ for $y \ge 1$ and summing nonnegative terms gives

$$\sum_{0 \ne v \in L_u^{\mathrm{rat}}} \|v\|_\infty^{-p} \le p \, 3^{D_0} \int_1^\infty R^{D_0 - p - 1} \, \mathrm{d}R = (k+1) 3^{2fk} \le 25^{f(k+1)}.$$

For the last inequality use $k+1 \le 2^k$, by induction, and $2^k 9^{fk} \le 18^{fk}$ since $f \ge 1$.

For $s > 0$ choose one ordered normal basis from Lemma 200.3 for each T. Distinct subspaces give different tuples because their orthogonal equations recover T. Raise that lemma's inequality to power $k+1$ and divide by the product of vector lengths to power p. Summing the resulting upper bounds for $d_u^0(T)^{-(k+1)}$ is bounded by summing over all ordered s-tuples of nonzero vectors. The product of the s one-vector sums and $s \le k$ give the formula. For $s = 0$ the only subspace is the whole space, whose finite lattice has discriminant one by Lemma 200.2 and the zero-space convention. □

201. The subspace discriminant at the column level

Lemma 201.1. *Let u represent a class for K^0 in Definition 13.1, and $a \in K^0$. The class represented by ua at the column level has the following lattice on X_0:*

$$\widehat{\Lambda}_{ua} \cap X_{0,f} = \{ v \in L_u : \ e_k^{\mathsf{T}} a_{\mathfrak{p}}^{-1} u_{\mathfrak{p}}^{-1} v_{\mathfrak{p}} \equiv 0 \bmod \mathfrak{p} \text{ for every chosen } \mathfrak{p} \mid N \}.$$

Here $e_k^{\mathsf T}$ is the last standard row in the k definite coordinates. On the finite right coset space K^0/K_c with uniform measure, the covectors $e_k^{\mathsf T}a_{\mathfrak p}^{-1} \bmod \mathfrak p$ are independent and uniformly distributed among the nonzero covectors of $\mathbf F_N^k$.

Proof. At the identity class write the middle lattice in the basis of Definition 15.1. A linear combination of $v_* = x_k/\bar\pi_N + z/N$ and z with integral coefficients b, c has zero z-coordinate exactly when $b = -Nc$. Its resulting x_k-coordinate is $-\pi_N c$ because $\pi_N\bar\pi_N = N$. Thus on X_0 the first $k-1$ coordinates are arbitrary integral elements and the last is in $\pi_N\widehat O$. Lemma 13.2 identifies this ideal as the unit ideal off the chosen primes above N, and as the maximal ideal at each of those primes. Applying the isometry ua, which fixes z, gives the displayed lattice. At N the right cosets of K_c record the nonzero covectors in precisely these components by Definition 13.3. In a component $aK_c \mapsto e_k^{\mathsf T}a^{-1}$ gives a bijection onto the nonzero covectors, since its stabilizer is the last-row stabilizer and every nonzero row extends to a basis. The finite factors at the two other marked primes are independent factors in the product of coset spaces from Proposition 13.5. Counting in the product proves all distribution assertions. □

Theorem 201.2. *For $0 \le s \le k$ write*

$$\theta_s = \frac{N^s - 1}{N^k - 1} + \frac{N^k - N^s}{N^k - 1}\, N^{-(k+1)}.$$

Let A_u be the finite stabilizer $\Gamma^u \cap uK^0u^{-1}$, where Γ^u means the rational isometries of X_0. Let $I_c = [K^0 : K_c]$. The exact sum for the marked lattice of Definition 199.1 is

$$\sum_{c\in\mathcal C(K_c)} \sum_{\operatorname{codim} T=s} d_c(T)^{-(k+1)} = I_c\,\theta_s^f \sum_{[u]\in\mathcal C(K^0)} \frac{1}{|A_u|} \sum_{\operatorname{codim} T=s} d_u^0(T)^{-(k+1)}.$$

Both inner sums run over all rational subspaces of X_0 of the given codimension.

Proof. Fix u, T on the right. In the $u_{\mathfrak p}$-frame the lattice $L_u \cap T_f$ reduces at a chosen $\mathfrak p \mid N$ to a space of dimension $k - s$: the intersection with a rational subspace is saturated, and extension of a local basis as in Theorem 196.4 preserves its rank on reduction. Among the $N^k - 1$ nonzero covectors, exactly $N^s - 1$ vanish on this space, by extending its basis over the residue field. In this case imposing the row congruence has index one in the local intersection. Every other covector maps it onto $\mathbf F_N$ and gives index N. An index multiplies its discriminant by that index by the finite basis determinant of Definition 196.1. The mean of the factor in its inverse $(k+1)$-st power is therefore θ_s; the f independent factors multiply by Lemma 201.1.

The finer classes above $[u]$ are the orbits of A_u on K^0/K_c, acting via conjugation by u on the left. Each orbit has $|A_u|$ elements since all fine stabilizers are trivial by Proposition 13.5. The inner sum over T is constant on any such orbit: a rational isometry changes T bijectively and preserves the finite and infinite hermitian determinants. Thus the sum over the finer classes is obtained by multiplying the coset mean by $I_c/|A_u|$. All terms are nonnegative, so one may exchange the mean and the subspace sums by increasing finite sums. Theorem 200.4 gives their finiteness. Summing over the finite set of coarse classes proves the formula. □

Corollary 201.3. *Put $C = |\mathcal C(K_c)|$. For $k \ge 2,\ 0 \le s \le k$,*

$$\sum_{c,T:\,\operatorname{codim} T=s} d_c(T)^{-(k+1)} \le C\,(2N^{s-k})^f (25fk\,\mathrm{rd}_E)^{f(k+1)s}.$$

At the actual arithmetic scales, for every fixed $\eta > 0$ and all sufficiently large x, uniformly on the window and in the integer s, the sum is at most

$$CN^{-kf}\, b^{\eta kf(s+1)}.$$

In particular it is at most $CN^{-kf}T_0^{s+1}$ for $T_0 = b^{kf/4}$. At $s = 0$ its exact value is $CN^{-(k+1)f}$.

Proof. Since $k \ge 2,\ N \ge 2$, one has

$$\theta_s \le \frac{N^s + N^{-1}}{N^k - 1} \le N^{s-k}\frac{1 + N^{-1}}{1 - N^{-k}} \le 2N^{s-k}.$$

Counting the coarse fibres in the proof of Theorem 201.2 gives $\sum_{[u]} I_c/|A_u| = C$. That theorem and Theorem 200.4 give the first inequality. For the second take logarithms after removal of CN^{-kf} and divide by $kf(s+1)\log b$. The result is bounded above by

$$\frac{\log 2}{k \log b} + \frac{\log N}{k \log b} + (1 + 1/k)\frac{\log(25 f k \operatorname{rd}_E)}{\log b}.$$

By Lemma 55.1 one has $\log N \le 60n \log b$ with $k/n^9 \to \infty$. The final quotient tends to zero since f, k grow at most polynomially in x by Proposition 10.5, and $\log \operatorname{rd}_E = O(1 + \log x) = o(\log b)$ by Proposition 12.2. These estimates are uniform throughout its prescribed window. For $s = 0$ the single coarse discriminant is one and $\theta_0 = N^{-(k+1)}$, giving the exact result. □

202. Arithmetic representatives with bounded real frames

Definition 202.1. In the scaled real coordinates of Proposition 20.5 let F_0 denote the matrix of the $n = \nu + 1$ standard frame columns. Use indices $1, \dots, n$, so the rational scales are $d_{\tau a} = \kappa_\tau^{n-a}/\sqrt{N}$. For $y \in G_\infty$ write $Y_{\tau b}(y) = y_\tau^{-1} F_{0,b}$ and $S_\tau(y) = S(y_\tau)$ of Theorem 124.2. Put

$$R = b^{6n+2}, \qquad \rho = b^{2n^2}, \qquad \mathcal{E} = \{y : S_\tau(y) \le R \text{ for every } \tau\}.$$

Let $\mathcal{B}$ consist of $y \in \mathcal{E}$ with $\|Y_{\tau b}(y)\| \le \rho$ for all τ, b, in the ordinary Euclidean norm of the scaled coordinates. Denote by Γ_c^0 the subgroup of Γ_c fixing every rational frame vector; it is the kernel of the action of Γ_c on the frame.

Lemma 202.2. *Uniformly in c, for $1 \le i < j \le n$ and $a \in gNO_E$ there is an element of Γ_c acting on the rational frame by $1 + aE_{ij}$. For every $y \in G_\infty$ and all sufficiently large arithmetic scales there is $v \in \Gamma_c$ such that*

$$\|Y_{\tau b}(v_\infty^{-1} y)\| \le b^{60n+26} S_\tau(y) \qquad \text{for every } \tau, b.$$

Consequently every $y \in \mathcal{E}$ has a Γ_c-translate in $\mathcal{B}$.

Proof. For $j < n$ use the indicated elementary matrix on P, its inverse adjoint on the dual and identity on the middle. The matrix and its inverse preserve the class lattice and are identity modulo g. For $j = n$ use the i-th layer of Lemma 20.4 with row l equal to a on z and zero on the other intervening basis vectors. Its adjoint column there is $(\bar{a}/N)z$ after the sign in that lemma, and its top-right entry can be taken to be $\eta a \bar{a}/N$, where $\eta + \bar{\eta} = 1,\ \eta \in O_E$ as in Lemma 70.1. The trace condition holds since the form at z is $-N$. The z-coordinates of integral vectors of the middle lattice belong to $N^{-1}\widehat{O}$ by its basis in Definition 15.1, unchanged under the class isometry on X_0. Thus this row takes values in $g\widehat{O}$, the adjoint column lies in $g\widehat{\Lambda}_c$, and the top-right entry is a multiple of g. This gives the required principal arithmetic element, inducing precisely the indicated addition on the frame.

Every tuple $(z_\tau)_{\tau \in \Phi} \in \mathbb{C}^\Phi$ can be approximated in each coordinate to within $f \max(1, |\omega|)$ by a tuple $(\tau a_0)_\tau$ with $a_0 \in O_E$. Indeed the $2f$ elements $\omega^e \zeta_{N_1}^j$ in the proof of Lemma 12.1 give a real basis of the embedding space by the nonzero embedding determinant. Round its real coefficients to nearest integers. Each basis element has norm at most $\max(1, |\omega|)$ at every place, giving the bound.

At each τ take u_τ in the real unipotent with $\|u_\tau y_\tau\|_{\mathrm{op}} \le S_\tau(y) + 1$ by the infimum and the compact-middle argument in Theorem 124.2. Let A_τ be its upper unitriangular matrix on the scaled frame. Right multiplication of A_τ by a rational elementary shear changes its ij-entry by $(d_{\tau j}/d_{\tau i}) g N \tau a_0$. The other changes are confined to rows $< i$ in column j. For each column in turn, processing the entries above its diagonal from bottom to top,

approximate the negative entry after division by the nonzero scaled step. Lemma 55.1 gives $|d_{\tau j}/d_{\tau i}| \le 1$; also $N \le b^{60n}$, $g \le b^{22}$ and $f\max(1,|\omega|) \le b$ eventually by Propositions 12.2 and 10.5. Thus every final entry of $A_\tau A(v_\infty)_\tau$ has absolute value at most b^{60n+23}, with v the product of the chosen shears. The frame at $v_\infty^{-1}y$ equals the frame at uy times this product of triangular matrices, since $y^{-1}v_\infty F_0 = (uy)^{-1}uv_\infty F_0$. Every column of the frame at uy has norm at most $S_\tau(y)+1$, by equality of the operator norms of a J-unitary and its inverse. The triangle inequality, $S_\tau(y) \ge 1$ and $2n \le b^3$ give the result. Finally $b^{60n+26}R \le \rho$ eventually, and S_τ is unipotent-invariant. □

Proposition 202.3. *There are Borel functions $\omega_c, \beta_c : G_\infty \to [0,1]$ such that ω_c is invariant on the left by the real unipotents fixing F_0 pointwise, $\beta_c \le \omega_c$, and*

$$\sum_{\gamma\in\Gamma_c} \beta_c(\gamma_\infty y) = 1, \qquad y \in \mathcal{E},\ \omega_c(y) > 0 \implies y \in \mathcal{B}.$$

For any nonnegative Borel function H invariant under the full real unipotent U_∞,

$$\int_{G_\infty} \beta_c(y)H(y)\,\mathrm{d}y = \mathfrak{v}\int_{U_\infty\backslash G_\infty} H(y)\,\mathrm{d}\bar{y}.$$

With these cutoffs, the cut matrix integral summed over each arithmetic double orbit still has the value in Lemma 193.2.

Proof. Let $J(y)$ indicate the frame bounds of Lemma 202.2 at y itself. Set

$$m(y) = \sum_{v\in\Gamma_c^0\backslash\Gamma_c} J(v_\infty y).$$

This is independent of coset representatives and invariant under Γ_c by right multiplication on the left cosets. It is at least one by that lemma. For fixed y finiteness follows from

$$(v_\infty y)^{-1}F_0 = y^{-1}F_0A(v_\infty)^{-1}.$$

The frame $y_\tau^{-1}F_0$ is injective, so its matrix has a left inverse at each place. Hence the bound in J, with the fixed $S_\tau(y)$, bounds every entry of $A(v_\infty)^{-1}$. In rational coordinates these entries are integral since v^{-1} preserves the class lattice and the frame is part of its local basis. The scaled entries differ by fixed nonzero factors across the embeddings. The embedding lattice O_E is discrete by Lemma 12.1, so only finitely many rational frame matrices occur. Their fibres are exactly the left cosets of Γ_c^0.

Put $\omega_c = J/m$, Borel by the countable sum and continuity of the frames and S_τ. Both J and each summand of m are invariant under a real unipotent that fixes the frame, since it and its conjugate by a real unipotent fix the frame by Lemma 129.1. The sum of ω_c over the cosets is one. Choose a Borel domain of left translates for the discrete group Γ_c^0 on G_∞ by the construction of Lemma 21.4, and let β_c be its indicator times ω_c. Summing first over Γ_c^0 proves the partition assertion. Its support implication follows from Lemma 202.2 and the definition of J.

For fixed y, tile U_∞ by the left translates of a domain for Γ_c. Summing β_c at the translates gives $\int_{U_\infty}\beta_c(uy)\,\mathrm{d}u = \mathfrak{v}$ by Proposition 20.5. Integrating over the unipotent quotient, using Lemmas 123.2 and 123.1, gives the stated formula. For an arithmetic double orbit apply the same two tilings as in Lemma 193.2, first to the norm. Its quotient integral there is finite. The two new partitions sum to one everywhere, so the integrals for the new weights converge absolutely on that orbit and unfolding gives its previous value. The same argument for the norm shows that an orbit of zero absolute mass before the change still has zero absolute mass. □

203. Separation of bounded rational frames

Lemma 203.1. *Let c,d,d' be column classes, $s \in \mathcal{G}$, $\gamma \in \mathcal{T}(c,s,d)$, $\delta \in \mathcal{T}(c,s,d')$. If $\gamma f_b = \delta f_b$ for all $1 \le b \le n$, then $d = d'$ and $\gamma^{-1}\delta \in \Gamma_d^0$.*

Proof. The element $v = \gamma^{-1}\delta$ has $h_d^{-1} v_f h_{d'} \in K_Y$ by Definition 188.1. It fixes the frame, so it stabilizes $P \subset P^\perp$ and acts on $P^\perp/P = M_0$ by an isometry fixing z. This is the extension of a rational isometry a of $X_0 = z^\perp \cap M_0$. Projection of $P^\perp \cap \widehat{\Lambda}_d$ on M_0 gives exactly its full middle lattice, since the two outer summands split integrally by Definition 15.1. The induced map therefore carries the middle lattice for d' onto that for d, with principal congruence in representative coordinates modulo g. The same follows for the inverse. Extending it by the identity on the outer pairs places $h_d^{-1} a_f h_{d'}$ in the column level by Proposition 66.1. The two double classes consequently coincide. For a single class the rational isometry a lies in its stabilizer, which is trivial by Proposition 13.5. The action of v on the flag and its middle quotient is now the identity, including every outer graded line. Hence $v \in U(F)$ by Definition 20.3. Its finite position lies in K_Y and it fixes the frame, so $v \in \Gamma_d^0$ by Definition 202.1. □

Proposition 203.2. *In the situation of Lemma 203.1, suppose $y \in G_\infty$ and both $\gamma_\infty^{-1} y$ and $\delta_\infty^{-1} y$ belong to $\mathcal{B}$. Unless $d = d'$ and $\gamma^{-1}\delta \in \Gamma_d^0$,*

$$\max_{\tau,b} \left\| Y_{\tau b}(\gamma_\infty^{-1} y) - Y_{\tau b}(\delta_\infty^{-1} y) \right\| \;\geq\; \varepsilon_0, \qquad \varepsilon_0 = (N\rho)^{-1}.$$

This holds uniformly for all sufficiently large arithmetic scales.

Proof. Write $z_0 = \gamma_\infty^{-1} y$, $v = \gamma^{-1}\delta$ and $\Delta_{\tau b} = z_{0,\tau}^{-1}(v_{\infty,\tau} - 1) F_{0,b}$. Suppose the displayed maximum is smaller than ε_0. Both $v f_b$ and f_b belong to the integral row lattice for d, since v_f carries the lattice for d' onto it. Thus every $[f_a, (v-1) f_b]$ belongs to O_E by the basis in Definition 15.1. The scaled hermitian matrix has operator norm one: it exchanges the outer paired coordinates and acts by signs on the middle. Invariance under $z_{0,\tau}$ and the scales of Definition 202.1 imply

$$|\tau[f_a, (v-1) f_b]| \leq |d_{\tau a} d_{\tau b}|^{-1} \|Y_{\tau a}(z_0)\| \, \|\Delta_{\tau b}\| < N\rho\varepsilon_0 = 1.$$

Here $|\kappa_\tau| \geq 1$ by Lemma 55.1. An algebraic integer with this bound at every CM embedding is zero: the absolute value of its rational norm is the product of the squared moduli and, for a nonzero integer of E, is at least one by Lemma 12.1. Hence $w_b = (v-1) f_b$ is orthogonal to P and z, and so belongs to $P \oplus X_0$. Also $[w_b, w_b] = 0$ by $[v f_b, v f_b] = [f_b, f_b]$ and $[w_b, f_b] = 0$. Positivity on X_0 gives $w_b \in P$.

Write $w_b = \sum_i a_{ib} e_i$. Each $a_{ib} \in O_E$ because the outer summand has the standard lattice in every class. For any nonzero column choose its largest index i with $a_{ib} \neq 0$. Choose $u_\tau \in \mathcal{U}$ with $\|u_\tau z_{0,\tau}\|_{\mathrm{op}} \leq R + 1$ as in Theorem 124.2. By the upper unitriangular frame action of Lemma 129.1, the i-th coordinate of $u_\tau z_{0,\tau} \Delta_{\tau b}$ is $(d_{\tau b}/d_{\tau i}) \tau a_{ib}$. Therefore

$$|\tau a_{ib}| \leq b^{23n} (R+1) \varepsilon_0 < 1$$

at every embedding. The first inequality uses $|\kappa_\tau| \leq b^{23}$; the second holds eventually since $n \to \infty$, $R = b^{6n+2}$, $\rho = b^{2n^2}$, and $N \geq 1$ by Lemma 55.1 and Proposition 10.5. The norm argument gives $a_{ib} = 0$, a contradiction. Every column is therefore fixed by v, and Lemma 203.1 applies. □

Theorem 203.3. *Fix c, s, two real points $y, y' \in G_\infty$, and $x \geq 0$. Take distinct pairs (d_p, δ_p) of a column class and a rational group element, with weights $w_p \geq 0$. For each positive weight suppose there is $\gamma_p \in \mathcal{T}(c, s, d_p)$ with $\delta_p \in \gamma_p \Gamma_{d_p}$, $\gamma_{p,\infty}^{-1} y \in \mathcal{B}$, and $w_p \leq x \beta_{d_p}(\delta_{p,\infty}^{-1} y')$, using Proposition 202.3. Suppose all their frames $(Y_{\tau b}(\gamma_{p,\infty}^{-1} y))_{\tau,b}$ belong to a single translate of a real subspace of dimension t. Then, uniformly for all sufficiently large scales,*

$$\sum_p w_p \leq b^{6n^2 t} x.$$

Proof. Use the maximum of the Euclidean column norms. If any positive weight occurs, fix its frame as center. Every frame is within 2ρ of this one. By Proposition 203.2 distinct frames have distance at least ε_0, since membership of $\gamma_p^{-1}\gamma_q$ in the indicated pointwise

stabilizer gives identical frames. In the affine t-plane take disjoint open norm balls of radius $\varepsilon_0/2$ about any finite set of distinct frames. They fit inside the ball of radius $2\rho+\varepsilon_0/2$. Lebesgue measure in linear coordinates scales as the t-th power of the radius by change of variables. The unit ball has finite positive measure since the coordinate and the column norms are equivalent in finite dimension, as follows by taking the maximum and positive minimum on a Euclidean unit sphere. Thus there are at most $(1+4N\rho^2)^t\le b^{6n^2t}$ distinct frames, using $N\le b^{60n}$ in Lemma 55.1. For $t=0$ there is at most one.

In the fibre of any frame all column classes equal d by Lemma 203.1, and, fixing one representative γ_*, every δ_p belongs to $\gamma_*\Gamma_d$. Distinct pairs in this fibre give distinct elements of that coset. Their weights sum to at most $x\sum_{u\in\Gamma_d}\beta_d(u_\infty^{-1}\gamma_{*,\infty}^{-1}y')=x$ by Proposition 202.3. Adding over the frames proves the bound, including the empty family. □

204. Full Poincaré columns and the error of archimedean truncation

Definition 204.1. Let $F_\infty(y)=\bigotimes_{\tau\in\Phi}F_\tau(y_\tau)$ for the normalized vectors of Theorem 123.3. Write V_∞ as in Definition 188.1. For $W=F_\infty$ or V_∞ set $K_\delta^W(y)=W(y)W(\delta_\infty^{-1}y)^*$ and $z_\delta=\delta_\infty^{-1}y$. For any set B of rational group elements put

$$A_B^W(c,s,d)=\sum_{\delta\in B\cap\mathcal{T}(c,s,d)}\int_{G_\infty}\beta_c(y)\beta_d(z_\delta)K_\delta^W(y)\,\mathrm{d}y$$

whenever the integrals and sum converge absolutely in the Hilbert–Schmidt norm. Here and below β_c is chosen by Proposition 202.3. For a family of matrices $A(c,s,d)$ write

$$\mathcal{E}_\Omega(A)=\mathfrak{v}^{-2}\sum_{c,d}\sum_{s\in\Omega}\|A(c,s,d)\|_{\mathrm{HS}}^2\qquad(\Omega\subset\mathcal{G}).$$

Proposition 204.2. *Let A_τ,λ_τ be as in Theorem 124.2. Put $A(y)=\prod_\tau A_\tau(y_\tau)$, $\lambda=\bigotimes_\tau\lambda_\tau$. For all sufficiently large scales and all c,s,y,*

$$\sum_d\sum_{\delta\in\mathcal{T}(c,s,d)}\beta_d(z_\delta)A(z_\delta)\le b^{12rn^3f}\lambda(G_\infty).$$

The integral of the left side against $\beta_c(y)A(y)\,\mathrm{d}y$ is at most $\mathfrak{v}J_c\,b^{(12rn^3+2r^3)f}$.

Proof. The upper bound in Theorem 124.2 and the rectangle formula for the product measure give $A(z_\delta)\le\int\mathbf{1}_{\mathcal{E}}(z_\delta h)\,\mathrm{d}\lambda(h)$. Fix h. Each nonzero summand of weight $w_{d,\delta}=\beta_d(z_\delta)\mathbf{1}_{\mathcal{E}}(z_\delta h)$ has a representative $u\in\Gamma_d$ with $u_\infty z_\delta h\in\mathcal{B}$ by Lemma 202.2. The rational matrix $\gamma=\delta u^{-1}$ belongs to $\mathcal{T}(c,s,d)$ and has $\gamma_\infty^{-1}yh\in\mathcal{B}$. Apply Theorem 203.3 at the two points yh,y, with $x=1$ and the whole real frame space. There are fn complex columns of length r, so its dimension is $2frn$. This bounds the sum for this h by b^{12rn^3f}. The weights themselves are measurable in h. Integrating the pointwise bound and using Lemma A.2 gives the first assertion. For the second, Proposition 202.3 integrates β_cA to $\mathfrak{v}\prod_\tau\int A_\tau\,\mathrm{d}\bar{y}_\tau$ since A is unipotent invariant. Multiply by the mass of λ and use the integral bound of Theorem 124.2 in each factor. The product of the dimensions there is J_c by Definition 69.1. □

Theorem 204.3. *The amplitudes of Definition 204.1 converge absolutely. Uniformly in c,s, for all sufficiently large scales,*

$$\sum_d\sum_{\delta\in\mathcal{T}(c,s,d)}\int\beta_c(y)\beta_d(z_\delta)\|K_\delta^{F_\infty}(y)-K_\delta^{V_\infty}(y)\|_{\mathrm{HS}}\,\mathrm{d}y\le\tfrac12\mathfrak{v}b^{-k^3f}.$$

For each fixed $D>0$, uniformly in B,Ω, eventually

$$\mathcal{E}_\Omega(A_B^{F_\infty}-A_B^{V_\infty})\le E_1b^{-D}.$$

Proof. For column vectors a,b', summing the squared absolute values of the entries gives $\|ab'^*\|_{\mathrm{HS}}=\|a\|\,\|b'\|$. The norm of a pure tensor is the product of its factor norms by the same coordinate calculation. Each kernel is bounded by $A(y)A(z_\delta)$, which gives absolute

convergence by Proposition 204.2. If the kernels differ then at least one real place has different full and cut vectors at one of y, z_δ. At such an argument both local norms are at most $b^{-3r^3 f} A_\tau$ by Theorem 124.2. Thus the norm of the difference is at most $2b^{-3r^3 f} A(y)A(z_\delta)$, everywhere. The integrated bound becomes

$$2\mathfrak{v} J_c b^{(12rn^3 - r^3)f}.$$

Use $J_c \le b^{2kf}$ from Lemma 69.2. One has $r = k + 2n - 1$, $k/n^9 \to \infty$, $n \to \infty$ by Proposition 10.5. Thus $r^3 - k^3 = (2n-1)(r^2 + rk + k^2)$ exceeds $12rn^3 + 2k + 1$ eventually. This gives the first bound since $b^f \ge 4$ eventually.

For fixed c, s the sum over d of the squared norms of the differences is bounded by the square of the sum of the norms, hence by $\mathfrak{v}^2 b^{-2k^3 f}/4$. There are C choices of c. The finite level groups embed in the product of the f matrix spaces of rank r over each of $\mathbf{F}_{g_0}, \mathbf{F}_{g_1}$ by Proposition 15.2, so $|\mathcal{G}| \le g^{fr^2} \le b^{22fr^2}$ by Lemma 55.1 and the level-prime bounds. Since $E_1 = |\bar{S}| C J_c \ge C$ by Definition 69.1, the claim follows from $(2k^3 - 22r^2)f \to \infty$. □

Theorem 204.4. *Periodize each coordinate of the seed $\mathbf{1}_{h_c K_Y} \beta_c F_\infty$ by the formula of Theorem 21.5. This gives bounded square-integrable automorphic columns. Their matrix of translated inner products at reduction s is $A_\Gamma^{F_\infty}(c, s, d)$, with counting measure on G_f/K_Y. The full columns are reproduced by right convolution with the product of the kernels $e_{J,\tau}$ of Theorem 122.2.*

Proof. Fix a finite argument hK_Y. If no rational element takes it to $h_c K_Y$, the series vanishes. Otherwise fix one such γ_0; all the other elements are $v\gamma_0$ with $v_f h_c K_Y = h_c K_Y$. These v constitute $\mathcal{T}(c, 1, c)$, a group closed under inversion. The sum of the norms of the seed terms is bounded uniformly in h, y by Proposition 204.2 at $\gamma_{0,\infty} y$. Absolute convergence gives measurability and left rational invariance by reindexing. To obtain the square integral, periodize the norm of the seed, expand its square on a domain as a nonnegative double sum, and unfold the first index by Lemma 21.4. The seed has support in a single finite component. The resulting single sum of norm overlaps is precisely the $d = c, s = 1$ subfamily in the second bound of Proposition 204.2, with each A replaced by the smaller $\|F_\infty\|$. Thus the periodization belongs to L^2. The measurable-seed assertion of Theorem 21.5, with absolute integrability from Theorem 204.3, gives the stated matrix.

Group the original series by left Γ_c-cosets of the rational elements taking hK_Y to $h_c K_Y$. The character is trivial on Γ_c by Proposition 21.2, and the cutoff sums to one by Proposition 202.3. Thus each coset gives the corresponding full vector, and its norm equals the sum of the seed norms on that coset. The sum of these norms is uniformly bounded by the preceding argument even with any real right translate of y. The local kernels are integrable by Corollary 96.4. Multiplying their product by this uniform bound permits interchange of convolution and the absolutely summable coset series by Lemmas A.2 and 123.1. The local reproduction identity in Theorem 122.2 applies also after normalization, hence to every coset tensor. This proves reproduction. □

Corollary 204.5. *Let $S(F) = U(F)H(F)$, where $H(F)$ consists of the rational isometries of X_0 extended by the identity on z and the outer pairs. Then*

$$A_{S(F)}^{F_\infty}(c, 1, d) = \mathbf{1}_{c=d}\, \mathfrak{v} I.$$

For either choice of W, the amplitudes on a set stable under rational unipotents on both sides vanish at every nonagreeing finite reduction.

Proof. If $\delta = ua \in S(F)$ belongs to $\mathcal{T}(c, 1, d)$, the map it induces on $P^\perp/P$ is a. The integral projection and stabilizer argument of Lemma 203.1 gives $c = d$, $a = 1$. The terms for $c = d$ are exactly Γ_c. Summing the second cutoff and using triviality of its phase gives $\int \beta_c(y) F_\infty(y) F_\infty(y)^* \, \mathrm{d}y$. Apply Proposition 202.3 entrywise, by decomposition into real and imaginary parts, which are integrable by Theorem 123.3. The last theorem gives the identity endomorphism in each factor after quotient integration. This proves the formula.

For finite disagreement take the two rational unipotents used in Proposition 198.2. On each arithmetic double orbit unfold the two cutoffs on their left domains as in Lemma 193.2. For the full kernel the identical tiling is justified first for its norm, whose sum of weighted integrals is finite by Theorem 204.3. The rational unipotents permute the double orbits and multiply their integrals by the same character value different from 1. Absolute convergence permits summing over the permutation. Thus the total vanishes. □

205. Archimedean sizes of the arithmetic frame coordinates

Definition 205.1. Call a rational element γ near if there exists $y \in \mathcal{B}$ with $\gamma_\infty^{-1} y \in \mathcal{B}$. For $\gamma \in \mathcal{T}(c, s, c')$ choose one representative of each double coset $\Gamma_c^0 \gamma \Gamma_{c'}^0$, fixing the choice for each pair c, c'. Nearness is constant on these cosets: left multiplication of a real argument by an element of $\mathcal{U}_0$ preserves its frame and its size S by Definition 202.1 and Lemma 129.1. Fix also a section r of the finite set of left cosets $\mathcal{G}/\bar{S}$. The label of s is $t = r(s\bar{S})^{-1} s \in \bar{S}$. At level j consider the near chosen representatives of the levels in Definition 197.1. Thus $j = n$ means $\mathfrak{d} \neq 0$, and at a singular level $0 \le j < n$ we use the agreeing flags of Theorem 194.3.

Lemma 205.2. *For a place τ and $y \in \mathcal{B}$ the matrix $M = y_\tau^{-1} F_0$ has operator norm at most $\sqrt{n}\,\rho$. On its range its inverse into $\mathbb{C}^n$ has operator norm at most*

$$(R+1)^n (\sqrt{n}\,\rho)^{n-1}.$$

Proof. The sum of the squared column norms is at most $n\rho^2$. Cauchy–Schwarz gives the first assertion. Choose $u \in \mathcal{U}$ with $\|u y_\tau\|_{\mathrm{op}} \le R+1$ by Theorem 124.2. The map $M' = (u y_\tau)^{-1} F_0$ satisfies $\|M' a\| \ge \|a\|/(R+1)$, since F_0 is an isometric inclusion in the Euclidean norm. By Lemma 129.1, $M' = M A(u)^{-1}$ with $\det A(u) = 1$. Consequently $\det(M^* M) = \det(M'^* M') \ge (R+1)^{-2n}$. Indeed the positive Gram matrix on the right has all its eigenvalues at least $(R+1)^{-2}$ by the norm inequality. A positive hermitian matrix is unitarily diagonalizable: maximize its quadratic form on the unit sphere to obtain an eigenvector by differentiation in orthogonal directions and apply induction on the invariant orthogonal complement. The square roots of the Gram eigenvalues of M have product at least $(R+1)^{-n}$ and each is at most $\sqrt{n}\,\rho$. This bounds the smallest below by $(R+1)^{-n} (\sqrt{n}\,\rho)^{1-n}$ and proves the claim. □

Proposition 205.3. *Uniformly for near arithmetic terms, at all sufficiently large arithmetic scales and for all $\tau \in \Phi,\ a, b,\ 0 \le j \le n$,*

$$|\tau Z_{ab}(\gamma)| \le b^{5n^2}, \qquad |\tau D_j(\gamma)|^2 \le b^{11 n^3}.$$

At singular level $j < n$, the coordinates of $D_j v_b$ in the original frame, for every $b > j$ in Proposition 195.4, have modulus at most $b^{10 n^3}$ at each embedding. The same bound holds for these vectors after multiplication of γ on the left by any rational unipotent fixing the original frame pointwise.

Proof. Take $y, \gamma_\infty^{-1} y \in \mathcal{B}$. Pair their scaled frame columns as in Proposition 203.2. This gives $|\tau Z_{ab}| \le N\rho^2 \le b^{5n^2}$ eventually, since $\log N \le 60 n \log b$ by Lemma 55.1. Expansion in permutations gives $|\tau D_j| \le j! b^{5jn^2}$; $j \le n$ and $\log(n!) \le n \log n = o(n^3 \log b)$ give the asserted square bound, uniformly and including $j = 0$.

Write a' and a for the coordinate columns of $D_j v_b$ in the transported and original frames. By Proposition 195.4, $a' = \mathrm{adj}(P_j)\varepsilon_b$. The same permutation estimate bounds the modulus of every conjugate entry of a' by b^{6n^3}. For each τ, put $M = y_\tau^{-1} F_0$ and $M' = (\gamma_\infty^{-1} y)_\tau^{-1} F_0$. The equality of the two rational expressions for $D_j v_b$ gives, in scaled coordinates,

$$M(\tau a_i / d_{\tau i})_i = M'(\tau a'_i / d_{\tau i})_i.$$

Lemma 205.2 applied to M and M' therefore bounds each $|\tau a_i|$ by

$$b^{23n}\, n\rho\, (R+1)^n (\sqrt{n}\,\rho)^{n-1} b^{6n^3}.$$

Here b^{23n} bounds every ratio $|d_{\tau i}/d_{\tau l}|$ by Lemma 55.1. The logarithm to base b of this expression is $8n^3 + O(n^2 + n\log n)$ since $R = b^{6n+2}$, $\rho = b^{2n^2}$. This proves the bound. A left unipotent fixing every f_i preserves Z and fixes v_b, since the latter belongs to their span. Thus it preserves all the coordinates in question. □

Corollary 205.4. *Among near arithmetic terms there are at most b^{12fn^4} different matrices Z, eventually. Put $M_b = \lceil 3b^{10n^3} \rceil$. If two such terms of singular level j have the same Z and, after possibly different left frame-fixing unipotents, the outer coordinates of $D_j v_b$ for the two frames agree modulo $M_b O_E$ for $b > j$, then all these outer coordinates are equal over E.*

Proof. Two distinct elements of O_E have distance at least one in the maximum of their conjugate moduli, by the norm argument of Proposition 203.2. The real dimension of the space of conjugates is $2f$. The disjoint-ball argument of Theorem 203.3, now centered at zero with radius b^{5n^2}, bounds the number of choices for one entry by $(1+2b^{5n^2})^{2f}$. There are n^2 entries. Integrality from Theorem 192.2 and the first bound of Proposition 205.3 prove the first assertion. For the second, each difference is $M_b a$ with $a \in O_E$, and all conjugate moduli of a are at most $2b^{10n^3}/M_b < 1$ by the same proposition. Its norm forces $a = 0$. □

206. Counting integral maps by their finite images

Lemma 206.1. *Let $R_0 = \widehat{O}$, let $K \subset R_0^h$ be an R_0-submodule containing aR_0^h for a nonzero integral finite idele a, and write $Q_0 = R_0^h/K$. Let $L = L_1 \oplus L_2$ be a free R_0-module with the two summands free of ranks e, v, respectively. Consider an indexed family of linear maps $Y_p : L \to R_0^h$ and integers $m_p \geq 1$ divisible by $|(Y_p(L_1) + K)/K|$. Suppose that at most A_0 indices induce each particular map $L \to Q_0$. For every integer $\ell \geq e$,*

$$\sum_p m_p^{-\ell} \leq A_0 \, |Q_0|^{h+v} \leq A_0 \, I(a)^{h(h+v)}.$$

Proof. Every submodule $H \subset Q_0$ is generated over R_0 by at most h elements. To see this, work at the finitely many primes of a. The inverse image of H in each local free module contains $aO_{E,\mathfrak{p}}^h$; diagonal elimination of Lemma 25.1 provides h generators there. More explicitly the inverse image is finitely generated by lifts of its finite quotient together with the h generators of $aO_{E,\mathfrak{p}}^h$. Projection onto each prime factor is an R_0-idempotent, so the submodule is the product of its local components. Combine their generator lists coordinatewise, padding by zeros. It follows that there are at most $|Q_0|^h$ possible H, by counting ordered h-tuples. For a fixed image H of L_1, there are at most $|H|^e |Q_0|^v$ quotient maps from L: record the values on bases of its two summands. Every such map occurs at most A_0 times, and every associated weight is at most $|H|^{-\ell}$. Since $\ell \geq e$, the contribution per image is at most $A_0 |Q_0|^v$. Finally Q_0 is a quotient of R_0^h/aR_0^h, which has order $I(a)^h$ by Definition 196.1. □

Definition 206.2. Fix c, a level j, and an occurring Gram matrix Z as in Definition 205.1. Decompose the rational matrix of the transported frame as

$$(\gamma f_1, \ldots, \gamma f_n) = \begin{pmatrix} A_\gamma^{\mathrm{top}} \\ X_\gamma \\ t_Z \\ P_Z \end{pmatrix} \quad \text{in } P,\ X_0,\ Ez,\ P^*.$$

Here $P_Z = Z_{1:\nu,*}$, $t_Z = -Z_{n,*}/N$. Write $Y = P_j$ as in Proposition 195.4, also using $Y = Z$ at $j = n$, and set $d = \min(j, \nu)$, $D = \det Y = D_j \neq 0$. Put $L_M = M_{0,f} \cap \widehat{\Lambda}_c$, $L_X = X_{0,f} \cap \widehat{\Lambda}_c$, and define

$$K_Z = gP_Z^* \widehat{O}^\nu + gY^* \widehat{O}^n \ \subset \widehat{O}^n.$$

Since Z is integral by Theorem 192.2, the cofactor identity gives $g\bar{D}\widehat{O}^n \subset K_Z$. For a term $\gamma \in \mathcal{T}(c, s, c')$ use the integral map $L_M \longrightarrow \widehat{O}^n$, $v \mapsto F_\gamma(v)$ from Definition 199.1. Its

integrality follows by pairing two vectors in the integral row lattice. On P_O its image is $P_Z^* \widehat{O}^\nu$.

Lemma 206.3. *The columns of $P_Z Y^{-1}$ of index $b > d$ are zero. Its first d rows are $(I_d \;\; 0)$. For every such $b > d$, $X_\gamma Y^{-1} \varepsilon_b = 0$ at a singular level. At $j = n$, for $b = n$, this vector has Euclidean norm at most $N^{-1/2}$ at each embedding.*

Proof. The first d rows agree with those of Y, giving the second assertion. At a singular level $d = j$ and $(\gamma f_i)_i Y^{-1} \varepsilon_b$ belongs to the original frame span by Proposition 195.4. Both its P^*-coordinates and its X_0-coordinates therefore vanish. At $j = n$, P_Z is the first ν rows of $Y = Z$, giving the claim about its last column. For $a = Y^{-1}\varepsilon_n$ the vector $(\gamma f_i)_i a$ has all its P^*-coordinates zero and its z-coordinate equal to $-1/N$, since $Za = \varepsilon_n$. Its norm for the row form is $-N a_n \bar{a}_n$ because the transported frame has the same Gram matrix as the original one. Thus at every embedding $\|\tau(X_\gamma a)\|^2 = 1/N - N|\tau a_n|^2 \le 1/N$. □

Proposition 206.4. *Let γ, δ have the same row class c, level j and matrix Z, with possibly different column classes and reductions. If $F_\delta - F_\gamma$ sends L_M into K_Z, then*

$$X_\delta - X_\gamma = gLP_Z$$

for a rational matrix L whose columns belong to L_X at the finite places. There is $u \in \Gamma_c^0$ for which the frames $u\gamma f$ and δf agree on $X_0 \oplus Ez \oplus P^*$. Here f denotes the matrix of the original frame columns.

Proof. Set $X = X_\delta - X_\gamma$, considered also as a middle matrix with zero z-coordinate. For the form h_0 the middle lattice L_M is self-dual by Proposition 15.2 and its decomposition into outer and middle parts. In a local basis the difference of evaluation maps on that lattice is $X^* h_0$, evaluated on the chosen basis vectors. Express each of these columns in $gP_Z^* \widehat{O}^\nu + gY^* \widehat{O}^n$. Taking adjoints, the rows of coefficients give vectors in the dual middle basis. That basis consists of integral vectors since the Gram matrix is unimodular, also at conjugate local pairs. In particular, for some finite adelic matrices with columns in L_M one has $X/g = L_0 P_Z + L_0' Y$. There is an integral projection of L_M onto L_X: locally the latter is the intersection with a vector subspace and thus saturated. The basis extension by diagonal elimination in Lemma 25.1 extends a basis of this intersection to one of the whole lattice. Project along the remaining vectors, at each place. Its linear extension fixes X_0, so applying it gives the same equality with all columns of L_0, L_0' in L_X.

Choose a rational X_0-valued matrix μ, with columns in L_X, so close to L_0 at finite places that $(L_0 - \mu)P_Z Y^{-1}$ also has columns in L_X. These are open lattice conditions containing L_0. For clarity, clear denominators by an integer, approximate the integral coordinates at the finitely many exceptional places by Chinese remaindering in O_E, and divide back by that integer, including every prime of the denominator among the exceptional places. At all other places integrality suffices. This is the additive approximation in the proof of Lemma 21.1. Now $B = (X/g - \mu P_Z)Y^{-1}$ is rational with columns in L_X. For $b > d$, Lemma 206.3 gives $B\varepsilon_b = 0$ at singular levels. At $j = n$ the only such column has norm at most $2/(g\sqrt{N}) < 1$ at every embedding by the same lemma. Its squared hermitian length lies in O_E, since the column belongs to the integral middle lattice. The norm argument of Proposition 203.2 forces the length to vanish, hence the column is zero by definiteness. Since the first d rows of Y are the first d rows of P_Z, BY is a combination of those rows with coefficients the first d columns of B. Adding them to the corresponding columns of μ gives the required rational L.

In the three blocks P, M_0, P^*, write $C_0 = gL$ as a middle matrix with zero z-entry, put $C_0^\dagger = C_0^* h_0$, and take

$$u = \begin{pmatrix} I & -C_0^\dagger & -\eta C_0^\dagger C_0 \\ 0 & I & C_0 \\ 0 & 0 & I \end{pmatrix}, \qquad \eta + \bar{\eta} = 1, \quad \eta \in O_E,$$

using Lemma 70.1. Multiplying by the Gram matrix of the hyperbolic and middle summands verifies $u^*Ju = J$: the middle cross terms cancel and the remaining corner is $(1-\eta-\bar{\eta})C_0^\dagger C_0$. Each off-diagonal block takes the class lattice into g times that lattice by integrality of h_0 there. Its inverse does likewise by the finite series for the inverse of a unitriangular matrix. It fixes P, z since $C_0^\dagger z = 0$, so $u \in \Gamma_c^0$ by Definition 202.1. Its effect on the middle frame rows is addition of $C_0 P_Z$. The z- and dual rows already agree by their formulas in Definition 206.2. □

207. Congruence classes of rational frame orbits

Lemma 207.1. *Fix c, j, t as in Definition 205.1. Let $(c', \gamma), (c'', \delta)$ be chosen representatives with this label, and $u, v \in \Gamma_c^0$. If $u\gamma f = v\delta f$, then $c' = c''$ and $\gamma = \delta$.*

Proof. Write k, k' for the two finite positions of $u\gamma, v\delta$ from Definition 188.1. All the class representatives $h_c, h_{c'}, h_{c''}$ fix the frame, so $k^{-1}k'$ fixes the standard finite frame and preserves the full integral row lattice. Its action on the quotient $P^\perp/P$ fixes z and preserves the middle lattice. The diagonal extension of that quotient action, identity on the outer pairs, is integral with integral inverse by the direct lattice decomposition of Definition 15.1. Removing it leaves an integral unipotent by the flag characterization of Definition 20.3. Hence the reduction of $k^{-1}k'$ is in $\bar{S}$ by Proposition 66.1. If s, s' are the reductions of k, k', they have the same left coset and the same label t, so $s = r(s\bar{S})t = s'$. The left corrections have principal congruence, hence the original reductions are also s, s'. Apply Lemma 203.1 to $u\gamma, v\delta$. It gives $c' = c''$ and $(u\gamma)^{-1}v\delta \in \Gamma_{c'}^0$. Thus the two original terms have the same arithmetic double coset. The choice in Definition 205.1 identifies them. □

Proposition 207.2. *Fix Z, j and put $a = M_b g D\bar{D}$, with M_b from Corollary 205.4 and D from Definition 206.2. Take two near rational terms of these data and apply arbitrary left corrections in Γ_c^0 to them. Suppose their corrected middle rows agree and the difference Ξ of their corrected top rows has entries in aO_E. There exists a further left correction in Γ_c^0 making their entire frames equal.*

Proof. The lower rows of both corrected frames are P_Z: a correction fixes the original frame, hence preserves Z. Equality of their Gram matrices therefore gives

$$\Xi^* P_Z + P_Z^* \Xi = 0.$$

Write $P = P_Z$ in this proof. Its rank is $d = \min(j, \nu)$ by Lemma 206.3, since PY^{-1} has its first d rows $(I_d\ 0)$ and its final $n-d$ columns zero. In particular $\ker P$ is spanned by the last $n-d$ columns of Y^{-1}.

We claim that $\Xi \ker P = 0$. For $j = n$, $d = \nu$ and P^* is injective, so the Gram identity applied to $\ker P$ gives this claim. For $j < n$ the columns $\Xi\ \mathrm{adj}(Y)\varepsilon_b$, $b > j$, are differences of the outer coordinates of $D_j v_b$ for the two corrected frames. They lie in $M_b O_E^\nu$ since $\Xi \in aO_E^{\nu\times n}$, $gD\bar{D} \in O_E$, and Y is integral. Corollary 205.4 makes them zero. Division by D proves the claim in this case too.

Define $C_0 : E^\nu \to E^n$ by first retaining only the first d coordinates, padding them to length n by zero, and then multiplying by Y^{-1}. For each $x \in E^n$ the first d coordinates of $Y(x - C_0 Px)$ vanish. Thus $PC_0P = P$ by the preceding description of $\ker P$, and $\Xi C_0 P = \Xi$. Put $J_0 = PC_0,\ L = \Xi C_0$, and set

$$H = L - L^* - J_0^* L.$$

Since $J_0^* L = C_0^* P^* \Xi C_0$, this last summand is skew-hermitian by the Gram identity. Thus $H^* = -H$. Also $L^*P = -J_0^*\Xi$, and consequently $HP = \Xi$. Every entry of DC_0 is integral. The entries of L, L^* are multiples of g in O_E, because Ξ is divisible by $gD\bar{D}$. The same is true of $J_0^* L$, whose possible denominator is $D\bar{D}$. In the three blocks $P_{\mathrm{out}}, M_0, P_{\mathrm{out}}^*$, where P_{out} denotes the outer isotropic space, the matrix with identity diagonal and sole off-diagonal block H in the top right is therefore integral and principal in every class. Skewness is

precisely its isometry condition, by multiplying the hyperbolic Gram matrix. It fixes the original frame and adds $HP = \Xi$ to the top rows. Choosing its sign to match the order of subtraction aligns the two frames. □

Theorem 207.3. *Fix a row class c, a finite label t, and a level j. Let $\mathcal{A}$ be any subset of the pairs (c', γ) of near chosen representatives with this label and level as in Definition 205.1. Fix an E-subspace $T \subset X_0$ of codimension at most s, $0 \le s \le k$. Assume $o_{\gamma,g}(T)$ is finite for every member and choose a positive integer m'_γ divisible by it. Then at all sufficiently large arithmetic scales, uniformly in these choices,*

$$\sum_{(c',\gamma)\in\mathcal{A}} (m'_\gamma)^{-k} \le b^{100fn^4(n+s+1)}.$$

Proof. By Corollary 205.4 there are at most b^{12fn^4} choices of Z. Fix one of them and use $a = M_b g D \bar{D}$, $K = K_Z$. On the full middle lattice L_M use the maps F_γ of Definition 206.2. Its submodule $L_1 = L_c(T)$ is saturated locally, so its local bases extend to bases of L_M by Lemma 25.1. Combining these yields a free summand over $\widehat{O}$ of rank $e = \dim T$ with a complementary free summand of rank $k + 1 - e \le s + 1$. Since K contains $gF_\gamma(P_O)$, the order of the image of L_1 modulo K divides $o_{\gamma,g}(T)$: the former is the order of a quotient of the finite group in Definition 199.1.

For each nonempty fibre of the induced maps on L_M modulo K, fix one member. Use Proposition 206.4 to translate every member on the left by Γ_c^0 to have the same middle frame rows as this member. Its top rows are integral over O_E since the full frame is integral in the class lattice at all finite places and its outer lattice is standard. There are at most $I(a)^{\nu n}$ choices of their residue matrix modulo a, by Definition 196.1. Proposition 207.2 and Lemma 207.1 give at most one original pair for each such residue matrix. Thus we may use $A_0 = I(a)^{\nu n}$ in Lemma 206.1. Indeed $a\widehat{O}^n \subset K$ by Definition 206.2. With $\ell = k \ge e$ the weighted count for this Z is at most

$$I(a)^{\nu n + n(n+s+1)}.$$

There is a uniform bound $I(a) \le b^{48fn^3}$ eventually: at every τ one has $M_b \le b^{11n^3}$, $g \le b^{22}$ by the level-prime bounds of Lemma 55.1, and $|\tau D|^2 \le b^{11n^3}$ by Proposition 205.3. Multiply the squares of the resulting f bounds on $|\tau a|$, using Definition 196.1. Since $\nu = n - 1$, the total exponent after summing over Z is at most $12fn^4 + 48fn^4(2n + s) \le 100fn^4(n + s + 1)$. □

208. Summing the weights of congruent pairs

Lemma 208.1. *Suppose $\gamma, \delta \in \mathcal{T}(c, s_0, c')$ have levels as in Definition 197.1. Let T be the first projection of $\mathcal{L}_\gamma \cap \mathcal{L}_\delta \cap \ker t$ as in Lemma 196.2. Write $d = \min(j, \nu)$ for the level j of γ, and put $T' = T \cap \ker(F_\gamma|_{X_0})$. Then*

$$o_{\gamma,g}(T) \mid g^{2fd} o_{\gamma,1}(T), \qquad o_{\delta,g}(T') \mid s_\delta,$$
$$\operatorname{codim}_{X_0} T' \le \operatorname{codim}_{X_0} T + n.$$

The saturation index s_δ uses the full sum of the two isotropic spaces as in Definition 197.1.

Proof. Write $B = F_\gamma(P_O)$, $A = F_\gamma(L_c(T))$. The image B has local rank d by Lemma 206.3 and its expression in Definition 206.2. It is locally free, by Lemma 25.1. Hence $[B : gB] = I(g)^d = g^{2fd}$ by Definition 196.1. The index $[A : A \cap gB]$ is $[A : A \cap B]$ times the order of the image of $A \cap B$ in B/gB. They are finite: for every $x \in T$, the defining common pair has a lift $p + x$ with $p \in P$ and $\gamma^{-1}(p + x) \in P \oplus X_0$. Hence $F_\gamma(x) = -F_\gamma(p)$. Choose such p on a rational basis of T and extend linearly. After multiplication by a fixed positive integer this map sends $L_c(T)$ into P_O, by the bounded denominators of these finite lattices in Definition 196.1. Thus $A \cap B$ has finite index in A. The order of a subgroup of a finite group divides the group order, as its cosets partition the group into sets of that size. This

proves the first divisibility. Intersecting T with the kernels of the n coordinates of F_γ proves the asserted codimension bound.

For $x \in T'$ one has $\gamma^{-1}x = p_\gamma + x'$ with $p_\gamma \in P$, $x' \in X_0$, since $F_\gamma x = 0$ means orthogonality to γP and γz. Thus $x \in W_\gamma$ and $(x, x') \in \mathcal{L}_\gamma \cap \ker t$. The first projection of this last space is injective by Lemma 196.2 applied with two copies of γ. By the definition of T, the pair over x in the common relation is therefore (x, x'). Its membership in $\mathcal{L}_\delta$ gives $p, p' \in P$ with $p + x = \delta(p' + x')$ by Definition 195.1. Consequently

$$\delta\gamma^{-1}x - x = p + \delta(p_\gamma - p') \in P + \delta P.$$

This subspace inclusion is rational and persists under extension to the finite adeles by applying it to a rational basis of T'.

For $x \in L_c(T')$ the finite vector $z_x = (x - \delta\gamma^{-1}x)/g$ thus belongs to $P_f + \delta P_f$. It belongs to $\widehat{\Lambda}_c$ because $\delta\gamma^{-1}$ acts as identity modulo g there by Proposition 196.3. Moreover $F_\delta(\delta\gamma^{-1}x) = F_\gamma(x) = 0$. It follows that $F_\delta(x) = gF_\delta(z_x)$. On $P_O + \delta P_O$ the image of F_δ is exactly $F_\delta(P_O)$, since the original frame is orthogonal to P. Thus the image of $F_\delta(L_c(T'))$ modulo $gF_\delta(P_O)$ is a subgroup of the image of the finite quotient

$$((P_f + \delta P_f) \cap \widehat{\Lambda}_c)/(P_O + \delta P_O)$$

under gF_δ modulo $gF_\delta(P_O)$. Its order divides that quotient's order s_δ, as claimed. □

Definition 208.2. Use the level j and near representatives of Definition 205.1. In each sum below take the ordered pairs γ, δ having that level and lying in the same $\mathcal{T}(c, s_0, c')$. Both representatives use the same chosen double-coset section. Write $T = T(\gamma, \delta)$ for the subspace of Lemma 208.1, $h = k - \dim_E T$, $d = \min(j, \nu)$, and set

$$M_h = |\bar{S}|\, C\, N^{-kf} (2T_0^2)^h,$$

$$w(\gamma, \delta; c) = \frac{1}{M_h d_c(T)^{k+1} \big(g^{2fd} o_{\gamma,1}(T) s_\delta\big)^k}.$$

Here $T_0 = b^{kf/4}$ as in Definition 69.1.

Theorem 208.3. *For each fixed $\epsilon > 0$, at all sufficiently large scales, uniformly in the level j,*

$$\sum_{c,c'} \sum_{s_0 \in \mathcal{G}} \sum_{\gamma,\delta} w(\gamma, \delta; c) \le 2b^{\epsilon kf}.$$

In particular the sum is at most $2T_0^2$.

Proof. Group by the finite label of s_0, by the row class c, and by the rational subspace T of codimension h. Fix these data. For a fixed first pair (c', γ), the second term has the same label and the same c', and its reduction equals that of the first. The space $T' = T \cap \ker(F_\gamma|_{X_0})$ is fixed throughout this inner sum. By Lemma 208.1, $o_{\delta,g}(T') \mid s_\delta$. The inner sum of s_δ^{-k} is therefore at most $B_{\min(k,h+n)}$, where $B_v = b^{100fn^4(n+v+1)}$, by Theorem 207.3. For first terms having at least one partner the lemma gives $o_{\gamma,g}(T) \mid g^{2fd} o_{\gamma,1}(T)$. The same theorem bounds the sum of the corresponding inverse k-th powers by B_h. Each pair (c', γ) at fixed c determines its reduction uniquely by Definition 188.1, so no multiplicity in s_0 is incurred. All weights are nonnegative, allowing these successive sums by Lemma A.2.

For any fixed $\eta > 0$, Corollary 201.3 gives, eventually,

$$\sum_{c} \sum_{\operatorname{codim} T = h} d_c(T)^{-(k+1)} \le CN^{-kf} b^{\eta kf(h+1)}.$$

Summing over the $|\bar{S}|$ labels and dividing by M_h bounds the contribution at h by

$$(2T_0^2)^{-h}\, B_h B_{\min(k,h+n)} b^{\eta kf(h+1)}.$$

Uniformly for $0 \le h \le k$, the logarithm to base b of the two B-factors is at most $100fn^4(3n + 2h + 2) = o(kf(h+1))$ by $k/n^9 \to \infty$ in Proposition 10.5. Take $\xi = \min(\epsilon, 1/2)$ and $\eta = \xi/2$,

and take the scale large enough to absorb the two B-factors into $b^{\xi kf(h+1)/2}$. The resulting bound is at most $2^{-h}b^{\xi kf}$, since $T_0^2 = b^{kf/2}$. The geometric series has sum two. Taking $\epsilon = 1/2$ proves the last assertion. □

209. A determinant integral for the real frame volumes

Definition 209.1. At one infinite place use $F = F_0,\ J,\ G_1 = U(J)$ from Definition 126.1. Let X be the isometric inclusion of the k definite middle coordinates, and let $F_- = (I - J)F$. For $y \in G_1$ put

$$V_y = F_-^* y^{-1} F, \qquad v(y) = |\det V_y|.$$

For $j \le n$ write $E_j : \mathbb{C}^j \to \mathbb{C}^n$ for the first-coordinate inclusion, and $I'_j = FE_j$. For $g_* \in G_1$ let $Z = F^* J g_* F,\ z_j = E_j^* Z E_j$. Say the frames have the required shared columns at j if for each $b > j$ there exist $a_b, x_b \in \mathbb{C}^n$ vanishing in coordinates larger than b, with $(a_b)_b = 1,\ |(x_b)_b| = 1,\ Fx_b = g_* F a_b$.

Lemma 209.2. *The function v is positive and continuous, and invariant under left multiplication by $\mathcal{U}_0$. Suppose z_j is invertible and the required shared columns exist at j. Set*

$$B = V_y^{-1} F_-^* y^{-1}, \qquad t = g_* I'_j z_j^{-1}, \qquad p = F^* J t.$$

Then

$$\frac{v(g_*^{-1} y)}{v(y)} = |\det(p^* B t)|\,|\det z_j|.$$

At $j = 0$ the determinants on the right are one.

Proof. The columns $e_i - e_i^*,\ 2z$ of F_- span the negative eigenspace of the hermitian involution J. If $V_y a = 0$, then $y^{-1} F a$ belongs to its positive eigenspace. Its J-norm is $-|a_n|^2$, so the vector vanishes and $a = 0$. Continuity follows by inversion and determinants. An element of $\mathcal{U}_0$ fixes F pointwise, giving left invariance. We have $BF = I_n,\ E_j^* p = I_j$. Put $M = B g_* I'_j - E_j$. For each $b \le j$, $I_n + Mp^*$ on ε_b agrees with $B g_* F$ there by $p^* E_j = I_j$. For a shared column at $b > j$, one has $p^* x_b = t^* J F x_b = z_j^{-*} I_j'^* J F a_b = 0$: when $j < n$ these j isotropic columns are orthogonal to all of F. Hence $(I_n + Mp^*)x_b = x_b = B g_* F a_b$. The matrices with columns ε_b up to j, then x_b or a_b, are upper triangular with determinants of modulus one. Therefore

$$|\det(B g_* F)| = |\det(I_n + Mp^*)| = |\det(I_j + p^* M)| = |\det(p^* B t z_j)|.$$

The middle identity follows by eliminating either off-diagonal block in the matrix with diagonal identities and off-diagonal blocks $M, -p^*$. The left side is precisely the ratio of the volumes by the definition of B. □

Lemma 209.3. *For $0 \le d \le n$, $k \ge 1$, an invertible complex d-square matrix S, and a complex k-by-d matrix A_0, one has*

$$\int_{\mathbb{C}^{k\times d}} \det\bigl(I_d + (AS + A_0)^*(AS + A_0)\bigr)^{-(k+n)}\ \mathrm{d}A \ \le\ |\det S|^{-2k}\bigl(\pi^d/d!\bigr)^k.$$

Here measure is Lebesgue measure in the complex entries, and a zero-dimensional space has mass one.

Proof. Translation and right multiplication on each row remove A_0, S; the real Jacobian for a complex invertible matrix S is $|\det S|^2$, by triangular elimination into shears, swaps and complex diagonal scalings. For an unshifted matrix let H be I_d plus the sum of row outer products except the last one. Writing u for that last row gives $\det(H + u^* u) = \det H(1 + uH^{-1}u^*)$, by the block determinant identity in Lemma 209.2. With the change $u = wH^{1/2}$ the Jacobian is $\det H$, as follows by the positive matrix diagonalization in Lemma 205.2. Integration thus lowers the exponent by one and extracts $\int_{\mathbb{C}^d}(1+\|w\|^2)^{-a}\,\mathrm{d}w$ at the current exponent a. In one complex coordinate, polar coordinates give $\int_{\mathbb{C}}(h+|z|^2)^{-a}\,\mathrm{d}z =$

$\pi h^{1-a}/(a-1)$ for $h>0,\ a>1$. Iterating gives $\pi^d/\prod_{i=1}^{d}(a-i)$ at $a>d$. The smallest exponent used in removing the k rows is $n+1$, so each such factor is at most $\pi^d/d!$. All successive integrals are valid for nonnegative functions by Lemma 123.1. □

Theorem 209.4. *Fix $y\in G_1,\ \theta>0$ with $\|uy\|_{\mathrm{op}}\le\theta$ for some $u\in\mathcal{U}$. Fix $Z\in\mathbb{C}^{n\times n}$ with $z_j=E_j^*ZE_j$ invertible, $0\le d\le j\le n$, an invertible d-square matrix R_1, and a k-by-j matrix V. Let $E:\mathbb{C}^d\to\mathbb{C}^j$ be the first-coordinate inclusion. There is a nonnegative continuous function K on $\mathbb{C}^{k\times d}$ with*

$$\int K(A)\,\mathrm{d}A\le\left(\frac{(2\theta^2)^j}{|\det z_j|}\right)^{k+n}\left(\frac{\pi^d}{d!\,\theta^{2d}|\det R_1|^2}\right)^{k}.$$

For every $g_*\in G_1$ with $F^*Jg_*F=Z$, the required shared columns at j, and $X^*g_*I_j'=V+AR_1E^*z_j$, one has $\big(v(y)/v(g_*^{-1}y)\big)^{k+n}\le K(A)$.

Proof. Let B,p be as in Lemma 209.2; $p=ZE_jz_j^{-1}$ is already fixed by Z. Use the upper unitriangular frame action $A(u)$ of Lemma 129.1, and put

$$T=\theta E_j^*A(u)E_j,\quad C_0=Tp^*B,\quad Q_1=\theta^{-1}(uy)^{-1}I_j'.$$

Since J is an ordinary unitary involution, a J-isometry h has $h^{-1}=Jh^*J$ and $\|h^{-1}\|_{\mathrm{op}}=\|h\|_{\mathrm{op}}$. Thus $\|Q_1\|_{\mathrm{op}}\le1$. Upper triangularity gives $Q_1T=y^{-1}I_j'$, and therefore $C_0yQ_1=I_j$ by $BF=I_n,\ p^*E_j=I_j$. It follows that $(C_0y)(C_0y)^*\ge I_j$, since taking adjoints gives $\|z\|\le\|(C_0y)^*z\|$ for every z. Here and below matrix inequalities refer to the nonnegative hermitian order. As F_- is negative for J, one has $BJB^*=-(By)(By)^*$ directly from $y^{-1}Jy^{-*}=J,\ JF_-=-F_-$. Consequently $C_0JC_0^*=-(C_0y)(C_0y)^*\le-I_j$.

Write $t=g_*I_j'z_j^{-1}$ and set $Y_0=JtT^*-C_0^*,\ W=X^*Y_0$. Then $F^*Y_0=0$ since $C_0F=Tp^*$. Its columns have zero outer e_i coordinates and zero z coordinate. On such columns the J-Gram matrix is just that of their definite components, so $Y_0^*JY_0=W^*W$. Also $t^*Jt=z_j^{-*}I_j'^*JI_j'z_j^{-1}\le0$. Expanding $Y_0^*JY_0$ and applying these inequalities gives

$$-C_0tT^*-(C_0tT^*)^*\ \ge\ I_j+W^*W.$$

For a complex matrix with positive hermitian part H, write it as $H^{1/2}(I+iS_1)H^{1/2}$ with S_1 hermitian. Its determinant has modulus at least $\det H$ by diagonalizing S_1. If $H\ge H_0>0$, conjugation by $H_0^{-1/2}$ gives $\det H\ge\det H_0$ by the same diagonalization. Applying both facts to $-C_0tT^*$, and using $|\det T|=\theta^j$, now gives by Lemma 209.2

$$\frac{v(y)}{v(g_*^{-1}y)}\le\frac{(2\theta^2)^j}{|\det z_j|\,\det(I_j+W^*W)}.$$

This derivation also holds for $j=0$ by empty determinants.

Since $X^*J=X^*$, the hypothesis on the definite components gives $WE=AS+A_0$, where

$$S=R_1E^*T^*E,\qquad A_0=(Vz_j^{-1}T^*-X^*C_0^*)E.$$

These are fixed, and $|\det S|=|\det R_1|\theta^d$ by triangularity. Since $WW^*\ge(WE)(WE)^*$, the determinant identity of Lemma 209.2 and the preceding determinant monotonicity show $\det(I_j+W^*W)\ge\det(I_d+(AS+A_0)^*(AS+A_0))$. In the ratio bound replace the denominator determinant by this smaller one and raise to $k+n$. The resulting continuous function $K(A)$ has the claimed integral by Lemma 209.3. □

210. Averaging the cut orbital integrals

Definition 210.1. For $\gamma \in \mathcal{T}(c, s, c')$ let $\mathcal{O} = \Gamma_c^0 \gamma \Gamma_{c'}^0$. Write $\beta_c^0 = \mathbf{1}_{\mathcal{B}} \beta_c$, and use the positive function v of Definition 209.1 at each real place. For $\delta \in \mathcal{O}$ put

$$q_\delta(y) = \prod_\tau \left(\frac{v(y_\tau)}{v((\delta_\infty^{-1} y)_\tau)} \right)^{k+n},$$

$$H_{\mathcal{O}}(y) = \sum_{\delta \in \mathcal{O}} \beta_{c'}^0(\delta_\infty^{-1} y) q_\delta(y),$$

$$S(\mathcal{O}; c) = \int_{G_\infty} \beta_c^0(y) \|V_\infty(y)\|^2 H_{\mathcal{O}}(y) \, \mathrm{d}y.$$

Use $S(\mathcal{O}^{-1}; c')$ for the same expression on the inverse orbit. The quantities are nonnegative extended integrals.

Lemma 210.2. *Let $a_\delta(c, c') = \int \beta_c(y) \beta_{c'}(\delta_\infty^{-1} y) \|K_\delta^{V_\infty}(y)\|_{\mathrm{HS}} \, \mathrm{d}y$. If the two S-integrals are finite, then*

$$\left(\sum_{\delta \in \mathcal{O}} a_\delta(c, c') \right)^2 \le S(\mathcal{O}; c) S(\mathcal{O}^{-1}; c').$$

If no member is near, all a_δ vanish.

Proof. Where the norm integrand is positive, both arguments lie in $\mathcal{E}$ by the cut support in Theorem 124.2. Positive cutoffs there force them to belong to $\mathcal{B}$ by Proposition 202.3. The cutoffs may therefore be replaced by β^0, and such a point witnesses nearness. By the rank-one norm computation of Theorem 204.3 the norm is $\|V_\infty(y)\| \, \|V_\infty(\delta_\infty^{-1} y)\|$. On the space of (δ, y) with counting times Haar measure multiply the first norm by $q_\delta^{1/2}$, the second by $q_\delta^{-1/2}$, and each by the square root of the product of the two cutoffs. Cauchy–Schwarz gives the claimed product: for the second squared integral substitute $(\delta^{-1}, \delta_\infty^{-1} y)$ for (δ, y). This change preserves the measure and exchanges the two ratios. □

Lemma 210.3. *Put $U_{0,\infty} = \prod_\tau \mathcal{U}_0$. Let H be a nonnegative Borel function on G_∞ invariant under left Γ_c^0. Suppose A is a measurable parameter set with a finite measure m of mass at least one, and $h : A \to U_{0,\infty}$ is measurable. If*

$$\int_A H(uh(x)y) \, \mathrm{d}m(x) \le D_0 \quad (u \in U_{0,\infty}, \ y \in \mathcal{B}),$$

then

$$\int \beta_c^0(y) \|V_\infty(y)\|^2 H(y) \, \mathrm{d}y \le D_0 \, \mathfrak{v} J_c.$$

Proof. Write $\beta_c = \omega_c \mathbf{1}_{D_c}$ as in Proposition 202.3, with D_c a Borel domain for Γ_c^0 on the real group. The factor $p = \mathbf{1}_{\mathcal{B}} \omega_c \|V_\infty\|^2$ is left $U_{0,\infty}$-invariant: the frame and size are unchanged, ω_c is invariant there by the cutoff construction, and the vector transforms by unitary scalars. For a fixed y, $D(y) = \{u : uy \in D_c\}$ is a domain for Γ_c^0 acting by left translation on $U_{0,\infty}$. Haar measure on $U_{0,\infty}$ is invariant on the right too: the triangular group multiplication in the coordinates of Lemma 126.2 has unit Jacobian on both sides. The translate $D(y)h(x)$ is again a domain. Partition one domain by translates of the other as in Lemma 21.4. Left Γ_c^0-invariance of H gives

$$\int_{D(y)} H(uy) \, \mathrm{d}u = \int_{D(y)} H(uh(x)y) \, \mathrm{d}u.$$

Integrate in x. If $y \in \mathcal{B}$, the hypotheses and nonnegative integration give an upper bound $D_0 \int_{D(y)} 1 \, \mathrm{d}u$ for the left integral, since $m(A) \ge 1$. If $D_0 = 0$, the left integral is zero by integrating the zero function in u. Multiplication by $p(y)$, first on the set $p(y) > 0$, makes the same comparison valid everywhere.

To integrate the orbitwise inequality, choose a positive Borel density α on G_∞ with $\int_{U_{0,\infty}} \alpha(uy)\,\mathrm{d}u = 1$. It is obtained by dividing a Gaussian in the matrix norm by its orbit integral as in Lemma 123.2: the subgroup is closed and the polynomial volume bound from Lemma 118.1 yields finiteness. By inversion, left Haar invariance on G_∞ and Lemma 123.1, for every nonnegative L, $\int \alpha(y) \int L(uy)\,\mathrm{d}u\,\mathrm{d}y = \int L(y)\,\mathrm{d}y$. Use $L = \mathbf{1}_{D_c} pH$ and $L = \mathbf{1}_{D_c} p$ on the two sides. It remains to bound $\int \beta_c^0 \|V_\infty\|^2$. Drop the box indicator and use Proposition 202.3 and $\int \|V_\tau\|^2\,\mathrm{d}\bar{y}_\tau \le \dim \pi_\tau$ from Theorem 124.2, giving $\mathfrak{v} J_c$. □

Proposition 210.4. *Suppose γ has level j, and put $d = \min(j, \nu)$. All $\delta \in \mathcal{O}$ have the same Z and level. They can be partitioned into at most i_γ^k subsets such that within each, for a fixed member δ_0, one has*

$$X_\delta - X_{\delta_0} = g L_\delta (P_Z)_{1:d,*}.$$

Here X_δ, P_Z are the rational frame blocks of Definition 206.2. The k-by-d matrix L_δ is unique, rational, with finite columns in L_X. For every real place τ, the difference of the scaled definite blocks of the frames is

$$g\,(\tau L_\delta)\,\mathrm{diag}_{i \le d}(1/\bar{d}_{\tau i})\,(Z_\tau^{\mathrm{sc}})_{1:d,*}, \quad Z_\tau^{\mathrm{sc}} = \mathrm{diag}(\bar{d}_{\tau i})\,(\tau Z)\,\mathrm{diag}(d_{\tau i}).$$

Proof. Write $\delta = u\gamma v$ with u, v in the two pointwise stabilizers. The second fixes f; the first fixes the original frame and consequently preserves all its pairings with γf. It also fixes all the shared vectors themselves; the level is unchanged by Definition 197.1 and Theorem 194.3. The action of u in coordinates P, M_0, P^* has identity diagonal, and its middle frame correction is $C_0 P_Z$, where $C_0 : P^* \to M_0$ is its middle-right block. The isometry condition gives the upper-middle block $-C_0^\dagger$ as in Proposition 206.4. Since u fixes z, the columns of C_0 are orthogonal to z; they belong to gL_X by the principal congruence of u on each integral dual basis vector. The matrix is rational.

Consider the modules of matrices with rows in the full span $R_\nu(\gamma)$ or the initial span $R_d(\gamma)$ of Definition 197.1, with coefficient columns in L_X. In local bases of L_X these are respectively k copies of the two row modules; hence their quotient has order i_γ^k. This description commutes with change of the local basis by an invertible integral matrix. Record the class of $(X_\delta - X_\gamma)/g$ in this quotient. For equal classes the difference is a combination of the first d rows with coefficient columns locally integral in L_X. Those rows are independent by Lemma 206.3. Solving a nonsingular minor makes the coefficients rational and unique and so identifies them with the local coefficients. The scaled definite block at τ is $(\tau X_\delta)\,\mathrm{diag}(d_{\tau i})$: the definite coordinates have scale one by Proposition 20.5. Pairing the scaled frames gives the displayed Z_τ^{sc}. Substitution gives the final formula, including the empty set of rows when $d = 0$. □

211. The cut amplitude at small discriminant

Proposition 211.1. *For any fixed $\eta > 0$, all sufficiently large scales, and every orbit $\mathcal{O}$ of level j,*

$$S(\mathcal{O}; c) \le \mathfrak{v} J_c\, b^{\eta k f} (b^4 g)^{-dkf} i_\gamma^k m_\gamma^{-(k+1)/2}, \qquad d = \min(j, \nu).$$

Proof. The function $H_{\mathcal{O}}$ is left Γ_c^0-invariant since its terms are permuted and v is left $\mathcal{U}_0$-invariant by Lemma 209.2. In Lemma 210.3 use entrywise complex unit disks for the parameters $A = (A_\tau)$, $A_\tau \in \mathbb{C}^{k \times d}$, with product Lebesgue measure of mass $\pi^{kdf} \ge 1$. Set $\theta = R + 1$ and

$$R_\tau = \mathrm{diag}_{i \le d} \frac{\pi g}{b \bar{d}_{\tau i}}.$$

Choose $h(A)_\tau \in \mathcal{U}_0$ so that the middle-right block of its inverse has first d columns $A_\tau R_\tau$ in the definite subspace and all other entries zero. In the formula for $u(B, T)$ of Definition 126.1, take $T = 0$ and choose B by its adjoint, then invert if required. This gives a continuous choice.

Fix $y \in \mathcal{B},\ u \in U_{0,\infty}$ and one of the subsets of Proposition 210.4, with base δ_0. Put $L'_\delta = ((b/\pi)\tau L_\delta)_\tau$. The frame of $g_{*,\tau} = h(A)_\tau^{-1} u_\tau^{-1} \delta_{\infty,\tau}$ has Gram pairings Z_τ^{sc}. Its first j definite columns are

$$V_\tau + (A_\tau + L'_{\delta,\tau}) R_\tau E^* z_{\tau,j}, \qquad z_{\tau,j} = (Z_\tau^{\mathrm{sc}})_{1:j,1:j},$$

where V_τ consists of the first j definite columns of $u_\tau^{-1}\delta_{0,\infty,\tau}F_0$ and $E : \mathbb{C}^d \to \mathbb{C}^j$ is the initial inclusion. Indeed u_τ^{-1} adds a fixed matrix times the common dual rows. The difference of definite columns for δ, δ_0 is therefore the scaled difference in Proposition 210.4. The inverse of $h(A)_\tau$ adds exactly the remaining term with A_τ. At a singular level the shared columns of Theorem 194.3 remain shared under both left operations since both fix the original frame. The change to the scaled frame multiplies both leading coefficients at a shared height by the same factor, so their ratio has modulus one by that theorem. At $j = n$ there are no required shared columns. Also y_τ has a representative of norm at most θ as in Theorem 124.2. Thus Theorem 209.4 supplies a kernel K_τ, independent of δ in this subset, for which

$$q_\delta(uh(A)y) \le \prod_\tau K_\tau(A_\tau + L'_{\delta,\tau}).$$

Here the numerator volumes are unchanged under $u, h(A)$.

In each summand translate A by L'_δ, with new variable A'. At a fixed A' at most one value of L_δ occurs among the terms whose translated unit disks contain A'. Indeed if two occur, the difference of any corresponding column of L_δ belongs to L_X and has squared norm at most $4\pi^2 k/b^2 < 1$ at every embedding. Its hermitian length is a rational element of O_E by the integral class lattice and vanishes by the norm argument of Proposition 203.2. Definiteness then forces columnwise equality. For this single L_δ, the point $Y = uh(A' - L'_\delta)y$ is fixed. The remaining weights are $\beta^0_{c'}(\delta_\infty^{-1}Y)$. In the support all the frames $\delta_\infty^{-1}Y$, evaluated by Definition 202.1, belong to the required box. Their pairwise column differences belong to $Y_\tau^{-1}P_\tau$ at each place since the definite, z- and dual coordinates of δf agree throughout this fibre. Thus the real dimension of the difference space is at most $2f\nu n$. Theorem 203.3 bounds the sum of the weights by $b^{12f\nu n^3}$. Summing and integrating are justified for nonnegative functions by Lemmas A.2 and 123.1. Including the number of subsets gives the uniform bound

$$\int H_{\mathcal{O}}(uh(A)y)\,\mathrm{d}A \le i_\gamma^k b^{12f\nu n^3} \prod_\tau \int K_\tau(A_\tau)\,\mathrm{d}A_\tau,$$

where the left integral is over the unit disks and those on the right are unrestricted. More precisely, when different subsets use different K_τ, use their common numerical bound from Theorem 209.4 on the right.

Write $\ell_{\tau i} = |d_{\tau i}|^2$, $e = j - d$. At each τ this numerical bound is at most

$$\left(\frac{(2\theta^2)^j}{|\tau D_j| \prod_{i\le j} \ell_{\tau i}}\right)^{k+n} \left(\frac{(b/g)^{2d} \prod_{i\le d} \ell_{\tau i}}{\theta^{2d}}\right)^k.$$

We used $\pi^{-d}/d! \le 1$. When $e = 1$ the only additional factor in the first product is $\ell_{\tau n} = N^{-1}$. Thus after taking the product over τ, the factors with exponent k give

$$m_\gamma^{-k/2} \big(2^j \theta^{2e} (b/g)^{2d}\big)^{kf}$$

by Definition 197.1. The base in parentheses is at most $(b^4 g)^{-d}$ eventually. For $j = d$ this follows from $2^d \le (g/b^6)^d$. For $j = n,\ d = n-1$, it follows from $2^n\theta^2 \le b^{14(n-1)} \le (g/b^6)^{n-1}$, since $n \to \infty,\ \theta = b^{6n+2} + 1,\ b \to \infty$ and $g \ge b^{20}$ by Theorem 11.4.

Multiplying the remaining product with exponent n by $m_\gamma^{1/2}$ contributes at most $b^{O(fn^3)}$. In fact $\prod_\tau |\tau D_j| \ge 1$ by integrality, and its exponent after this multiplication is $1 - n$. The additional norm factor N^{-ef} is at most one. Each $\ell_{\tau i}^{-1} \le N \le b^{60n}$ by Lemma 55.1. The bound follows from $j \le n$ and $\theta \le b^{6n+3}$. Finally the packing loss $b^{12f\nu n^3}$ and this last

$b^{O(fn^3)}$ together are at most $b^{\eta kf}$ eventually by $k/n^9 \to \infty$. Apply Lemma 210.3 to obtain the result. □

Definition 211.2. With M_0 here denoting the number M_h at $h = 0$ in Definition 208.2, put $d = \min(j, \nu)$. For $x > 0$ define

$$D_j^\circ = \begin{cases} T_0^{13-32j} E_1, & 0 \le j \le \nu, \\ E_2, & j = n, \end{cases} \qquad a_j(x) = T_0^2 \left(\frac{D_j^\circ}{M_0 g^{2dkf} x^{k+1}} \right)^{1/2}.$$

Theorem 211.3. *Suppose γ has level j and $b_\gamma \le N^f m_\gamma$, with the quantities of Definition 199.1. At all sufficiently large arithmetic scales its entire pointwise frame-stabilizer orbit has*

$$\sum_{\delta \in \mathcal{O}} a_\delta(c, c') \le \mathfrak{v} a_j(G_\gamma).$$

In particular this holds for every singular orbit.

Proof. Inversion exchanges the two frame matrices by conjugate transpose; hence $D_j(\gamma^{-1}) = \bar{D}_j(\gamma)$ and $m_{\gamma^{-1}} = m_\gamma$. The level is unchanged by the symmetric shared-flag condition of Theorem 194.3 and the invariant of Definition 192.1. Apply Proposition 211.1 with $\eta = 1/2$ in both orientations. The product bound of Lemma 210.2, after taking its square root and using $i_\gamma, i_{\gamma^{-1}} \ge 1$, gives at most

$$\mathfrak{v} J_c T_0^2 (b^4 g)^{-dkf} \left(\frac{i_\gamma i_{\gamma^{-1}}}{m_\gamma} \right)^{(k+1)/2}.$$

The ratio in parentheses equals N^f / G_γ by hypothesis. To bound its square by $\mathfrak{v}^2 a_j(G_\gamma)^2$ it suffices after cancellation to prove $E_1 J_c N^f T_0^{-32d} \le D_j^\circ$, since $M_0 = |\bar{S}| C N^{-kf}$ and $E_1 = |\bar{S}| C J_c$. Lemma 69.2 gives $J_c \le T_0^8$, and $N^f \le T_0$ eventually by Lemma 55.1. At a singular level $d = j$ this gives the prescribed power. At $j = n$ use $E_2 \ge T_0^{44} E_1$ from Theorem 69.3. Finally $\mathfrak{d} = 0$ at each singular level, hence $b_\gamma = 0$ and the hypothesis holds there. □

212. Normalization of the test on the regular real orbit

Lemma 212.1. *Suppose γ is near and $\mathfrak{d}(\gamma) \ne 0$. For each τ use $\mathcal{T}_\tau, t_\tau, \lambda_\tau$ of Theorem 132.2 for $\gamma_{\infty,\tau}$ with $\rho = b^{2n^2}$. Under the product of the probability laws of the t_τ, for every bounded Borel function ϕ on G_∞,*

$$\mathbb{E}_t \left\| \int_{\prod_\tau \mathcal{T}_\tau} \phi(y) F_\infty(\gamma_\infty t y) F_\infty(t y)^* \, \mathrm{d}y \right\|_{\mathrm{HS}}^2 \le \int |\phi(y)|^2 \, \mathrm{d} \bigotimes_\tau \lambda_\tau(y).$$

Proof. With $\ell_{\tau i} = |d_{\tau i}|^2$ put $l_\tau = \prod_{i=1}^n \ell_{\tau i}$. Taking determinants in the scaled pairings of Proposition 210.4 gives $\Delta(Z_\tau^{\mathrm{sc}}) = l_\tau^2 \, \tau \mathfrak{d}(\gamma) > 0$: the missing last factor in the penultimate minor is N^{-1}. Positivity follows from Theorem 192.2. Nearness supplies the two bounded frames required by Theorem 132.2 at every place.

Regard each integral in that theorem as a linear operator into the Hilbert space of square-integrable Hilbert–Schmidt matrices over the law of t_τ. Its norm on bounded tests in $L^2(\lambda_\tau)$ is at most one. On a finite sum of products of bounded one-place tests, the operator of this lemma is their tensor product by Lemma 123.1. A contraction acting in one tensor factor remains contractive on finite sums: in an orthonormal basis of the span of the other factors, expand and add the squared norms of its coefficients. Applying the one-factor operators successively proves the inequality on these finite sums.

For passage to a general bounded test, on each group take the finite measure $\lambda_\tau + \mathbf{1}_{\mathcal{T}_\tau} \, \mathrm{d}y_\tau$. Finite linear combinations of indicators of rectangles are dense in L^2 of their product. Indeed the sets approximable in measure by finite unions of rectangles form a sigma-algebra:

complements and finite unions preserve approximation, and for countable unions first truncate the increasing finite unions using finiteness of the measure and countable additivity. They contain all rectangles and hence all Borel sets, since the finite product has a countable base of rectangles. Approximate simple functions first and then bounded functions by subdividing their real and imaginary ranges. This gives simultaneous L^2 approximation for the two relevant product measures. The actual integrals against Haar on the compact product converge uniformly in t, by boundedness of F_∞, Cauchy–Schwarz and the finite Haar mass of that compact set. This identifies the tensor-product limit and proves the stated inequality. □

Proposition 212.2. *In Lemma 212.1, write $B = b_\gamma$. Uniformly for the near regular terms at all sufficiently large scales one has*

$$b^{-k^3 f} \le a_n(B), \qquad \prod_\tau \lambda_\tau(G_1) \le \tfrac{1}{4}\,\mathfrak{v}^2 a_n(B)^2.$$

Proof. Integrality and total positivity of the nonzero $\mathfrak{d}$ give $B \ge 1$ by the integral norm argument of Theorem 192.2. Also $\tau\mathfrak{d} \le |\tau D_n|^2$, so $B \le b^{11fn^3}$ by Proposition 205.3. In Definition 211.2, $E_2/M_0 \ge E_1/M_0 = J_c N^{kf} \ge 1$ by Theorem 69.3. Since $g \le b^{22}$, the negative base-b logarithm of $a_n(B)$ is at most $f(22\nu k + 11n^3(k+1)/2) \le k^3 f$ eventually. Here $k/n^9 \to \infty$ by Lemma 55.1. This proves the first assertion.

Use $l_\tau, \ell_{\tau i}$ of the preceding proof, put $L = \prod_\tau l_\tau$, and $h_* = 15000n^3$. Theorem 132.2 and $\prod_\tau \Delta(Z_\tau^{\mathrm{sc}}) = L^2 B$ bound the product of the measures by

$$\frac{J_c^2\, 2^{(k-1)f}}{\mathcal{D} C_{\mathrm{op}}^f}\, \rho^{9nfh_*} (L^2 B)^{-(k-1+h_*)}.$$

The degree and dimension products here are those of Definitions 68.1 and 69.1. Divide by $\mathfrak{v}^2 a_n(B)^2$, using $E_2 = (CJ_c)^2/(\mathcal{D}\mathfrak{m}_r)$. The resulting bound is

$$\frac{\mathfrak{m}_r M_0 g^{2\nu kf}}{C^2 C_{\mathrm{op}}^f \mathfrak{v}^2}\, T_0^{-4} 2^{(k-1)f} \rho^{9nfh_*} L^{-2(k-1+h_*)} B^{2-h_*}.$$

Since $B \ge 1,\ h_* \ge 2$, the last B-factor is at most one. By the definitions in Theorem 68.2, the initial fraction is exactly

$$\frac{P_{\mathrm{pkt}}}{Z_{\mathrm{pkt}}}\, \frac{V_E^n Z_{\mathrm{sc}}^2 g^{2\nu kf}}{u\, c_* N^{kf}}.$$

Here $M_0 = |\bar{S}| C N^{-kf}$, so this identity follows by canceling $C, |\bar{S}|$ in the displayed packet formulas. That theorem bounds $P_{\mathrm{pkt}}/Z_{\mathrm{pkt}} \le 2$ eventually. One has $g^{2\nu kf}/u = g^{-\nu(2\nu+1)f} \le 1$. The formula for c_* in Theorem 67.3 gives

$$\frac{V_E^n}{c_* N^{kf}} = \frac{V_E^{n+k}}{V_F} \left(\frac{(k-1)!}{\pi^k}\right)^f \le b^{o(kf)}.$$

Indeed $V_E = (\mathrm{rd}_E/2)^f,\ V_F = \mathrm{rd}_E^{f/2} \ge 1$ by Theorem 68.2, and $\log \mathrm{rd}_E, \log k = O(1+\log x) = o(\log b)$ by Propositions 12.2 and 10.5. Also $(k-1)! \le k^k$ and $n \le k$.

The remaining scale factor is

$$Z_{\mathrm{sc}}^2 L^{-2(k-1+h_*)} = \prod_{\tau,i} \ell_{\tau i}^{2(1+2(n-i)-h_*)}.$$

Every exponent is negative. Since $\ell_{\tau i}^{-1} \le N \le b^{60n}$, this product is at most $b^{O(fn^5)}$. Also $\rho^{9nfh_*} = b^{O(fn^6)}$. These are $b^{o(kf)}$ by Lemma 55.1, and $2^{(k-1)f} = b^{o(kf)}$. Finally $T_0^{-4} = b^{-kf}$. The ratio under consideration is at most $b^{-(1-o(1))kf}$ and thus at most $1/4$ eventually. □

213. Bounds for the cut amplitudes on a fixed level

Lemma 213.1. *Fix c, s, c' and a set B of regular rational elements stable under left and right rational unipotents. For $\delta \in B \cap \mathcal{T}(c, s, c')$ one has $\Gamma_c \cap \delta \Gamma_{c'}^0 \delta^{-1} = \{1\}$. In every $\Gamma_c \delta \Gamma_{c'}^0$ choose one element δ_R, taking a near representative of Definition 205.1 if the orbit contains a near element. On $B \cap \mathcal{T}(c, s, c')$ set*

$$p(\delta, y) = \mathbf{1}_{\delta = \delta_R} \omega_{c'}(\delta_\infty^{-1} y)$$

with δ_R chosen on the orbit of δ and $\omega_{c'}$ as in Proposition 202.3. Then for every (δ, y),

$$\sum_{u \in \Gamma_c,\, v \in \Gamma_{c'}} p(u \delta v^{-1}, u_\infty y) = 1.$$

Moreover $\omega_{c'}(hy) = \omega_{c'}(y)$ whenever $h \in \prod_\tau H_0$, the pointwise real frame stabilizer.

Proof. If u is in the stated intersection, it fixes δf pointwise. Let A be its upper unitriangular action on the original rational frame. Pairing the two frames gives $A^{-*} Z(\delta) = Z(\delta)$. Since $\mathfrak{d}(\delta) > 0$ at infinity, $Z(\delta)$ is invertible by Theorem 192.2. Thus $A = I$. The middle-right block C_0 of u as in Proposition 210.4 now satisfies $C_0 P_Z = 0$ since $u \delta f = \delta f$. The matrix P_Z has full row rank as it consists of the first ν rows of an invertible matrix. Hence $C_0 = 0$ and the only possible off-diagonal block of u is the top-right block H. The same fixed-frame equation gives $H P_Z = 0$, so $u = 1$.

Fix a left coset $\Gamma_{c'}^0 v_* \subset \Gamma_{c'}$. As u, v vary with v in this coset, $u \delta v^{-1}$ runs over precisely one orbit of the type used for δ_R, each element exactly once by the trivial intersection just proved, applied at δv_*^{-1}. Thus there is one contributing pair and its weight is $\omega_{c'}(v_{*,\infty} \delta_\infty^{-1} y)$ by the pointwise invariance in Proposition 202.3. Summing over the cosets gives one by its construction. Nearness is constant on the smaller double cosets of the two arithmetic pointwise stabilizers, so the specified choice is possible.

For the last assertion recall that $\omega_{c'}$ is the indicator J divided by the sum of its translates in that proposition. At each place $h \in H_0$ fixes the frame pointwise and belongs to the graph projection defining S by Lemma 126.2. Consequently $S(hy) = S(y)$. For every arithmetic unipotent v, $v^{-1} F_0$ is a linear combination of F_0 by Lemma 129.1. Thus $h^{-1} v^{-1} F_0 = v^{-1} F_0$, so $J(vhy) = J(vy)$; also $S(vhy) = S(vy) = S(y)$. These identities prove the invariance of the quotient defining $\omega_{c'}$. □

Proposition 213.2. *With B as in Lemma 213.1, one has*

$$\|A_B^{V_\infty}(c, s, c')\|_{\mathrm{HS}} \le \mathfrak{v} \sum_{\substack{\gamma \in B \cap \mathcal{T}(c,s,c') \\ \gamma \text{ near chosen representative}}} a_n(b_\gamma).$$

Proof. Write $p_0(\delta, y) = \beta_c(y) \beta_{c'}(\delta_\infty^{-1} y)$. It sums to one under the same action as p by the two partitions of Proposition 202.3. Counting measure times Haar measure is invariant under the action. For any nonnegative invariant $K(\delta, y)$, insert the sum of translates of p into $\int p_0 K$, change variables termwise, and sum the translates of p_0. This gives $\int pK = \int p_0 K$ by Lemma A.2. It also holds for integrable vector-valued K by applying it to its norm first. Full and cut kernels are invariant under these arithmetic unipotents by their transformation laws and Proposition 21.2. Apply the identity to the two kernels, their norms and the norm of their difference. The relevant integrals are finite by Theorem 204.3. For the new weight the total difference integral is still at most $\mathfrak{v} b^{-k^3 f}/2$, so this bounds the nonnegative contribution of any one chosen δ_R as well.

If the cut integrand with weight p is nonzero at y, then $y \in \mathcal{E}$ and $\delta_{R,\infty}^{-1} y \in \mathcal{B}$. Indeed the latter point is in $\mathcal{E}$ by cut support and its positive $\omega_{c'}$ enforces the frame bounds of Proposition 202.3. Choose $u \in \Gamma_c$ so that $u_\infty y \in \mathcal{B}$ by Lemma 202.2. Thus $u \delta_R$ is near and in the same left-right orbit, which by the rule of choice means that δ_R itself is a near chosen representative. Only these terms need to be estimated.

Fix one such $\gamma = \delta_R$. Apply Lemma 212.1 with $\phi = \omega_{c'}$. Its values lie in $[0,1]$, so some t in the probability construction makes the norm of the displayed integral at most $\big(\prod_\tau \lambda_\tau(G_1)\big)^{1/2}$. Indeed otherwise its squared norm would exceed this bound squared on a set of full probability, and integration would contradict the lemma. The compact-support implication in Theorem 132.2 holds for every such t.

In the cut integral at γ change variables to $y = \gamma_\infty t z$. The weight becomes $\omega_{c'}(tz) = \omega_{c'}(z)$ by Lemma 213.1. Wherever the resulting integrand is nonzero, $tz \in \mathcal{E}$ and z has all real frame columns bounded by ρ, since t fixes the frame and the weight is positive. Also $S_\tau((\gamma_\infty t z)_\tau) \le R \le \rho$. Hence $z \in \prod_\tau \mathcal{T}_\tau$ by the cited compact-support implication. On this product replace the cut kernel by the full one. The norm of the integrated error is at most $\mathfrak{v} b^{-k^3 f}/2$ by the difference integral at this representative, since restricting an integral of a norm only decreases it. The full integral there has the bound just obtained from the choice of t. Proposition 212.2 bounds the sum of these two contributions by $\mathfrak{v} a_n(b_\gamma)$. Finally, the weighted series giving $A_B^{V_\infty}$ is absolutely convergent. Sum its terms in the new weight and enlarge the chosen representative list to all the near representatives of the smaller double cosets, which includes it. □

Theorem 213.3. *Suppose a set B of rational terms is stable under rational unipotents on both sides, and every member has a fixed level j in Definition 197.1. For agreeing s and all sufficiently large arithmetic scales,*

$$\|A_B^{V_\infty}(c,s,c')\|_{\mathrm{HS}} \le \mathfrak{v} \sum_{\substack{\gamma \in B \cap \mathcal{T}(c,s,c') \\ \gamma \text{ near chosen representative}}} a_j(G_\gamma).$$

At nonagreeing s the left side is zero.

Proof. Finite disagreement is covered by Corollary 204.5. Split B at $b_\gamma = N^f m_\gamma$, assigning equality to the smaller side. The discriminant and the leading minor, and hence the two sides, are invariant under rational unipotents by Theorem 192.2. On the smaller side partition by $\Gamma_c^0, \Gamma_{c'}^0$ double cosets and apply Theorem 211.3 to their absolute sums. Cosets without near representatives give zero by Lemma 210.2. The larger side consists entirely of regular terms; there $d = \nu,\ j = n$ and both row indices are one by Definition 197.1. Hence $G_\gamma = b_\gamma$. Proposition 213.2 applied to this stable subset completes the proof by the triangle inequality. □

214. The contribution from pairs outside the regular diagonal

Lemma 214.1. *Suppose two near chosen representatives γ, δ of level j belong to the same $\mathcal{T}(c,s,c')$, with s agreeing. Use i, l, h, T from Lemma 196.2 and Definition 208.2, so $h = k - \dim T$. Let $X \ge 1$ be a lower bound on the product of the norms of distinct marked primes at each of which s fixes a nonzero reduced frame vector. Set $X_l = X$ for $l = 2$, and $X_l = 1$ otherwise. Put $e = 1$ at $j = n$ and $e = 0$ otherwise. Then*

$$\frac{a_j(G_\gamma) a_j(G_\delta)}{\sqrt{w(\gamma,\delta;c) w(\delta,\gamma;c)}} \le T_0^4 D_j^\circ (2T_0^2)^e Z_0^i X_l^{-(k+1)},$$

where $Z_0 = 2(b/g)^{kf} T_0^2$ as in Definition 69.1.

Proof. The common subspace T is symmetric in the two terms. Abbreviate $d_T = d_c(T),\ P_1 = P_\gamma(T),\ P_2 = P_\delta(T)$ from Definition 199.1. Multiplying the two ordered weights gives

$$\sqrt{w(\gamma,\delta;c) w(\delta,\gamma;c)} = \frac{1}{M_h\, g^{2fdk}\, d_T (P_1 P_2)^{k/2}}.$$

All indices in P_1, P_2 are positive integers, so $d_T \le \sqrt{P_1 P_2}$. Substitution of Definition 211.2 bounds the ratio in the statement by

$$T_0^4 D_j^\circ (2T_0^2)^h \left(\frac{P_1 P_2}{G_\gamma G_\delta} \right)^{(k+1)/2} .$$

By Theorem 199.4 the last power is at most $(2/g)^{fi(k+1)} X_l^{-(k+1)}$, and $h \le i + e$. Here Theorem 198.4 applies with the chosen prime set, whose product is at least X. Since $g \ge 2,\ b \ge 2$, we may increase the factor $(2/g)^{fi(k+1)}$ to $(b/g)^{kfi}$. Combining its i powers with $2T_0^2$ gives the claimed Z_0^i. □

Theorem 214.2. *Let B be a set of rational group elements stable under rational unipotents on both sides. Fix $\Omega \subset \mathcal{G}$ and $X \ge 1$. At every agreeing $s \in \Omega$ assume the marked fixed-vector condition of Lemma 214.1 with lower bound X. Let $\mathcal{R}$ be a collection of middle relations, and let $\mathcal{L}$ denote the collection of all rational middle relations. Choose $H \ge E_1$ satisfying the following two conditions.*

1. *If an agreeing rational term of B at level zero lies in $\mathcal{T}(c, s, c')$ for an agreeing $s \in \Omega$, then $H \ge T_0^{20} E_1$.*
2. *If $\mathcal{R} \ne \mathcal{L}$, then $H \ge T_0^{12} E_2 / X^k$.*

For $\Lambda \in \mathcal{L}$ write $B_\Lambda = \{\gamma \in B : \mathfrak{d}(\gamma) \ne 0,\ \mathcal{L}_\gamma = \Lambda\}$. Suppose $w_0 \ge 0$ satisfies

$$\sum_{\Lambda \in \mathcal{R}} \mathcal{E}_\Omega(A_{B_\Lambda}^{V_\infty}) \le w_0.$$

Then, for every fixed $D > 0$, at all sufficiently large scales uniformly in the specified families,

$$\mathcal{E}_\Omega(A_B^{F_\infty}) \le (1 + b^{-D}) w_0 + b^{-D} H.$$

Proof. At nonagreeing reductions both amplitudes vanish by Corollary 204.5. For the cut amplitudes discard the rationally disagreeing orbits: their individual integrals are zero by Theorem 193.3 and the orbit identity in Proposition 202.3. It is enough to discard such terms at singular levels. For the remaining singular terms the levels $j = 0, \dots, \nu$ are exhaustive by Theorems 194.3 and 192.2. These level sets are rationally unipotent stable: left and right flag action is upper unitriangular, so preserves the heights and leading coefficients of shared vectors. Rational agreement is preserved since the two characters are homomorphisms and conjugating their common unipotents just conjugates within their respective groups. The regular level and every middle relation are stable by Theorem 192.2 and Proposition 195.2.

At any one level group the cut amplitude at each c, s, c' by its middle relation. All these coefficient series are absolutely convergent by Theorem 204.3. Expand the squared norm of their sum into inner products and bound each off-diagonal term by the product of the two norms. At the regular level keep the squares with equal relation in $\mathcal{R}$ exactly. At singular levels keep none. For every remaining ordered pair of relations, Theorem 213.3 bounds the product by the sum over corresponding near chosen term pairs of $\mathfrak{v}^2 a_j(G_\gamma) a_j(G_\delta)$. All upper bounds are nonnegative and sum by Lemma A.2. Each pair of terms determines its relations and its finite reduction once the two column classes are fixed. Thus it occurs at most once in this upper bound, even on an omitted diagonal.

The factor multiplying the geometric mean in Lemma 214.1 is at most H / T_0^3 on these pairs, eventually. At $j = 0$ it is at most $T_0^{17} E_1$ by $Z_0 \le 1$ in Theorem 69.3; if such a pair occurs the first assumption applies. For $1 \le j \le \nu$ it is at most $T_0^{17 - 32j} E_1$. At $j = n$ with unequal relations $i \ge 1$ by Theorem 199.4, so it is at most $2T_0^6 E_2 Z_0 \le 2T_0^{-12} E_1$ by Theorem 69.3. On an omitted equal regular relation $i = 0,\ l = 2$ by Theorem 199.4. Here the bound is $2T_0^6 E_2 / X^{k+1} \le H / T_0^3$ eventually by the second assumption and $X \ge 1$.

Use $\sqrt{uv} \le (u + v)/2$ from $(\sqrt{u} - \sqrt{v})^2 \ge 0$. The sum of the geometric means over the relevant ordered pairs is at most the sum over all ordered pairs of one level of their weights.

By Theorem 208.3 with $\epsilon = 1/4$ this is at most $2T_0$. After dividing by $\mathfrak{v}^2$, the extra energy at each level is therefore at most $2H/T_0^2$.

The square root of $\mathcal{E}_\Omega$ is a Euclidean norm on the entries restricted to Ω, divided by $\mathfrak{v}$. Adding the n singular pieces to the regular piece and using $\sqrt{w_0 + a} \le \sqrt{w_0} + \sqrt{a}$ gives an upper bound $\sqrt{w_0} + (n+1)\sqrt{2H}/T_0$ for the cut norm. Add the full-to-cut error using Theorem 204.3 with exponent $4D+8$. Since $E_1 \le H$ and $T_0 = b^{kf/4}$, the full norm is at most $\sqrt{w_0}+b^{-2D-2}\sqrt{H}$ eventually, using Proposition 10.5. Finally $(a+b')^2 \le (1+u)a^2+(1+u^{-1})b'^2$ for $u > 0$ by the nonnegativity of $(\sqrt{u}\,a - b'/\sqrt{u})^2$. Use $u = b^{-D}$ to obtain the result. □

215. The second moment of the full Poincaré columns

Definition 215.1. Give the quotient of Definition 21.3 the measure of Lemma 21.4, with counting measure on the finite components. Let $\mathcal{H}$ be its complex L^2-space, with ρ the unitary right action of $\mathcal{G}$ through the finite component. This is well-defined by normality of K_Y in K_Y^0 and preserves the measure since it permutes counting measure and commutes with rational left translation. Let $Q : \mathbb{C}^{CJ_c} \to \mathcal{H}$ take the coordinate vector indexed by a class and a tensor coordinate to its column in Theorem 204.4 divided by $\sqrt{\mathfrak{v}}$. Put $a(s) = \|Q^*\rho(s)Q\|_{\mathrm{HS}}^2$.

Theorem 215.2. *For the vector conditions and sets $\Omega(S, D)$ of Definition 19.3, uniformly on the arithmetic window and in the specified conditions, for every fixed $A > 0$ and all sufficiently large scales,*

$$\sum_{s \in \Omega(S,D)} a(s) \le (1 + b^{-A}) E_2\, b^{-6k|S|}.$$

The same statement holds with $a(s)$ defined using the inverse translate.

Proof. Set $Y = b^{-6k|S|}$, $H = YE_2$. Since $|S| \le f$, $Y \ge T_0^{-24}$ and Theorem 69.3 gives $H \ge T_0^{20}E_1$. The prime product in Theorem 214.2 can be taken to be $X = g_0^{|S|}$, choosing at each prescribed pair the prime on the side of the fixed vector. The inverse-transpose condition specifies the partner, as in Definition 19.3.

If S is the whole set of f pairs, take no retained relations and $w_0 = 0$ in that theorem. Indeed $T_0^{12}E_2/X^k \le b^{-7kf}E_2 \le H$ since $g_0 \ge b^{10}$ by Theorem 11.4. Otherwise take all relations. For each $s \in \Omega(S, D)$ there is a marked component at which all leading minors are nonzero. Hence every rational term above s satisfies the open flag condition by Corollary 192.4. By Theorem 195.3, on this open set the relation is the graph of the inverse middle isometry σ^{-1}. In the decomposition of Theorem 192.3, the common unipotents may be removed from the discriminant. At the representative $w_p\sigma$ the Gram matrix is diagonal in the outer pivots and has last entry $[z, \sigma z]$. Thus $\mathfrak{d} = |D_\nu|^2([z, \sigma z]\overline{[z, \sigma z]} - N^2)$, where $|D_\nu|^2$ denotes $D_\nu \bar{D}_\nu$. Regularity implies $|\tau[z, \sigma z]| > N$ for every τ by Theorem 192.2. The cut amplitudes for each regular relation are precisely the sums over the pivots in Theorem 191.3 by Corollary 192.4. Changing cutoffs has preserved these stable orbit integrals by Proposition 202.3. Consequently the sum of their diagonal energies is at most $w_0 = (1 + b^{-A-3})H$ by Theorem 191.3.

Apply Theorem 214.2 in both cases, with $B = \Gamma$ and exponent $A + 3$. In the second case the resulting multiplier of H is $(1 + b^{-A-3})^2 + b^{-A-3} \le 1 + b^{-A}$ eventually. The first case satisfies the same bound. By Theorem 204.4 the unfolded matrix entries are the inner products with conjugate linearity in the first variable. Written as $\langle \rho(s)Qe_{c',j}, Qe_{c,i}\rangle$, each is the complex conjugate of the corresponding entry of $Q^*\rho(s)Q$. Thus the squared Hilbert–Schmidt norm is the sum of the squared absolute entries in either convention. Finally $Q^*\rho(s^{-1})Q = (Q^*\rho(s)Q)^*$, so the value of a is the same at s^{-1} and s. □

Corollary 215.3. *For all sufficiently large scales on the window,*

$$\sum_{s\in\mathcal{G}} \|Q^{*}\rho(s)Q\|_{\mathrm{HS}}^{2} \le (1+9b^{-4})E_{2}.$$

More generally, for every nonnegative weight of positive type dominated pointwise by a and supported on $V(S,D)$, its total mass is at most $(1+9b^{-4})E_{2}b^{-6k|S|}$.

Proof. The weight a is of positive type by the finite-rank Hilbert–Schmidt Gram calculation in Corollary 19.6, applied to this Q. Theorem 215.2 with $A=4$ supplies the hypothesis of Theorem 19.5 with $D_{*}=E_{2}$ since $1+b^{-4}\le(1-b^{-4})(1+9b^{-4})$ eventually. That theorem proves both assertions. □

Proposition 215.4. *Let $S(F)=U(F)H(F)$ as in Corollary 204.5. For $a=0,1$ let Ω_{a} consist of the $s\in\mathcal{G}$ whose components above g_{a} are those of a single member of $\bar{S}$. Then for every fixed $A>0$, eventually*

$$\mathcal{E}_{\Omega_{a}}(A^{F_{\infty}}_{\Gamma\backslash S(F)}) \le b^{-A}E_{1}.$$

Proof. At every prime above g_{a} all members of $\bar{S}$ fix the first frame vector e_{1}: the definite group acts on X_{0} and the unipotent frame action is upper unitriangular, also at the conjugate split component by the rational flag definition of the group in Proposition 66.1. Hence $X=g_{a}^{2f}$ is allowed for every $s\in\Omega_{a}$. No rationally agreeing term of level zero above this set can belong to the complement. Indeed Theorem 194.3 at $j=0$ says γ preserves the entire frame flag with each graded scalar equal to λ, including $\gamma e_{1}=\lambda e_{1}$, and $\lambda\bar{\lambda}=1$. The same scalar occurs in the finite position because the two class matrices fix e_{1}. It is integral by the integral row lattices, and congruent to one at all primes above g_{a}. As the prime splits completely by Lemma 13.2, $(\lambda-1)/g_{a}\in O_{E}$. This algebraic integer has every conjugate of modulus at most $2/g_{a}<1$, so vanishes by Proposition 203.2. Thus $\lambda=1$. The action on $P^{\perp}/P$ fixes z and restricts to an isometry of X_{0}. Removing its diagonal extension leaves a unipotent by Definition 20.3, so $\gamma\in S(F)$.

Take $H=E_{1}$, retain no regular relations, and take $w_{0}=0$ in Theorem 214.2. Its first condition is vacuous by the preceding argument. By Theorem 69.3 and the formula for Z_{0},

$$E_{2}/E_{1} \le \tfrac{1}{2}(g/b)^{kf}T_{0}^{-20} \le \tfrac{1}{2}b^{16kf}.$$

The last inequality uses $g\le b^{22}$ by Theorem 11.4. Since $X^{k}\ge b^{20kf}$ and $T_{0}^{12}=b^{3kf}$, the second condition also holds. The complement is stable on both sides since $U(F)\subset S(F)$, so the asserted bound follows. □

216. Translation by the finite Bessel subgroup

Definition 216.1. Write $c_{0}=CJ_{c}$ for the dimension of the domain of Q in Definition 215.1. For a union B of double cosets of $S(F)$ in the rational row group Γ, let $D_{B}(s)$ be the c_{0}-by-c_{0} matrix whose (c,d)-block is $\mathfrak{v}^{-1}A_{B}^{F_{\infty}}(c,s,d)$ in Definition 204.1. The sums converge absolutely by Theorem 204.3. Write D_{out} for $D_{\Gamma\backslash S(F)}$.

Lemma 216.2. *For each $t\in\bar{S}$ there is a permutation p_{t} of the column classes with the following property. For every d there is an element $\sigma_{d}\in S(F)\cap\mathcal{T}(d,t,p_{t}(d))$. All choices for this pair of classes lie in the same right coset $\sigma_{d}\Gamma_{p_{t}(d)}$, and $\sigma_{d}\Gamma_{p_{t}(d)}\sigma_{d}^{-1}=\Gamma_{d}$.*

Proof. Use H for the definite middle group and choose an integral finite lift $k_{t}\in S_{f}$ of t by Proposition 66.1. Factor $h_{d}k_{t}=u_{f}a_{f}$, with $u_{f}\in U_{f}$ and $a_{f}\in H_{f}$. Indeed two such factors multiply by the semidirect product law, since H_{f} normalizes the unipotent flag, and $h_{d}\in H_{f}$. The double coset definition of the column classes in Proposition 13.5 gives $a_{f}=\eta_{f}h_{e}k$, with $\eta\in H(F)$ and $k\in K_{c}$, where K_{c} denotes the column level. Its extension lies in K_{Y} by Proposition 66.1. Hence $h_{d}^{-1}u_{f}\eta_{f}h_{e}=k_{t}k^{-1}\in k_{t}K_{Y}$. This membership is

open as a condition on u_f. Lemma 21.1 supplies a rational unipotent u in the same open set. Then $\sigma_d = u\eta$ belongs to $\mathcal{T}(d,t,e)$.

The finite condition for $\mathcal{T}$ in Definition 188.1 respects multiplication with matching middle class and inversion with the classes interchanged. This follows by multiplying the level cosets, since K_Y is normal in K_Y^0. If two choices lead to e, e', their quotient $\sigma_d^{-1}\sigma_d'$ belongs to $S(F) \cap \mathcal{T}(e,1,e')$. The proof of Corollary 204.5 identifies this set as empty for unequal classes and as Γ_e for equal classes. Thus $e = p_t(d)$ is unique, with the stated right coset. If the images of d, d' agree, use $\sigma_d \sigma_{d'}^{-1} \in S(F) \cap \mathcal{T}(d,1,d')$ to deduce $d = d'$. Injectivity implies surjectivity since the class set is finite. Finally conjugation by σ_d preserves $U(F)$ and carries the level condition at $(e,1,e)$ to that at $(d,1,d)$. Conjugation by its inverse gives the reverse inclusion. □

Proposition 216.3. *For each $t \in \bar{S}$ there is a unitary matrix U_t on the column coordinate space such that for every B in Definition 216.1 and every $s \in \mathcal{G}$,*

$$D_B(st) = D_B(s) U_t.$$

Proof. For $\sigma_d = u\eta$ in Lemma 216.2 put $L_d = \chi_\infty(u_\infty) \bigotimes_\tau \pi_\tau(\eta_\tau)$. Here the definite isometry acts on the positive definite middle space in the real coordinates of Theorem 122.2. Its transformation law implies $F_\infty(\sigma_d y) = L_d F_\infty(y)$, also after the scalar normalizations of Theorem 123.3. The character has absolute value one and the tensor factors are unitary. Put the single nonzero block in block row d of U_t in column $e = p_t(d)$, with value L_d. This matrix is unitary by the permutation property.

Right multiplication by σ_d bijects $B \cap \mathcal{T}(c,s,d)$ with $B \cap \mathcal{T}(c,st,e)$, because $BS(F) = B$. In the latter sum write a term as $\delta\sigma_d$ and put $z = \delta_\infty^{-1} y$. The kernel in Definition 204.1 then becomes $F_\infty(y) F_\infty(z)^* L_d$. The second cutoff is $\beta_e(\sigma_{d,\infty}^{-1} z)$. Its sum over the left action of Γ_d on z is one by the conjugation identity of Lemma 216.2 and the partition of Proposition 202.3. The same holds for $\beta_d(z)$. Within each right Γ_d-coset of δ the kernel is constant because the character is trivial on Γ_d by Proposition 21.2. Summing the nonnegative cutoffs on that coset therefore gives the same integral with either cutoff. Absolute convergence permits the regrouping. This proves the asserted equality block by block. □

217. The first moment at the marked primes

Lemma 217.1. *Let $i \in \{0,1\}$, put $J_i = \prod_{w \mid g_i} \mathrm{GL}_r(\mathbf{F}_{g_i})$, with w running through the f split pairs, and let $\iota_i : J_i \to \mathcal{G}$ insert this factor. For a self-adjoint idempotent $x = \sum_{t \in J_i} x_t[t] \in \mathbb{C}[J_i]$ put $Z_x = \sum_t x_t D_{\mathrm{out}}(\iota_i(t)^{-1})$. For every fixed $A > 0$, at all sufficiently large scales, uniformly in i, x,*

$$\|Z_x\|_{\mathrm{HS}}^2 \le c_0 b^{-A}.$$

Proof. Give matrix-valued functions on $\mathcal{G}$ the squared norm $\sum_s \|f(s)\|_{\mathrm{HS}}^2$. Define the convolution by

$$L_x f(s) = \sum_t x_t f(\iota_i(t)^{-1} s).$$

Each translation preserves counting measure, its adjoint is its inverse, and translations respect multiplication. Thus L_x is a self-adjoint idempotent. Its range and kernel are orthogonal, so $\|L_x f\| \le \|f\|$. The set Ω_{1-i} in Proposition 215.4 contains $\bar{S}$ and is invariant under left translation by $\iota_i(J_i)$. Apply the contraction to $f = \mathbf{1}_{\Omega_{1-i}} D_{\mathrm{out}}$. For every $s \in \bar{S}$, Proposition 216.3 gives $L_x f(s) = Z_x U_s$. Its Hilbert–Schmidt norm is $\|Z_x\|_{\mathrm{HS}}$, since the columns of a unitary matrix form an orthonormal basis. Summing just over $s \in \bar{S}$ on the left and using Proposition 215.4 on the right gives

$$|\bar{S}| \|Z_x\|_{\mathrm{HS}}^2 \le \sum_{s \in \Omega_{1-i}} \|D_{\mathrm{out}}(s)\|_{\mathrm{HS}}^2 \le b^{-A} E_1 = b^{-A} |\bar{S}| c_0$$

by Definitions 204.1 and 69.1. □

Lemma 217.2. *With J_i, ι_i as in Lemma 217.1, let $U_i \subset J_i$ be the image of the integral unipotents at the split pairs above g_i, and let ξ_i denote the character induced there by χ_g of Definition 160.3. Define $h_i(t) = \xi_i(t)$ on U_i, and $h_i(t) = 0$ elsewhere. Then*

$$c_0^{-1} \operatorname{tr} D_{S(F)}(\iota_i(t)^{-1}) = h_i(t), \qquad \sum_{t \in J_i} \left| |J_i|^{-1} \sum_{j \in J_i} h_i(jtj^{-1}) \right|^2 \le 2$$

at all sufficiently large scales.

Proof. For a diagonal block d suppose $\delta = u\eta \in S(F)$ lies above $(d, \iota_i(t)^{-1}, d)$. Projection to $P^\perp/P$ as in Lemma 203.1 shows that the definite η stabilizes the middle lattice in class d. Its conjugate by h_d lies in K^0 by Proposition 66.1. At every marked pair above g_{1-i} it is the identity modulo g_{1-i}, by the same projection. This element of the finite definite stabilizer is trivial by the odd-prime torsion argument of Proposition 13.5, applied at g_{1-i}. It follows that $\delta \in U(F)$ and $t \in U_i$.

Conversely, for $t \in U_i$, lift t^{-1} to an integral finite unipotent with identity at all places outside g_i. The open level condition on the variable u_f, $h_d^{-1} u_f h_d \in k_{\iota_i(t)^{-1}} K_Y$, holds at the conjugate of this lift by h_d. This conjugation preserves U_f, so Lemma 21.1 gives a rational unipotent δ in the same open set. As the simple-root entries are invariant under conjugation by h_d, the character of δ_f is $\xi_i(t^{-1})$. Indeed it is trivial on integral unipotents off g and on the principal unipotent kernel above g, by the calculation in Lemma 188.3. Consequently $\chi_\infty(\delta_\infty) = \xi_i(t)$ by Proposition 21.2. For this translate the permutation of Lemma 216.2 fixes every d, and the corresponding block of $U_{\iota_i(t)^{-1}}$ in Proposition 216.3 is $\chi_\infty(\delta_\infty) I$. Apply that proposition with $B = S(F)$, $s = 1$, using $D_{S(F)}(1) = I$ from Corollary 204.5. Summing the diagonal entries proves the trace formula, including vanishing off U_i.

For the squared sum take T to consist of all places over $g_0 g_1$, and $Z = K_T$ in Corollary 162.3. Its quotient is $J_0 \times J_1$ by Proposition 15.2. Membership and the phase in its integral unipotent image are products of the independent local conditions at the split pairs. Hence at $\iota_i(t)$ the conjugation average in that corollary is precisely the displayed average of h_i. The other factor has argument 1 and phase one. Restrict the nonnegative integral sum to this slice. It is bounded by $\mathcal{E}_T(K_T)/\mu(K_T)$, since the remainder is nonnegative. The product estimate in the proof of Theorem 177.3 bounds this ratio by $1 + 2^{-n-1} \le 2$. $\square$

Theorem 217.3. *With Q, ρ of Definition 215.1, let $R_i = \rho(\iota_i(e_i))$ for the one-prime good idempotents of Corollary 74.3, extending ι_i linearly. At all sufficiently large scales, uniformly on the window,*

$$\sum_{j=1}^{c_0} \|(Q^*Q - I)e_j\| \le c_0 b^{-4}, \qquad \operatorname{tr}(Q^*(1 - R_i)Q) \le 2c_0 b^{-4} \quad (i = 0, 1).$$

Here e_j in the sum denotes the standard coordinate basis.

Proof. By Theorem 215.2's inner product calculation $D_\Gamma(s) = \overline{Q^* \rho(s) Q}$, with the bar taken entrywise. At $s = 1$ subtract $D_{S(F)}(1) = I$ from Corollary 204.5. The result is $D_{\mathrm{out}}(1)$, whose squared norm is at most $c_0 b^{-8}$ by Lemma 217.1 at $x = [1]$. The sum of the column norms is at most $\sqrt{c_0}$ times the Hilbert–Schmidt norm by Cauchy–Schwarz. This proves the first bound.

For the second let $x = 1 - e_i$. Its coefficients are invariant under conjugacy, and $\sum_t |x_t|^2 = x_1$ by taking the identity coefficient of $x^* x = x$. The trace identity

$$\operatorname{tr}(Q^*(1 - R_i)Q) = \sum_{t \in J_i} x_t \operatorname{tr} D_\Gamma(\iota_i(t)^{-1})$$

follows because $Q^* \rho(s^{-1}) Q = (Q^* \rho(s) Q)^*$. The complementary part has absolute value at most $\sqrt{c_0} \|Z_x\|_{\mathrm{HS}} \le c_0 b^{-4}$ by Lemma 217.1 at $A = 8$ and Cauchy–Schwarz on the diagonal entries. For the $S(F)$-part use Lemma 217.2. In the sum $\sum_t x_t h_i(t)$ one can replace $h_i(t)$ by

its conjugation average since conjugation permutes J_i and fixes x_t. Cauchy–Schwarz bounds the result in absolute value by $\sqrt{2x_1}$. The one-prime estimate in the proof of Corollary 74.3 gives $x_1 \le b^{-8}/2$. The $S(F)$-part therefore contributes at most $c_0 b^{-4}$. Adding the two bounds proves the theorem. □

218. Independent translates of the ramified columns

Definition 218.1. Let $D = |\mathcal{G}|\mathfrak{m}_r\mathcal{D}$, a positive real number, and put $\epsilon = b^{-4}$, $\theta = f/(N-1)$. Let $R = R_0R_1$ be the product of the two projections of Theorem 217.3. Identify the columns with $\mathbb{C}^{c_0}$ by evaluation at the representatives and by the orthonormal tensor coordinates of Definitions 215.1 and 204.1. Write V_{ram} for the subspace $\mathrm{Col}_{\mathrm{ram}}$ of Theorem 13.6 in these coordinates and P for its orthogonal projection. Write $Q_s = \rho(s)Q$ and

$$W_* = \sum_{s\in\mathcal{G}} RQ_s(V_{\mathrm{ram}}), \qquad W = R\mathcal{P}L^2(Y),$$

where $\mathcal{P}$ is the real convolution projection of Proposition 99.2. The quotient has counting measure at the finite level as in Definition 110.1.

Theorem 218.2. *At every sufficiently large scale on the window one has $W_* \subseteq W$, $\dim W < \infty$, and*

$$\dim W_* \;\ge\; \frac{(1-5\epsilon-\theta)^2}{1+9\epsilon}\,D \;\ge\; (1-19\epsilon-2\theta)D.$$

Proof. The full columns are reproduced by $\mathcal{P}$ by Theorem 204.4. The finite and real translations commute. Each two-prime good tuple is primitive in at least one coordinate over g_0, so its projection is contained in $\mathcal{E}$ of Definition 110.1. Thus $W_* \subset W \subset \mathcal{P}\mathcal{E}L^2(Y)$, which is finite dimensional by Theorem 109.3.

All the following operators on L^2 have range in the finite-dimensional invariant space spanned by W and the $Q_s(\mathbb{C}^{c_0})$; the traces can be taken there. Put

$$A = |\mathcal{G}|^{-1}\sum_s Q_sQ_s^*, \qquad A' = |\mathcal{G}|^{-1}\sum_s RQ_sPQ_s^*R.$$

Cyclic permutation in the trace and reindexing pairs (s,t) by $s^{-1}t$ give, by Corollary 215.3 and Definition 69.1,

$$\|A\|_{\mathrm{HS}}^2 = |\mathcal{G}|^{-1}\sum_t a(t) \le (1+9\epsilon)c_0^2/D.$$

One has $0 \le A' \le RAR$ as quadratic forms since P is a projection. For positive semidefinite H, K their product has nonnegative trace, as seen in an orthonormal eigenbasis of H. Expanding $\mathrm{tr}((A' + (RAR - A'))^2)$ therefore gives $\|A'\|_{\mathrm{HS}} \le \|RAR\|_{\mathrm{HS}} \le \|A\|_{\mathrm{HS}}$. The last inequality follows in a basis adapted to R by deleting matrix entries.

Theorem 13.6 gives $\dim V_{\mathrm{ram}} \ge (1-\theta)c_0$. Expanding in the coordinate basis,

$$|\,\mathrm{tr}(P(Q^*Q - I))| \le \sum_j \|(Q^*Q-I)e_j\|\,\|Pe_j\| \le \epsilon c_0$$

by Theorem 217.3. Since R_0, R_1 are commuting projections, $1-R \le (1-R_0)+(1-R_1)$. The same theorem gives $\|(1-R)Q\|_{\mathrm{HS}}^2 \le 4\epsilon c_0$. Commutation with translations and orthogonality now give

$$\mathrm{tr}\,A' = \|RQP\|_{\mathrm{HS}}^2 = \mathrm{tr}(PQ^*Q) - \|(1-R)QP\|_{\mathrm{HS}}^2 \ge (1-5\epsilon-\theta)c_0.$$

The last factor is positive eventually since $\theta \le b^{-8}$ by the bounds in Corollary 19.6. The range of A' lies in W_*. Cauchy–Schwarz on its diagonal in a basis adapted to W_* gives $(\mathrm{tr}\,A')^2 \le \dim(W_*)\|A'\|_{\mathrm{HS}}^2$. This proves the first inequality. The second follows by expanding both sides after multiplication by $1+9\epsilon$. □

Proposition 218.3. *For all sufficiently large scales and any $a \in \mathbb{C}^{\times}$ there is a set $T \subseteq \mathcal{G}$ with*

$$|T| c_0 \le 3D \log(1 + b^2)$$

such that at least $\lceil (1 - 4/b^2)D \rceil$ of the vectors $aRQ_s z$, $s \in T$, with z from the short integral family of Corollary 14.5, are linearly independent.

Proof. Fix a subspace $F \subseteq W$ and write $l = \dim F$. From the orthogonal-projection argument in Lemma 15.3, applied in the finite invariant space used in Theorem 218.2,

$$\dim (RQ_s)^{-1} F \le 5\epsilon c_0 + \|P_F Q_s\|_{\mathrm{HS}}^2.$$

The average of the last term is $\mathrm{tr}(P_F A)$, at most $\sqrt{l}\,\|A\|_{\mathrm{HS}} \le c_0 \sqrt{(1 + 9\epsilon) l / D}$ by the proof of that theorem. Choose s at most this average, and apply $2\sqrt{(1 + 9\epsilon) l / D} \le 1 + 9\epsilon + l/D$. The map L onto V_{ram} of Theorem 13.6 has kernel of dimension at most θc_0. Pullback by L adds at most this amount to an inverse-image dimension. Multiplication by a preserves F. Consequently the map $aRQ_s L$ has inverse image of F of dimension at most

$$c_0 \left((1 + l/D)/2 + \eta \right), \qquad \eta = b^{-2},$$

since $5\epsilon + 9\epsilon/2 + \theta \le \eta$ eventually.

For the rounding cost we verify $c_0 \le D$. The proof of Proposition 215.4 gave $E_2 / E_1 \le b^{16kf}/2$. Every element of $\bar{S}$ fixes the first frame vector at each of the $2f$ chosen factors. The full group at each factor acts transitively on nonzero vectors by basis extension. Thus

$$[\mathcal{G} : \bar{S}] \ge \prod_{i=0}^{1} (g_i^r - 1)^f \ge b^{18rf}$$

eventually by Theorem 11.4. The identities of Definition 69.1 give $D / c_0 = [\mathcal{G} : \bar{S}] E_1 / E_2 \ge 1$.

Start with $F = 0$. While $u = D - \dim F > 4\eta D$, adjoin to F the whole image of the supplied $aRQ_s L$. The dimension gain is at least $c_0(u/(2D) - \eta) > 0$, by the inverse-image bound. The new deficit u' therefore satisfies

$$u' - 2\eta D \le (1 - c_0/(2D))(u - 2\eta D).$$

Each step selects a new s. By $1 - v \le \exp(-v)$, at most $\lceil (2D/c_0) \log((1 - 2\eta)/(2\eta)) \rceil$ steps suffice when $0 < \eta \le 1/8$. Using $c_0 \le D$, their cost is at most $2D \log(1/(2\eta)) + D \le 3D \log(1 + 1/\eta)$. The images of the indicated short vectors span the resulting space by Corollary 14.5. Extract a basis of it by successively adjoining vectors outside the span. Its dimension is an integer at least $(1 - 4\eta)D$. □

Corollary 218.4. *If the actual row space W of Definition 218.1 satisfies $\dim W \le D$, Proposition 15.7 holds for the normalized columns Q, these good projections and $d_0 = \dim W$, with its cost measured in units of d_0.*

Proof. Work in $W + \sum_s Q_s(\mathbb{C}^{c_0})$, a finite invariant space by Theorem 218.2. Its R-image is W, since R fixes W and every Q_s is in the real projection. The mass hypothesis in Corollary 19.6 holds with $D_* = E_2$ by Theorem 215.2 and the comparison with the required constant in Corollary 215.3. The two first-moment hypotheses hold by Theorem 217.3. Finally $d_0 E_2 \le D E_2 = |\mathcal{G}| c_0^2$ by Definition 69.1. The corollary gives the asserted basis and cost. □

219. Ray characters and Frobenius at primes of good reduction

Definition 219.1. Let K be a number field of degree d over $\mathbb{Q}$, with $O = O_K$, and let $m \ge 2$ be an integer. Let I_m be the group of fractional ideals with valuation zero at every prime over m. Let C_m be its quotient by principal ideals (z) with $\tau(z) > 0$ at every real embedding and $z - 1 \in m O_{\mathfrak{p}}$ for each $\mathfrak{p} \mid m$. The numerator of an ideal norm in I_m and its denominator, in lowest terms, are prime to m, by Proposition 82.3. We write $N\mathfrak{a} \bmod m$ for the resulting unit residue.

Proposition 219.2. *The group C_m is finite. The map $\mathfrak{a} \mapsto N\mathfrak{a} \bmod m$ on I_m factors through C_m. For every nontrivial complex character λ of C_m, the series*

$$L_m(s,\lambda) = \sum_{\substack{\mathfrak{a}\subseteq O,\ \mathfrak{a}\neq 0\\ (\mathfrak{a},m)=1}} \lambda([\mathfrak{a}])(N\mathfrak{a})^{-s}$$

has an absolutely convergent Euler product for $\Re s > 1$ and extends holomorphically to $\Re s > 1 - 1/d$. The trivial character has a simple pole at one of positive residue.

Proof. For z integral at the primes of m, the denominator ideal in Proposition 82.3 has index prime to m. It contains its index, so $z = a/k$ with $a \in O$ and a positive integer k prime to m. The local congruences in Definition 219.1 say $a - k \in mO$, by comparing valuations above m and observing integrality at the other primes. The matrix of multiplication by z on an integer basis of O therefore reduces to the identity modulo m when these congruences hold. Its determinant is $N_{K/\mathbb{Q}}z$ by Lemma 82.2. This determinant is positive under the sign condition, because the nonreal embeddings occur in conjugate pairs. Hence $N((z)) \equiv 1 \bmod m$. This proves factorization of the norm. Finiteness follows from finiteness of the ordinary class group in Lemma 82.5: among principal ideals in I_m, two generators of equal signs and equal invertible residue have quotient satisfying the ray conditions. There are finitely many possible pairs of these data.

Choose an integral ideal $\mathfrak{b} \in I_m$ representing the inverse of any prescribed ray class. To obtain one, clear the denominators of an integer basis of a fractional representative by an integer prime to m, using the denominator argument above, then raise this positive integer to the order of its residue in $(\mathbb{Z}/m\mathbb{Z})^\times$. Its principal ideal represents the identity in C_m. Now $\mathfrak{b} + mO = O$ since no maximal ideal contains both. Thus the members $a \in \mathfrak{b}$ with $a \equiv 1 \bmod mO$ form one translate of $\mathfrak{b} \cap mO = m\mathfrak{b}$, by prime factorization. Take $U \subset O^\times$ to consist of the units with this same residue and positive real signs. It has finite index by the finite sign and residue maps. The ideals in the class of $\mathfrak{b}^{-1}$ being counted are $(a)\mathfrak{b}^{-1}$ for the nonzero elements of this translate with positive real signs, modulo U. This description works with empty sign conditions for a totally imaginary field.

Apply Lemmas 83.1 and 83.2 to that translate in the positive-sign sector at $YN\mathfrak{b}$. If v is the sector volume, U_0 its unit basis subgroup, and V_K the covolume of O, the leading constant in the ideal count is $v/(m^d V_K[U : U_0])$. Indeed the lattice covolume is $m^d V_K N\mathfrak{b}$, and each admissible generator orbit gives exactly $[U : U_0]$ representatives in the sector as in Theorem 83.3. The error is $O_{K,m}(1 + Y^{1-1/d})$, uniformly over the finitely many classes. Summing with the character weights and integrating the counting function as in Corollary 83.4 gives the continuations and the stated pole. Unique factorization and the unweighted count justify the Euler products there. □

Lemma 219.3. *Let L/K be a finite Galois extension of number fields and $G = \mathrm{Gal}(L/K)$. Outside a finite set of rational primes the following hold for every prime $\mathfrak{p}$ of K. Put $k = O_K/\mathfrak{p}$, $q = |k|$, and choose a homomorphism $h : O_L \to \overline{k}$ extending reduction. There is a unique $\sigma_h \in G$ with*

$$h(\sigma_h x) = h(x)^q \quad (x \in O_L).$$

All such h form one free transitive G-set by composition. The elements σ_h, called the arithmetic Frobenius elements above $\mathfrak{p}$, form a conjugacy class. Writing e for their common order, $\mathfrak{p}O_L$ is a product of $|G|/e$ distinct primes of norm q^e. For a Galois intermediate extension K'/K, restriction gives the Frobenius at the restriction of h, outside its finite exceptional set as well.

Proof. Choose an integral generator u of $L/\mathbb{Q}$ by Lemma 82.2. The minimal polynomial P of u over K has coefficients in O_K, being symmetric functions of integral conjugates lying in K. Since the extension is normal, its distinct roots are the γu for $\gamma \in G$. Indeed an embedding over K is obtained by choosing any root of P, and normality maps L onto

itself. Thus $\deg P = |G|$. Let $M > 0$ be the index of $\mathbb{Z}[u]$ in O_L, finite by the same lemma. After inverting M, the rings O_L and $O_K[u]$ coincide. Exclude the divisors of M and of the nonzero integer norm of the discriminant of P. For every other $\mathfrak{p}$ we have

$$O_L/\mathfrak{p}O_L \simeq k[X]/(\bar{P}),$$

by monic division in $O_K[X]$, localization and reduction. The reduction $\bar{P}$ has distinct roots since the product of the squared root differences remains a unit. Homomorphisms h correspond to the roots of $\bar{P}$ in $\bar{k}$. For any one h the values $h(\gamma u)$ are distinct and exhaust the roots. This proves freeness and transitivity. The q-power of h is another such map, which defines σ_h, and composition with γ replaces σ_h by $\gamma^{-1}\sigma_h\gamma$.

Factor $\bar{P}$ into irreducibles over k. Coprime factors give a product of fields by the polynomial Euclidean algorithm. Each factor of degree a has as its roots one orbit of length a under the q-power, as in Lemma 73.2: successive Frobenius powers fix its coefficients, and the product over a full root orbit is fixed by Frobenius and hence has coefficients in k. That fixed field is k since $X^q - X$ has at most q roots. The orbit through h has length e by freeness. The kernels of the projections to the field factors are all the primes above $\mathfrak{p}$, and their intersection is their product by pairwise coprimality. This proves the claimed factorization and norms. Finally restrict the defining equation for σ_h to $O_{K'}$ and use uniqueness in K'/K. □

220. Degree-one primes with prescribed Frobenius

Lemma 220.1. *Let K be any number field, p a rational prime, ζ a primitive p-th root of unity, and $H = \mathrm{Gal}(K(\zeta)/K) \subset (\mathbb{Z}/p\mathbb{Z})^\times$ by its action on ζ. For every $a \in H$, after deleting any finite set of rational primes, one has as $\sigma \downarrow 1$*

$$\sum_{\substack{\mathfrak{q} \text{ over the remaining primes} \\ N\mathfrak{q} \text{ prime},\ N\mathfrak{q} \equiv a \bmod p}} (N\mathfrak{q})^{-\sigma} = |H|^{-1} \log\left(\frac{1}{\sigma - 1}\right) + O_{K,p}(1).$$

The bounded term may also depend on the deleted set.

Proof. The extension $L = K(\zeta)$ is Galois since it splits $X^p - 1$. Its group acts faithfully on ζ, and H is cyclic by the finite multiplicative subgroup argument of Lemma 13.2. If $B \leq H$, its fixed field L^B has $[L : L^B] = |B|$. Indeed for an integral primitive element v of $L/\mathbb{Q}$, its $|B|$ distinct images under B are roots of its minimal polynomial over L^B, and the product of the corresponding monic linear factors belongs to $L^B[X]$. It has exactly that degree, and v generates L over L^B. Since H is abelian, restriction to L^B has kernel B. Its image has size $[L^B : K]$, so L^B/K is Galois with group H/B. Here equality of the automorphism count with the degree gives all embeddings by the separability argument of Lemma 82.2.

Take m divisible by p, the excluded primes and all exceptional rational primes in Lemma 219.3 for L/K and its subextensions L^B/K. At every $\mathfrak{q} \nmid m$, reduction of ζ has order p, since $\prod_{j=1}^{p-1}(1 - \zeta^j) = p$ by evaluating $(X^p - 1)/(X - 1)$ at 1. Hence the arithmetic Frobenius acts on ζ by $h_\mathfrak{q} = N\mathfrak{q} \bmod p$. In particular this norm belongs to H. The elements $h_\mathfrak{q}$, $\mathfrak{q} \nmid m$, generate H. Otherwise, if their subgroup were $B \subsetneq H$, all these primes would split completely in L^B by the restriction and norm formulas of Lemma 219.3. The Euler products after omitting the divisors of m would give

$$\zeta_{L^B}^{(m)}(s) = \left(\zeta_K^{(m)}(s)\right)^{[L^B:K]}.$$

Both zeta functions have simple poles of positive residue at one by Corollary 83.4. Removing a finite set of Euler factors multiplies by a function holomorphic and nonzero at one. The equality on real $s > 1$ contradicts these pole orders.

For every character χ of H let $\lambda_\chi(\mathfrak{a}) = \chi(N\mathfrak{a} \bmod p)$ on I_m. This is a character of the signed ray group by Proposition 219.2 and $p \mid m$. The generation just proved makes it

nontrivial if $\chi \neq 1$. For $\Re s > 1$ one has

$$\prod_\chi L_m(s, \lambda_\chi) = \zeta_L^{(m)}(s).$$

In fact at $\mathfrak{q} \nmid m$ with Frobenius of order e, its values under all characters of the cyclic H run through the e-th roots of unity, each $|H|/e$ times. Multiplying the factors gives $(1 - (N\mathfrak{q})^{-es})^{-|H|/e}$, the local product for L by Lemma 219.3. Now divide by $L_m(s,1) = \zeta_K^{(m)}(s)$. The ratio of zeta functions has a finite positive limit at one. Every remaining factor on the left is holomorphic there by Proposition 219.2. Thus each such factor has a nonzero value at one.

For real $\sigma > 1$ expand the Euler logarithms and use orthogonality of the characters of H, obtained by summing the geometric series on a generator as in Lemma 7.1. The terms with exponent at least two are bounded uniformly by $2\sum_{\mathfrak{q}} (N\mathfrak{q})^{-2} \leq 2\zeta_K(2)$. The principal logarithm is $\log(1/(\sigma - 1)) + O(1)$. All nonprincipal Euler logarithms are bounded as $\sigma \downarrow 1$ by their nonzero values and the branch argument in Corollary 7.5. This proves the formula with all prime ideals of K prime to m. Above a rational prime ℓ there are at most $[K : \mathbb{Q}]$ primes by $N(\ell O_K) = \ell^{[K:\mathbb{Q}]}$ and factorization. Those with norm other than ℓ have norm at least ℓ^2, so their whole sum is bounded by $[K : \mathbb{Q}] \sum_{\ell \geq 2} \ell^{-2}$. A finite change of the omitted primes changes the result by a bounded amount. □

Theorem 220.2. *Let L/K be a finite Galois extension of number fields, and fix $g \in \mathrm{Gal}(L/K)$. Infinitely many primes $\mathfrak{q}$ of K of rational prime norm admit a residue map on O_L whose arithmetic Frobenius is g in the sense of Lemma 219.3. All these primes can be required to avoid any specified finite set of rational primes. The same assertion holds for geometric Frobenius, defined as the inverse of arithmetic Frobenius.*

Proof. Write $G = \mathrm{Gal}(L/K)$ and n for the order of g. Choose a finite normal extension $L_1/\mathbb{Q}$ containing L by adjoining the conjugates of a primitive generator from Lemma 82.2. Choose an odd prime $p \equiv 1 \bmod n$ outside the exceptional set for $L_1/\mathbb{Q}$ in Lemma 219.3, using Corollary 7.5. In particular p has valuation one at each prime of L_1 over it. The polynomial

$$\frac{(1+X)^p - 1}{X}$$

is irreducible over L_1. At one such valuation it is monic, has all lower coefficients divisible by p, and constant term p of valuation one. In a monic factorization over the field both factors would have integral coefficients, since the least coefficient valuation is additive on multiplication by reduction of primitive polynomials over the residue field. Both reductions would then be powers of X of positive degree, making the constant term divisible by the square of the uniformizer. This proves irreducibility and hence $[L(\zeta_p) : L] = p - 1$.

Put $E' = L(\zeta_p)$ and $J = (\mathbb{Z}/p\mathbb{Z})^\times$. Restriction embeds $\mathrm{Gal}(E'/K)$ in $G \times J$. The compositum is normal and separable over K and its degree is $|G|(p-1)$, so this inclusion is an isomorphism. Choose $a \in J$ of order n by cyclicity as in Lemma 13.2. Put $\Delta = \langle (g, a) \rangle$, $K_0 = (E')^\Delta$. The primitive-element fixed-field argument in Lemma 220.1 gives $[E' : K_0] = |\Delta| = n$ and identifies its Galois group with Δ. The images of ζ_p under Δ are distinct, so it has degree at least n over K_0. Thus $E' = K_0(\zeta_p)$, and its cyclotomic Galois group over K_0 identifies with $\langle a \rangle$ by the second coordinate.

By Lemma 220.1 for K_0, p, a, infinitely many primes $\mathfrak{Q}$ of K_0 have prime norm $\ell \equiv a \bmod p$. Remove the specified primes and the finitely many exceptions for all three Galois extensions $E'/K_0, E'/K, L/K$ in Lemma 219.3. The contraction $\mathfrak{q}$ of $\mathfrak{Q}$ to K also has residue field $\mathbf{F}_\ell$, since its residue field injects into that of $\mathfrak{Q}$ and contains $\mathbf{F}_\ell$. Take a residue map for E'/K_0 at $\mathfrak{Q}$. Its Frobenius in Δ has second coordinate a and so equals (g, a). As both residue fields have the same size, the very same equation $h(\gamma x) = h(x)^\ell$ identifies this element as Frobenius over K too. Restriction to L gives g. Each ℓ supports finitely many $\mathfrak{Q}$, hence

the result gives infinitely many distinct $\mathfrak{q}$. Using g^{-1} proves the statement for geometric Frobenius. □

221. Determination of a characteristic polynomial at inertia

Definition 221.1. Fix an algebraic closure $\overline{K}$ of a number field K, write $O_{\overline{K}}$ for its algebraic integers, and put $G_K = \mathrm{Aut}(\overline{K}/K)$. Give G_K the topology whose basic neighborhoods prescribe the restriction to a finite normal subextension of $\overline{K}/K$. A Frobenius datum is a triple (v, h, γ) where v is a prime of K of rational prime norm q, $h : O_{\overline{K}} \to \overline{\mathbf{F}}_q$ extends reduction at v, and $\gamma \in G_K$ satisfies

$$h(\gamma^{-1}x) = h(x)^q \qquad (x \in O_{\overline{K}}).$$

The exponent -1 specifies geometric Frobenius.

Lemma 221.2. *Every nonempty open $V \subset G_K$ contains geometric Frobenius elements at degree-one primes outside any prescribed finite set of primes of K. More precisely, infinitely many such primes v admit a residue map h for which there is at least one datum (v, h, γ), and every such datum has $\gamma \in V$.*

Proof. We first give the lifting argument at arbitrary finite extensions of number fields. If K_2/K_1 is finite, each residue map on O_{K_1} at a prime $\mathfrak{p}$ extends to O_{K_2}, with values in the same algebraic closure of the residue field. Indeed localize O_{K_2} at $O_{K_1} \setminus \mathfrak{p}$. It is a nonzero finite module over the local ring: an integer basis of O_{K_2} generates it there by Lemma 82.2. The maximal ideal of the base cannot generate the unit ideal in this algebra. Otherwise for module generators z one would have $z = Bz$ with B a matrix over that maximal ideal. The adjugate of $I - B$ and its unit determinant would give $z = 0$. A maximal ideal containing the extended ideal therefore lies over $\mathfrak{p}$. Its quotient is a finite field extension of the base residue field, which embeds in the chosen algebraic closure by adjoining roots successively.

If K_2/K_1 is Galois with group H, then H is transitive on residue maps extending a fixed one, including at ramified primes. To see this, take an integer basis $z_1, \ldots, z_m$ of O_{K_2} and independent indeterminates $T_1, \ldots, T_m$. The coefficients of

$$\prod_{\tau \in H} \left(X - \sum_{j=1}^{m} T_j \tau z_j \right)$$

belong to $O_{K_1}[T_1, \ldots, T_m]$. They are integral and fixed by H, and its fixed field is K_1 by the primitive-element argument in Lemma 220.1. For two such maps b, b' reduction therefore gives the same polynomial. Substitute $X = \sum_j T_j b'(z_j)$. The product using b' vanishes. The product using b must then have a zero factor in the polynomial ring over the residue field closure, an integral domain. Comparing all coefficients of T_j gives $b' = b \circ \tau$ on the basis for some single $\tau \in H$.

Now choose a finite normal L/K and $g \in \mathrm{Gal}(L/K)$ such that all extensions of g lie in V, by the topology in Definition 221.1. Theorem 220.2 gives infinitely many v of prime norm q with a residue map b on O_L whose arithmetic Frobenius is g^{-1}. We may exclude the underlying rational primes of the given finite set. Take an increasing sequence of finite normal fields $L = L_0 \subseteq L_1 \subseteq \cdots$ exhausting $\overline{K}$. Such a sequence exists since the polynomials over the countable field K form a countable set and adjoining their splitting fields successively exhausts the closure. Extend b along the sequence by the first argument, giving h. At each level the two maps h and h^q agree on O_K. By transitivity choose an automorphism realizing h^q by composition with h on that level. Extend it to G_K by the algebraic embedding extension in Lemma 16.2. From these choices at successively higher levels extract an infinite subsequence with constant restriction to L_0, then one constant on L_1, and continue. This works since the restriction sets are finite. The eventual restrictions on the diagonal subsequence are compatible and define an automorphism of the union, with

inverse given by their inverses. Its inverse γ gives the desired datum. Every such γ must restrict to g by uniqueness of finite Frobenius with residue map b in Lemma 219.3. □

Definition 221.3. A property of the data in Definition 221.1 holds for most Frobenius data if for each nonempty open V in G_K there are a nonempty open $U \subseteq V$ and a finite set S of primes of K such that the property holds whenever $\gamma \in U$ and $v \notin S$. Let ℓ be a rational prime. Give the algebraic closure K_ℓ of $\mathbb{Q}_\ell$ the absolute value of Lemma 16.2. Fix polynomials $P_v \in K_\ell[X]$ at the degree-one primes outside a finite set, assigning arbitrary polynomials at those exceptions. A continuous homomorphism $r : G_K \to \mathrm{GL}_d(K_\ell)$, with the entrywise topology, has Frobenius polynomials P_v in the approximation sense if, for every $j \geq 0$ and $\eta > 0$, for most Frobenius data

$$\left| [X^j] \det(XI - r(\gamma)) - [X^j] P_v \right|_\ell \leq \eta.$$

Here $[X^j]$ denotes extraction of one coefficient.

Theorem 221.4. *Suppose r, r' have the same Frobenius polynomials in the approximation sense of Definition 221.3. Then $\det(XI - r(s)) = \det(XI - r'(s))$ for every $s \in G_K$. In particular this holds for the marked inertia element of Theorem 113.2 when $K = E$.*

Proof. If the coefficients at degree j differ at s, let their difference have absolute value $\delta > 0$. Each coefficient is a finite polynomial in the matrix entries, hence continuous. There is a nonempty open neighborhood V of s where the absolute difference exceeds $\delta/2$. Take $\eta = \delta/4$ in Definition 221.3. First apply the condition for r within V, then that for r' within the resulting open subset. On the resulting nonempty open U both conditions hold away from the union of their finite exceptions. Lemma 221.2 supplies a datum there. Subtraction of the two estimates at its common P_v bounds the absolute difference by η by the ultrametric inequality. This contradicts the choice of V. The argument applies coefficient by coefficient, including the leading coefficients, so it also identifies the degrees. □

Corollary 221.5. *For a fixed actual Hecke eigenspace in Proposition 112.2, fix its chosen marked prime, and use s of Theorem 113.2 in G_E. Suppose every occurring primitive label c supplies a continuous $r_c : G_E \to \mathrm{GL}_{d_c}(K_2)$, $d_c > 0$, with the same family P_v in the approximation sense of Definition 221.3. Suppose $r_c(s)$ obeys the annihilating polynomial and the determinant identity of Proposition 112.2, for that label and the same T, η, A there. Then the primitive label is unique if one occurs.*

Proof. Theorem 221.4 gives the same characteristic polynomial on all the matrices $r_c(s)$. Apply Proposition 112.2 with $L = K_2$. Its image of the finite cyclic residue group can be taken in K_2 by composing the chosen T into $\mathbb{C}^\times$ with the fixed field isomorphism of Lemma 16.2. □

222. Reconstruction from traces

Definition 222.1. Fix an algebraically closed field k of characteristic zero, a group G, and a positive integer d. A function $t : G \to k$ satisfies the trace identities of degree d if, for every finite list $g_1, \ldots, g_m \in G$ and $B \in k[X_1, \ldots, X_m]$,

$$B(\operatorname{tr} r(g_1), \ldots, \operatorname{tr} r(g_m)) = 0 \quad \text{for every } r : G \to \mathrm{GL}_d(k)$$

implies $B(t(g_1), \ldots, t(g_m)) = 0$. Here and below such r are group homomorphisms. Write $k[G]$ for the algebra of finite sums of symbols $[g]$ with $[g][h] = [gh]$, and τ for the linear extension of t to this algebra.

Lemma 222.2. *For t of Definition 222.1 satisfying the identities, let $I = \{x \in k[G] : \tau(xy) = 0 \text{ for every } y \in k[G]\}$. Then I is a two-sided ideal and $A = k[G]/I$ has dimension at most d^2. The functional τ descends to A, takes 1 to d, and has nondegenerate symmetric pairing $(x, y) \mapsto \tau(xy)$. Its value on each nilpotent is zero. On each idempotent it takes a value in $\{0, 1, \ldots, d\}$.*

Proof. Cyclicity $\tau(xy) = \tau(yx)$ and $\tau(1) = d$ follow from the corresponding matrix trace identities by linear expansion. Right multiplication preserves I by associativity and left multiplication by cyclicity. Taking $y = 1$ gives $I \subset \ker \tau$. These facts and the definition give the claimed pairing. For $m = d^2 + 1$ and any $x_i, y_j \in k[G]$, $\det[\tau(x_i y_j)]_{i,j=1}^m = 0$. This is a polynomial in finitely many values of t. For any representation it vanishes because the m matrices $r(x_i)$, with r extended linearly, are dependent in the d^2-dimensional matrix space. If the images of the x_i were independent in A, the m linear functionals $y \mapsto \tau(x_i y)$ would be independent by nondegeneracy. Their joint map onto k^m is surjective, since a proper subspace of k^m has a nonzero annihilating linear form by basis extension. Choosing preimages of the coordinate vectors would yield a determinant of one. This proves the dimension bound.

For $x \in A$ form the formal power series

$$D_x(Z) = \exp\left(- \sum_{m \geq 1} \tau(x^m) Z^m / m \right).$$

Each coefficient is a polynomial in finitely many traces of powers, since the exponent has zero constant term. Its coefficients past degree d vanish by the trace identities applied to a lift of x. Indeed for a d-by-d matrix M, choose an eigenvector over k and triangularize by induction on the quotient dimension. The resulting diagonal λ_j gives trace powers $\sum_j \lambda_j^m$ and $\det(I - ZM) = \prod_j (1 - \lambda_j Z)$. Formal differentiation and the constant term one identify this determinant with the displayed exponential for matrix trace, degree at most d. Uniqueness here follows by equating the successive coefficients in the logarithmic derivative, since the field has characteristic zero. Thus D_x is a polynomial with constant term one.

If x is nilpotent, $S(Z) = \sum_{m \geq 1} \tau(x^m) Z^{m-1}$ is a polynomial and $D_x' = -S D_x$. A nonzero S makes the degree of the right side at least $\deg D_x$, incompatible with the derivative, including when the derivative is zero. Therefore $S = 0$, in particular $\tau(x) = 0$. If $x^2 = x$, put $u = \tau(x)$. Then $(1 - Z) D_x' = -u D_x$. Starting with constant term one, the coefficient of Z^{d+1} is $(-1)^{d+1} u(u-1) \cdots (u-d)/(d+1)!$, by comparison of successive coefficients. Its vanishing gives the assertion. □

Lemma 222.3. *Let B be a unital k-algebra and $S_1, \ldots, S_m$ pairwise nonisomorphic, nonzero finite-dimensional simple left B-modules. Simplicity means there is no proper nonzero invariant subspace. The joint action maps B onto $\prod_i \mathrm{End}_k(S_i)$.*

Proof. A nonzero homomorphism between two simple modules is an isomorphism, by its kernel and image. Any endomorphism of one of them has an eigenvalue over k. Subtracting that scalar leaves an endomorphism with kernel, hence zero. Choose a basis $v_{i1}, \ldots, v_{im_i}$ of each S_i. The lists $(bv_{ij})_{i,j}$ form a B-submodule of $V = \bigoplus_i S_i^{m_i}$. If it were proper, choose a proper submodule containing it of largest dimension. The quotient is simple, and at least one coordinate summand maps nontrivially to it. Identify this quotient with that S_i. The quotient map on V is then zero on the summands of other types, and on the copies of S_i it is a list of scalars, at least one nonzero, by the preceding argument. Taking $b = 1$ gives a nontrivial dependence among the basis vectors of S_i, since the list for $b = 1$ maps to zero. This is impossible. Thus all lists of images are realized, which gives all tuples of endomorphisms. □

Theorem 222.4. *Every t satisfying Definition 222.1 is the trace of a semisimple d-dimensional representation of G over k, unique up to change of basis among semisimple representations. Here semisimple means a direct sum of simple representations. If k is a topological field, G a topological group, and t continuous, this representation is continuous in the matrix entries.*

Proof. Take A of Lemma 222.2. Choose a chain $0 = A_0 \subset \cdots \subset A_h = A$ of left ideals with simple successive quotients, by taking a proper submodule of largest dimension repeatedly.

Let S_i represent the distinct isomorphism classes of its quotients. Their joint action has kernel J, a two-sided ideal. For all $x \in J$ one has $xA_j \subset A_{j-1}$, so $J^h A = 0$ and hence $J^h = 0$. For $x \in J$, $y \in A$, the product xy is in J and nilpotent. Lemma 222.2 gives $\tau(xy) = 0$ for every y, hence $x = 0$. By Lemma 222.3 the action identifies A with $\prod_i \operatorname{End}_k(S_i)$.

On a matrix factor, cyclicity makes τ zero on every off-diagonal unit E_{ab}, since $E_{aa}E_{ab} = E_{ab}$ and $E_{ab}E_{aa} = 0$ for $a \neq b$. Its values on the diagonal units agree by comparing $E_{ab}E_{ba}$ and $E_{ba}E_{ab}$. Call this common value m_i. The lemma on the trace pairing makes m_i an integer between zero and d, and nondegeneracy excludes zero, since that entire factor would then pair to zero. Consequently

$$\tau((a_i)_i) = \sum_i m_i \operatorname{tr}(a_i), \qquad \sum_i m_i \dim S_i = \tau(1) = d.$$

The action of $[g]$ on $\bigoplus_i S_i^{m_i}$ is invertible with inverse the action of $[g^{-1}]$. Its trace is $t(g)$. Each summand is simple under G because the images of $[g]$ span A.

Conversely, in any semisimple representation with trace t, apply Lemma 222.3 to the distinct constituents. The span of the group has this full matrix product as its action algebra. Its trace pairing is nondegenerate: in each factor a nonzero entry is detected by multiplying by the opposite matrix unit, and the positive multiplicity is nonzero in k. The kernel of the map from $k[G]$ is therefore exactly I. After identifying this algebra with A, the possible simple modules are the defining modules of its factors. Indeed the central factor idempotents split any module, and in a simple module precisely one acts nontrivially. For a matrix factor choose a nonzero vector fixed by some E_{aa}, possible since their sum is the identity. Its images under E_{ba} span an invariant copy of the defining module, independent by applying E_{ab} one at a time. Simplicity makes it the whole module. The values of τ on its diagonal units now prescribe the same multiplicities m_i, proving uniqueness.

Finally choose a basis of A from the spanning images of group elements $[g_j]$. The matrix $(\tau([g_i][g_j]))_{i,j}$ is invertible by nondegeneracy. The coordinates of $[g]$ in this basis are fixed linear combinations of $\tau([g][g_j]) = t(gg_j)$ by inversion of that matrix. These functions are continuous when t is, so the action of $[g]$ on the stated direct sum is continuous. □

223. Uniform approximation of eigenvalues and traces

Definition 223.1. Put $k = K_\ell$ with the absolute value of Lemma 16.2. Let I_j be a finite set for each $j \geq 1$, $I = \coprod_j I_j$. An element of $k \times k^I$ has coordinates z_*, z_i for $i \in I$. Give this set the product topology and coordinatewise multiplication. We use a closed submonoid N containing the all-ones tuple, weights $w_i \in k$, and real numbers $\epsilon_j > 0$ tending to zero such that

$$\left| z_* - \sum_{i \in I_j} w_i z_i \right|_\ell \leq \epsilon_j \qquad (z \in N,\ j \geq 1).$$

Lemma 223.2. *Let Y be a set, let $f : Y \to k$ be nonzero, and suppose f is uniformly approximable by finite linear combinations of functions from a family $\mathcal{F}$. There exist data of Definition 223.1 and choices $f_i \in \mathcal{F}$ for $i \in I$ with the following property. Whenever a k-linear operator B on k^Y preserves every uniform upper bound for the absolute values of a function and $Bf = z_* f$, $Bf_i = z_i f_i$ for all i, one has $z \in N$. Zero functions can be omitted from the chosen lists.*

Proof. Fix y_0 with $f(y_0) \neq 0$ and put $F = f/f(y_0)$. For $\epsilon_j = 2^{-j}$, choose I_j and scalars a_i so that $|F(y) - \sum_{i \in I_j} a_i f_i(y)|_\ell \leq \epsilon_j$ at every y. Multiplying a uniformly approximating sum for f by $f(y_0)^{-1}$ gives the required sums. Let N_0 consist of the tuples obtained from operators as in the statement and set $N = \overline{N_0}$. The identity operator gives the all-ones tuple and composition gives products since all the indicated functions are simultaneous eigenvectors. The closure is still a monoid, since continuity of multiplication gives product neighborhoods mapping into any given neighborhood of a product, and each meets N_0. Apply B to the

uniform error and evaluate at y_0. This gives the inequality in Definition 223.1 on N_0, with $w_i = a_i f_i(y_0)$. The inequality defines a closed set, since it uses finitely many coordinates, so it holds on N. □

Lemma 223.3. *For the data of Definition 223.1, let $B \in k[X_1, \dots, X_m]$, and $z^{(1)}, \dots, z^{(m)} \in N$. Write $z_u = (z_u^{(1)}, \dots, z_u^{(m)})$ for $u = *$ or $u \in I$. Then*

$$\left| B(z_*) - \sum_{i \in I_j} w_i B(z_i) \right|_\ell \leq \epsilon_j \max_\alpha |B_\alpha|_\ell,$$

where B_α are its coefficients and the maximum for the zero polynomial is zero.

Proof. For each monomial X^α, the coordinate tuple of its values is the product of the corresponding powers of the $z^{(m')}$, hence belongs to N. This includes the constant monomial by the identity requirement. Apply the defining bound to each such tuple, multiply by the coefficient and sum. The ultrametric triangle inequality gives the displayed estimate. □

Theorem 223.4. *Write $y = (v, h, \gamma)$ for the Frobenius data of K in Definition 221.1 and use* most *in the sense of Definition 221.3. Suppose $b(y)$ and $b_i(y)$ are k-valued functions and $t_i : G_K \to k$ are continuous. Assume the following conditions.*

1. *For most data one has $(b(y), (b_i(y))_{i \in I}) \in N$ and $b(y) \in M$, where $M \subseteq k$ is complete in the metric of k.*
2. *For each $i \in I$ and each $\delta > 0$, for most data $|b_i(y) - t_i(\gamma)|_\ell \leq \delta$.*

Then there is a continuous $t : G_K \to k$ with values in M such that $(t(g), (t_i(g))_{i \in I}) \in N$ for every g. For each $\delta > 0$, for most data $|b(y) - t(\gamma)|_\ell \leq \delta$. These last approximation conditions determine t uniquely among continuous functions. Every finite-dimensional $\mathbb{Q}_\ell$-subspace of k is an eligible complete M.

Proof. Finite intersections of properties holding for most data still hold for most data, by passing to the supplied open subset successively and taking the union of finite exclusions. A bound $|u(\gamma)|_\ell \leq a$ for most data, for continuous $u : G_K \to k$ and $a \geq 0$, holds everywhere. Otherwise start in the nonempty open set where the bound fails, refine once using the property, and use Lemma 221.2 to find a datum there outside the exceptions.

Set $t'_j(g) = \sum_{i \in I_j} w_i t_i(g)$, a continuous function. For fixed j, finite intersection of the second hypothesis at precision $\epsilon_j / (1 + \max_{i \in I_j} |w_i|_\ell)$ gives, by the first hypothesis and the bound on N,

$$|b(y) - t'_j(\gamma)|_\ell \leq \epsilon_j$$

for most data. An empty list uses maximum zero. By subtraction and the preceding continuity argument, for any j, l one has

$$|t'_j(g) - t'_l(g)|_\ell \leq \max(\epsilon_j, \epsilon_l) \qquad (g \in G_K).$$

For fixed g, j take a neighborhood in which t'_j varies by at most ϵ_j from its value at g. Refinement using the estimates and Lemma 221.2 gives some $m_j = b(y) \in M$ with $|m_j - t'_j(g)|_\ell \leq \epsilon_j$. These m_j form a Cauchy sequence by the uniform inequality. Let $t(g)$ be their limit in M. The same inequality gives $|t(g) - t'_j(g)|_\ell \leq \epsilon_j$ at every g, by letting l tend to infinity. This also proves independence of choices. The uniform limit is continuous. For a prescribed tolerance use a single sufficiently large j and continuity of t'_j. The bound on $b(y) - t(\gamma)$ for most data follows by the same choice of j. Two continuous candidates agree by applying the first paragraph to their difference with arbitrarily small tolerance.

For a neighborhood of $(t(g), (t_i(g))_i)$ in the product topology it suffices to control the first and finitely many other coordinates to a given positive precision. By continuity start with a neighborhood of g controlling these finitely many continuous functions. Inside it impose their prime approximations and the first hypothesis, then take a datum by Lemma 221.2. Its

tuple in N approximates the chosen coordinates by the ultrametric inequality. Closedness now proves membership in N.

Finally in a finite $\mathbb{Q}_\ell$-basis of M the field norm is at most a fixed multiple of the maximum coordinate norm. On the coordinate unit sphere it is continuous and takes a positive minimum by the compactness argument of Lemma 113.1 and linear independence. Scaling gives the reverse inequality up to a fixed factor. Completeness of the coordinates proves completeness of M, including the zero subspace. □

224. Characteristic polynomials from approximating representations

Theorem 224.1. *Use k, N, I_j, w_i of Definition 223.1, a number field K, and $d \geq 1$. At its degree-one primes let P_v and $P_{i,v}$ for $i \in I$ be monic of degree d over k. Suppose the following properties hold.*

1. *Each i supplies a continuous $r_i : G_K \to \mathrm{GL}_d(k)$ with Frobenius polynomials $P_{i,v}$ in the approximation sense of Definition 221.3.*
2. *For each $a = 0, \ldots, d-1$, for most Frobenius data one has*

$$\left(-[X^a]P_v,\ (-[X^a]P_{i,v})_{i\in I}\right) \in N.$$

For most data $-[X^{d-1}]P_v$ belongs to a fixed complete subset of k.

Then a continuous semisimple $r : G_K \to \mathrm{GL}_d(k)$ has Frobenius polynomials P_v in the same sense. It is unique up to isomorphism among such semisimple representations. For every $s \in G_K$ one has

$$(\operatorname{tr} r(s), (\operatorname{tr} r_i(s))_i) \in N.$$

Proof. Apply Theorem 223.4 to $b = -[X^{d-1}]P_v$, $b_i = -[X^{d-1}]P_{i,v}$, and $t_i = \operatorname{tr} r_i$. The coefficient of X^{d-1} in a characteristic polynomial is minus the trace by expansion of the determinant, so the theorem applies. It gives a continuous t with the stated tuple condition. If a polynomial B in the traces at $s_1, \ldots, s_h \in G_K$ vanishes for every degree-d representation, it vanishes for all r_i. Lemma 223.3, applied to the trace tuples at these s_j, bounds $|B(t(s_1), \ldots, t(s_h))|_\ell$ by $\epsilon_j \max_\alpha |B_\alpha|_\ell$ for every approximation index j. Sending j to infinity gives zero. Thus t is the trace of a continuous semisimple r by Theorem 222.4.

For $1 \leq h \leq d$ let B_h be the coefficient of Z^h in $\exp(-\sum_{n=1}^{h} Y_n Z^n/n)$, a fixed polynomial in $Y_1, \ldots, Y_h$. On traces of powers of a matrix this gives its characteristic coefficient of degree $d-h$, by the triangularization in Lemma 222.2. Put $C_h = \max_\alpha |(B_h)_\alpha|_\ell$ and write $c_h(r, s) = [X^{d-h}] \det(XI - r(s))$. At every $s \in G_K$, Lemma 223.3 at $s, s^2, \ldots, s^h$ gives

$$\left| c_h(r,s) - \sum_{i\in I_j} w_i c_h(r_i, s) \right|_\ell \leq C_h \epsilon_j.$$

Fix a positive tolerance δ. Take j so large that $\max(1, C_h)\epsilon_j \leq \delta$. For most data, the sum over the finite I_j on the left at $s = \gamma$ differs from $\sum_i w_i [X^{d-h}]P_{i,v}$ by at most δ, since the first assumption can be imposed simultaneously to tolerance $\delta/(1 + \max_{i\in I_j} |w_i|_\ell)$. The second assumption and the defining inequality for N compare this last sum with $[X^{d-h}]P_v$ to within ϵ_j. Their minus signs occur on both sides of the inequality. The ultrametric inequality gives the required estimate. Leading coefficients already coincide, and coefficients beyond degree d vanish on both sides. Finally any two such representations have the same characteristic polynomials by Theorem 221.4. Their traces agree and uniqueness follows by Theorem 222.4. □

Proposition 224.2. *In Theorem 224.1 fix $s \in G_K$. Let $Q_*, Q_i \in k[X]$ of degree at most m be given. For each $0 \leq n \leq m$ suppose there is $u_n \in k^\times$ such that*

$$(u_n[X^n]Q_*, (u_n[X^n]Q_i)_i) \in N.$$

If $Q_i(r_i(s)) = 0$ for every i, then $Q_*(r(s)) = 0$. If, separately, $\det r_i(s) = b_i$ for every i and $(ub_*, (ub_i)_i) \in N$ for some $u \neq 0$, then $\det r(s) = b_*$.

Proof. For each $g \in G_K$ and $0 \leq n \leq m$ take the trace tuple at $s^n g$ and the indicated coefficient tuple. The polynomial in their coordinates given by $\sum_n u_n^{-1} Y_n Z_n$ has value $\operatorname{tr}(Q_i(r_i(s))r_i(g)) = 0$ on each i-coordinate. By Lemma 223.3 it has value zero also on the distinguished coordinate on letting j tend to infinity. Thus $\operatorname{tr}(Q_*(r(s))r(g)) = 0$ for all g. The span of $r(G_K)$ contains $Q_*(r(s))$ and its trace pairing is nondegenerate by the construction in Theorem 222.4. This yields the first conclusion. For the determinant, use the polynomial $(-1)^d B_d$ in the traces at $s, \ldots, s^d$ from the preceding theorem, subtracting u^{-1} times the supplied last coordinate. The same transfer gives the second conclusion. □

Corollary 224.3. *Keep the representation and trace tuple of Theorem 224.1, and fix $s \in G_K$. Let $S \subset k^\times$ be finite and $a \geq 1$. If for every i all eigenvalues of $r_i(s)$ belong to S and $(\det r_i(s))^a = c$ for the same $c \in k$, then all eigenvalues of $r(s)$ belong to S and $(\det r(s))^a = c$.*

Proof. Put $H(X) = \prod_{\alpha \in S} (X - \alpha)^d$. Each monic degree-$d$ characteristic polynomial of $r_i(s)$ divides H. For division of a fixed H by any monic degree-d polynomial, long division makes the coefficients of the remainder polynomials in the divisor coefficients: at each step one subtracts the current leading coefficient times a monomial times the divisor. This also covers $\deg H < d$. The divisor coefficients are themselves the polynomials B_h of the preceding theorem in the traces of the powers of s. Each remainder coefficient is therefore a polynomial vanishing on every approximating trace tuple. By Lemma 223.3 and $\epsilon_j \to 0$, it vanishes on the reconstructed traces. Thus the characteristic polynomial of $r(s)$ divides H, proving root containment. The polynomial $((-1)^d B_d)^a - c$ in the same traces vanishes on every i-coordinate, hence on the distinguished one by the identical argument. This proves the determinant identity. □

225. Polynomial approximation and a flag over a local extension

Lemma 225.1. *Let $L \subset K_\ell$ be a finite extension of $\mathbb{Q}_\ell$. Write $O = \{x \in L : |x|_\ell \leq 1\}$. Its maximal ideal has a generator π, and the integer valuation v normalized by $v(\pi) = 1$ has $v(\ell) \geq 1$. Its residue field is finite. Fix $h \geq 1$, a complete set $A \subset O$ of representatives modulo π^h, and $a \in A$. Put*

$$H_a(X) = \prod_{\substack{c \in A \\ c \neq a}} \frac{X - c}{a - c}.$$

For every $x \in O$, $H_a(x)$ is integral and congruent modulo π to the indicator of $a + \pi^h O$. The polynomials $H_a(X)^{\ell^j}$ converge uniformly on O to that indicator as $j \to \infty$.

Proof. For $x \neq 0$ in L, all its $\mathbb{Q}_\ell$-conjugates in K_ℓ have the same absolute value by Lemma 113.1 and the extension of embeddings in Lemma 16.2. Their product with multiplicity is the determinant of multiplication by x on L: in the power basis of $\mathbb{Q}_\ell(x)$ its characteristic polynomial is its minimal polynomial, repeated $[L : \mathbb{Q}_\ell(x)]$ times on L by a basis over this smaller field. Hence $-\log_\ell |x|_\ell \in [L : \mathbb{Q}_\ell]^{-1}\mathbb{Z}$. These values form a subgroup containing one, so its positive generator is attained by some π. Division by powers of π proves the assertions about the maximal ideal and valuations. The coordinate-norm equivalence and compact unit sphere argument in Theorem 223.4 make O compact: it is a closed subset of a bounded coordinate box, and scalar multiples of $\mathbb{Z}_\ell$ are compact by digit extraction. The quotient modulo π is finite because its open cosets cover a compact set. The same argument gives finiteness modulo every π^h.

If $x \equiv a \bmod \pi^h$, each factor in the displayed product is $1 + (x - a)/(a - c)$, congruent to one modulo π. If $x \equiv b \bmod \pi^h$ for $b \in A \setminus \{a\}$, then $v(x - c) = v(b - c)$ for $c \neq b$ and $v(x - b) \geq h$, with $v(0) = +\infty$. The sums $\sum_{c \in A,\ c \neq b} v(b - c)$ and $\sum_{c \in A,\ c \neq a} v(a - c)$ are

equal. Each counts, for $1 \le t < h$, the number of other residues in the same class modulo π^t, a number independent of its center by translation. Subtracting the common term $v(a-b)$ from both sums yields $v(H_a(x)) \ge h - v(a-b) \ge 1$. If $v(u) \ge 1$, then $v(u^{\ell^j}) \ge \ell^j$. If $v(u-1) \ge 1$, the binomial expansion of u^ℓ gives $v(u^\ell - 1) \ge v(u-1)+1$, since the middle binomial coefficients are divisible by ℓ. Iteration proves the uniform convergence on both sets of residues. □

Proposition 225.2. *Let $C \subseteq L^m$ be compact. For any continuous $\phi : C \to K_\ell$ and $\delta > 0$, there is a polynomial $P \in K_\ell[X_1, \dots, X_m]$ with $\sup_{x \in C} |\phi(x) - P(x)|_\ell \le \delta$.*

Proof. The empty set or a product with zero coordinates is immediate. After rescaling each coordinate by a power of π, compactness puts C in O^m of Lemma 225.1. For each point of C, continuity gives a product of residue balls on whose intersection with C the oscillation is at most δ. By finite subcover, there is a common $h \ge 1$ for which the boxes modulo π^h refine the chosen local balls on C. Here two such balls that meet are nested, by the ultrametric inequality. On each box meeting C, take the value of ϕ at a representative point of C. The resulting finite linear combination of box indicators approximates ϕ within δ. By the preceding lemma each box indicator is the uniform limit of the products of the one-variable polynomials there, which are bounded by one. Finite linear combinations also converge uniformly by the ultrametric inequality. Choose one polynomial approximation within δ of the indicator combination and undo the coordinate rescaling. □

Lemma 225.3. *For every $n \ge 1$ there is $Y \in \mathrm{GL}_n(K_\ell)$ with entries in a single finite extension L'/L such that for every $g \in \mathrm{GL}_n(L)$ and $1 \le j \le n$ the upper left j-by-j minor of gY is nonzero. If u is the first column of Y, then $g \mapsto gu$ is injective on $M_n(L)$.*

Proof. There are elements α in K_ℓ of arbitrarily large degree over L. Indeed a generator of $\mathbb{Q}_\ell(\mu_{\ell^a-1})$ in Lemma 113.1 has degree at least $a/[L:\mathbb{Q}_\ell]$ over L, by comparing the degrees of the compositum over $\mathbb{Q}_\ell$. Choose α of degree greater than 2^{n^2}, put $L' = L(\alpha)$, and set

$$Y_{ab} = \alpha^{\,2^{(a-1)n+b-1}} \qquad (1 \le a, b \le n).$$

In any j-row minor of the first j columns of Y, a determinant monomial has exponent whose binary digits give exactly its chosen positions. Distinct terms have distinct exponents, and the exponent sets for distinct row subsets are disjoint. All exponents are smaller than 2^{n^2}, so these minors, for fixed j, are linearly independent over L by the degree of α. In particular $\det Y \ne 0$. The first j rows of an invertible g have rank j: elimination gives some nonzero j-column minor. Expanding the leading minor of gY by multilinearity as in Proposition 76.2 gives a linear combination of the independent minors of Y with these minors of g as coefficients. It cannot vanish. At $j = 1$ the entries of u are independent over L. Each row of a matrix over L is therefore determined by its product with u, giving the last assertion. □

226. Density of algebraic coefficients with prescribed weight gaps

Definition 226.1. For $n \ge 1$, call $u_1, \dots, u_n \in \mathbb{Z}$ *free up to degree B* if for each $0 \le h \le B$ the integers $\sum_i \alpha_i u_i$ are distinct as $\alpha \in \mathbb{Z}^n_{\ge 0}$ ranges over $\sum_i \alpha_i = h$. For a dominant integral weight $\lambda_1 \ge \cdots \ge \lambda_n$ put

$$u_i(\lambda) = -(\lambda_i - \lambda_n + n - i).$$

If $m_j = \lambda_j - \lambda_{j+1}$, this is $-\sum_{i \le j < n} (m_j + 1)$. A matrix coefficient of its algebraic representation is $g \mapsto \beta(W_\lambda(g)\xi)$ for a vector ξ and a linear functional β. Over K_ℓ we use the exterior tensor realization in Theorem 77.2, transported algebraically through Lemma 16.2. Its tensor basis, minors and action formulas have integer coefficients, allowing determinant inverses for negative weights.

Lemma 226.2. *Fix $n \geq 2$, $B = (n+1)^2$, and put $y_i = (B+1)^{n-i}$ for $i < n$, $y_n = 0$. Take $m_i = y_i - y_{i+1} - 1$ for $1 \leq i < n$. For any $t \geq 0$, $t \in \mathbb{Z}$, increase m_1 by t, leaving the other m_i fixed. These nonnegative multiplicities of diagram columns, with $\lambda_n = 0$, give weights $u_i(\lambda)$ free up to degree B. For $n = 1$ every integral weight has this property.*

Proof. The negatives of the shifted weights are $y_1 + t, y_2, \ldots, y_n$ by summation in Definition 226.1. Each entry except the last exceeds B times its successor. For two distinct exponent lists α, α' of equal degree h, let j be their first different entry, ordered so that $\alpha_j > \alpha'_j$. This index is less than n, since the total degrees agree. The gain at j on pairing with the positive weights is at least that entry, and the loss at later indices is at most h times its successor since the total of the negative differences in absolute value is at most h. Thus the pairings differ. In one variable there is exactly one exponent list at each given degree. □

Theorem 226.3. *Let X be a compact topological space, $L/\mathbb{Q}_\ell$ finite inside K_ℓ, T a finite set, and $z : X \to \prod_{\tau \in T} \mathrm{GL}_n(L)$ continuous. For every continuous $\phi : X \to K_\ell$ constant on the fibres of z, and every $\delta > 0$, there is a finite linear combination of functions*

$$x \longmapsto \beta\Big(\Big(\bigotimes_{\tau \in T} W_{\lambda^\tau}(z_\tau(x))\Big)\xi\Big)$$

within δ of ϕ uniformly on X. At each τ the weights can all be taken from the one-parameter family of Lemma 226.2 when $n \geq 2$, and from the nonnegative integers when $n = 1$. In particular the shifted weights at every τ are free up to $(n+1)^2$.

Proof. For empty X there is no approximation condition. For empty T the given function is constant and a scalar multiple of the coefficient of the empty tensor. In the other cases take Y, L', u of Lemma 225.3. Put

$$z'(x) = (z_\tau(x)u)_{\tau \in T} \in (L')^{n|T|}, \qquad C = z'(X).$$

This has precisely the same fibres as z by that lemma. Every continuous function on X constant on these fibres defines a continuous function on C. Indeed inverse images of closed sets are compact in X, and their images in the Hausdorff space C are closed.

For $n \geq 2$ write $D_j(A)$ for the leading minor of order j and form the continuous function

$$b(x) = \prod_{\tau \in T} \prod_{1 \leq j < n} D_j(z_\tau(x)Y)^{m_j}$$

with the baseline multiplicities of Lemma 226.2. For $n = 1$ put $b = 1$. It never vanishes by Lemma 225.3. Thus ϕ/b is continuous and factors through C. The maximum $M = \max_X |b|_\ell$ is finite and positive. Proposition 225.2 gives a polynomial H in the entries of z' such that $|\phi - b\,H(z')|_\ell \leq \delta$ everywhere, by approximating ϕ/b within δ/M.

Consider one monomial of H. Its factors at τ are entries of $z_\tau u$, hence minors of height one of $z_\tau Y$ in its first column. Let t_τ be their total degree. After multiplication by b the factor at τ is a product of flag minors with the baseline numbers of columns and t_τ extra columns of height one. For $n \geq 2$ this is the diagram of precisely the weight of Lemma 226.2 at t_τ. More explicitly, for the exterior tensor representation T_λ and distinguished tensor v_λ of Theorem 77.2, this product evaluates a tensor coordinate of

$$T_\lambda(z_\tau(x))T_\lambda(Y)v_\lambda.$$

The vector $T_\lambda(Y)v_\lambda$ lies in W_λ by that theorem's equality of the group orbit spans. The coordinate restricted to W_λ is a linear functional, so this is a coefficient of that representation over K_ℓ. For $n = 1$ the product is a constant times $z_\tau(x)^{t_\tau}$, a coefficient of the one-dimensional weight t_τ. Products over τ are coefficients of the tensor product by tensoring the vectors and functionals. Summing the finitely many monomials proves the assertion. □

227. Local-field coordinates for definite automorphic forms

Definition 227.1. Let H be the standard definite unitary group of Definition 78.1 in any positive rank n, and put $\Gamma_H = H(F)$. Fix a rational prime ℓ, set $k = K_\ell$, and fix a field isomorphism $\iota : \mathbb{C} \to k$ of Lemma 16.2. At the embedding $\tau \in \Phi$ use $\sigma_\tau(a) = \overline{\tau(a)}$ on E, in agreement with the conjugation in Definition 78.1. Put $L_\tau = \mathbb{Q}_\ell(\iota\sigma_\tau(E)) \subseteq k$. Write $z_\tau(h)$ for the matrix of the component of $h \in H_f$ at the place of E induced by $\iota\sigma_\tau$, with entries in L_τ. Put $z = (z_\tau)_{\tau\in\Phi}$. For a tuple of dominant weights κ write $W_\kappa = \bigotimes_{\tau\in\Phi} W_{\kappa^\tau}$ over k, with the action in Definition 226.1.

An algebraic form of weight κ over k is a function $f : H_f \to W_\kappa$ fixed on the right by some open subgroup of the integral stabilizer K^0, with $f(\gamma g) = W_\kappa(z(\gamma))f(g)$ for $\gamma \in \Gamma_H$. Its local-field realization is

$$F_f(g) = W_\kappa(z(g)^{-1})f(g).$$

Given an open subgroup $J \subseteq K^0$ and a character $\theta : J \to k^\times$ with open kernel, forms of level (J, θ) satisfy $f(gj) = \theta(j)f(g)$.

Lemma 227.2. *Each z_τ in Definition 227.1 is a well-defined continuous homomorphism. All its entries lie in one finite extension $L/\mathbb{Q}_\ell$ for all g, τ, and $z_\tau(\gamma)$ is the entrywise application of $\iota\sigma_\tau$ on rational matrices. Transporting coefficients by ι identifies the complex definite finite-level forms of those weights with the corresponding forms over k. For every such f, F_f is continuous and left Γ_H-invariant. Some finite extension $L_f/\mathbb{Q}_\ell$ contains every coordinate of $f(g)$ and of $F_f(g)$, for every g, in algebraic flag bases.*

Proof. The field generated by the image of an integral generator of $E/\mathbb{Q}$ over $\mathbb{Q}_\ell$ has finite degree and is L_τ. The closure of $\iota\sigma_\tau(E)$ there is L_τ: it contains the $\mathbb{Q}_\ell$-span of the powers of the generator by density of rational coefficients. The finite extension is complete by Theorem 223.4. The preimage in O_E of its maximal ideal is a prime over ℓ, since integral elements have integral images by their monic equations. Its normalized valuation is a positive multiple of the number-field valuation there: elements outside the prime are units in the local field and denominators outside the prime can be inverted, while a local uniformizer has positive valuation by Proposition 82.3. Consequently L_τ is canonically the completion used for the indicated component. Evaluation at one finite place is continuous in the restricted product topology and multiplicative, giving the asserted homomorphism. The compositum L of the finitely many L_τ has finite degree by repeated basis multiplication. Rational matrices evaluate as claimed.

The exterior-power formulas in Theorem 77.2 and the basis in Proposition 76.2 are algebraic. Their matrices in rational bases have entries polynomial in the matrix entries and determinant inverses, with rational coefficients: the ambient tensor entries have this form, and choosing independent rational orbit coefficients or the standard minor basis only requires inversion of a fixed rational matrix. Thus applying ι carries the complex rational-point law to the displayed left law. The finite-level law is transported by the same scalar isomorphism, also in the reverse direction. Right invariance by an open subgroup makes f locally constant. The formula and continuity of z give continuity of F_f; the factors of $W_\kappa(z(\gamma))$ cancel on the left.

For any open $J_0 \subseteq K^0$ fixing f, the set $\Gamma_H\backslash H_f/J_0$ is finite. Indeed the trace lattice argument of Lemma 13.4 applies with the standard rank n in place of the column rank. It counts the isometry classes of the fixed-discriminant positive definite trace lattices with their O_E-action, hence the classes at K^0. The subgroup J_0 has finite index by the compactness of K^0 and its open cosets. Choose one c in each fine class. Adjoin all coordinates of $f(c)$ to L. There are finitely many and each belongs to $k = \overline{\mathbb{Q}}_\ell$, so the resulting L_f is finite. At $g = \gamma cj$ the transformation law applies a matrix over L to $f(c)$. Applying $W_\kappa(z(g)^{-1})$ also uses entries in L. This proves the assertion on the whole group. □

Proposition 227.3. *Fix J, θ, κ in Definition 227.1. For each representative c of $\Gamma_H \backslash H_f / J$ put $D_c = J \cap c^{-1}\Gamma_H c$. Evaluation $f \mapsto (F_f(c))_c$ identifies the space of these forms with the direct sum over c of the spaces*

$$\{v \in W_\kappa : W_\kappa(z(t))v = \theta(t)v \text{ for every } t \in D_c\}.$$

The projection onto the summand condition on a single copy of W_κ is

$$v \longmapsto |D_c|^{-1} \sum_{t \in D_c} \theta(t) W_\kappa(z(t)^{-1})v.$$

Proof. The double classes are finite in number by Lemma 227.2. The stabilizer D_c is finite by the stabilizer part of Lemma 13.4: after conjugation it preserves a definite trace lattice, giving only finitely many choices for the images of its basis. Writing $ct = \gamma c$ in the two transformation laws gives the claimed condition on $F_f(c)$. Conversely for one admissible v define

$$f(\gamma c j) = W_\kappa(z(\gamma c))\, \theta(j) v$$

and set $f = 0$ on the other classes. If $\gamma c j = \gamma' c j'$, then $t = j' j^{-1} = c^{-1}\gamma'^{-1}\gamma c \in D_c$; inserting its law on v proves equality of the two proposed values. This gives all prescribed values and fixes the form uniquely. Its kernel of right invariance contains $\ker \theta$, which is open. Reindexing the finite sum by right multiplication shows the indicated average satisfies the required law, and it fixes every already admissible vector. This proves the projection formula. □

228. Changing weight and level on the definite group

Theorem 228.1. *Use f, κ, J, θ, z of Definition 227.1, with f an algebraic form of weight κ, initially of any finite level. Fix $T \subseteq \Phi$ and put $U = \Phi \setminus T$. Suppose for each $d \in J$ with $z_T(d) = 1$ that*

$$\theta(d) = 1, \qquad f(gd) = f(g) \quad (g \in H_f).$$

For every $\delta > 0$, F_f is uniformly within δ of a finite sum of functions of the form

$$g \longmapsto (A \otimes I) W_\lambda(z(g)^{-1}) f'(g).$$

Here f' is an algebraic form of level (J, θ); $\lambda_U = \kappa_U$, and at every $\tau \in T$ the weights λ^τ come from Theorem 226.3. The linear map $A : W_{\lambda_T} \to W_{\kappa_T}$ is allowed to depend on the summand. Norms of vectors are the maxima of the absolute values of their coordinates in the tensor flag bases.

Proof. Choose representatives c at level J as in Proposition 227.3. On the compact group J put $F_c(j) = F_f(cj)$ and

$$B_c(j) = \theta(j)^{-1}\big(I \otimes W_{\kappa_U}(z_U(j))\big) F_c(j).$$

This is continuous since θ has open kernel. Expanding $F_c(j)$ leaves $\theta(j)^{-1}(W_{\kappa_T}(z_T(j)^{-1}) \otimes I) W_\kappa(z(c)^{-1}) f(cj)$. Thus B_c is constant on fibres of $j \mapsto z_T(j^{-1})$: two points in the same fibre differ by multiplication on the right by an element d in the hypothesis.

Fix a positive provisional tolerance ε. Apply Theorem 226.3 component by component to B_c and $j \mapsto z_T(j^{-1})$. Its matrix entries are in a common finite local extension by Lemma 227.2. The result is a finite sum approximating B_c within ε, with terms

$$(A W_{\lambda_T}(z_T(j)^{-1}) \otimes I)v, \qquad v \in W_{\lambda_T} \otimes W_{\kappa_U}.$$

Indeed a coefficient $\beta(W_{\lambda_T}(\cdot)\xi)$ in an output coordinate $e_a \otimes e_b$ has this form with $A = e_a \beta$ and $v = \xi \otimes e_b$. Multiply back by $\theta(j)(I \otimes W_{\kappa_U}(z_U(j)^{-1}))$. Its operator norm is bounded by a constant $C \geq 1$ on the compact J, independent of the approximant. We obtain for F_c an error at most $C\varepsilon$ and terms

$$G_v(j) = \theta(j)(A \otimes I) W_\lambda(z(j)^{-1}) v.$$

Use D_c of Proposition 227.3. For $t \in D_c$ one has $F_c(tj) = F_c(j)$, by the left Γ_H-invariance of Lemma 227.2. Averaging the approximating sum at tj therefore leaves a uniform error at most $|\,|D_c|\,|_\ell^{-1} C\varepsilon$. For each term the average changes $G_v(j)$ to $G_{v^\circ}(j)$, where v° is precisely the finite stabilizer projection of Proposition 227.3. Indeed $z(tj)^{-1} = z(j)^{-1} z(t)^{-1}$ and $\theta(tj) = \theta(t)\theta(j)$. The form f' in that proposition supported on $\Gamma_H cJ$ with $F_{f'}(c) = v^\circ$ satisfies $F_{f'}(cj) = \theta(j) W_\lambda(z(j)^{-1}) v^\circ$. This realizes each averaged term by the required automorphic form.

Choose ε with $C\varepsilon \max_c |\,|D_c|\,|_\ell^{-1} \le \delta$, possible by finiteness of the classes. Sum the constructions on the distinct classes, which have disjoint supports, and use left Γ_H-invariance to extend each estimate from cJ to all of $\Gamma_H cJ$. The map $A \otimes I$ commutes with the action on the unchanged U-factors, as follows directly from the tensor product. This proves all the assertions. □

Corollary 228.2. *Let $\mathcal{V}$ be a set of pairwise distinct finite places of F split in E at which J contains the full local integral unitary factor, and suppose θ is trivial on each of those factors. In Theorem 228.1 the f' can all be chosen as simultaneous eigenvectors for the spherical Hecke algebras at the places in $\mathcal{V}$, transported to k by ι.*

Proof. At each fixed weight the complex form space is finite dimensional by evaluation and the finite stabilizers in Proposition 227.3, transported by Lemma 227.2. We give the unitary inner product explicitly. Pass to the open subgroup $J_1 = \ker \theta$. Its index in J is finite, since J is compact, so θ has finite image and under ι^{-1} takes values on the complex unit circle. At level J_1 take the sum of the complex positive definite coefficient pairings over the double classes, each divided by the order of its stabilizer, as in Theorem 78.3. These pairings are unitary on the definite real group by Theorem 77.2. Equivalently this is integration with Haar mass one on J_1, counting a rational orbit with the reciprocal stabilizer. Partitioning into J_1-classes gives precisely the sum by the orbit count in Theorem 78.3. On the full space before level restriction each finite right translation preserves that pairing by invariance of the quotient measure, as in Lemma 21.4. Hence convolution has adjoint given by conjugated inversion. The spherical operators at each place and their adjoints preserve the desired (J, θ)-space: remove that local factor from any $j \in J$ by multiplication by its inverse component, reducing the remaining level action to disjoint factors. At the local factor the operators are bi-invariant. They commute there and across distinct places by Lemma 111.1, whose local diagonalization and transpose proof holds in any rank. Thus they are commuting normal endomorphisms. If one is nonscalar, diagonalize it and restrict all the others to its mutually orthogonal proper eigenspaces. The adjoints commute there too. Induction on dimension gives a simultaneous eigenbasis for the whole family. For a normal endomorphism this diagonalization follows by taking an eigenvector, which is also an eigenvector of the adjoint with conjugate eigenvalue by equality of the two squared norms for the shifted normal operator, and proceeding on its orthogonal complement. In the scalar case any basis works. Apply ι to this decomposition of each approximating form and distribute the finite sum. The spaces and the specified weights and level are unchanged by taking eigencomponents. □

229. An auxiliary local character for the definite eigenforms

Proposition 229.1. *Let ℓ be a prime, $n \ge 1$, $J_1 = 1 + \ell M_n(\mathbb{Z}_\ell) \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$, and $J_2 = 1 + \ell^2 M_n(\mathbb{Z}_\ell)$. There is a finite-order character $\theta : J_1 \to K_\ell^\times$, trivial on J_2, with the following cuspidality property. For every idempotent $e \in M_n(\mathbb{Q}_\ell)$ other than zero and one there is a $j \in J_1 \cap (1 + eM_n(\mathbb{Q}_\ell)(1-e))$ with $\theta(j) \ne 1$.*

Proof. Choose $A \in M_n(\mathbf{F}_\ell)$ to be multiplication by a generator of $\mathbf{F}_{\ell^n}^\times$ in a field basis. The field and generator exist by Lemma 73.2. There is no nonzero proper A-invariant subspace: stability under A gives stability under all polynomials in A, and these include multiplication

by every element of the field. Choose $\zeta \in K_\ell$ of order ℓ, possible by Lemma 16.2. Put

$$\theta(1+\ell X) = \zeta^{\operatorname{tr}(A\bar{X})} \qquad (X \in M_n(\mathbb{Z}_\ell)).$$

The exponent is read modulo ℓ. Multiplication adds X mod ℓ, since the cross term has a further factor of ℓ. Hence this is a character, trivial on J_2.

Let $W = \operatorname{im} e$ have dimension a, $0 < a < n$. The subspace $eM_n(1-e)$ consists exactly of maps into W vanishing on W. In fact e is the identity on its image, and any such map B satisfies $eB(1-e) = B$. Choose a basis matrix $P \in \mathrm{GL}_n(\mathbb{Z}_\ell)$ whose first a columns span W over $\mathbb{Q}_\ell$. To construct it, scale a nonzero vector in W until it has integral entries and a unit coordinate. Elementary elimination using that unit extends it to an integral basis. In this basis W splits as the first line and a subspace of the last $n-1$ coordinates by subtracting the first coordinate. Induct on a. The lower left block of $\bar{P}^{-1}A\bar{P}$ is nonzero by the invariant-subspace property. Choose an elementary matrix D in the opposite upper right block for which $\operatorname{tr}(\bar{P}^{-1}A\bar{P}\bar{D}) \neq 0$, using the transposed position of one nonzero entry. Then $B = PDP^{-1}$ is integral, takes values in W, and annihilates W. The element $j = 1 + \ell B$ is invertible with inverse $1 - \ell B$ and has nontrivial character by the displayed formula. For $n = 1$ there is no idempotent to test. □

Lemma 229.2. *For the field E, imaginary quadratic subfield K_0 and CM type Φ of Proposition 10.3, and any finite set of rational primes, there is an odd prime ℓ outside the set split completely in E. For such primes outside finitely many further exceptions, the maps $\iota\sigma_\tau : E \to K_\ell$ of Definition 227.1 have images in $\mathbb{Q}_\ell$. They specify exactly one place from each conjugate pair above ℓ, all above one of the two places of K_0.*

Proof. Apply Theorem 220.2 to the identity of a normal closure of $E/\mathbb{Q}$. Omit 2, the specified primes, and those dividing the discriminant or power-basis index of an integral primitive element u of E as in Lemma 82.2. At the resulting primes all conjugates of u reduce to distinct elements of $\mathbf{F}_\ell$ by Lemma 219.3. Each simple root of its monic minimal polynomial modulo ℓ lifts to $\mathbb{Z}_\ell$ by the digit correction proved in Lemma 113.1. Thus all embeddings of E into K_ℓ take values in $\mathbb{Q}_\ell$, with distinct prime kernels of reduction on O_E, by the power-basis index condition. Each completion is $\mathbb{Q}_\ell$ by Lemma 227.2. The chosen complex embeddings all restrict to the same embedding of K_0 and select one from each conjugate pair by Proposition 10.3. Applying conjugation and then the same ι to all of them preserves those two properties. This proves the claims. □

Theorem 229.3. *Use the arithmetic field of Lemma 229.2 with $|\Phi| \geq 2$. Fix a finite family of complex algebraic forms on the standard definite groups, of any positive finite ranks and finite levels. Choose any finite set of rational primes to exclude. There is one auxiliary prime ℓ outside this set, and two distinct places w_0, w_1 above the same place of K_0 at ℓ, with the following property. For each given form f, transported to K_ℓ, its realization F_f is uniformly approximable by finite sums of the forms and coefficient maps of Theorem 228.1. Each approximating form is of the cuspidal type of Proposition 229.1 at w_0, is spherical at w_1, and is a simultaneous spherical Hecke eigenform away from a single finite set of places. Its shifted weights at the embeddings belonging to the two places are free up to $(n+1)^2$ in rank n.*

Proof. An open subgroup fixing one form contains a product neighborhood of the identity whose local groups are the full integral group outside a finite set of rational primes, by the restricted product topology. At its finitely many smaller factors such a neighborhood contains compact open subgroups, for example sufficiently small principal lattice congruence groups. Thus there is a product level fixing the form. Omit the union of the exceptional primes for all the given forms when choosing ℓ by Lemma 229.2. Choose w_0, w_1 from the distinct places in that lemma and let T consist of the two corresponding embeddings. For each form choose J equal to its product level except at the split factor for w_0, where it is J_1. The full split integral group there and at w_1 is $\mathrm{GL}_n(\mathbb{Z}_\ell)$: with the standard definite form a

matrix on one member of a pair determines the inverse transpose on the conjugate member, as in Lemma 13.2. Extend the local θ of Proposition 229.1 to J by trivial characters on the other factors. Its value is one when $z_T(d) = 1$, because the component at w_0 is then the identity. Also all of J fixes f on the right. Theorem 228.1 gives the required approximation, with the prescribed freedom of weights. The full factor at w_1 fixes each approximant since its level character is trivial there. Apply Corollary 228.2 at every split place outside the union of the finitely many exceptional places, and also at w_1. Zero summands can be discarded. These choices impose the given wild character on every eigenform and give the claimed common exceptional set. □

230. Hecke polynomials of the definite approximants

Definition 230.1. Let v be a finite place of E distinct from its conjugate, and put $q_v = |O_E/v|$. Use the factor of the unitary group identified with $\mathrm{GL}_n(E_v)$ by its component at v. Put $K_v = \mathrm{GL}_n(O_v)$ for the valuation ring O_v of E_v. For $0 \le h \le n$ let $T_{v,h}$ denote the sum of right translations by representatives of the cosets uK_v in

$$K_v \,\mathrm{diag}(\pi I_h, I_{n-h})\, K_v,$$

where π generates the maximal ideal of O_v, and put the identity at every other unitary factor. The cosets form a finite set, since the double coset is compact and K_v open. The sum on right-K_v-invariant forms is independent of these choices. On a nonzero spherical eigenform f with $T_{v,h}f = t_{v,h}f$ define

$$P_v(X) = \sum_{h=0}^{n} (-1)^h q_v^{h(h-1)/2}\, t_{v,h} X^{n-h}.$$

Here $T_{v,0} = 1$ on the invariant forms, so P_v is monic.

Proposition 230.2. *Let f over $k = K_\ell$ be a nonzero eigenform for the operators of Definition 230.1 at all degree-one places outside a finite set. For its approximants from Corollary 228.2 use any family with the same property outside a common finite set. In their common output coordinates and at each such $v \nmid \ell$, the eigenvalues $-[X^a]P_v$ and $-[X^a]P_{i,v}$ arise from one linear contraction on scalar functions, for each $0 \le a < n$. All coefficients of the P_v for f lie in a single finite extension of $\mathbb{Q}_\ell$ inside k.*

Proof. For an approximating form f' the function in question is $(A \otimes I)F_{f'}$ of Theorem 228.1, and the target function is F_f. View all as scalar functions on $H_f \times \mathcal{B}_0$, where $\mathcal{B}_0$ is the finite set indexing a basis of W_κ, by evaluating one coordinate. For $a = n - h$, fix representatives u of the finite cosets in Definition 230.1 and use the operator

$$B_{v,a}F(g,j) = -(-1)^h q_v^{h(h-1)/2} \sum_u F(gu, j).$$

The ultrametric inequality and $|q_v|_\ell = 1$ show that it preserves every uniform pointwise upper bound in absolute value. The components $z_\tau(u)$ are the identity since u is supported over a rational prime different from ℓ. Thus $z(gu)^{-1} = z(g)^{-1}$. The common coefficient map and this weight action can be taken outside the finite sum. Applying the eigenform relation inside it gives the eigenvalue $-[X^a]P_{i,v}$ on each approximating function and $-[X^a]P_v$ on the target. This also verifies the signs when $h = 1$.

Choose g, j with $f(g)_j \neq 0$. The numerator $(T_{v,h}f)(g)_j$ and denominator $f(g)_j$ belong to L_f from Lemma 227.2. Their quotient is $t_{v,h}$; all the rational normalizing factors belong to L_f. Finitely many exceptional polynomials, if specified over k, can be included by a finite extension since each has finitely many algebraic coefficients over $\mathbb{Q}_\ell$. □

Theorem 230.3. *Fix a nonzero spherical eigensystem of a definite form in Theorem 229.3, with polynomials P_v at the degree-one primes of E outside a finite set. There is a countable list of definite eigenforms of the same rank n, with the cuspidal type at w_0, spherical at w_1,*

and the free shifted weights specified there, for which the following implication holds. If each form in the list admits a continuous $r_i : G_E \to \mathrm{GL}_n(K_\ell)$ with its spherical polynomials $P_{i,v}$ in the approximation sense of Definition 221.3, then the given form admits a continuous semisimple representation with polynomials P_v in that sense. Its trace at every $s \in G_E$ obeys the tuple condition of Theorem 224.1 for the same list.

Proof. Outside finitely many rational primes, degree-one places of E split over F. More precisely the place below one such v has the same prime norm by its residue subfield. Lemma 219.3 for E/F shows that above this place an inert prime would have squared norm and a split one has a distinct conjugate. Delete the ramified primes, the rational primes of the common exceptional levels, and ℓ. Use the full spherical algebras at every remaining split factor for the approximating eigenforms. Choosing either member of a conjugate pair just chooses local coordinates on that same split factor. Thus the simultaneous eigenbasis of Corollary 228.2 provides both orientations.

The nonzero scalar function given by the coordinates of F_f in Proposition 230.2 is uniformly approximable by finite linear combinations of the corresponding coefficient functions of actual eigenforms by Theorem 229.3. Take the finite lists for errors tending to zero in Lemma 223.2, discarding zero functions. For each listed function fix a form realizing it. The countable union of these lists is the required one. For every remaining v and $a < n$, Proposition 230.2 and that lemma put $(-[X^a]P_v, (-[X^a]P_{i,v})_i)$ in the resulting closed monoid N. The target trace coefficient belongs to the fixed finite local extension in that proposition, which is complete by Theorem 223.4. Conditions holding at all places outside a finite set hold for most Frobenius data by Definition 221.3, taking the same open set in its definition. Hence the representation hypotheses on the listed forms allow Theorem 224.1 to be applied with $K = E, d = n$. It gives the asserted representation and the trace tuple at each s. □

Corollary 230.4. *In Corollary 221.5 the field K_2 may everywhere be replaced by K_ℓ for any prime $\ell \neq g_0$, with T transported by the corresponding ι. For its uniqueness conclusion it suffices that, for each occurring label c, the representation supplied there has, at the specified s, all eigenvalues in $Z_c \cup \mu_A(K_\ell)$ and satisfies*

$$(\det r_c(s))^A = \left(\prod_{z \in Z_c} z \right)^A .$$

When the representation is obtained by Theorem 230.3, these two conditions follow if they hold on every approximating representation for that label.

Proof. Theorem 221.4 applies in the field K_ℓ, giving all occurring labels the same multiset of eigenvalues, provided their target spherical polynomials agree as in Corollary 221.5. The displayed identity puts the quotient of the two products in $\mu_A(K_\ell)$. Lemma 112.1 now gives uniqueness. For the last assertion the union of the stated root sets is finite and its prescribed determinant power is fixed for the particular label. Apply Corollary 224.3. □

231. Powers of a matrix along a diagonal family

Lemma 231.1. *Let k be a field, $R = k[y_1, \dots, y_n]$, K its fraction field, and Y a square M-by-M matrix of homogeneous linear forms over R. Its monic polynomial p of least degree in $K[X]$ with $p(Y) = 0$ belongs to $R[X]$. It divides $\det(XI - Y)$ in $R[X]$ and is homogeneous of total degree $\deg_X p$ in $X, y_1, \dots, y_n$.*

Proof. The adjugate identity $(XI - Y)\operatorname{adj}(XI - Y) = \det(XI - Y)I$ yields $\det(XI - Y)|_{X=Y} = 0$: write the adjugate in powers of X and evaluate each coefficient with the powers of Y on its left. The two terms on the left then cancel. Division with remainder by the least-degree monic annihilator shows that it divides every annihilator in $K[X]$.

We give the descent of monic factors. In a unique factorization domain A, remove from each list of coefficients the common factor with the minimum exponent at each prime.

The resulting polynomial is primitive. Products of primitive polynomials are primitive since reduction modulo each prime gives two nonzero polynomials over a domain. Every polynomial over the fraction field is a fractional scalar times a primitive polynomial over A, uniquely up to a unit of A, by those minimum exponents. The one-variable polynomial ring over a field has unique factorization: induction on degree gives existence, and division with remainder gives a Bezout identity with each irreducible coprime to a given polynomial, so irreducibles are prime. Apply this factorization over $\mathrm{Frac}(A)$ to the primitive part of a polynomial in $A[X]$, clearing each factor to its primitive representative. The product is primitive, so the remaining scalar for the primitive part is a unit. Uniqueness over the fraction field, primitive normalization and uniqueness on constant factors give unique factorization in $A[X]$. Starting with k proves it for R. Apply primitive normalization to both monic factors of $\det(XI - Y)$ in $K[X]$. Their primitive representatives have product equal to the determinant up to a unit. The product of their leading coefficients is thus a unit, so each can be made monic over R. They are then the original monic factors.

The determinant is homogeneous of total degree M. In a product of nonzero polynomials over a domain the lowest and highest total degrees both add, by multiplying the respective nonzero homogeneous parts. A homogeneous product thus has only homogeneous factors. The term $X^{\deg_X p}$ determines the total degree of p. □

Definition 231.2. For a square matrix A of size M and an integer $m \geq 1$, form the M^2-by-m matrix whose columns list the entries of $I, A, \ldots, A^{m-1}$ in a fixed common order. Its m-row determinants are called the power minors of order m. If $m > M^2$ the set of such minors is empty. Over a field their simultaneous vanishing means that the powers are linearly dependent: elimination finds m pivot rows exactly when the columns are independent.

Theorem 231.3. *Fix a field k, integers $M \geq n \geq 1$, and a surjection $c : \{1, \ldots, M\} \to \{1, \ldots, n\}$. Write $D(y)_{aa} = y_{c(a)}$ with other entries zero. Let $x \in \mathrm{GL}_M(k)$ and $1 \leq m \leq n+1$. Suppose $e_1, \ldots, e_n$ are free in the sense of Definition 226.1 through degree $m(m-1)/2$. If all power minors of order m of $D(T^{e_1}, \ldots, T^{e_n})x$ vanish in $k[T, T^{-1}]$, then $m = n + 1$. Moreover, for each i there is a scalar $\lambda_i \in k$ with*

$$(x^{-1})_{ab} = \lambda_i \delta_{ab} \qquad (c(a) = c(b) = i).$$

Proof. Set $Y = D(y)x$ with independent indeterminates y_i. Every entry of Y^j is homogeneous of degree j, so each power minor is homogeneous of degree $h = 0 + 1 + \cdots + (m-1)$. Distinct monomials of that degree have distinct Laurent exponents under $y_i \mapsto T^{e_i}$. Their coefficients cannot cancel. Hence the minors already vanish over $k[y_1, \ldots, y_n]$. Definition 231.2 implies that the degree r of the least monic annihilator over $k(y_1, \ldots, y_n)$ is less than m.

Write this annihilator as $p(X) = X^r + a_1 X^{r-1} + \cdots + a_r$, with $a_0 = 1$. By Lemma 231.1, a_j is homogeneous of degree j. Its constant term a_r divides $(-1)^M \det(x) \prod_a y_{c(a)}$, hence is a nonzero scalar times a monomial by unique factorization. If $y_i = 0$, a row with $c(a) = i$ is zero in every positive power of Y. The (a, a)-entry of $p(Y) = 0$ at this specialization shows $y_i \mid a_r$. Thus $n \leq r < m \leq n + 1$, giving $r = n$ and $a_n = u\, y_1 \cdots y_n$ with $u \in k^\times$.

Multiply $p(Y) = 0$ on the right by x^{-1}. For row a and column b of the same block i, each entry $(Y^j x^{-1})_{ab}$ with $j \geq 2$ is divisible by y_i^2, since $Y^j x^{-1}$ starts and ends with $D(y)$. The term with $j = 1$ equals $a_{n-1} D(y)$ after multiplying by its polynomial coefficient. Extracting the coefficient of $y_1 \cdots y_n$ gives

$$u(x^{-1})_{ab} + \delta_{ab} \left[\textstyle\prod_{h \neq i} y_h \right] a_{n-1} = 0.$$

Here brackets denote coefficient extraction, including the constant coefficient for an empty product. This proves the assertion for every block. □

232. The rank determined by spherical annihilators

Definition 232.1. For a number field E, let G_E be the group of Definition 221.1. Let $k_0 \subset k$ be Hausdorff topological fields with continuous inclusion and $r : G_E \to \mathrm{GL}_M(k_0)$

continuous. Fix $Q \in \mathrm{GL}_M(k)$ and write $s(g) = Qr(g)Q^{-1}$. Integers κ_a, $1 \le a \le M$, give a diagonal family in the local polynomial closure in this basis if the following holds. For every polynomial F in the M^2 matrix entries over k, vanishing of $F(s(g))$ throughout some neighborhood of the identity implies the Laurent polynomial identity

$$F(\mathrm{diag}(T^{\kappa_a})_{a=1}^{M}) = 0.$$

A representation has diagonal matrices in its linear span if in some basis that span contains all diagonal matrices over its coefficient field.

Lemma 232.2. *For finitely many finite-dimensional continuous representations of a group G over an algebraically closed topological field k_0, form their direct sum W. In a chain of invariant subspaces from 0 to W with simple successive quotients, keep one copy of each isomorphism class of quotient. Their direct sum V is continuous, and its image has diagonal matrices in its linear span. For each $g \in G$, every polynomial annihilating the action of g on all the original representations annihilates its action on V.*

Proof. Repeatedly taking a largest-dimensional proper subrepresentation gives the chain. Bases adapted to the chain give block triangular matrices with continuous entries on every diagonal block. Their values there are the quotient actions. Polynomials act on these blocks by the same polynomials of the quotient actions, proving annihilation. Apply Lemma 222.3 to the action of the group algebra on one representative of each simple type. The resulting image is the full product of the endomorphism algebras. In concatenated bases of the simple modules its span thus contains every diagonal matrix, including each separate coordinate projection. □

Theorem 232.3. *Let $M, n \ge 1$, and let E, k_0, k, r, Q, κ be as in Definition 232.1. Suppose that r has diagonal matrices in its linear span and that the indicated diagonal family belongs to the local polynomial closure. Suppose that among the κ_a occur n weights e_i free through degree $n(n+1)/2$ in Definition 226.1. At the degree-one primes v fix polynomials $P_v \in k_0[X]$. Suppose that for most Frobenius data (v, h, γ), in the sense of Definition 221.3, P_v is monic of degree n and annihilates $r(\gamma)$. Then $M = n$ and*

$$P_v(X) = \det(XI - r(\gamma)) \quad \textit{for most Frobenius data.}$$

Proof. By Definition 231.2, all power minors of order $n+1$ of $s(\gamma)$ vanish on these data. Their vanishing is a closed condition on G_E. Every nonempty open set contains a Frobenius element satisfying it: first take the smaller open set in Definition 221.3, then apply Lemma 221.2 with the indicated finite exclusions. Thus the minors vanish on G_E. For any $g \in G_E$, substitute $As(g)$ for the variable matrix in each minor. At $A = s(h)$ the resulting polynomial vanishes for every h. Definition 232.1 puts $A = \mathrm{diag}(T^{\kappa_a})$ in this identity.

Take $g = 1$. The n prescribed weights are distinct by freeness in degree one. If there were a further distinct weight, use the $n + 1$ diagonal positions with distinct weights as rows of the power minor. In the field $k(T)$ the resulting matrix is invertible: a polynomial of degree at most n vanishing at $n + 1$ distinct powers of T is zero by division by its linear factors. Consequently $\kappa_a = e_{c(a)}$ for a surjection $c : \{1, \dots, M\} \to \{1, \dots, n\}$. Theorem 231.3 applied at every g now says that the compression of $s(g)^{-1}$ to each weight block is scalar. Inversion permutes the group elements. Thus the compression of every matrix in their k-linear span is scalar on that block.

In a basis giving the diagonal-span hypothesis, the M coordinate projections have rank one and sum to I. After extension to k and change to the s-basis they still belong to the span and retain those properties. On a block of dimension at least two, the compression of any rank-one matrix has rank at most one. Being scalar, each of these compressions would therefore vanish, whereas their sum is the block identity. All blocks have dimension one, giving $M = n$.

Finally the set B where all order-n power minors of $s(g)$ vanish is closed. If it contained a nonempty open set, choose g in that open set. For all h near 1, hg belongs to it, and the same polynomial substitution now puts $\mathrm{diag}(T^{\kappa_a})s(g)$ in all the order-n zero sets. Theorem 231.3 excludes $m = n$. Every nonempty open set therefore contains a nonempty open subset disjoint from B. On the latter the first n powers are independent. Applying the hypothesis on most Frobenius data within this open subset gives a monic annihilator P_v of degree n. Its difference from the monic characteristic polynomial, also an annihilator by the adjugate argument of Lemma 231.1, has degree less than n if nonzero. Independence forces this difference to vanish. The choices of open subsets give the asserted quantifier for most data. □

Corollary 232.4. *In Theorem 230.3, it suffices for each listed eigenform to produce a continuous representation over K_ℓ of some positive finite dimension, with diagonal matrices in its linear span, having its spherical polynomials as annihilators for most Frobenius data and a diagonal family as in Theorem 232.3 over an extension of K_ℓ containing among its weights the n free shifted weights of that eigenform.*

Proof. The bound $(n + 1)^2$ in Lemma 226.2 dominates $n(n + 1)/2$. The polynomial of Definition 230.1 is monic of degree n. Theorem 232.3 therefore gives rank n and exact characteristic polynomials on most data. Exact equality gives every coefficient tolerance in Definition 221.3, as required in Theorem 230.3. □

233. Fixed elements and cyclotomic periods

Definition 233.1. Let p be a prime, K_p the algebraic closure of $\mathbb{Q}_p$ of Lemma 16.2, and C_p its metric completion. Put $G_p = \mathrm{Aut}(K_p/\mathbb{Q}_p)$ with the subgroups fixing finite normal extensions as a neighborhood basis at the identity. The absolute value is invariant under these automorphisms by Lemma 113.1. Each acts on C_p by applying it to a Cauchy sequence, with the same norm and with its inverse supplied by the inverse automorphism. The completion is a field since the norms of a Cauchy sequence with nonzero limit eventually have fixed positive value, making the inverse sequence Cauchy. Choose primitive roots ζ_j of order p^j, $j \geq 1$, with $\zeta_{j+1}^p = \zeta_j$. Such a choice follows by taking successive p-th roots in K_p. The cyclotomic character $\chi_p : G_p \to \mathbb{Z}_p^\times$ is determined by $g(\zeta_j) = \zeta_j^{\chi_p(g) \bmod p^j}$. The compatible exponents on these cyclic groups multiply under composition.

Theorem 233.2. *For every subgroup $J \subseteq G_p$ one has $C_p^J = \overline{K_p^J}$ inside C_p. More precisely if $x \in C_p^J$ and $z \in K_p$, there is a $b \in K_p^J$ with $|x - b|_p \leq 16|x - z|_p$.*

Proof. The J-orbit of z is a finite set, contained in the roots of its minimal polynomial over $\mathbb{Q}_p$. Start with this set S and the selected $z \in S$. Suppose at a step that $S \subset K_p$ is nonempty and J-invariant with $|S| \leq 2^k$, $k \geq 1$, and a point $z \in S$ has been selected. Write $\delta = |x - z|_p$ and $U = \{t \in S : |x - t|_p = \delta\}$. It is nonempty and J-invariant by Definition 233.1. Set $m = 2^{k-1}$. The polynomial

$$f(X) = \sum_{t \in U} (X - t)^m$$

is fixed by J, has degree m, and has leading coefficient $|U|$ as an integer. Also $|f(x)|_p \leq \delta^m$. Factoring f over K_p, one root z' therefore satisfies

$$|x - z'|_p \leq \delta \, \big| \, |U| \, \big|_p^{-1/m} \leq \delta \, 2^{k/2^{k-1}},$$

since for any positive integer b one has $|b|_p^{-1} \leq b$. This includes $f(x) = 0$. The distinct roots form a nonempty J-invariant set of size at most 2^{k-1}. Continue from that set and z'. After k steps the set has one element b, necessarily fixed by J. The factors accumulated have product at most $2^{\sum_{i \geq 1} i/2^{i-1}} = 16$, since summing the geometric tails $\sum_{i \geq j} 2^{1-i}$ over $j \geq 1$

gives 4. This proves the bound, including the case of an initial singleton. Density of K_p gives one inclusion for fixed fields. Continuity of each $g \in J$ gives the other. $\square$

Lemma 233.3. *Put $L_j = \mathbb{Q}_p(\zeta_j)$, $L_\infty = \bigcup_{j\geq 1} L_j$ and let $\widehat{L}_\infty$ be its closure in C_p. Then $K_p^{\ker \chi_p} = L_\infty$. There are continuous $\mathbb{Q}_p$-linear maps $R_j : \widehat{L}_\infty \to L_j$ commuting with G_p, with*

$$|R_j(x)|_p \leq p^{1/(p-1)}|x|_p, \qquad \lim_{j\to\infty} R_j(x) = x.$$

Proof. The kernel fixes precisely the tower among the algebraic elements. Indeed an element outside L_∞ has a separable minimal polynomial of degree greater than one over L_∞. Sending it to another root extends to an automorphism over L_∞ by the extension argument of Lemma 16.2, and that automorphism lies in the kernel.

Let $d_s = p^{s-1}(p-1)$ and $\pi_s = \zeta_s - 1$. Every primitive p^s-th root has the form ζ_s^a with a prime to p. Factoring $X^a - 1$ and then using an inverse exponent modulo p^s shows that all their differences from one have the same absolute value. The product of these differences, up to sign, is p, by evaluating $(X^{p^s} - 1)/(X^{p^{s-1}} - 1)$ at 1 using the finite geometric sum. Hence $|\pi_s|_p = p^{-1/d_s}$. The d_s terms $c_i\pi_s^i$, $0 \leq i < d_s$, have pairwise different valuations when nonzero and $c_i \in \mathbb{Q}_p$. In particular the powers are independent, and the cyclotomic equation gives the matching upper degree bound. They are a basis of L_s. The norm of a sum in this basis is the largest term norm, which also proves that the valuation group is $d_s^{-1}\mathbb{Z}$ in the normalization $v(p) = 1$. All maps $\zeta_s \mapsto \zeta_s^a$ for a prime to p are therefore automorphisms: the d_s distinct primitive roots are the entire set of conjugates and all lie in L_s.

For $s \geq j$ put $q = p^{s-j}$. The maps of L_s fixing L_j act on ζ_s by exponents $1 + up^j$, $0 \leq u < q$. On L_s define R_j as the average of these q maps. This agrees with the average on every intermediate layer since restriction has equal-sized fibers. The binomial formula and the geometric sum on the q-th roots give $R_j(\pi_s^i) = (-1)^i$ for $0 \leq i < q$. The powers π_s^i in this range are a basis over L_j, since their valuations are distinct modulo $d_j^{-1}\mathbb{Z}$ and $[L_s : L_j] = q$. Thus for $z = \sum_{i<q} c_i\pi_s^i$ the norm bound is

$$|R_j(z)|_p \leq \max_i |c_i|_p \leq p^{(q-1)/d_s}|z|_p \leq p^{1/d_j}|z|_p.$$

All values lie in L_j. The finite field extension L_j is complete by the coordinate-norm proof of Theorem 223.4. This gives the continuous extensions. They commute with G_p since the finite cyclotomic actions commute. Each R_j fixes L_j. Approximating any x by one element of a finite layer and using the uniform norm bound for all larger layers proves convergence. $\square$

Theorem 233.4. *Let $a \neq 0$ be an integer. If $x \in C_p$ satisfies $g(x) = \chi_p(g)^a x$ for all g in a neighborhood of 1 in G_p, then $x = 0$.*

Proof. First assume the equation holds for all g. By Theorem 233.2 for $\ker \chi_p$ and Lemma 233.3, $x \in \widehat{L}_\infty$. For each j, the assignments $\zeta_s \mapsto \zeta_s^{1+p^j}$ are compatible automorphisms of all the layers by the same lemma and extend to K_p by Lemma 16.2. The resulting g fixes L_j and has $\chi_p(g) = 1 + p^j$. Applying R_j to the equation gives $R_j(x) = (1+p^j)^a R_j(x)$. The rational multiplier differs from one for every nonzero a, so each $R_j(x) = 0$. Their convergence proves $x = 0$.

In the general case, $g * x = \chi_p(g)^{-a} g(x)$ defines a group action on the underlying set of C_p, and its stabilizer at x contains the subgroup fixing some finite normal extension by Definition 233.1. The latter has finite index since restriction to the extension takes values in a finite permutation group of its conjugates. Thus the twisted orbit B is finite and nonempty. If $x \neq 0$ its members are all nonzero. The product $y = \prod_{b\in B} b$ satisfies $g(y) = \chi_p(g)^{a|B|} y$ for every g, by the permutation of B under the twisted action. The first case applied to $a|B| \neq 0$ gives $y = 0$, a contradiction. $\square$

234. A single local frame and the spherical rank

Lemma 234.1. *Let $w_1, \dots, w_b \in \mathbb{Z}$ and $W \subset C_p^b$ be a linear subspace. Suppose an open subgroup $H \subseteq G_p$ preserves W under the transformations*

$$S_g(v)_i = \chi_p(g)^{w_i} g(v_i) \qquad (g \in H).$$

There is a basis of W whose individual vectors are supported on positions of one weight and have all coordinates in K_p fixed by H. The projection of every $v \in W$ onto any one weight still belongs to W.

Proof. Choose positions $I \subseteq \{1, \dots, b\}$ such that projection $W \to C_p^I$ is an isomorphism, by selecting independent rows in a matrix for any basis of W. This includes $I = \varnothing$ when $W = 0$. Let $v^{(j)}$ be the inverse image of the j-th unit vector for $j \in I$. On I, the vector $S_g(v^{(j)})$ agrees with $\chi_p(g)^{w_j} v^{(j)}$, hence the two agree everywhere. A nonzero entry $v_i^{(j)}$ therefore has $w_i = w_j$ by Theorem 233.4. It is then fixed by H. Every element of C_p fixed by H is algebraic over $\mathbb{Q}_p$: choose a finite normal extension $L/\mathbb{Q}_p$ whose fixing subgroup lies in H. Its fixed elements in K_p are precisely L by the extension-of-embeddings argument of Lemma 16.2. The field L is complete by Theorem 223.4, so is closed in C_p. Apply Theorem 233.2 to its fixing subgroup. The resulting basis of W has the stated support, which also proves the assertion about projections. □

Proposition 234.2. *Let $x_i \in C_p$ and $a_i \in \mathbb{Z}$ for $1 \le i \le b$. For a polynomial $F \in C_p[X_1, \dots, X_b]$, if*

$$F\big((\chi_p(g)^{-a_i} g(x_i))_i\big) = 0 \qquad (g \text{ near } 1 \text{ in } G_p),$$

then $F((T^{a_i} x_i)_i) = 0$ in $C_p[T, T^{-1}]$.

Proof. Expand $F = \sum_\alpha f_\alpha X^\alpha$ over its finite set of monomials and set $v_\alpha = \sum_i \alpha_i a_i$. Choose an open subgroup H on which the hypothesis holds. In the coordinate space indexed by the monomials, take the span of $(\chi_p(g)^{-v_\alpha} g(x^\alpha))_\alpha$ over all $g \in H$. It is preserved by the transformations of Lemma 234.1 with weights $-v_\alpha$, since h carries the vector indexed by g to the one indexed by hg. The linear form with coefficients f_α vanishes on this span. Project the vector indexed by 1 to the positions with $v_\alpha = d$. It remains in the span by that lemma, so $\sum_{v_\alpha = d} f_\alpha x^\alpha = 0$. These are the coefficients of the required Laurent identity. □

Definition 234.3. Fix a continuous homomorphism $j : G_p \to G_E$. In particular, for a chosen embedding $E \subset \mathbb{Q}_p$ at a completely split prime, view its algebraic closure as the elements of K_p algebraic over E, and use restriction. That subfield is algebraically closed by transitivity of algebraicity and the algebraic closure in Lemma 16.2. Restriction is continuous since fixing a finite list in that subfield follows from fixing its finite normal closure over $\mathbb{Q}_p$. For a representation $r : G_E \to \mathrm{GL}_M(K_p)$, a local Hodge–Tate frame of weights $\kappa_1, \dots, \kappa_M$ at j is an invertible matrix P over C_p satisfying

$$r(j(g))\, g(P) = P \operatorname{diag}(\chi_p(g)^{-\kappa_i}) \qquad (g \text{ near } 1).$$

The action $g(P)$ is applied to the entries.

Theorem 234.4. *A frame as in Definition 234.3 gives the local polynomial closure of Definition 232.1 in the basis $Q = P^{-1}$ over $k = C_p$. The multiset of its weights depends only on the representation up to K_p-isomorphism. For a G_E-invariant subspace over K_p, the subspace and quotient admit such frames, and their multisets together are the multiset of r.*

Proof. The frame identity gives $Qr(j(g)) = \operatorname{diag}(\chi_p(g)^{-\kappa_i}) g(Q)$. If $F(Qr(t)Q^{-1}) = 0$ for t near 1 in G_E, continuity of j gives this vanishing at $t = j(g)$ near 1 in G_p. Apply Proposition 234.2 to the entries of Q, assigning weight κ_i to every entry of its i-th row, with polynomial $B \mapsto F(BQ^{-1})$. Its Laurent identity is $F(\operatorname{diag}(T^{\kappa_i})) = 0$.

For frames P, P' of the same representation, $A = P^{-1}P'$ satisfies $g(A_{ij}) = \chi_p(g)^{\kappa_i - \kappa'_j} A_{ij}$ near 1. Theorem 233.4 kills entries linking unequal weights. Invertibility of A makes each

weight block an isomorphism between its column and row coordinate spaces, giving equality of multiplicities. An algebraic change of basis and its inverse have finitely many entries fixed by an open subgroup of G_p, so the same conclusion applies across a K_p-isomorphism.

For an invariant subspace U choose a K_p-basis adapted to U, shrinking the subgroup to fix the entries of the change of basis. Write U_C for its C_p-span. The space $P^{-1}U_C$ is stable under $z \mapsto \operatorname{diag}(\chi_p(g)^{-\kappa_i})g(z)$ on this subgroup, since under P this becomes $v \mapsto r(j(g))g(v)$. Take the basis supplied by Lemma 234.1 with its pivot positions I. Complete it by the coordinate vectors at the nonpivot positions, which span the kernel of projection onto I. Each basis vector of $P^{-1}U_C$ belongs to the weight of its pivot and has entries fixed on the subgroup. Multiplying these vectors by P gives a frame of U. The images of P times the nonpivot vectors form a frame in the quotient. The frame identities follow on applying $r(j(g))g$ to these columns, in the adapted coordinates. Each original position supplies exactly one weight in the resulting two frames. □

Corollary 234.5. *For each form in Theorem 230.3, suppose there exists a finite-dimensional continuous K_ℓ-representation of G_E annihilated by its monic spherical polynomials on most Frobenius data. Suppose it has a single local Hodge–Tate frame at the chosen split embedding for w_1, containing among its weights the n specified shifted weights of the form. Then the given definite eigensystem in that theorem has a continuous semisimple rank-n representation with its spherical polynomials in the approximation sense.*

Proof. Take a chain of invariant subspaces for the representation of each form and keep one copy of each distinct simple quotient as in Lemma 232.2. The annihilators persist. By Theorem 234.4 the successive frames account for the original multiset. Discarding repeated isomorphism classes retains every weight that occurs, since isomorphic factors have the same weights. The frames of the retained factors form a block diagonal frame of their direct sum, on the intersection of the finitely many neighborhoods of 1. It yields the required diagonal family by the same theorem. The sum has diagonal matrices in its span and positive dimension, as at least one prescribed weight occurs. Apply Corollary 232.4. □

235. Local weights in a simultaneous Hecke quotient

Lemma 235.1. *Let D be a finite-dimensional vector space over a field k, let $(T_s)_{s\in S}$ be commuting endomorphisms, and fix $\theta_s \in k$. Polynomials in these variables always use only finitely many of them. Then*

$$D\Big/\sum_{s\in S}(T_s-\theta_s I)D\neq 0 \quad\Longleftrightarrow\quad F(T)|_D=0 \implies F(\theta)=0 \quad \text{for every } F\in k[(X_s)_{s\in S}].$$

Proof. On the quotient each T_s acts by θ_s since the subspace being divided out is stable under all T_s by commutativity. Thus a nonzero quotient gives the implication. If $D = 0$, the constant polynomial one violates it. In positive dimension write $B = k[(T_s)_s] \subseteq \operatorname{End}(D)$. Under the indicated implication evaluation at θ descends to a unital algebra map $B \to k$. Let $\mathfrak{m}$ be its kernel. If the quotient were zero, express every basis vector d_i of D as a sum of $(T_s - \theta_s I)$ applied to vectors, expanding those vectors in the same basis. This gives $d_i = \sum_j a_{ij} d_j$ for $a_{ij} \in \mathfrak{m}$. The adjugate identity over the commutative algebra B gives $\det(I - (a_{ij}))d_i = 0$ for all i. Hence that determinant is the zero endomorphism in B, whereas its image in k is 1. This contradiction proves the converse. □

Definition 235.2. Given a frame P of Definition 234.3 for V, put $D_u(V) \subseteq C_p \otimes_{K_p} V$ equal to the C_p-span of its columns of weight u, using the given basis of V to view these columns as vectors. For a space over $\mathbb{Q}_p$ first extend it to K_p.

Proposition 235.3. *The spaces of Definition 235.2 are intrinsic. A G_E-equivariant K_p-linear endomorphism preserves each $D_u(V)$. For an invariant K_p-subspace $U \subseteq V$ and quotient V/U, the map on the weight u spaces is onto $D_u(V/U)$ with kernel $D_u(U)$.*

Proof. The change-of-frame calculation in Theorem 234.4 makes the matrix linking any two frames zero between different weights. This gives the same column spans. An algebraic change of basis transports them as subspaces of the tensor product by the same proof. For an equivariant endomorphism A, restrict to an open subgroup fixing its finitely many entries and satisfying the frame identity. For $B = P^{-1}AP$, commuting A past $r(j(g))$ gives $g(B_{ij}) = \chi_p(g)^{\kappa_i - \kappa_j} B_{ij}$, as in the change-of-frame calculation. Thus the same vanishing by Theorem 233.4 shows preservation of each $D_u(V)$.

Use the basis for the subspace and its coordinate complement constructed in Theorem 234.4. The subspace has a frame by vectors in the respective $D_u(V)$, and the remaining frame vectors project to a frame of the quotient. In each weight these vectors together are a basis of $D_u(V)$, by the basis decomposition with pivot positions within the same weight. Quotienting gives the stated kernel and image. □

Theorem 235.4. *Let $V = \bigoplus_i V_i$ be a finite sum of finite-dimensional K_p-spaces with continuous G_E-actions. Let T_s, $s \in S$, be a commuting family of K_p-linear operators preserving each summand and commuting with G_E. For $\theta_s \in K_p$ there is a continuous G_E-quotient V_θ universal among linear quotients on which every T_s acts by θ_s. Assume V has a frame as in Definition 234.3. The quotient admits a frame, and u occurs in its weights exactly when*

$$F(T)D_u(V) = 0 \implies F(\theta) = 0 \quad \text{for every } F \in C_p[(X_s)_{s\in S}].$$

Proof. Use $V_\theta = V/U$ with $U = \sum_s (T_s - \theta_s I)V$. This is an invariant subspace under the group and all the operators. Every linear map turning the operators into their scalar values vanishes on U and hence factors uniquely through this quotient. In a basis adapted to U the quotient action is a diagonal block of the original continuous matrices, proving continuity.

Over C_p, the span U_C equals $\sum_s (T_s - \theta_s I)(C_p \otimes V)$: both sides are the span of the images of any original basis for all s. Each operator preserves $D_w(V)$ by Proposition 235.3. Splitting a vector into its weight components therefore gives

$$U_C \cap D_u(V) = \sum_s (T_s - \theta_s I) D_u(V).$$

The quotient has a frame by Theorem 234.4, and its weight space is the quotient of $D_u(V)$ by this intersection by Proposition 235.3. Apply Lemma 235.1 over C_p. □

Corollary 235.5. *For an approximating form in Theorem 230.3, put $p = \ell$. Let $V, (T_s)_s, \theta$ satisfy the hypotheses of Theorem 235.4 with a frame at w_1. Assume its polynomial test there holds for each of the n specified shifted weights. At almost all degree-one primes v let $T_{v,h}$, $1 \le h \le n$, be among the operators with $\theta_{v,h}$ the eigenvalues defining the form's P_v in Definition 230.1. Write $T_{v,0} = I$, q_v for the norm of v, and r_V for the action. If for most Frobenius data*

$$\sum_{h=0}^{n} (-1)^h q_v^{h(h-1)/2} T_{v,h}\, r_V(\gamma)^{\,n-h} = 0 \quad \text{on } V,$$

then V_θ supplies the representation for this form in Corollary 234.5.

Proof. Theorem 235.4 puts all the indicated weights in a frame of V_θ and gives the continuous quotient action. On this quotient each term in the sum descends with $T_{v,h}$ replaced by $\theta_{v,h}$. The result by Definition 230.1 is exactly $P_v(r_{V_\theta}(\gamma)) = 0$. Removing finitely many exceptional primes respects the quantifier in Definition 221.3. □

236. Integral polarization data for the cyclotomic compositum

Lemma 236.1. *Let K, E, N_1, f be as in Definition 10.1 and Proposition 10.3. Write $F = E^{\mathfrak{c}}$ for the fixed field of complex conjugation, $\beta = (1 + \sqrt{-d})/2$, and $b = (d+1)/4$. Fix an embedding $\sigma : E \to \mathbb{C}$, a finite set B of rational primes, and P the product of the distinct*

primes in $B\cup\{2\}$. There are a set Ψ of one embedding from each conjugate pair, an integer $R\ge 3$ and $u\in O_F$ with the following properties.

1. *$\sigma\in\Psi$ and each other $\tau\in\Psi$ restricts to the conjugate of $\sigma|_K$. In particular $\tau(\beta)=\sigma(\bar\beta)$ there, while $\sigma(\beta)\ne\sigma(\bar\beta)$.*
2. *$u\equiv 1 \bmod PO_F$, $R\equiv 1 \bmod P$, and $\Im\tau(u/\varpi)>0$ at every $\tau\in\Psi$, with ϖ the different generator of Proposition 18.4.*
3. *For each $c\in\{1,\beta\}$ there exist $R-3$ elements $a_t\in O_E$ spanning O_E over $\mathbb{Z}$ and satisfying*

$$c\bar c+3+\sum_{t=1}^{R-3}a_t\bar a_t=R.$$

Proof. The two embeddings in each pair restrict differently to the imaginary quadratic field K, giving exactly the required choice outside the pair of σ. The two values of β differ by $\sigma(\sqrt{-d})\ne 0$. Conjugation has fixed field F of degree f: every $x\in E$ is the sum of $(x+\bar x)/2$ and $\sqrt{-d}$ times $(x-\bar x)/(2\sqrt{-d})$, both coefficients in F. The restrictions of the f pairs give its f real embeddings, since $E=F(\sqrt{-d})$. Under these embeddings PO_F is a full lattice in $\mathbb{R}^f$ by Lemma 82.2. Every point is within a fixed distance of $1+PO_F$: subtract 1, expand in a lattice basis, and round its coordinates. The nonzero $\tau(1/\varpi)$ are purely imaginary by Proposition 18.4. Choose u near a vector whose coordinates are a common large positive length times the signs of $\Im\tau(1/\varpi)$. Taking the length larger than the error proves the second assertion about u.

Take any $R\equiv 1 \bmod P$ with $R\ge 3+b+f(1+b)$. The basis $\zeta_{N_1}^j,\beta\zeta_{N_1}^j$, $0\le j<f$, of Theorem 18.2 has $2f$ entries and sum of relative norms $f(1+b)$, since $\beta\bar\beta=b$. Start with this basis and append $R-3-c\bar c-f(1+b)$ entries equal to one. This number is nonnegative. The remaining number of positions equals $c\bar c+f(b-1)\ge 0$. Filling them with zeros proves the exact length, span and norm. □

Theorem 236.2. *Fix $n\ge 1$ and the choices of Lemma 236.1. Put $\delta=R^2u\in O_F$, $\xi=\delta/\varpi\in E$, $h=|N_{F/\mathbb{Q}}(\delta)|$, and $L=(\prod_p O_p)^n\subseteq\mathbb{A}_{E,f}^n$, where the rings and the character ψ_f are as in Proposition 18.5. For finite adelic column vectors define*

$$e(x,y)=\psi_f\left(\xi\sum_{a=1}^{n}\bar y_a x_a\right).$$

This pairing is biadditive with multiplicative values and is alternating. The scalar ξ is purely imaginary and has positive imaginary part throughout Ψ. Writing $L^\perp=\{x:e(x,L)=1\}$, one has

$$L^\perp=\delta^{-1}L,\qquad T=L^\perp/L,\qquad |T|=h^{2n},\qquad hT=0.$$

Here h and R are prime to P. There is a subgroup Λ with $R^{-1}L\subseteq\Lambda\subseteq L^\perp$ and $\Lambda^\perp=\Lambda$. Its image in $\mathbb{A}_{E,f}^n/L$ contains the full R-torsion.

Proof. The character is additive, so e is multiplicative under addition in either argument. Locally the trace of $\xi\sum_a\bar x_a x_a$ is its own negative by conjugation, hence zero. This gives $e(x,x)=1$, and evaluation at $x+y$ gives $e(y,x)=e(x,y)^{-1}$. The sign of ξ follows from Lemma 236.1 and $R^2>0$. Proposition 18.5, applied in one coordinate at a time and allowing the inverse character, identifies the dual of L as $\delta^{-1}L$. It contains L since δ is integral.

Multiplication by δ on a $\mathbb{Z}$-basis of O_F has determinant of absolute value h, by Lemma 82.2. Its adjugate shows that $h/\delta\in O_F$, by applying the inverse map times h to 1. As $\delta\equiv 1 \bmod PO_F$, the multiplication matrix is congruent to the identity modulo P. Thus h is prime to P. Multiplication by δ identifies T with $L/\delta L$, and h kills it. To count it, the diagonal O_E surjects onto $(\prod_p O_p)/h(\prod_p O_p)$ with kernel hO_E by the Chinese remainder theorem on each coordinate of the integral basis in Theorem 18.2. As h/δ is integral, reducing further gives $(\prod_p O_p)/\delta(\prod_p O_p)\simeq O_E/\delta O_E$. Indeed modulo h, multiplication by

δ corresponds under this isomorphism on both sides. The index in O_E is $|N_{E/\mathbb{Q}}\delta| = h^2$: multiplication scales its lattice covolume by its real determinant and the embeddings of E occur in pairs giving the same value on δ. Taking the n coordinates proves the asserted order.

The pairing on $L^\perp$ descends to T by its definition and skew symmetry. The subgroup $R^{-1}L/L$ is isotropic: on $R^{-1}L$ both denominators cancel the R^2 in ξ, and $uL \subseteq L$ in the pairing of Proposition 18.5. Take a largest isotropic subgroup of the finite T containing it. Any class pairing trivially with that subgroup could be adjoined with all its integer multiples and preserve isotropy by alternation. Thus it already belongs to the subgroup. Its preimage Λ contains L, so any vector pairing trivially with Λ is in $L^\perp$. The argument in T gives $\Lambda^\perp = \Lambda$. Vectors representing R-torsion modulo L are precisely $R^{-1}L$. □

237. Degree, level and good primes of the polarized data

Lemma 237.1. *Let J be a finite set, $R \geq 1$ an integer, and $\sigma : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ any function. A monomial ideal of exponents is a subset $I \subseteq \mathbb{Z}_{\geq 0}^J$ closed under addition by all nonnegative exponent vectors. There is an integer $i \geq 1$ such that, for every such I,*

$$\#\{\alpha \notin I : \textstyle\sum_t \alpha_t = R^j\} \leq \sigma(j) \quad (0 \leq j \leq i)$$

implies the same inequalities for every $j \geq 0$.

Proof. Every such ideal is the union of finitely many translates of the nonnegative orthant. Otherwise choose successively an element outside the union of the translates of all earlier choices. Any infinite sequence of natural numbers has an infinite nondecreasing subsequence: some value repeats infinitely or the sequence eventually exceeds each fixed bound. Selecting successively in each of the finitely many coordinates gives two selected vectors comparable in their original order, a contradiction.

If the asserted cutoff failed, choose for every $t \geq 1$ an ideal I_t satisfying the counts through t and violating one subsequently. There are at most countably many exponent vectors. Passing to a diagonal subsequence makes membership of each vector eventually constant. The vectors eventually present form a monomial ideal I. At any one degree the possible exponent vectors form a finite set, so I satisfies every count, by eventual agreement and the increasing indices. All finitely many generators of I are eventually present simultaneously in the subsequence. Every later ideal there contains I and hence satisfies all counts, a contradiction. □

Proposition 237.2. *Use $E, n, P, R, \delta, h, L, T, \Lambda$ of Theorem 236.2, and set $A = \mathbb{A}_{E,f}^n/L$. For all $j \geq 0$,*

$$\#\{y \in A : R^j y \in T\} = h^{2n} R^{2fnj}.$$

There exist $i \geq 1$ and a positive integer $N = R^i h$ prime to P with both of the following properties. The cutoff of Lemma 237.1 holds with $J = T$, $\sigma(j) = h^n R^{fnj}$, and this i. For any nonzero $\mathfrak{n} \in NO_E$, every $y \in A$ with $R^j y \in T$ for some $0 \leq j \leq i$ is killed by $\mathfrak{n}$.

Proof. Multiplication by R^j is onto A, since division by the rational integer is possible in the finite adeles. Its kernel $R^{-j}L/L$ has cardinal R^{2fnj}: in the $2f$ integral basis coordinates of Theorem 18.2 use the Chinese remainder argument of Theorem 236.2 with R^j. All fibres have this size. The same theorem gives $|T| = h^{2n}$, hence the count and its integer square root $\sigma(j)$. Choose i by Lemma 237.1. Since $hT = 0$, $R^j y \in T$ implies $hR^j y = 0$. For $j \leq i$ the integer hR^j divides N, proving the claim for all its integral multiples. The coprimality follows from Theorem 236.2. □

Lemma 237.3. *Let w be any finite prime of E, above a prime u_0 of $K = \mathbb{Q}(\sqrt{-d})$ over a rational prime p. There exists a prime $v \neq w$ of E above the same u_0.*

Proof. Choose two distinct primes from $3, 5, 7$ different from p and denote their product by m. Then $m \mid N_1$ by Definition 10.1. The group $(\mathbb{Z}/m\mathbb{Z})^\times$ has at least four elements of square one by independent choices of signs at the two primes. No cyclic group has more than two such elements, since in a cyclic group of order a the solutions correspond to $2t = 0 \bmod a$. Choose a unit class outside the subgroup generated by $p \bmod m$, and lift it to a unit $c \bmod N_1$ by the surjectivity proved in Lemma 3.2. Proposition 10.3 gives $\gamma \in \mathrm{Gal}(E/K)$ acting on ζ_m by exponent c.

In the residue field k_w the image of ζ_m has order exactly m. Indeed reduction is injective on the m-th roots for $p \nmid m$ by Proposition 44.1. Every automorphism of the finite residue field is a power of the p-power map. To see this for a field of p^a elements, each nonzero element has $(p^a - 1)$-st power one by multiplying over its permutation of the multiplicative group. The first a powers of Frobenius are distinct by the root bound for $X^{p^b} - X$, $0 < b < a$. A degree-a extension has at most a embeddings, by adjoining generators one by one and counting roots of their minimal polynomials. This proves the assertion, including $a = 1$. If γ preserved w, its induced automorphism would send the primitive residue root to both its c-th and its p^j-th power for some j, contrary to the choice of c. Thus $v = \gamma w \neq w$, with contraction u_0. □

Theorem 237.4. *Fix the choices of Proposition 237.2 and $0 \neq \mathfrak{n} \in NO_E$. For a rational prime p split into two distinct primes $u_0, \bar{u}_0$ in K, call w over u_0 safe if $\mathfrak{n} \notin w$ and $\mathfrak{n} \notin v$ for every v over $\bar{u}_0$. For every such w there is an integer b_0 satisfying*

$$p,\ \beta - b_0 \in w, \qquad (\beta - b_0, \bar{\beta} - b_0) = O_E, \qquad R,\ |T| \notin w,$$
$$(\mathfrak{n}) + w\,(p, \bar{\beta} - b_0) = O_E.$$

A second prime $v \neq w$ contains $p, \beta - b_0$. Here parentheses denote the ideal generated by their entries.

Proof. By Lemma 11.3, $O_K = \mathbb{Z}[\beta]$, with polynomial $X^2 - X + (d+1)/4$. Modulo p its two distinct prime factors correspond to two distinct roots in $\mathbf{F}_p$, since the quotient is the polynomial quotient and has degree two. In particular $p \neq d$, since the discriminant is $-d$. Take b_0 congruent to $\beta \bmod u_0$ and to $0 \bmod d$, by Lemma A.1. A common prime divisor of $\beta - b_0$ and $\bar{\beta} - b_0$ would contain their difference $\sqrt{-d}$, so would be above d. There $2\beta = 1 + \sqrt{-d}$ reduces to 1, whereas $2b_0$ reduces to zero. This contradiction gives the unit ideal, since a proper ideal lies in a maximal ideal as in Proposition 82.3. The other root mod p belongs to the conjugate prime, so every prime of E containing $p, \bar{\beta} - b_0$ lies over $\bar{u}_0$. By safety neither these primes nor w contains $\mathfrak{n}$. A prime containing a product of two ideals contains one of them, as seen by multiplying two elements outside it. Thus the indicated ideal sum is O_E. Since $\mathfrak{n}$ is a multiple of $R^i h$ with $i \geq 1$, its absence from w excludes both R and $|T| = h^{2n}$. Finally Lemma 237.3 gives v over u_0, with both required elements in its contraction. □

238. Weights at conjugate embeddings and scalar twists

Definition 238.1. For a dominant weight $\lambda = (\lambda_1, \dots, \lambda_n)$ of a definite form as in Definition 227.1, write $m_h = \lambda_h - \lambda_{h+1}$ for $h < n$. At the chosen CM representative τ of the pair the oriented jumps are $j - 1 - \lambda_j$, $1 \leq j \leq n$. At its conjugate $\bar{\tau}$ they are $\lambda_j + n - j$. Equivalently set

$$v_j = \lambda_n + \sum_{h=j}^{n-1} m_h + n - j.$$

The multiset at τ consists of the $n - 1 - v_j$; at $\bar{\tau}$ it consists of the v_j. The determinant exponent of the weight is λ_n.

Lemma 238.2. *For any $B \geq 0$, a list of integers is free through degree B in Definition 226.1 if and only if its image under $t \mapsto at + c$ is, for any fixed integers c and $a \neq 0$. In particular*

both oriented lists of Definition 238.1 at either modified place of Theorem 229.3, with any one integer added to all entries, are free through $(n+1)^2$.

Proof. For an exponent list of total degree h, the pairing with the transformed weights equals a times the original pairing plus ch. Equality at fixed degree is unchanged since $a \neq 0$. A permutation also preserves freeness by permuting the exponent lists. For $u_j = u_j(\lambda)$ in Definition 226.1, telescoping the m_h gives

$$j - 1 - \lambda_j = u_j + n - 1 - \lambda_n, \qquad \lambda_j + n - j = -u_j + \lambda_n.$$

Apply the first assertion to the free u_j of Theorem 226.3. This includes $n = 1$, where each total degree has one exponent list. □

Theorem 238.3. *Let* $P_v \in K_p[X]$ *be monic of degree* $n \geq 1$ *at the degree-one primes of a number field* E *outside a finite set, and take* $c_v \in K_p^\times$ *there. Suppose a continuous finite-dimensional* K_p*-representation* V *of* G_E *has a frame at some* $j : G_p \to G_E$ *in the sense of Definition 234.3. Suppose some* n *weights in that frame are free through* $n(n+1)/2$. *Assume*

$$Q_v(X) = c_v^{-n} P_v(c_v X)$$

annihilates the action on V *for most Frobenius data. Then the direct sum* r_0 *of one copy of each distinct simple constituent of* V *has rank* n *and characteristic polynomial* Q_v *for most data. If a continuous character* $a : G_E \to K_p^\times$ *satisfies* $a(\gamma) = c_v$ *for most data, the representation* $r(g) = a(g) r_0(g)$ *has characteristic polynomial* P_v *for most data. For every* $s \in G_E$ *its eigenvalues are those of* $r_0(s)$ *multiplied by* $a(s)$, *and* $\det r(s) = a(s)^n \det r_0(s)$.

Proof. Lemma 232.2 yields a continuous direct sum with diagonal matrices in its span and the prescribed annihilators. The proof of Corollary 234.5 gives a frame retaining each occurring weight. The proof applies to any list of weights since it only uses the frames of subspaces and quotients and their uniqueness from Theorem 234.4. That theorem supplies the diagonal family, so Theorem 232.3 gives the claimed rank and characteristic polynomial. For the second assertion intersect the two conditions on most data: refine a nonempty open set first for one condition, then its nonempty open subset for the other, and take the union of the two finite exclusions in Definition 221.3. For $a(\gamma) = c_v$ factoring c_v out of each row of $XI - c_v r_0(\gamma)$ gives $c_v^n Q_v(X/c_v) = P_v(X)$. This determinant identity at any single group element also gives both assertions at s. Scalar multiplication gives a continuous group homomorphism since a is a continuous character. □

Corollary 238.4. *For each approximating form of Theorem 230.3, use the situation of Corollary 235.5, allowing as the* n *tested weights either of its oriented jump lists at* w_1 *with an integer* μ *added throughout. Allow the Hecke eigenvalues of the geometric quotient at* v *to be* $c_v^{-h} t_{v,h}$ *for* $0 \leq h \leq n$, *where* $t_{v,h}$ *are the coefficients' eigenvalues in Definition 230.1. If a continuous character with the Frobenius values* c_v *is given for each form, these conditions suffice for the attachment conclusion of Theorem 230.3.*

Proof. For each form, Lemma 238.2 gives the freeness bound at w_1. The polynomial test of Theorem 235.4 supplies all these weights in the quotient. Descending the operator congruence as in Corollary 235.5 gives coefficients $(-1)^h q_v^{h(h-1)/2} c_v^{-h} t_{v,h}$ at X^{n-h}. These form precisely $c_v^{-n} P_v(c_v X)$. Apply Theorem 238.3 before using Theorem 230.3. Exact equality for most data implies each tolerance of Definition 221.3. □

239. Principal levels with specified local conditions

Definition 239.1. Continue with the CM representatives Ψ and distinguished σ in Lemma 236.1. Put $\widehat{O} = \prod_p O_p$ and $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. For a matrix write $a^* = \bar{a}^{\mathsf{T}}$. For $0 \neq \mathfrak{n} \in O_E$ define

$$K(\mathfrak{n}) = \{a \in \mathrm{GL}_n(\widehat{O}) : a^* a = tI \text{ for some } t \in \widehat{\mathbb{Z}}^\times,\ a - I \in \mathfrak{n} M_n(\widehat{O})\}.$$

A signed frame consists of $x \in \mathrm{GL}_n(\mathbb{A}_{E,f})$, $J \in M_n(\mathbb{C})$, and a matrix $A \in M_n(E)$ with $x^*Ax = \varepsilon I$ for some $\varepsilon \in \widehat{\mathbb{Z}}^\times$, such that $\tau(A)$ is positive definite Hermitian for all $\tau \in \Psi \setminus \{\sigma\}$ and $\sigma(A)J$ is positive definite Hermitian. Its stabilizer at this level consists of the $y \in \mathrm{GL}_n(E)$ with $x^{-1}yx \in K(\mathfrak{n})$ and $\sigma(y)J = J\sigma(y)$.

Proposition 239.2. *If $\mathfrak{n}$ belongs to any prime w of E above an odd rational prime ℓ with $E_w = \mathbb{Q}_\ell$, every stabilizer in Definition 239.1 is trivial.*

Proof. For y in the stabilizer set $a = x^{-1}yx$. Multiplying the two frame identities gives $y^*Ay = tA$ at the finite adeles. Some entry of the invertible A is nonzero. The quotient of the corresponding entries in this equation is an element of E equal diagonally to t. In a rational basis of E starting with 1, equality in $E \otimes \mathbb{Q}_p$ forces all its other rational coordinates to vanish. Thus $t \in \mathbb{Q}^\times$ and is a unit at every prime, giving $t = \pm 1$ by integer factorization. There is a $\tau \neq \sigma$ in Ψ, since $f = \varphi(N_1) > 1$ by Proposition 10.3. Positivity of $\tau(A)$ there implies $t = 1$. Thus y is unitary for each $\tau(A)$ at these embeddings. At σ, commuting with J gives $\sigma(y)^*\sigma(A)J\sigma(y) = \sigma(A)J$, so it is also unitary for this positive matrix.

For a fixed positive definite form the entries of a unitary matrix are bounded: its columns have fixed squared lengths for that form, and the minimum of that form on the Euclidean unit sphere is positive by compactness. Hence the entries of y at all embeddings lie in a bounded box depending on the frame. Choose a positive integer b clearing the entries of both x and x^{-1} into $\widehat{O}$, possible by their finite adelic denominators. Then all entries of $b^2y = (bx)a(bx^{-1})$ belong diagonally to $\widehat{O}$ and hence to O_E, by rational coordinates in the basis of Theorem 18.2. There are only finitely many such matrices in the box, by the lattice property of Lemma 82.2. The stabilizer is a group, so each of its elements has finite order. At w, its conjugate a_w belongs to $I + \ell M_n(\mathbb{Z}_\ell)$, which is torsion-free for odd ℓ by the binomial argument in Proposition 13.5. Thus $a_w = I$. The embedding $E \to E_w$ is injective, giving $y = I$. □

Lemma 239.3. *Let D be a finite set of finite primes of E, and assign an integer $k_w \geq 1$ to each $w \in D$. For any positive integer N there is a nonzero $\mathfrak{n} \in NO_E$ such that*

$$\mathfrak{n} \in w^{k_w} \ (w \in D), \qquad \{v : \mathfrak{n} \in v\} = D \cup \{v : N \in v\}.$$

Let K^0 be the standard definite lattice stabilizer of Definition 78.1. If a subgroup H contains every element $g \in K^0$ congruent to I modulo w^{k_w} at the places of D, then for every finite standard unitary isometry g with $g^{-\mathsf{T}} \in K(\mathfrak{n})$ one has $g \in H$.

Proof. The product $\mathfrak{a} = \prod_{w \in D} w^{k_w}$ has a positive power principal by Lemma 82.5. Write $\mathfrak{a}^m = zO_E$, $z \neq 0$. Integrality of the ideal gives $z \in O_E$. Use $\mathfrak{n} = Nz$; its support and the lower bounds on its valuations follow by Proposition 82.3, also for the unit ideal when D is empty. If $g^{-\mathsf{T}}$ is integral and invertible over $\widehat{O}$, so is g; since it is a unitary isometry, $g \in K^0$. Taking inverse transpose preserves congruence to I over every quotient ring, hence g satisfies the required conditions in D. □

Corollary 239.4. *For one approximating form f of Theorem 229.3, choose the coefficient prime ℓ odd, as permitted by Lemma 229.2. Use w_0 for the place of its cuspidal character and w_1 for the free-weight spherical place above the same prime u_0 of K. Choose another completely split prime $p_* \neq \ell$ at which f is spherical at every place, and a place $w_* \mid p_*$. There are polarization data of Theorem 236.2 and a principal level $\mathfrak{n}$ as in Proposition 237.2 for which all signed-frame stabilizers are trivial and w_1 is safe. The level is divisible by w_0^2 and w_* and avoids $\bar{w}_0, \bar{w}_*$. Every finite unitary g with $g^{-\mathsf{T}} \in K(\mathfrak{n})$ fixes f by right translation.*

Proof. Lemma 229.2 supplies p_* avoiding the finitely many exceptional rational primes and ℓ. Include p_*, ℓ in B of Lemma 236.1, so that N from Proposition 237.2 is a unit above both. At ℓ, the level of f in Theorem 229.3 contains the full integral groups except for the

factor indexed by the pair of w_0; its character of Proposition 229.1 is trivial on $1+\ell^2 M_n(\mathbb{Z}_\ell)$. Impose congruence of depth two at w_0, and of depth one at w_*. At each of the finitely many other exceptional factors impose a sufficiently deep principal congruence fixing f, possible by openness of its stabilizer in the restricted product. Use one representative of each split pair, since for a unitary matrix its other component is the inverse transpose after conjugating the local field, as in Lemma 13.2. At a nonsplit prime impose the congruence in its one factor. By taking a product neighborhood inside the stabilizer, these conditions and integrality elsewhere suffice to fix f. Apply Lemma 239.3 with these places and depths. Over ℓ the only selected prime is w_0. In particular the level meets neither w_1 nor a prime over $\bar{u}_0$; it is also a unit at $\bar{w}_*$. Proposition 239.2 applies at w_0. Theorem 237.4 applies at w_1 because ℓ splits in K. □

240. Extension of a definite vector to the general linear group

Definition 240.1. Let U denote the finite adelic isometries of the standard Hermitian form in rank n, with rational subgroup $U(F)$ as in Definition 78.1, and set $G=\mathrm{GL}_n(\mathbb{A}_{E,f})$, $\Gamma=\mathrm{GL}_n(E)$. Write $\alpha(g)=g^{-\mathsf{T}}$. It is an involutive automorphism, and on U it equals entrywise conjugation. Use $C=K(\mathfrak{n})$ from Definition 239.1. For a representation $\Xi:\Gamma\to\mathrm{GL}(W)$ on a finite-dimensional complex space, a definite function at this level is a function $\phi:G\to W$ with $\phi(ygk)=\Xi(y)\phi(g)$ for $y\in\Gamma,\ k\in C$, whose nonzero values occur only at g admitting a signed frame at $J=I$. On either group write $(R(a)\phi)(g)=\phi(ga)$ for right translation.

Lemma 240.2. *For a complex standard definite form f of weight $\lambda=(\lambda^\tau)_\tau$ as in Definition 227.1 before coefficient transport, there is a representation Ξ such that*

$$f(tu)=\Xi(\alpha(t))f(u)\qquad (t\in U(F),\ u\in U).$$

It may be taken to act by a scalar on all scalar rational matrices. More generally choose, for each conjugate pair of embeddings, its original representative τ. In the τ-factor one may use either $\tau(y)$ or $(\bar\tau(y))^{-\mathsf{T}}$ as the argument of W_{λ^τ}. One may also multiply the tensor representation by $N_{E/\mathbb{Q}}(\det y)^m$ for any $m\in\mathbb{Z}$.

Proof. Use the algebraic representations on the flag tensors in Theorem 77.2, with integral determinant powers as in Definition 226.1. Each displayed matrix operation gives a group homomorphism. For $y=\alpha(t)=\bar t$ with $t\in U(F)$, both arguments are $\overline{\tau(t)}$, since $\tau(t)^{-\mathsf{T}}=\overline{\tau(t)}$. Also $\det(t)\overline{\det(t)}=1$, giving rational norm one by pairing the embeddings of Lemma 82.2. Thus the tensor acts by exactly the prescribed covariance of Definition 78.1. A scalar zI acts in W_{λ^τ} by $z^{\sum_j \lambda_j^\tau}$: each column of height h in the exterior tensor contributes z^h, and the last determinant power contributes its exponent times n to the total exponent. Scalar rational matrices therefore act by scalars in every indicated choice. □

Theorem 240.3. *Let $f:U\to W$ be a nonzero form with the rational covariance of Lemma 240.2. Let Ξ be any representation giving that covariance. Suppose $R(u)f=f$ whenever $u\in U$ and $\alpha(u)\in C$. There is a definite function ϕ of Definition 240.1 such that*

$$\phi(y\alpha(u)k)=\Xi(y)f(u)\qquad (y\in\Gamma,\ u\in U,\ k\in C),$$

with value zero on the complement of $\Gamma\alpha(U)C$. In particular it is nonzero.

Proof. Suppose $y_1\alpha(u_1)k_1=y_2\alpha(u_2)k_2$. For $y=y_2^{-1}y_1$ this gives $y=\alpha(u_2)k_2k_1^{-1}\alpha(u_1)^{-1}$. Since both outside factors are unitary, $y^*y=tI$ at the finite adeles for some $t\in\widehat{\mathbb{Z}}^\times$. As in Proposition 239.2, a diagonal entry gives $t\in\mathbb{Q}^\times\cap\widehat{\mathbb{Z}}^\times$, and positivity of y^*y at any complex embedding gives $t=1$. Thus y is a rational isometry and $k_2k_1^{-1}$ is unitary. Applying α gives $\alpha(y)u_1=u_2\alpha(k_2k_1^{-1})$. The hypothesis makes the last factor fix f. Rational covariance yields $f(u_2)=\Xi(y)f(u_1)$, proving equality of the two proposed values. The support condition and the formula give left covariance and right invariance since Γ and C are groups. On a coset represented by $g=y\alpha(u)k$, use the rational matrix $A=(y^{-1})^*y^{-1}$. It is positive definite at

every complex embedding, and $g^*Ag = k^*k$ is a profinite scalar unit. This gives the signed frame at $J = I$. Finally $\phi(\alpha(u)) = f(u)$. □

Proposition 240.4. *At a split place $w \neq \bar{w}$ put $u_w(a) \in U$ equal to $a \in \mathrm{GL}_n(E_w)$ at w, the conjugate inverse transpose at its mate and the identity elsewhere. Put $b_w(a) = \alpha(u_w(a)) \in G$. If $\mathfrak{n} \notin \bar{w}$, then for each $g \in G$ there exists a complex linear map L_g from functions $U \to W$ to W such that*

$$\phi(gb_w(a)) = L_g(R(u_w(a))f) \qquad (a \in \mathrm{GL}_n(E_w)).$$

Consequently every finite complex linear relation among the local translates of f is a relation among the corresponding translates of ϕ.

Proof. The finite scalar similitudes normalize $U = \alpha(U)$, by the matrix identity $k^*k = tI$. Thus $\alpha(U)C$ is a subgroup, and $\Gamma\alpha(U)C$ and its complement are stable under right multiplication by $b_w(a)$. Outside this set take $L_g = 0$. Inside it choose $g = y\alpha(u)k$ as in Theorem 240.3. Write $d = k_w^{-\mathsf{T}}$. One has $kb_w(a)k^{-1} = b_w(dad^{-1})$. At w this follows by applying inverse transpose, and at $\bar{w}$ it follows from the unitary relation for the conjugated element; all other components are the identity. Also $b_w(d) \in C$: its component at w equals k_w, its component at the mate is integral and invertible with no congruence condition there, and it is unitary. Hence $R(u_w(d)^{-1})f = f$. The extension formula now gives

$$\phi(gb_w(a)) = \Xi(y)f(u\,u_w(dad^{-1})) = \Xi(y)(R(u_w(a))f)(u\,u_w(d)).$$

Evaluation at the last point followed by $\Xi(y)$ is linear. Applying it at each g to a linear relation proves the last assertion. □

241. The central character of the definite test vector

Proposition 241.1. *Write $\mathbb{A}_{K,f}^\times$ for the finite ideles of the imaginary quadratic field K. Define $s(t) = \bar{t}I$ in G via the inclusion of its finite adeles in those of E. Let ϕ be as in Theorem 240.3, with $\Xi(\bar{z}I) = \eta(z)I$ for $z \in K^\times$. The span of the functions $R(s(t))\phi$ is finite dimensional and consists of definite functions at level $\mathfrak{n}$. It contains a nonzero ψ and a character $\omega : \mathbb{A}_{K,f}^\times \to \mathbb{C}^\times$ with open kernel such that*

$$R(s(t))\psi = \omega(t)\psi, \qquad \omega(z) = \eta(z) \quad (z \in K^\times).$$

For the choices of Ξ in Lemma 240.2, $\eta(z)$ is a product of integral powers of the two complex embeddings of z. All local linear relations of Proposition 240.4 hold for ψ.

Proof. Choose a positive rational integer m with $m \in \mathfrak{n}O_E$, for example $|N_{E/\mathbb{Q}}\mathfrak{n}|$ by the adjugate argument of Theorem 236.2. Let H_m be the units of $\prod_u O_{K_u}$ congruent to one modulo m in this product. Their scalar images lie in $C = K(\mathfrak{n})$: the finite multiplier $t\bar{t}$ is a rational profinite unit and $\bar{t} \equiv 1 \bmod m$. Thus H_m fixes ϕ, and the rational z act by $\eta(z)$ by left covariance and centrality.

The quotient $\mathbb{A}_{K,f}^\times/(K^\times H_m)$ is finite. Taking the valuations at all primes maps finite ideles onto fractional ideals, by choosing local uniformizers at the finite support, using the factorization of Proposition 82.3. The kernel is the product of local units. The ordinary class group is finite by Lemma 82.5. The product of units modulo H_m is also finite: only the factors over m occur, and each has finitely many residues modulo m by Lemma 225.1. Hence finitely many central translates span the orbit, since a change by $K^\times H_m$ acts by a scalar.

These translates are right C-invariant and left Ξ-equivariant by centrality. If $\phi(gs(t)) \neq 0$, let A be a positive definite rational matrix giving its frame at $J = I$. Then $g^*Ag = \varepsilon/(t\bar{t})$ times the identity, with ε a profinite unit. Choose $a = \prod_p p^{v_p(t\bar{t})} \in \mathbb{Q}_{>0}$, a finite product since t is an idele. The matrix aA stays positive and gives a profinite unit multiplier at g. This proves the support requirement of Definition 240.1.

A commuting family of invertible endomorphisms on a nonzero finite-dimensional complex space has a common eigenvector. Indeed, if some endomorphism is nonscalar choose a

nonzero proper eigenspace of it, using the complex root property of Lemma 16.2; all other operators preserve that eigenspace. Iterate in smaller dimension. The scalar case is immediate. Applied to the central orbit this gives ψ; the eigenvalues form a character since the operators are invertible and compose multiplicatively. Its kernel contains H_m, open in the restricted product. The scalar calculation of Lemma 240.2 gives the asserted description of η, including the rational norm factor by the product of embeddings. Finally each central translate commutes with the local translations, so all the homogeneous linear relations persist under their finite linear combinations. □

Theorem 241.2. *Take $f, \mathfrak{n}, w_0, w_1, w_*$ from Corollary 239.4, transporting the form back to $\mathbb{C}$ and using a rational weight extension of Lemma 240.2. The corresponding ψ of Proposition 241.1 obeys the cuspidal character relation at w_0 through b_{w_0} and is fixed by $b_{w_*}(\mathrm{GL}_n(O_{w_*}))$. The inverse image $b_{w_0}^{-1}(C)$ is contained in $I + \ell M_n(\mathbb{Z}_\ell)$; the inverse image $b_{w_*}^{-1}(C)$ is contained in the subgroup whose reduction is upper triangular with every diagonal entry one.*

Let w be safe in the sense of Theorem 237.4, with contraction u to K. Put $z_w \in \mathbb{A}_{K,f}^{\times}$ equal to $N_{E_w/K_u}(\pi_w)$ at u and to one elsewhere, for a uniformizer π_w. Set $c_w = \omega(z_w)$. The approximants may be chosen so that at every such w the following finite sum acts on ψ through the indicated eigenvalue:

$$\sum_a R\big(s(z_w^{-h})\, b_w(a)\big)\psi = c_w^{-h} t_{w,h}\psi, \qquad 0 \le h \le n.$$

Here a runs over the right-factor cosets aK_w of Definition 230.1, $K_w = \mathrm{GL}_n(O_w)$. The scalar c_w is independent of the choice of uniformizer.

Proof. Choose simultaneous spherical eigenvectors at every split factor where the level character is trivial on its full integral group by Corollary 228.2. Mark in the support choice of Corollary 239.4 one member of every other exceptional prime pair. If w is safe, neither w nor its conjugate is marked: the former avoids the level and the latter lies over $\bar{u}$, where the level is a unit. The pair is split since its contractions to K differ. Thus f has all the indicated spherical eigenvalues at w. Theorem 240.3 applies to f by the level property of Corollary 239.4. That corollary excludes $\bar{w}_0, \bar{w}_*$ from the level. Applying Proposition 240.4 at w_0, w_*, and then Proposition 241.1, transfers both local relations. At w_0 the character satisfies the idempotent test of Proposition 229.1. At each of w_0, w_* the level exponent is positive, so $a^{-\mathsf{T}}$ at that place in $b_w(a) \in C$ reduces to I. Taking the inverse transpose over the residue field proves both assertions about inverse images.

At each safe w, the conjugate place also avoids the level. The same two propositions transfer the K_w-invariance and the relation for the sum of $R(u_w(a))$ on f. Right multiplying its terms by the same central scalar multiplies the eigenvalue by $\omega(z_w)^{-h}$. This proves the displayed formula and its independence of the right coset representatives. A unit idele supported at u has s-image integral, with unit multiplier, and supported above $\bar{u}$, where no level congruence is imposed. It fixes ψ, so ω is trivial on these units. Replacing π_w by a unit multiple changes its local norm by a unit: in the product of local conjugates each has absolute value one by the invariance proved in Lemma 113.1. This proves the independence of c_w. □

242. The coefficient representation and a marked eigenvalue

Lemma 242.1. *For strictly increasing nonnegative integers $l_1 < \cdots < l_n$, put*

$$S_l(z) = \frac{\det[z_j^{l_i}]_{i,j=1}^n}{\det[z_j^{i-1}]_{i,j=1}^n}.$$

It is a symmetric polynomial with integer coefficients. As a polynomial in the elementary symmetric functions $e_1, \ldots, e_n$ of the roots $z_1, \ldots, z_n$, it has a representative of ordinary

total degree at most $l_n - n + 1$. For distinct nonzero complex numbers z_j put $L_i(Y) = \prod_{k\neq i}(Y - z_k)/(z_i - z_k)$. Then

$$\sum_{i=1}^{n} L_i(Xz_i) S_l(z_1, \dots, z_{i-1}, Xz_i, z_{i+1}, \dots, z_n) = S_l(z) \sum_{j=1}^{n} X^{l_j}.$$

Proof. The denominator is $\prod_{a<b}(z_b - z_a)$: each collision forces a zero, its total degree is $n(n-1)/2$, and the coefficient of $\prod_j z_j^{j-1}$ is one. Each linear factor divides the numerator by monic division and specialization at the collision. It is prime because quotienting identifies two variables in an integral polynomial ring, and the factors for different pairs do not divide one another. Thus their product divides the numerator over $\mathbb{Z}$. Both determinants change by the same sign under permutations. The quotient has degree at most $l_n - (n-1)$ in each individual variable, by additivity of degrees in a product.

For a homogeneous symmetric polynomial with this bound D, take its lexicographically largest monomial. Its exponents $b_1, \dots, b_n$ must be nonincreasing by symmetry. With $b_{n+1} = 0$, the product $\prod_{h=1}^{n} e_h^{b_h - b_{h+1}}$ has that leading monomial with coefficient one, since $z_1 \cdots z_h$ is the leading monomial of e_h. It uses $b_1 \leq D$ factors, and its other monomials also have degree at most D in each variable. Subtracting the leading coefficient times this product and iterating terminates, since there are finitely many monomials of the given homogeneous degree. This proves the coefficient bound for S_l, which is homogeneous by its determinant formula.

Upon replacing z_i by Xz_i, the ratio of the new denominator to the old is $L_i(Xz_i)$. After cancellation the i-th summand has just the i-th column of the numerator modified, over the original denominator. In each permutation term the successive column modifications use every exponent l_j once. Summing proves the identity in $\mathbb{C}(X)$, hence in $\mathbb{C}[X]$ by the polynomial assertion. □

Definition 242.2. Use the original CM representatives τ_v of the definite weight at the real places v of F. Fix a distinguished embedding $\sigma : E \to \mathbb{C}$, set $s_0 = \sigma|_K$, and let ν_v denote the embedding in the v-pair which restricts to s_0. Write v_0 for the place of σ. Let $k_{v,1} < \cdots < k_{v,n}$ be the oriented jumps of Definition 238.1 at ν_v, listed increasingly. The inequalities are strict by dominance of λ^v. Put

$$m_v = \max(0, -k_{v,1}), \qquad l_{v,j} = m_v + k_{v,j}, \qquad \mu = n \sum_v m_v.$$

In the v-factor of Lemma 240.2 take $H_v(y) = \nu_v(y)$ if $\nu_v = \tau_v$ and $H_v(y) = \nu_v(y)^{-\mathsf{T}}$ otherwise. Put

$$\Xi(y) = \left(\prod_v \left(\nu_v(\det y) \bar{\nu}_v(\det y) \right)^{-m_v} \right) \bigotimes_v W_{\lambda^v}(H_v(y)).$$

Define $A_{\nu_v}(z) = S_{l_v}(z)$ and $A_{\bar{\nu}_v}(z) = (\prod_i z_i)^{m_v}$.

Theorem 242.3. *The representation Ξ of Definition 242.2 has the rational unitary covariance of Lemma 240.2. On the scalars $\bar{a}I$, $a \in K^\times$, its character of scalar action is*

$$\eta(a) = s_0(a)^{-\mu} \bar{s}_0(a)^{A-\mu}, \qquad A = \sum_v \epsilon_v \sum_{j=1}^{n} \lambda_j^v,$$

where $\epsilon_v = 1$ at $\nu_v = \tau_v$ and -1 otherwise. Let $y \in \mathrm{GL}_n(E)$ have characteristic polynomial with distinct roots. Write z_τ for the list of its roots under any embedding τ, with conjugate lists numbered by complex conjugation. Then

$$\operatorname{tr} \Xi(y^{-1}) = \prod_\tau A_\tau(z_\tau).$$

Proof. The scalar factor defines a character on Γ. On unitary rational matrices each pair of determinant values has product one, so the proof of Lemma 240.2 applies separately at each v. At $\bar{a}I$ that pair contributes $(s_0(a)\bar{s}_0(a))^{-nm_v}$, and the v-representation contributes $\bar{s}_0(a)^{\epsilon_v \sum_j \lambda_j^v}$ by the scalar calculation there. Multiplication proves the central formula.

At $\nu_v = \tau_v$ the tensor factor on y^{-1} has eigenvalue arguments z_i^{-1} for $z = z_{\nu_v}$. Theorem 80.1 applies at these distinct nonzero numbers: its determinant identity on diagonal matrices is a Laurent-polynomial identity by the tableau proof there, and the matrix is diagonalizable since its roots are distinct. In both numerator and denominator of that formula multiply each column j by z_j^{n-1}. The new exponent list in the numerator is $i-1-\lambda_i^v$, and the denominator list is $i-1$ in increasing order. At $\nu_v \neq \tau_v$ the tensor argument on y^{-1} has eigenvalues z_i. Reverse the rows of both alternants. The denominator again has list $i-1$ and the numerator list is the increasing ordering of $\lambda_i^v + n - i$. These are precisely the two oriented jump lists. Multiplication by $(\prod_j z_j)^{m_v}$ adds m_v to the numerator exponents. The conjugate factor of the determinant twist is $(\prod_j z_{\bar{\nu}_v,j})^{m_v}$. Taking the product over all pairs gives the trace identity, since the trace of a tensor product is the product of its traces in product bases. □

Theorem 242.4. *In Theorem 242.3, for $1 \le i \le n$ form lists $z_\tau^{(i)}(X)$ as follows. At $\nu_{v_0} = \sigma$ multiply $z_{\sigma,i}$ by X, at $\bar{\sigma}$ multiply every root except $\overline{z_{\sigma,i}}$ by X. For $v \neq v_0$ multiply all roots at $\bar{\nu}_v$ by X and keep the list at ν_v. Take L_i of Lemma 242.1 on z_σ. Then*

$$\sum_i L_i(Xz_{\sigma,i}) \prod_\tau A_\tau(z_\tau^{(i)}(X)) = \operatorname{tr} \Xi(y^{-1}) \sum_{j=1}^{n} X^{\mu + k_{v_0,j}}.$$

In particular each indicated coefficient is exactly the trace. The exponents $\mu + k_{v_0,j}$ are nonnegative and pairwise distinct.

Proof. Each of the non-distinguished conjugate determinant factors contributes X^{nm_v} to the modification. At the conjugate of the distinguished embedding the factor contributes $X^{(n-1)m_{v_0}}$. These are independent of i. All remaining changes belong to the distinguished alternant. Lemma 242.1 gives its original value times $\sum_j X^{l_{v_0,j}}$. The total exponent outside this sum is $\mu - m_{v_0}$. Substitute $l_{v_0,j} = m_{v_0} + k_{v_0,j}$ and apply Theorem 242.3. The exponents are distinct by Definition 242.2 and nonnegative since $\mu \ge m_{v_0}$. □

243. A finite family of local terms for the jump coefficients

Lemma 243.1. *Let E be the field of Definition 10.1, $d_E = [E:\mathbb{Q}]$, and let $b_1, \dots, b_{d_E}$ be an integral basis. Let $n, \nu \ge 1$ and choose, for each $\tau : E \to \mathbb{C}$, a polynomial Q_τ in n variables of ordinary total degree at most ν. Define*

$$\mathcal{B}_n = \left\{ \sum_{t=1}^{d_E} h_t b_t : h_t \in \{0, \dots, nd_E\} \right\}.$$

There are complex constants d_α, indexed by $\alpha \in \mathcal{B}_n^\nu$, with the following simultaneous identity for all lists $x_{\tau,1}, \dots, x_{\tau,n}$ in any commutative complex algebra:

$$\prod_\tau Q_\tau(e_1(x_\tau), \dots, e_n(x_\tau)) = \sum_{\alpha \in \mathcal{B}_n^\nu} d_\alpha \prod_{a=1}^{\nu} \prod_\tau \prod_{j=1}^{n} (1 - 2\tau(\alpha_a) x_{\tau,j}).$$

The constants depend only on the polynomials, the basis, n and ν.

Proof. First take the $x_{\tau,j}$ as independent indeterminates. In the product on the right use independent variables $Y_{a,\tau}$ in place of $2\tau(\alpha_a)$. The coefficient of $\prod_{a,\tau} Y_{a,\tau}^{s_{a,\tau}}$, for $0 \le s_{a,\tau} \le n$, is $\prod_{a,\tau} (-1)^{s_{a,\tau}} e_{s_{a,\tau}}(x_\tau)$ with $e_0 = 1$, by finite multiplication. For each monomial in Q_τ, assign each of its at most ν elementary symmetric factors to a separate a, filling unused positions by $s = 0$. The assignments for distinct τ are independent. Expanding the product

of the Q_τ shows that the left side is a fixed complex linear combination of coefficients of this polynomial in the Y's.

For each a substitute $Y_{a,\tau} = 2\sum_t \tau(b_t) H_{a,t}$. The matrix $(\tau(b_t))_{\tau,t}$ is nonsingular over $\mathbb{C}$: a power basis of a primitive element has a nonsingular Vandermonde matrix by Lemma 82.2, and the change to the given rational basis is invertible. The substitution thus has a linear inverse, so each original coefficient is a complex linear combination of coefficients in the H's. For each a the total degree in its block is at most nd_E, as each of its d_E former variables had degree at most n. Every coefficient in the H's is recovered linearly from the grid $0, \dots, nd_E$ in each variable. Indeed in one variable the interpolation basis at j is $\prod_{0\le t\le nd_E,\ t\ne j}(X-t)/(j-t)$; subtracting the interpolant leaves a polynomial of bounded degree vanishing at more points than its degree. Apply this successively to each variable. The resulting identity of complex polynomials in the $x_{\tau,j}$ holds upon every indicated substitution into a complex algebra. □

Definition 243.2. Use the data of Definition 242.2 and take

$$\nu = \max\left(1,\ \max_v m_v,\ \max_v (l_{v,n} - n + 1)\right).$$

Let $y \in \mathrm{GL}_n(E)$ have distinct eigenvalues at each embedding. For its i-th marked root at σ, form the lists $z_\tau^{(i)}(X)$ of Theorem 242.4. For $\alpha \in O_E^\nu$ define the polynomial

$$B_{\alpha,y,i}(X) = L_i(X z_{\sigma,i}) \prod_{a=1}^{\nu} \prod_{\tau} \prod_{j=1}^{n} \left(1 - 2\tau(\alpha_a) z_{\tau,j}^{(i)}(X)\right).$$

The non-marked lists use the CM representatives Ψ of Lemma 236.1.

Theorem 243.3. *There are constants d_α for the finite grid in Lemma 243.1, the same for every regular y and every jump $k_{v_0,j}$, such that*

$$\operatorname{tr} \Xi(y^{-1}) = \sum_{\alpha \in \mathcal{B}_n^\nu} d_\alpha \sum_{i=1}^{n} [X^{\,\mu + k_{v_0,j}}] B_{\alpha,y,i}(X).$$

Here regular means that the characteristic polynomial of y is separable. In particular, for any finite complex weighted list (c_t, y_t) of such matrices, a nonzero sum $\sum_t c_t \operatorname{tr} \Xi(y_t^{-1})$ forces for each indicated jump a nonzero sum of the corresponding marked coefficients for at least one α in the grid.

Proof. At ν_v, use Lemma 242.1 to express A_{ν_v} in the elementary symmetric functions by a polynomial of ordinary total degree at most $l_{v,n} - n + 1$. At $\bar\nu_v$, take $Q_{\bar\nu_v} = e_n^{m_v}$, meaning the m_v-th power of the last elementary symmetric variable. All degrees are at most ν by Definition 243.2. Choose the constants of Lemma 243.1 for these polynomials. Substitute the modified root lists in $\mathbb{C}[X]$, multiply by $L_i(X z_{\sigma,i})$, and sum over i. By Theorem 242.4 the coefficient on the left at each stated degree is exactly $\operatorname{tr} \Xi(y^{-1})$. On the right the operations produce the stated finite sum. Summing this equality against c_t and using that a sum of zeros is zero proves the final assertion. □

244. An ordinary Iwahori operator with regular support

Definition 244.1. Let L be a finite extension of $\mathbb{Q}_p$, O its integers, π a uniformizer, $k_0 = O/\pi O$ its residue field of size q, and $v(\pi) = 1$, as in Lemma 225.1. Let $I \subset \mathrm{GL}_n(O)$ be the inverse image of the upper triangular subgroup over k_0. For a complex smooth representation ρ of $\mathrm{GL}_n(L)$, meaning each vector has an open stabilizer, denote by P_H the probability average over a compact open subgroup H. This means the finite average of the orbit of a vector, with equal weight on the cosets of its stabilizer in H. On the space V^I put $U_g = P_I \rho(g)|_{V^I}$.

Lemma 244.2. *If $IgI\,IhI = IghI$ then $U_g U_h = U_{gh}$. For every permutation matrix w, the operator U_w is invertible on V^I.*

Proof. For a compact subgroup, finitely many cosets occur in an open quotient by compactness, and passage to a smaller open subgroup refines every coset equally. The average in Definition 244.1 is therefore linear and a projection onto invariants. The operator U_g averages the right-factor cosets aI in IgI uniformly. On composing two such averages one obtains positive weights of total one on the right-factor cosets in $IgIhI$, invariant under left I. Indeed left multiplication permutes the cosets in the first average and the resulting changes of representative act on the second average through left I. If the product is one double coset, left I is transitive on these right-factor cosets. Thus the composite has exactly the uniform weights of U_{gh}.

Write s_i for the adjacent swap of positions $i, i+1$ and P_i for the subgroup of $\mathrm{GL}_n(O)$ preserving the standard flag modulo π with just its dimension-i subspace omitted. The quotient P_i/I parametrizes lines in the two-dimensional residue quotient between its neighboring flag members. Every such line is realized by an invertible two-by-two integral block lifting a residue matrix. One is the original flag, and the other q are represented by $(1+\tilde{a}E_{i,i+1})s_iI$, $a \in k_0$, where $\tilde{a}$ is any integral lift. Thus $P_i = I \sqcup Is_iI$ and on V^I one has $P_{P_i} = (1+qU_{s_i})/(q+1)$. Since P_{P_i} is a projection, U_{s_i} acts by 1 on its image and by $-1/q$ on its kernel. It is invertible.

If $w(i) < w(i+1)$, conjugation by w takes every $1+\tilde{a}E_{i,i+1}$ above into I, whence $IwI\,Is_iI = Iws_iI$ by the displayed cosets. A permutation with a descent loses one inversion on right multiplication by the corresponding adjacent swap. Induct on the number of inversions, applying the product formula to the shorter permutation followed by that swap. This proves invertibility for every w. □

Theorem 244.3. *For integers $a_1 \geq \cdots \geq a_n$ put $t_a = \mathrm{diag}(\pi^{a_1}, \ldots, \pi^{a_n})$. One has $It_aI\,It_bI = It_{a+b}I$ whenever both lists are nonincreasing. The operator U_{t_a} of Definition 244.1 is invertible on V^I, and $U_{t_a}^m = U_{t_{ma}}$ for every $m \geq 1$.*

Proof. Every $k \in I$ factors as u_+du_- with u_+ upper unitriangular integral, d diagonal with unit entries, and u_- lower unitriangular with entries below the diagonal in πO. Eliminate the last column above its diagonal using its unit last pivot by row operations, then the last row to the left using column operations with coefficients in πO. The remaining block stays upper triangular with nonzero diagonal modulo π, so induction gives the factorization. One has $t_au_+dt_a^{-1} \in I$ and $t_b^{-1}u_-t_b \in I$ by comparing the differences of the exponents. Hence $t_akt_b \in It_{a+b}I$, proving equality of the two sets. The power formula follows from Lemma 244.2.

Define $\Omega e_1 = \pi e_n$, $\Omega e_i = e_{i-1}$ for $i > 1$, on the standard column basis. This matrix normalizes I. Indeed it carries the lattice with its last h coordinates in πO and others in O to the analogous lattice with $h+1$ such coordinates for $0 \leq h < n$. For $h = n$ continue periodically by scaling all lattices by π. The group I consists exactly of simultaneous stabilizers of this periodic chain, by reduction of the intermediate lattices modulo π. Thus U_{Ω^j} is the invertible translation by Ω^j on V^I. For $1 \leq j < n$ a permutation w_j undoes the cyclic permutation in Ω^j, giving $w_j\Omega^j = \mathrm{diag}(\pi I_j, I_{n-j})$. Since Ω^j normalizes I, its product double coset with Iw_jI is single. Both factors act invertibly by the preceding lemma. Finally t_a is the scalar $\pi^{a_n}I$ times these diagonal matrices to powers $a_j - a_{j+1}$. The products have nonincreasing exponent lists throughout. The product formula proves invertibility, also for $n = 1$. □

Proposition 244.4. *If $a_1 > \cdots > a_n$, every element of It_aI has n distinct eigenvalues in L with valuations $a_1, \ldots, a_n$. This holds on $(It_aI)^m$ with these valuations multiplied by m.*

Proof. A product $k_1t_ak_2$ is conjugate to $t_ak_2k_1$. For $k \in I$, every principal minor of k reduces to the product of the corresponding nonzero diagonal entries, hence is a unit. In $\det(YI - t_ak) = \sum_{j=0}^{n} c_jY^{n-j}$, expanding the determinant gives $c_j = (-1)^j \sum_{|S|=j} \pi^{\sum_{i\in S} a_i} \det k_{S,S}$.

There is a unique summand of least valuation, indexed by the last j positions. Thus $v(c_0) = 0$ and $v(c_j) - v(c_{j-1}) = a_{n-j+1}$, strictly increasing in j.

For one of these successive differences b, substitute $Y = \pi^b Z$. The valuations of the terms' coefficients now decrease strictly to two adjacent equal minima and then increase strictly, since the new successive differences are $a_{n-j+1} - b$. Divide by the power of π of the minimum. The reduction is $Z^k(A + BZ)$ with both A, B nonzero, so its nonzero residue root is simple. Lift it to a unit root over O: at stage $l \geq 1$, if a polynomial value lies in $\pi^l O$ and its derivative is a unit, adding $\pi^l z$ corrects the next digit by choosing $z \bmod \pi$ from the linear term. Iterating converges in the complete field by Theorem 223.4. This gives an eigenvalue of valuation b. All n values are distinct and exhaust the roots by degree. The last assertion follows from Theorem 244.3. □

245. Regular tests for the definite eigensystem

Definition 245.1. For G, Γ, C of Definition 240.1 put

$$S = \{g \in G : g^* g = tI,\ t \in \mathbb{A}_{\mathbb{Q},f}^{\times}\}, \qquad S^+ = \{g \in S : g^{-1} \in M_n(\widehat{O})\}.$$

The multiplier is unique since $n \geq 1$; membership in S^+ is preserved by multiplication. The group C is compact open in S: it is the intersection with the integral congruence subgroup, and on this intersection the multiplier is a unit at every prime by invertibility. Compactness follows from the closed matrix equalities for a scalar in $\widehat{\mathbb{Z}}^{\times}$ inside the compact integral matrix group, using the finite residue quotients of Lemma 225.1. Let $\mathcal{H}^+$ be the complex functions on S/C of finite support in S^+/C invariant under left C. On the C-fixed vectors of a representation write their action as $\sum_{aC} H(aC)R(a)$. The product is composition of these finite sums, grouping identical right-factor cosets. This is well defined because changing a to ak, $k \in C$, permutes the second cosets preserving their coefficients. The product is again left C-invariant, and multiplication is associative by finite summation. We extend H by zero to G off the indicated cosets. Call H regular if $H(x^{-1}y^{-1}x) \neq 0$, $x \in G,\ y \in \Gamma$, implies that y has separable characteristic polynomial over E.

Lemma 245.2. *For the level of Corollary 239.4, the safe spherical sums of Theorem 241.2 give elements of $\mathcal{H}^+$. Each has representatives $g \in S^+$ with $d_{w_0}(g) = d_{w_*}(g) = I$, where $d_v(g) = g_v^{-\mathsf{T}}$. Finite products and linear combinations of the sums have expansions by representatives with the same two identities.*

Proof. Let w be safe, $u = w|K$, and $g = s(z_w^{-h})b_w(a)$ as in Theorem 241.2. Both factors are similitudes. The inverse of $b_w(a)$ is integral at w, since a is integral, and at its mate its denominators divide the conjugate uniformizer when $h \geq 1$. This follows by taking inverses in the integral double coset of $\mathrm{diag}(\pi_w I_h, I_{n-h})$. At $h = 0$ it is already integral. The inverse scalar $s(z_w^h)$ is integral everywhere and divisible at $\bar{w}$ by its uniformizer when $h \geq 1$: the local norm of π_w has positive valuation, by taking the product of its conjugates with the invariant local absolute value of Lemma 113.1. Thus $g \in S^+$. The safety condition excludes $w, \bar{w}$ from the level and excludes every prime above $\bar{u}$ from the level. Both markers divide the level, hence neither occurs in the support of this local factor or of its central scalar.

For $k \in C$ one has $d_w(k) \in K_w$ and $b_w(d_w(k)) \in C$, since both places of this pair have unit level. By the conjugation formula of Proposition 240.4, left multiplication by k acts on the right cosets of the sum by left multiplication on aK_w by $d_w(k)$. The scalar is central. This operation permutes the cosets, proving the required invariance. The projection from local cosets to global cosets is well defined since $b_w(K_w) \subset C$. In a product one may keep the ungrouped representatives, multiplying the chosen representatives of each factor. The two dual component maps are homomorphisms, giving the last assertion. □

Theorem 245.3. *Let ψ be the definite vector of Theorem 241.2. Write $J = 1 + \ell M_n(\mathbb{Z}_\ell)$ for the group of its character χ at w_0. Let c be a polynomial in the safe spherical operators, with*

constant terms allowed, such that $c\psi = \psi$. There is $b \in \mathcal{H}^+$ with the following properties for each $m \geq 1$. The element $H_m = (bc)^m$ is regular, $H_m\psi \neq 0$, and its coset function satisfies

$$H_m(b_{w_0}(j)g) = \chi(j)^{-1}H_m(g) \qquad (j \in J,\ g \in G).$$

It is a left multiple of c in $\mathcal{H}^+$.

Proof. In the group at w_*, use I of Definition 244.1, $\pi = p_*$ and $t = \mathrm{diag}(\pi^{n-1}, \pi^{n-2}, \dots, 1)$. In any smooth S-representation, let

$$E_\chi = \int_J \chi(j)^{-1} R(b_{w_0}(j))\, dj, \qquad U = P_I R(b_{w_*}(t)) P_I, \qquad z = p_*^{-(n-1)} I.$$

Here P_I uses the copy $b_{w_*}(I)$. Probability averages of these locally constant functions are the finite averages of Lemma 244.2; the character has open kernel by Proposition 229.1. Translation of the first sum by $b_{w_0}(j)$ multiplies it by $\chi(j)$, and E_χ fixes vectors with that character. All factors at distinct marked pairs commute. For $k \in C$, its action by conjugation on either local subgroup is conjugation by $d_v(k)$ in local coordinates by Proposition 240.4. Here $d_{w_0}(k) \in 1 + \ell^2 M_n(\mathbb{Z}_\ell) \subset \ker \chi$, and $d_{w_*}(k) \in I$ by Corollary 239.4. Thus $R(k)$ commutes with E_χ by invariance of the character under conjugation in J, and with U by its two I-averages. The scalar z commutes with all the operators.

Act with $R(z)E_\chi U$ on the basis vector $[C]$ in the permutation representation with basis S/C and left translation. This representation is smooth since the stabilizer of each basis vector is open and a vector uses finitely many basis vectors. The result is a finite sum of cosets fixed by left C, and is the desired b. Indeed all its terms have representatives in $z\, b_{w_0}(J) b_{w_*}(ItI)$. The inverse of the last local factor has denominators at worst p_*^{n-1} at the two places, since I is integral; the inverse of the scalar clears them. Elsewhere all inverse factors are integral. Thus $b \in \mathcal{H}^+$. For any C-fixed vector v, the assignment $[gC] \mapsto R(g)v$ intertwines the translations and their finite averages, so $bv = R(z)E_\chi U v$. Right translations of a definite function through S generate a smooth space since the function is C-fixed.

By Lemma 245.2 the indicated expansion of c commutes term by term with the local operators: a similitude with $d_v(g) = I$ centralizes b_v by the conjugation identity. On the C-fixed vectors the products therefore commute with U, E_χ and $R(z)$. Now $E_\chi\psi = \psi$ and $P_I\psi = \psi$ by Theorem 241.2. Consequently $H_m\psi = R(z)^m U^m \psi \neq 0$, since U is invertible on I-invariants by Theorem 244.3. Likewise the leftmost factor b sends C-invariants into the χ-space, since E_χ commutes with $U, R(z)$. Acting on $[C]$ gives exactly the stated coset identity. It also holds outside S since $b_{w_0}(J) \subset S$.

Every coset in H_m has d_{w_*} in $(I(p_*^{n-1}t)I)^m = I(p_*^{n-1}t)^m I$ by Theorem 244.3, including the final right factor from C. Its eigenvalues are distinct by Proposition 244.4. At $x^{-1}y^{-1}x$ the dual component is conjugate to the transpose of y_{w_*}. A repeated root of the rational characteristic polynomial would persist on embedding its coefficients in this local field, since a common factor with its derivative stays a common factor. This proves regularity. Finally $H_m = ((bc)^{m-1}b)c$. □

246. A nonzero definite orbital sum for each prescribed weight

Proposition 246.1. *Let $\mathcal{D}$ be the set of double cosets in $\Gamma\backslash G/C$ containing an x with a signed frame at $J = I$, as in Definition 239.1. At the level of Corollary 239.4, $\mathcal{D}$ is finite. Evaluation at one representative per class identifies the definite functions of Definition 240.1 with $\bigoplus_{i\in\mathcal{D}} W$. The algebra $\mathcal{H}^+$ of Definition 245.1 preserves this space.*

Proof. Write A_x for the rational positive Hermitian matrix with $x^* A_x x = \varepsilon I$, $\varepsilon \in \widehat{\mathbb{Z}}^\times$. It is unique: a second matrix is a scalar multiple of the first at the finite adeles, with their multiplier ratio a profinite unit. Taking the quotient of corresponding entries at a nonzero entry shows that the ratio belongs to E and is a rational profinite scalar. In a rational basis of E starting with one its other coordinates vanish, so it is a rational number. Positivity at infinity and the unit condition at every finite prime force it to be one, as in Proposition 239.2.

Associate to x the O_E-lattice $\Lambda_x = E^n \cap x\widehat{O}^n$ and the rational symmetric form $\langle v, w\rangle_x = \mathrm{Tr}_{E/\mathbb{Q}}(v^* A_x w)$. By the lattice and completion argument in Lemma 13.4 this is a full lattice with the prescribed completions. The form is positive because its squared lengths are the sums over the embeddings of positive Hermitian squared lengths. On each completion its Gram matrix in the basis transported by x is the standard trace Gram matrix multiplied by the unit ε_p. Thus the form is integral and its Gram determinant is the same positive integer for every x, by comparison at each prime as in that lemma. Its bounded-basis argument yields finitely many possible isometry classes of trace lattices. Multiplication by each b in the basis of O_E has operator norm at most $\max_\tau |\tau(b)|$, since the summands for squared lengths are scaled by $|\tau(b)|^2$. There are therefore only finitely many sets of multiplication matrices on the representative lattices, as proved in the same lemma. An isometry respecting these matrices extends by rational linearity to an E-linear map, and recovers the Hermitian form by varying an argument through all E-multiples and using nondegeneracy of the field trace. Consequently the pairs (Λ_x, A_x) have finitely many classes under $x \mapsto yx$, $A_x \mapsto y^{-*} A_x y^{-1}$ with $y \in \Gamma$, where $y^{-*} = (y^*)^{-1}$.

Within any one such class arrange that the lattice and the matrix agree by a rational change of frame. Two resulting frames differ on the right by $k \in \mathrm{GL}_n(\widehat{O})$ because their local lattices agree. Their Gram identities give $k^* k$ equal to a profinite scalar unit. The group of these integral similitudes has finite index over C: reduction modulo $\mathfrak{n}\widehat{O}$ has kernel C and finite image by Lemma 225.1 and Proposition 82.3. This proves the finiteness of $\mathcal{D}$. Each $\Gamma \cap xCx^{-1}$ is trivial by Proposition 239.2 at $J = I$. Arbitrary prescribed values on representatives therefore extend uniquely by the required two equivariances and by zero away from these classes.

For $a \in S$ with multiplier t, set $b = \prod_p p^{v_p(t_p)} > 0$, a positive rational number. If x has a positive signed frame A_x, the point xa has frame $b^{-1} A_x$, since t/b is a profinite unit. Applying this argument to a^{-1} proves the converse. Every coset operator in $\mathcal{H}^+$ preserves the support condition and left covariance, and its full sum preserves right C-invariance by Definition 245.1. □

Theorem 246.2. *For $H \in \mathcal{H}^+$ let $\mathcal{O}(H)$ denote the Γ-orbits of pairs (xC, y), where x has a positive signed frame, $y \in \Gamma$ and $H(x^{-1}y^{-1}x) \neq 0$. The action is $(xC, y) \mapsto (zxC, zyz^{-1})$. The set is finite, and on the definite space of Proposition 246.1 one has*

$$\operatorname{tr} H = \sum_{[(xC,y)] \in \mathcal{O}(H)} H(x^{-1}y^{-1}x) \operatorname{tr} \Xi(y^{-1}).$$

If H is regular, all characteristic polynomials in this sum are separable.

Proof. The summand is invariant: changing x on the right by $k \in C$ conjugates its argument by k, which preserves H by its two C-invariances. Simultaneous rational change of variables leaves this argument unchanged and conjugates the coefficient endomorphism. On representatives x_i from the preceding proposition the diagonal block of H at i includes exactly the cosets aC of its finite support with $x_i a = \gamma x_i k$, $\gamma \in \Gamma$, $k \in C$. Each contributes $H(a)\Xi(\gamma)$ by covariance. For a fixed such coset γ is unique by triviality of $\Gamma \cap x_i C x_i^{-1}$. The associated pair is $(x_i C, \gamma^{-1})$. Every orbit in the statement has a representative of this form, with $aC = x_i^{-1} y^{-1} x_i C$. Two such pairs coincide in the orbit set only if they coincide on the representatives, again by the trivial stabilizers. This identifies a finite set and summing traces of the diagonal blocks gives the formula. Regularity gives separability by Definition 245.1. □

Corollary 246.3. *For each c of Theorem 245.3 there is a single regular left multiple $H = (bc)^m$, with $1 \le m \le |\mathcal{D}| \dim W$, for which the following holds. For every jump $k_{v_0,j}$*

there is an $\alpha \in \mathcal{B}_n^\nu$ of Theorem 243.3 with

$$\sum_{[(xC,y)]\in\mathcal{O}(H)} H(x^{-1}y^{-1}x)\sum_{i=1}^{n}[X^{\,\mu+k_{v_0,j}}]B_{\alpha,y,i}(X) \neq 0.$$

Moreover $H(b_{w_0}(j')g) = \chi(j')^{-1}H(g)$ for all $j' \in J$.

Proof. The finite-dimensional operator $T = bc$ has nonzero positive powers by Theorem 245.3 and Proposition 246.1. Triangularize it over $\mathbb{C}$ by choosing an eigenvector using Lemma 16.2 and inducting on the quotient dimension. If all diagonal entries were zero, the power of order $|\mathcal{D}| \dim W$ of the strictly upper triangular matrix would be zero. Let $\lambda_1, \dots, \lambda_r$ be the distinct nonzero diagonal entries with multiplicities $h_i \geq 1$. If the traces of the first r powers vanished, $\sum_i (h_i\lambda_i)\lambda_i^k = 0$ for $0 \leq k < r$. The Vandermonde evaluation matrix is invertible: a polynomial of degree below r vanishing at the r distinct nodes is zero by division by linear factors. Thus every $h_i\lambda_i = 0$, a contradiction. Choose $m \leq r$ with nonzero trace and put $H = T^m$. The preceding theorem gives a finite weighted sum of traces of regular matrices with this nonzero value. Theorem 243.3 gives the asserted α for each jump. All other properties of H follow from Theorem 245.3. □

247. A positive trace form for each marked polynomial

Definition 247.1. For a separable characteristic polynomial P of degree n over E, order the roots z_i of σP by their real parts and then their imaginary parts. Use that ordering for the index of a root in Definition 243.2. For $y \in \Gamma$ with polynomial P, put

$$J_{0,i}(y) = I, \qquad J_{1,i}(y) = 2L_i(\sigma y) - I, \qquad L_i(T) = \prod_{j\neq i}(T - z_j)/(z_i - z_j).$$

The polynomial $L_i(\sigma y)$ acts by one on the i-th eigenline and by zero on the others: eigenvectors of distinct eigenvalues are independent, by successively applying the operators $\sigma y - z_j I$, and here these lines span by degree. Thus $J_{1,i}$ has sign $+1$ on that eigenline and -1 on its complement. Both matrices commute with σy and conjugate equivariantly when y does.

For a regular $H \in \mathcal{H}^+$ and $\epsilon \in \{0, 1\}$, let $\mathcal{O}_\epsilon(H)$ consist of the Γ-orbits of (xC, y, i) for which $H(x^{-1}y^{-1}x) \neq 0$ and x has a signed frame at $J_{\epsilon,i}(y)$. The action takes xC to zxC and y to zyz^{-1}, preserving the index. The definition is independent of representatives: the frame matrix is taken to $(z^{-1})^* A z^{-1}$ and right multiplication by C changes its finite multiplier by a unit.

Lemma 247.2. *For every term in Definition 247.1, put $g = x^{-1}y^{-1}x$ and let A be its signed-frame matrix. This matrix is unique, and $y^*Ay = qA$ for a positive integer q. The multiplier of g is q^{-1}. The polynomial P is monic in $O_E[T]$, and at every embedding of E each of its roots has absolute value $\sqrt{q}$. For a fixed H, the pairs (q, P) occurring in either orbit set form a finite set.*

Proof. The finite frame identity $x^*Ax = \varepsilon I$ implies $A = A^*$ by injectivity of E into its completions. As in Proposition 246.1, two frame matrices for x differ by a rational scalar which is a unit at every prime. Positivity at an embedding in $\Psi \setminus \{\sigma\}$, nonempty by Proposition 10.3, selects $+1$. Since $g \in S^+$, substitution of $yx = xg^{-1}$ gives $y^*Ay = t^{-1}A$, where $g^*g = tI$. Dividing corresponding entries at a nonzero entry of A makes $q = t^{-1}$ a rational scalar by the same rational basis argument. At an embedding other than σ the equation is a similitude of a positive matrix, giving $q > 0$. Also g^{-1} is integral, so $(g^{-1})^*g^{-1} = qI$ is integral and $q \in \mathbb{Z}$. The characteristic polynomial of g^{-1} equals P. Its coefficients, in E and integral at every finite place, belong to O_E by Proposition 82.3.

At $\tau \in \Psi$ write $J = J_{\epsilon,i}(y)$ if $\tau = \sigma$, and $J = I$ otherwise. Because J commutes with τy, the equation also exhibits τy as a similitude of the positive Hermitian matrix $\tau(A)J$. Evaluating at a nonzero eigenvector of eigenvalue r gives $|r|^2 = q$. The conjugate embeddings have the conjugate roots, so this holds at all embeddings. On a right coset g_0C, all prime

valuations of the similitude multiplier agree with those for g_0. They determine the positive rational q uniquely, and H has finite coset support. For fixed q, the coefficient of T^{n-j} in P has absolute value at most $\binom{n}{j}q^{j/2}$ at every embedding by the elementary symmetric formula. There are finitely many possibilities in the integral lattice O_E by Lemma 82.2. □

Proposition 247.3. *For each triple (q, P, i) in Lemma 247.2 and each ϵ, there is $D \in O_E[T]$, depending only on these data and ϵ, such that for every corresponding frame the matrix $A' = AD(y)$ is Hermitian positive definite at all embeddings. Moreover $D(y)$ is invertible, y is a similitude of A' of multiplier q, and $x^*A'x$ has integral entries at every finite place.*

Proof. Factor P over E into its distinct monic irreducible factors P_b by polynomial division, and put $L_b = E[T]/(P_b)$ and $\lambda_b = T \bmod P_b$. Every embedding of L_b in $\mathbb{C}$ takes λ_b to a number of squared absolute value q, by the preceding lemma. In one such embedding actual complex conjugation therefore sends λ_b to q/λ_b and preserves E. It preserves L_b, acts as an involution *, and has the same description under every embedding by its values on the generators. Its fixed field K_b is totally real, since every embedding extends to L_b by Lemma 16.2. As $i_0 = \sqrt{-1} \in E$ by Lemma 20.1, the two parts $(z + z^*)/2$ and $(z - z^*)/(2i_0)$ give $L_b = K_b \oplus i_0 K_b$. Each real embedding of K_b extends in exactly two ways, by the two images of i_0. Precisely one extends an embedding in the chosen type Ψ. Thus these real embeddings, as b varies, index exactly the roots at all the embeddings in Ψ.

At these embeddings prescribe the sign $+1$, except that for $\epsilon = 1$ prescribe -1 at the $n-1$ unmarked roots over σ. Choose $a_b \in O_{K_b}$ having the prescribed signs. Indeed the full lattice of Lemma 82.2 has a bounded fundamental parallelotope, so a point sufficiently far inside that orthant has a lattice point in its translate by this parallelotope. The polynomials P_b are pairwise coprime. For each b the product of the other factors has an inverse modulo P_b by the Euclidean algorithm. Multiplying by this inverse and summing gives a polynomial in $E[T]$ with residues a_b in all L_b. A common positive integer denominator clears its coefficients into O_E without changing any sign. Call the resulting polynomial D.

For a complex embedding in Ψ, the eigenlines of y are pairwise orthogonal for A: on two distinct roots r, s one has $\bar{r}s \neq q$ by $|r|^2 = q$, and the similitude equation forces their pairing to vanish. Nonsingularity makes each diagonal entry in an eigenbasis nonzero. Positivity of AJ means these entries have exactly the prescribed signs. The values of D are real, nonzero, and have these same signs. In this basis $AD(y)$ is a positive Hermitian form. The conjugate embeddings inherit positivity. Hermitian symmetry over E follows already at one embedding by injectivity. Each eigenvalue of $D(y)$ is nonzero and $D(y)$ commutes with y. This proves invertibility and $y^*A'y = qA'$. Finally $x^*A'x = \varepsilon D(g^{-1})$ is integral by the coefficient choice and $g^{-1} \in M_n(\widehat{O})$. □

248. Finiteness of the marked regular terms

Lemma 248.1. *In Proposition 247.3 let Δ_E be the positive Gram determinant of $\mathrm{Tr}_{E/\mathbb{Q}}(\bar{u}v)$ on O_E in any integral basis, and put $\Lambda_x = E^n \cap x\widehat{O}^n$. The positive trace lattice (Λ_x, B'), with $B'(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}(v^*A'w)$, is integral of rank $n[E : \mathbb{Q}]$ and its Gram determinant is*

$$\Delta_E^n \left| N_{E/\mathbb{Q}}\big(\det_E D(y)\big) \right|.$$

This number depends only on P, D. The endomorphism y preserves Λ_x by inclusion and multiplies its squared lengths by q.

Proof. The lattice, rank and completion statements follow exactly as in Lemma 13.4: the common integer denominators of x, x^{-1} bound the local lattices between two completions of rational lattices. The finite quotient of the two bounds decomposes into its prime parts, so imposing the required local submodules commutes with completion. The trace of an integral local element lies in $\mathbb{Z}_p$; in an integral basis it is the trace of its integral multiplication

matrix. Integrality of $x^*A'x$ in Proposition 247.3 now gives integrality of B' at every prime and hence over $\mathbb{Z}$.

Over $\mathbb{Z}_p$ use the x-transport of the product of n integral bases of O_E. Write $Y = x^{-1}yx$. Its trace Gram matrix is the standard trace Gram matrix multiplied on the right by the matrix over $\mathbb{Q}_p$ of $D(Y)$, and scaled by ε_p, because $x^*A'x = \varepsilon_p D(Y)$ at p. The scalar is a unit and the initial Gram determinant is Δ_E^n. On tensoring with $\mathbb{Q}_p$ the linear maps $D(Y)$ and $D(y)$ are conjugate by x, also as $\mathbb{Q}_p$-linear maps. The rational linear determinant of $D(y)$ equals $N_{E/\mathbb{Q}}(\det_E D(y))$: after extending scalars to $\mathbb{C}$, the basis of embeddings in Lemma 82.2 separates the map into one conjugate matrix for each embedding. Their determinants multiply to the field norm. Also $\det_E D(y)$ is determined by P, D, being the product of $D(r)$ over the roots of P in any splitting field. The local computation therefore gives the claimed prime valuation of the global Gram determinant at every prime. Both sides are positive rational numbers, so they are equal. The inclusion $y\Lambda_x \subset \Lambda_x$ follows at all completions from $Y \in M_n(\widehat{O})$, and the squared-length formula follows by taking the trace of $y^*A'y = qA'$. □

Theorem 248.2. *For any regular $H \in \mathcal{H}^+$ and either $\epsilon \in \{0, 1\}$, the entire orbit set $\mathcal{O}_\epsilon(H)$ of Definition 247.1 is finite. Every orbit has trivial stabilizer in the action of Γ on triples (xC, y, i).*

Proof. There are finitely many q, P, i by Lemma 247.2; fix one choice. Use the single polynomial D of Proposition 247.3 for this choice and ϵ. Lemma 248.1 gives the same determinant and rank for all the positive trace lattices under consideration. Lemma 13.4 gives finitely many isometry classes. On any fixed representative lattice a ball for its positive form meets the lattice in a finite set, by boundedness in a real basis. For each member b of the fixed integral basis of O_E, multiplication by b has norm at most $\max_\tau |\tau(b)|$: the positive embedding summands in the trace norm are each scaled by $|\tau(b)|^2$. Its values on each basis vector thus have finitely many possibilities. The same holds independently for y, since it preserves the lattice by inclusion and has operator norm $\sqrt{q}$ by the preceding lemma. Hence the trace lattice, its full O_E-action and its endomorphism y form a finite list up to isomorphism.

A lattice isometry respecting this data extends over $\mathbb{Q}$ to an E-linear map intertwining the two endomorphisms. It preserves the full E-valued form A', since one can multiply its second argument by each element of E and use nondegeneracy of field trace as in Proposition 246.1. It therefore also preserves the original form recovered by

$$h_A(v, w) = h_{A'}(v, D(y)^{-1}w), \qquad h_M(v, w) = v^*Mw.$$

The mark is transported because it is specified at a fixed root of σP and its reflection conjugates with y. After these rational identifications the same lattice, endomorphism and original form are fixed. Their finite frames differ on the right by an integral similitude, by their equal local lattices and their scalar unit Gram matrices. Such frames have finitely many right C-classes by the finite reduction argument of Proposition 246.1. This proves finiteness for the fixed data and hence for all the orbits. Finally a stabilizing rational matrix commutes with y, hence at σ with $J_{\epsilon,i}(y)$, and belongs to xCx^{-1}. Proposition 239.2 then gives the identity. □

249. Matching the irreducible marked terms

Lemma 249.1. *Fix q, P, i occurring on either side in Definition 247.1, and suppose P is irreducible over E. Put $L = E[T]/(P)$, let λ be the residue of T, and let K' be the fixed field of the involution of Proposition 247.3. There is a basis identifying each rational y with multiplication by λ on L. In these coordinates the nonsingular Hermitian forms A satisfying $y^*Ay = qA$ are exactly*

$$h_a(v, w) = \mathrm{Tr}_{L/E}(a\, v^*w), \qquad a \in (K')^\times.$$

The rational changes of coordinates commuting with y are precisely the multiplications m_z, $z \in L^\times$, and $m_z^* h_a m_z = h_{az^*z}$. The signs on the eigenlines are the signs of a at the corresponding real embeddings of K'.

Proof. The adjugate identity gives $P(y) = 0$ by Lemma 73.2. Evaluation defines a unital map of the field L into $\mathrm{End}_E(E^n)$. Its action on any nonzero vector identifies L with E^n, by injectivity and equality of dimensions. An endomorphism commuting with y is L-linear and hence is multiplication by its value on 1. The adjoint of y for the given Hermitian form is qy^{-1}. On E the adjoint is conjugation, so on L it is exactly the involution described in Proposition 247.3. For any one of these forms write $f(w) = h(1, w)$. Then $h(v, w) = f(v^*w)$. The field trace pairing identifies L with its E-linear dual because a nonzero b paired with b^{-1} has nonzero trace n. Thus $f(w) = \mathrm{Tr}_{L/E}(aw)$ for a unique $a \in L$. Taking a star commutes with trace up to conjugation on E, as seen by summing over the embeddings above a given embedding of E, using Lemma 82.2. Hermitian symmetry now gives $a^* = a$ by the same nondegenerate pairing. Nonsingularity gives $a \neq 0$. Conversely such a gives a nonsingular form and $\lambda^*\lambda = q$ gives the required multiplier. Multiplication by z acts by the displayed pullback. At an embedding in Ψ, the power basis of L evaluated at the n distinct roots has invertible Vandermonde matrix, as in Theorem 81.1. Trace in those coordinates is addition and the star is complex conjugation. The resulting Hermitian form is diagonal with entries the specified real values of a. This proves the signs. □

Theorem 249.2. *Assume n is odd. For each regular $H \in \mathcal{H}^+$ there is a bijection between the terms of $\mathcal{O}_0(H)$ and $\mathcal{O}_1(H)$ with irreducible P. It preserves P, i, q and the value of $H(x^{-1}y^{-1}x)$ at each term.*

Proof. Fix P, i, q occurring on either side. Take L, K', λ as in Lemma 249.1. The real embeddings of K' correspond exactly to the eigenlines in the CM type. The change from the signs of $J_{0,i}$ to those of $J_{1,i}$ reverses $n - 1$ signs at σ and none elsewhere. Their product is one because n is odd. The field L contains the twelfth root of unity of Lemma 20.1 and $L = K'(i_0)$ with K' totally real, by Proposition 247.3. Theorem 85.1 supplies $d \in (K')^\times$ with exactly those signs and a finite idele z of L such that $z^*z = d$ at every finite place. Fix this choice for the given data.

Use the coordinates of the lemma, in which a triple is represented by a and an adelic isomorphism $x : \mathbb{A}_{E,f}^n \to L \otimes_E \mathbb{A}_{E,f}$. The finite Gram equation reads $x^* h_a x = \varepsilon I$ with ε any rational profinite unit. Changing the rational coordinates of that fixed y and changing the frame representative identify

$$(a, x) \ \sim \ (a/(w^*w),\, m_w x k), \qquad w \in L^\times,\ k \in C.$$

These are all identifications by the centralizer calculation in the lemma. The matrix A, and hence its parameter a in a chosen coordinate system, is uniquely determined for a signed frame by Lemma 247.2. The signed condition says exactly that a has the designated signs, by the lemma and the definition of $J_{\epsilon,i}$.

Take (a, x) to $(da, m_z^{-1}x)$. Both finite Gram matrices are identical, by $z^*z = d$, and the real signs change as required. The finite Hecke argument is $x^{-1}m_\lambda^{-1}x$. It stays identical because the multiplications m_z, m_λ commute. The map descends to equivalence classes since it commutes with all the displayed changes of representative, as in Theorem 86.3. Its inverse uses d^{-1}, z^{-1}. Thus it matches the two sets for the given data and preserves every indicated weight. Taking the union over the data gives the assertion. □

250. Cancellation at the place of the cuspidal character

Lemma 250.1. *Let $G_0 \in \mathrm{GL}_n(\mathbb{Q}_\ell)$ have separable characteristic polynomial $P = P_1P_2$, with P_1, P_2 coprime and of positive degree. Write $V_a = \ker P_a(G_0)$ for $a = 1, 2$. These give a direct sum decomposition of $\mathbb{Q}_\ell^n$. Let M be the space of maps with image in V_1 vanishing*

on V_1, and write $u_B = 1 + B$ for $B \in M$. The map

$$\mathcal{L} : M \longrightarrow M, \qquad \mathcal{L}(B) = G_0 B - B G_0$$

is invertible, $u_B^{-1} G_0 u_B = G_0 + \mathcal{L}(B)$, and for $D \in M$ there exists a single $T \in M$ such that

$$u_{B+T}^{-1} G_0 u_{B+T} = u_D \, u_B^{-1} G_0 u_B \qquad (B \in M).$$

The set of B with $u_B^{-1} G_0 u_B$ integral is compact.

Proof. An identity $a_1 P_1 + a_2 P_2 = 1$ and $P(G_0) = 0$ decompose every vector into its two kernel components and show their intersection is zero. Each component is nonzero by evaluating at a root of its corresponding factor in a splitting field; extension of scalars preserves a matrix kernel's dimension by row elimination. The space M identifies with $\mathrm{Hom}(V_2, V_1)$. Its members have pairwise zero products, and G_0 preserves both subspaces. Thus both left and right multiplication by G_0 preserve M, and $B G_0 B' = 0$ for $B, B' \in M$. If $G_0 B = B G_0$, then $P_1(G_0) B = B P_1(G_0) = 0$. The restriction of $P_1(G_0)$ to V_2 is invertible by the displayed Bezout identity. Thus $B = 0$, giving invertibility of $\mathcal{L}$ in finite dimension. Use $u_B^{-1} = 1 - B$ and the zero products to obtain the conjugation formula. Taking $T = \mathcal{L}^{-1}(D G_0)$ gives the required translation since $D\mathcal{L}(B) = 0$. Finally $G_0 + M$ is closed, being an affine subspace defined by linear equations; its intersection with the compact integral matrices is compact. The inverse affine map to M is continuous by its finite matrix formula in bases. This proves the last assertion. □

Theorem 250.2. *Let $H \in \mathcal{H}^+$ be regular and suppose its coset function satisfies the character relation of Theorem 245.3. For $\epsilon \in \{0, 1\}$ and any complex weight $w(P, i)$,*

$$\sum_{\substack{[(xC, y, i)] \in \mathcal{O}_\epsilon(H) \\ P = \det(TI - y) \text{ reducible over } E}} H(x^{-1} y^{-1} x)\, w(P, i) = 0.$$

Proof. The whole sum is finite by Theorem 248.2. Fix one occurring reducible P and a root index i. Choose a factorization $P = P_1 P_2$ in $E[T]$ with both factors nonconstant and coprime, possible by separability. It remains coprime after base change to $E_{w_0} = \mathbb{Q}_\ell$, by the Bezout identity. Work before the rational quotient with the signed triples (xC, y, i) of this polynomial and index, allowing the weight to be zero, and imposing just $s = x^{-1} y^{-1} x \in S$ on the Hecke argument. All nonzero terms occur in this set. At one triple let $G_0 = d_{w_0}(s)$. It is conjugate to $y_{w_0}^{\mathsf{T}}$, hence has polynomial P. Use M, u_B of Lemma 250.1 for G_0, and write $b = b_{w_0}$. Consider the set of triples

$$(xb(u_B)C,\ y,\ i), \qquad B \in M.$$

The finite Gram matrix stays scalar with the same multiplier since $b(u_B)$ is unitary, and the rational signed form is unchanged. All new Hecke arguments lie in S.

These sets partition the triples under consideration. Indeed a change $x \mapsto xk$ with $k \in C$ conjugates G_0 and its two kernels by $d_{w_0}(k)^{-1}$, and so conjugates the corresponding M by the same matrix. The identity $k b(v) k^{-1} = b(d_{w_0}(k) v d_{w_0}(k)^{-1})$ from Proposition 240.4 makes the displayed set independent of the representative. Changing the base point within it by $b(u_B)$ conjugates its local group $1 + M$ by u_B^{-1}, leaving that group unchanged since it is abelian. The map $B \mapsto u_B$ is an isomorphism from the additive group of M to this group, with $u_B u_{B'} = u_{B+B'}$. Thus every member gives the same set. Rational change of coordinates conjugating y and multiplying x on the left leaves s unchanged, hence permutes these sets.

In a fixed set, the displayed triples are indexed without repetition by

$$M / M_C, \qquad M_C = \{B \in M : b(u_B) \in C\}.$$

The subgroup M_C is open by continuity and openness of C in S. For a nonzero term the dual component of $s_B = b(u_B)^{-1} s b(u_B)$ is integral because s_B^{-1} is integral. This component

is $u_B^{-1}G_0u_B$. Lemma 250.1 confines the representatives of nonzero terms to a compact subset of M. Its image in the discrete quotient by M_C is finite.

Take the projection e onto V_1 along V_2. It is neither zero nor the identity and $M = eM_n(\mathbb{Q}_\ell)(1-e)$. Proposition 229.1 gives $u_D \in J \cap (1+M)$ with $\chi(u_D) \neq 1$. For the T given by the preceding lemma one has $s_{B+T} = b(u_D)s_B$. Indeed, for a similitude s, $sb(v)s^{-1} = b(G_0vG_0^{-1})$ by the same two-place matrix identity as for k above. Consequently $s_B = b(u_B^{-1}G_0u_BG_0^{-1})s$, giving the claim from the local equation. The translation by T permutes M/M_C and multiplies every summand by $\chi(u_D)^{-1}$. The sum on this set has finite support, hence is zero.

It remains to pass to rational orbits. For a set in this partition meeting the nonzero support of H, its rational setwise stabilizer permutes the finite nonempty subset on which $H(s_B) \neq 0$, since H is invariant under a change of representatives. On that subset point stabilizers are trivial by Theorem 248.2. The setwise stabilizer is therefore finite and acts freely there. Rational orbits in the rational saturation of this set are in bijection with its orbits under that stabilizer: two of its elements in the same rational orbit differ by a map taking intersecting parts of the partition to each other, hence preserving that part. Each nonzero rational-orbit weight thus occurs exactly the stabilizer order times in the local sum. Dividing the zero sum by that order gives zero on its saturation. Only finitely many such saturations contain contributing rational orbits, again by Theorem 248.2. Summing over them and over P, i proves the assertion. □

251. The definite eigensystem in the marked orbital formula

Theorem 251.1. *Let n be odd and let $H \in \mathcal{H}^+$ be regular with $H(b_{w_0}(j)g) = \chi(j)^{-1}H(g)$ as in Theorem 245.3. For any complex function $\omega(P,i)$ of the characteristic polynomial and its marked root, the two finite sums*

$$\sum_{[(xC,y,i)]\in\mathcal{O}_\epsilon(H)} H(x^{-1}y^{-1}x)\,\omega(P,i), \qquad \epsilon = 0,1,$$

are equal.

Proof. Theorem 248.2 gives finiteness on both sides. On irreducible polynomials the bijection of Theorem 249.2 preserves each summand. For reducible polynomials each sum vanishes by Theorem 250.2. □

Definition 251.2. Let $\tau_{w,r} \in \mathcal{H}^+$ be the safe spherical operator of Lemma 245.2 for $0 \le r \le n$, extended by zero for $r > n$. A polynomial in these operators may be a finite complex linear combination of ordered words, including the empty word. For regular $H \in \mathcal{H}^+$, $\alpha \in O_E^\nu$ and integer $h \ge 0$, write

$$M_{\alpha,h}(H) = \sum_{[(xC,y,i)]\in\mathcal{O}_1(H)} H(x^{-1}y^{-1}x)\,[X^h]B_{\alpha,y,i}(X)$$

with $B_{\alpha,y,i}$ from Definition 243.2. Here $B_{\alpha,y,i}$ depends only on P, i: the unordered root lists at the embeddings and the marked root at σ and its conjugate determine every factor. Say that a system $\theta = (\theta_{w,r})$ occurs in the marked orbital formula in weight h if, for every polynomial F with $F(\theta) = 1$, there is $B \in \mathcal{H}^+$ with $BF(\tau)$ regular and $M_{\alpha,h}(BF(\tau)) \neq 0$ for some integral tuple α.

Theorem 251.3. *For any one approximating form, choose $\mathfrak{n}, \psi, \Xi$ with the oriented coefficient system of Definition 242.2, as in Theorems 241.2 and 242.3. Set $\theta_{w,r} = c_w^{-r}t_{w,r}$ at safe places for $0 \le r \le n$ and zero for $r > n$. Then θ occurs in the marked orbital formula in each weight $h_j = \mu + k_{v_0,j}$, $1 \le j \le n$. For a fixed F the same B can be used in all of these weights and each testing tuple can be taken in the finite grid $\mathcal{B}_n^\nu$ of Theorem 243.3.*

Proof. The weights h_j are nonnegative: $k_{v_0,j} + m_{v_0} \ge 0$ and $\mu = n\sum_v m_v \ge m_{v_0}$ by Definition 242.2. By Theorem 241.2, the vector $\psi \neq 0$ is a common eigenvector with the stated eigenvalues. Thus $c = F(\tau)$ fixes ψ when $F(\theta) = 1$, by applying each word successively. Corollary 246.3 yields a regular $H = (bc)^m = Bc$ whose definite trace is nonzero, with $B = (bc)^{m-1}b$, and the indicated character relation. That corollary gives for each h_j a grid tuple α for which the sum over $\mathcal{O}(H)$, with internal sum over marks of $[X^{h_j}]B_{\alpha,y,i}$, is nonzero. The expanded sum is exactly the sum on $\mathcal{O}_0(H)$: for each rational orbit of pairs, all n marks are preserved under a change of representative by their common ordering of σP. Apply Theorem 251.1 with $\omega(P,i) = [X^{h_j}]B_{\alpha,y,i}$. Its marked sum is $M_{\alpha,h_j}(H)$ and has the same nonzero value. The choices of H, B preceded the choice of j, proving the simultaneous assertion. □

252. The complex period family of a signed frame

Definition 252.1. Let $V_{\mathbb{R}} = \prod_{\tau\in\Psi}\mathbb{C}^n$ with the underlying real linear structure; embed E^n by its embeddings in Ψ. For $x \in G$ put $\Lambda_x = E^n \cap x\widehat{O}^n$ in $V_{\mathbb{R}}$, as in Lemma 248.1. A reflection J is a complex-linear involution on $\mathbb{C}^n$ whose $+1$ eigenspace is a line and whose -1 eigenspace has dimension $n-1$. Use triples $(x, J, (P_a)_{a=1}^{\nu})$ with $P_a \in V_{\mathbb{R}}/\Lambda_x$ and x, J admitting a signed frame in Definition 239.1. For the complex structure on $V_{\mathbb{R}}$ use multiplication by iJ at σ and by i at all other places of Ψ. Each of these real maps squares to $-I$, giving complex scalar multiplication by $(u+iv)w = uw + v(iJ)w$ at σ and the usual rule elsewhere. The finite level C identifies x and xk, $k \in C$, by the identity on the lattice and on the remaining data. A rational $t \in \Gamma$ acts by $x \mapsto tx$, $J \mapsto \sigma(t)J\sigma(t)^{-1}$ and $P_a \mapsto tP_a$. The lattice then becomes $t\Lambda_x$. Denote the classes by $\mathcal{X}(C,\nu)$.

Proposition 252.2. *Fix an x admitting some reflection and let A be its unique rational signed-frame matrix. At the other embeddings of Ψ it is positive definite, and $H = \sigma(A)$ has signature $(1, n-1)$. The allowable reflections are in bijection with the positive complex lines for H. In a basis with $H = \operatorname{diag}(1, -I_{n-1})$ they are parametrized by*

$$u_b = (1, \bar{b}), \qquad b \in \mathbb{B} = \{b \in \mathbb{C}^{n-1} : \|b\|^2 < 1\}.$$

For $w = (w_0, w')$ at σ there is a period map

$$Q_b(w) = (w_0 - b^{\mathsf{T}}w',\ \bar{w}' - b\bar{w}_0).$$

At the remaining embeddings use the identity in defining $Q_b : V_{\mathbb{R}} \longrightarrow \mathbb{C}^{fn}$. It is a real-linear isomorphism, carrying the complex structure of Definition 252.1 to the standard one. For fixed w it is holomorphic in b. In particular the action of $a \in E$ in these complex coordinates has characters $\sigma(a)$ on the first line, $\bar{\sigma}(a)$ with multiplicity $n-1$, and $\tau(a)$ with multiplicity n at each remaining $\tau \in \Psi$.

Proof. The uniqueness proof of Lemma 247.2 uses only $x^*Ax = \varepsilon I$ and positivity at another embedding; it applies to all allowable reflections for x. The form HJ is positive Hermitian, hence $J^*H = HJ$. The two eigenspaces of J are H-orthogonal. On the positive line of J the form H is positive and on its complement it is negative. Conversely a positive line determines J by acting as one there and minus one on its H-orthogonal complement. That complement is negative: in a diagonal basis with one positive coordinate, subtracting the projection onto a positive unit vector removes the only positive square, and the restriction is nondegenerate. A diagonal basis with entries $1, -1, \dots, -1$ exists from any initial reflection by orthogonalizing separately in its positive and negative eigenspaces and rescaling. In these coordinates each positive line has a nonzero first coordinate. Normalizing it to one gives u_b with $\|b\| < 1$, and every such vector spans a positive line. More explicitly its orthogonal complement consists of $w_0 = b^{\mathsf{T}}w'$ and is negative there by Cauchy–Schwarz and $\|b\| < 1$.

The first component of Q_b is complex linear in w and kills the negative complement. The second is conjugate linear and kills the positive line. They therefore carry iJ to ordinary

multiplication by i: on the negative summand complex conjugation changes the sign of i. If both components vanish, $w' = \bar{b}w_0$ and $w_0 = \|b\|^2 w_0$, hence $w = 0$. Equality of the real dimensions gives invertibility. The formula is affine in b at each fixed w. Its linearity in the positive coordinate and conjugate linearity in the negative coordinates give the displayed characters. These calculations also apply at $n = 1$, with $\mathbb{B}$ a single point. □

Theorem 252.3. *For fixed x, the triples in Definition 252.1 before rational identifications form a holomorphic family over $\mathbb{B}$ of ν-fold products of compact complex tori. In the coordinates of Proposition 252.2 the quotient manifold of dimension $n - 1 + \nu f n$ is*

$$\big(\mathbb{B} \times (\mathbb{C}^{fn})^{\nu}\big)/\Lambda_x^{\nu}, \qquad (b, z_a)_a \longmapsto (b, z_a + Q_b(\lambda_a))_a.$$

A rational change $x \mapsto tx$ induces a biholomorphic isomorphism of these families, with the induced transformation of the base. The equivalence classes $\mathcal{X}(C, \nu)$ have complex manifold charts locally induced by those families.

Proof. The lattice is full by Lemmas 13.4 and 82.2. Its image by the real isomorphism Q_b is full, so the quotient in a fiber is compact by its bounded closed fundamental parallelotope. All lattice translations in the statement are holomorphic by the preceding proposition, with inverses given by $-\lambda_a$. They act freely. For b in any compact subset, the inverse real matrices Q_b^{-1} are bounded by the cofactor formula and the nonvanishing of their determinants. On a compact subset of the total space, a translation meeting this subset again has bounded λ_a, hence there are only finitely many such translations. Near a point choose a relatively compact neighborhood, and then shrink it to avoid all its nonidentity translates on this finite list, using freeness and continuity. This gives the quotient complex charts. Two distinct orbits have disjoint image neighborhoods by the same compactness argument with a neighborhood of each point, proving the Hausdorff property.

A rational t takes the associated form to $(t^{-1})^* A t^{-1}$. The transformation $b \mapsto b'$ is holomorphic since $(1, b')$ is obtained from the image of $(1, b)$ by the conjugate matrix of $\sigma(t)$ in the chosen two bases, dividing by its nonzero first coordinate. It carries J_b to $J_{b'}$, so $Q'_{b'} t Q_b^{-1}$ is complex linear. This matrix is holomorphic in b: at any fixed b_0 choose real vectors v_j whose Q_{b_0}-images are a complex basis. Locally the matrix is the product of the matrix with columns $Q'_{b'} t v_j$ and the inverse of the one with columns $Q_b v_j$. Both column matrices are holomorphic and the latter stays invertible. Translation of lattices is respected since $t\Lambda_x = \Lambda_{tx}$. Applying the argument to t^{-1} gives the inverse.

It remains to pass to rational orbits. Take one xC in each rational orbit of such frames. For t with $x^{-1}tx \in C$, uniqueness of the frame matrix from the finite scalar Gram equation and positivity at a second embedding gives $t^* A t = A$. Indeed the transformed rational form has these same properties. Such t act properly on the ball in the following explicit sense. If both b and tb lie in a fixed compact subset, then $\sigma(t)^* H J_{tb} \sigma(t) = H J_b$. The matrices $H J_b$ vary continuously: the projection onto u_b is $w \mapsto u_b (u_b^* H w)/(u_b^* H u_b)$, with nonzero denominator. On this compact subset the maxima and the positive minima on unit vectors bound $\sigma(t)$. At all the other embeddings, preservation of the positive matrix $\tau(A)$ also bounds the entries of t. Its finite entries have common bounded denominators since $t \in xCx^{-1}$. Only finitely many rational matrices satisfy all these bounds by Lemma 82.2. The stabilizer of each b is trivial by Proposition 239.2. Thus these rational actions on the base, and hence on the torus families, are free and have the same locally disjoint neighborhood property as the lattice translations. Taking the union of the quotient charts over the frame representatives gives the claimed charts. This union is second countable. There are countably many possible rational matrices A, and for fixed A every finite frame differs from any given one on the right by an element of S, by the two scalar Gram identities. The set S/C is countable: the finite adeles have a countable base formed from finitely many local balls and the integral factors elsewhere. Local balls have a countable base by uniformizers and finite residue digit representatives as in Lemma 225.1. Finite matrices and their subspaces inherit a countable base, and each open coset in S contains a distinct basic set. Finally each ball and vector

space is second countable, and images of a countable base under an open quotient map form a base. □

253. Compactness and polarization of the signed family

Lemma 253.1. *In the setting of Theorem 252.3 write $A_\tau = \tau(A)$, $J_\sigma = J$ and $J_\tau = I$ for $\tau \neq \sigma$. On $V_{\mathbb{R}}$ put*

$$B_{A,J}(v,w) = 2\Re \sum_{\tau \in \Psi} v_\tau^* A_\tau J_\tau w_\tau .$$

It is a positive real inner product. In a $\mathbb{Z}$-basis of Λ_x its determinant is Δ_E^n of Lemma 248.1. Every nonzero lattice vector satisfies $B_{A,J}(v,v) \geq 2$.

Proof. Positivity and symmetry follow from the positive Hermitian matrices $A_\tau J_\tau$. Write B_A for the real form with each J_τ replaced by I. On the rational space it is the trace of $v^* A w$ by Lemma 82.2. The finite Gram equation $x^* A x = \varepsilon I$ shows that B_A is integral, with determinant of absolute value Δ_E^n, by the local basis proof of Lemma 248.1 applied directly to this Gram equation. In a real basis $B_{A,J}(v,w) = B_A(v, \mathcal{J} w)$ where $\mathcal{J}$ has components J_τ. Its real determinant is one, since the complex-linear involution J has eigenvalues $+1, -1$ on complex subspaces and each real multiplicity is even. This gives the assertion on the positive determinant.

For $0 \neq v \in \Lambda_x$ the element $a = v^* A v$ belongs to O_F: it is conjugation invariant and integral in every completion by the same finite Gram equation. Its value at any of the other places is strictly positive, since v has a nonzero image under each field embedding and A_τ is positive there. Thus $a \neq 0$ and $|N_{F/\mathbb{Q}} a|$ is a positive integer by Lemma 82.2. Some real embedding has $|\tau(a)| \geq 1$, since the product of these absolute values is at least one. At each place $|\tau(a)| \leq v_\tau^* A_\tau J_\tau v_\tau$: in the orthogonal decomposition from Proposition 252.2 the left side is the absolute difference and the right side the sum of two nonnegative squared lengths. This proves the bound. □

Theorem 253.2. *The complex manifold $\mathcal{X}(C,\nu)$ of Definition 252.1 is compact for every integer $\nu \geq 0$.*

Proof. The lattice basis bound proved in Lemma 13.4 applies to the Euclidean metric $B_{A,J}$: its covolume and a lower bound for its shortest length are fixed by Lemma 253.1. Choose for each point an ordered lattice basis with all lengths bounded by one constant. Put $D = n[E : \mathbb{Q}]$. In this basis the positive Gram matrices M have bounded entries by Cauchy–Schwarz, and determinant $\Delta_E^n > 0$. Hence their inverses also have bounded entries by cofactors. Coordinates of a vector of bounded length are bounded uniformly: its pairings with the basis vectors are bounded and its coordinate column is M^{-1} times the column of those pairings.

The integral trace form B_A also has bounded entries in the chosen basis, since $B_A(v,w) = B_{A,J}(v, \mathcal{J} w)$ and $\mathcal{J}$ is an isometry for the positive metric by the orthogonal decomposition. For each integral basis element $b \in O_E$, multiplication by b is integral on the lattice and has norm at most $\max_\tau |\tau(b)|$ by the positive embedding summands. Its integer coordinate matrix therefore has bounded entries. There are only finitely many combined possibilities for the matrix of B_A and this list of multiplication matrices. Identify any two ordered bases with identical data. The resulting rational linear isomorphism is E-linear and recovers A by nondegeneracy of the field trace, as in Proposition 246.1. Thus, on one fixed representative per list, the ordered lattice basis and rational form can be kept fixed. The possible x with this lattice and form have only finitely many right C-classes by the same finite reduction argument in that proposition, which uses only the local scalar unit Gram equations. We have made only rational and level identifications allowed in $\mathcal{X}(C,\nu)$.

For any one of these finitely many representatives, the possible positive matrices M lie in a compact set of positive matrices: take the closure with the uniform entry bounds and fixed positive determinant. Limits of positive matrices are positive semidefinite by evaluation on

real vectors, and a nonsingular such matrix is positive definite by orthogonal diagonalization, or completing squares. In the fixed real basis the operator $\mathcal{J}$ is $B_A^{-1}M$, with B_A here its fixed nonsingular Gram matrix. The conditions of being identity off σ, complex linear at σ, having $J^2 = I$, and $\operatorname{tr}_{\mathbb{C}} J = 2 - n$, are closed. The last two equations imply the required multiplicities since the projectors $(I \pm J)/2$ split the space. Positive definiteness of M implies positivity of $A_\sigma J$: restriction to this complex factor tests the real part of the form, and symmetry of M, also applied after multiplication by i, yields Hermitian symmetry. Hence the allowed reflections in this compact set form a compact set in the ball coordinates: $J \mapsto \operatorname{im}(I + J)$ is continuous locally by choosing a nonzero column, and the positive line has nonzero first coordinate in the coordinates of Proposition 252.2.

For each of the ν torus points use a representative in the fixed closed fundamental parallelotope of Λ_x in $V_{\mathbb{R}}$. The maps taking the reflection and vectors in these compact sets to the family are continuous by the period charts of Theorem 252.3. Their images, for the finitely many representatives, cover the whole manifold. It is compact. □

Proposition 253.3. *Let $\xi = \delta/\varpi$ be as in Theorem 236.2. For the complex torus at (x, J) the real extension of*

$$E_A(v, w) = \operatorname{Tr}_{E/\mathbb{Q}}(\xi\, w^* A v) \qquad (v, w \in E^n)$$

is an integral Riemann form: it is alternating, takes integer values on the period lattice, is invariant under simultaneous application of the complex structure $I_J = (iJ_\tau)_\tau$, and $E_A(v, I_J v) > 0$ for $v \neq 0$. The quotient of the dual lattice for this form by Λ_x has order h^{2n} and exponent dividing h, with $h = |N_{F/\mathbb{Q}}\delta|$.

Proof. Alternation follows from $\bar{\xi} = -\xi$, $A^* = A$ and invariance of field trace under conjugation. At each prime the form in frame coordinates has matrix pairing $\operatorname{Tr}(\xi\varepsilon_p w^* v)$. The unit $\varepsilon_p \in \mathbb{Z}_p^\times$ keeps the integral dual unchanged. Since ϖ generates the different, the integral dual of the local standard lattice is precisely $\delta^{-1}O_{E,p}^n$, by Proposition 18.4 or the trace-dual computation in Theorem 236.2. Thus E_A is integral on Λ_x. The global dual is determined by these local duals: apply the local Gram equations in a rational lattice basis, or the completion argument of Lemma 13.4. After transport by x, the finite quotients therefore have the same orders and exponents as for the standard lattice in Theorem 236.2. Their direct product is the global quotient by prime decomposition of a finite abelian group, giving the stated size and exponent.

In the real coordinates field trace is $2\Re\sum_{\tau\in\Psi}$. Since $J_\tau^* A_\tau J_\tau = A_\tau$, simultaneous application of iJ_τ in these coordinates preserves the alternating form. Finally

$$E_A(v, I_J v) = 2\sum_{\tau\in\Psi} \Im(\tau(\xi))\, v_\tau^* A_\tau J_\tau v_\tau.$$

All coefficients are strictly positive by Lemma 236.1, and each Hermitian form in the sum is positive definite. This is the required positivity. □

254. Hecke legs and their coincidence sets on the complex family

Proposition 254.1. *Let D_0 be a finite set of right cosets gC in S^+/C. For all sufficiently large integers m, put $C_m = \{k \in C : k \equiv I \bmod m!\widehat{O}\}$. It is normal of finite index in C, satisfies $g^{-1}C_m g \subseteq C$ for all these cosets, and forgetting the finer level is a holomorphic covering $\pi_m : \mathcal{X}(C_m, \nu) \to \mathcal{X}(C, \nu)$. The group C/C_m acts simply transitively on its fibers by right translation on the finite frame. For each $gC \in D_0$ and $\mathbf{a} = (a_j) \in O_E^\nu$ there is a holomorphic map*

$$u_{g,\mathbf{a}} : \mathcal{X}(C_m, \nu) \longrightarrow \mathcal{X}(C, \nu), \qquad [x, J, (P_j)] \longmapsto [xg, J, (a_j P_j)].$$

Proof. The reduction map has finite image by the finite local residue rings of Lemma 225.1, proving normality and finite index. Each $gCg^{-1} \cap C$ is an open neighborhood of one in C. The congruences C_m are a base there, since any such neighborhood contains given

congruences at finitely many primes and the ordinary integral conditions elsewhere, and $v_p(m!) \to \infty$ for each fixed p. This gives the inclusions simultaneously. They depend only on gC.

The construction of the charts of Theorem 252.3 applies also at C_m by openness, finite index and the inherited trivial point stabilizers. Changing x to xk for $k \in C$ induces an action on the finer classes by normality, with the same lattice and reflection. Every lift of one coarse class has a representative of this kind, with its original torus points, by undoing a coarse rational change. Two such lifts are equal only when the cosets kC_m are the same: a rational equivalence between them belongs to xCx^{-1} and fixes the reflection, so it is the identity by Proposition 239.2. The map is locally the identity in the disjoint period charts of Theorem 252.3. More explicitly, choose a neighborhood of the line meeting none of its other translates under the coarse rational stabilizer of xC. At each possible xkC_m this gives a copy of the same local chart, using the same lattice translates, and the preceding fiber description exhausts its inverse image. This proves the covering assertion.

For the second map one has $\Lambda_x \subseteq \Lambda_{xg}$ by $g^{-1} \in M_n(\widehat{O})$. Thus $a_j P_j$ defines a point mod Λ_{xg} since Λ_x is O_E-stable. The form at xg is a positive rational multiple of the form at x: divide by the positive rational with the valuations of the multiplier of g, as in Proposition 246.1. All signatures, the line and the complex structure are preserved. Replacing x by xk for $k \in C_m$ changes xg on the right by $g^{-1}kg \in C$; replacing g by gk' with $k' \in C$ also leaves its class fixed. Rational changes commute with the scalar endomorphisms a_j. This proves well-definedness. Use the same period coordinates at x and xg, since scaling the form by a positive scalar keeps its line and negative complement unchanged. On those charts the base map is the identity, and the fiber maps are the complex linear scalar actions in Proposition 252.2, followed by reduction modulo the larger lattice. Hence they are holomorphic. □

Lemma 254.2. *For a term of $\mathcal{O}_1(H)$ with H regular and any $\alpha \in O_E$, the endomorphism $D_\alpha = I - 2\alpha y$ preserves Λ_x by inclusion. Its kernel on $V_{\mathbb{R}}/\Lambda_x$ has order the positive odd integer*

$$d_\alpha(y) = \prod_{\tau : E \hookrightarrow \mathbb{C}} \det(I - 2\tau(\alpha)\tau(y)).$$

For $\boldsymbol{\alpha} = (\alpha_j) \in O_E^\nu$, $B_{\boldsymbol{\alpha},y,i}(1) = \prod_j d_{\alpha_j}(y)$.

Proof. The matrix $x^{-1}yx$ is integral because the inverse Hecke argument is integral, so $y\Lambda_x \subseteq \Lambda_x$. In an integral basis the endomorphism D_α reduces to the identity modulo two. Its determinant as an integer matrix is therefore odd, in particular nonzero. Over $\mathbb{R}$ it is invertible, and multiplication by D_α identifies its torus kernel $D_\alpha^{-1}\Lambda_x/\Lambda_x$ with $\Lambda_x/D_\alpha\Lambda_x$. This has order the absolute determinant by the lattice index argument of Lemma 20.2. The rational determinant is the displayed product over embeddings by Lemma 248.1, and is positive since the determinants occur in nonzero conjugate pairs. At $X = 1$, every root in Definition 243.2 takes its original value and $L_i(z_i) = 1$, so that product over all embeddings is precisely the contribution from each j. □

Theorem 254.3. *Let $H \in \mathcal{H}^+$ be regular, take $D_0 = \operatorname{supp} H$ in Proposition 254.1, and fix $\boldsymbol{\alpha} \in O_E^\nu$. The coincidences are the pairs*

$$(gC, p), \qquad gC \in D_0, \quad p \in \mathcal{X}(C_m, \nu), \quad u_{g,2\boldsymbol{\alpha}}(p) = \pi_m(p).$$

They map onto $\mathcal{O}_1(H)$. The fiber above $[(xC, y, i)]$ has size $[C : C_m]B_{\boldsymbol{\alpha},y,i}(1)$, and $H(g) = H(x^{-1}y^{-1}x)$ throughout it. In particular there are finitely many coincidences.

Proof. Write a finer point using x, J, P_j. Equality of the two coarse classes is equivalent to the existence of a rational y such that

$$gC = x^{-1}y^{-1}xC, \qquad \sigma(y)J = J\sigma(y), \qquad (I - 2\alpha_j y)P_j = 0 \pmod{\Lambda_x} \quad \text{for every } j.$$

Indeed y takes the frame xg to x modulo C, transports the reflection to itself and identifies the displayed torus points, exactly the equivalence in Definition 252.1. Such y is unique: for two choices $y_1 y_2^{-1} \in xCx^{-1}$, and it fixes J, so is trivial by Proposition 239.2. Regularity gives distinct roots of σy. An involution commuting with this matrix acts by a sign on each eigenline and a reflection takes sign $+1$ on exactly one. Thus $J = J_{1,i}(y)$ for a unique i. This defines the claimed orbit. Under rational change of the finer point y conjugates accordingly, and under right C_m change the same y works since $g^{-1}C_m g \subseteq C$.

Conversely fix x, y, i for an orbit. For each kC_m in C/C_m use the finer frame xk, reflection $J_{1,i}(y)$, and $gC = k^{-1}x^{-1}y^{-1}xkC$. This lies in D_0 with its original weight by the two C-invariances. Replacing k by kk' with $k' \in C_m$ keeps that coset: left multiplication on it by $(k')^{-1}$ fixes it by the choice of C_m for all D_0. The fiber coordinates can be any element of $\prod_j D_{\alpha_j}^{-1}\Lambda_x/\Lambda_x$, since $\Lambda_{xk} = \Lambda_x$. These choices give coincidences, and every one above the orbit has such a representative by a rational change. If two choices give the same pair, the rational equivalence between their points is in xCx^{-1} and fixes the common reflection. It is trivial by the same neatness, after which equality forces equal kC_m and equal torus points. The finite count is therefore as stated by Lemma 254.2 and Theorem 248.2. □

255. The tangent determinant at a Hecke coincidence

Theorem 255.1. *At a coincidence (gC, p) of Theorem 254.3 put $u = u_{g,2\boldsymbol{\alpha}}$, $z = \pi_m(p) = u(p)$ and $A_p = du_p(d\pi_{m,p})^{-1}$ on $T_z\mathcal{X}(C,\nu)$. Let x, y, i be the corresponding marked data. Then*

$$\det(I - XA_p) = B_{\boldsymbol{\alpha},y,i}(X)/B_{\boldsymbol{\alpha},y,i}(0), \qquad \det(I - A_p) \neq 0.$$

Proof. By Lemma 247.2 all the eigenvalues r of all conjugates of y have $|r|^2 = q$ for the same $q > 0$. At σ they are distinct, and their eigenlines are pairwise orthogonal for the rational signed form as in Proposition 247.3. Choose eigenvectors of squared lengths $+1, -1, \dots, -1$ for that form, the first for the marked root z_i. Such rescaling is possible since $AJ_{1,i}$ is positive. The fixed line then has ball coordinate zero of Proposition 252.2. Comparing u to π_m in the target chart entails the rational coordinate change y taking xg to x modulo C. On the positive line it sends $(1, \bar{b})$ to $(z_i, (z_j\bar{b}_j)_{j\neq i})$. Taking the conjugates of the normalized coordinates gives diagonal action on the base with eigenvalues $\bar{z}_j/\bar{z}_i = z_i/z_j$ for $j \neq i$. The last equality uses the common squared absolute value.

The derivative preserves the vertical tangent space by the base map calculation. On its a-th summand it is induced by $2\alpha_a y$ in the period coordinates. At zero $Q_0(w) = (w_0, \bar{w}')$ at σ, so on this factor the eigenvalues are $2\sigma(\alpha_a)z_i$ on the line and $2\bar{\sigma}(\alpha_a)\bar{z}_j$ for $j \neq i$ on the complement. At each $\tau \in \Psi \setminus \{\sigma\}$ they are all the $2\tau(\alpha_a)r$ for roots r of τP. This uses the maps of Proposition 254.1 and their local charts. Changing a torus lift by a lattice vector adds the holomorphic section $Q_b(\lambda)$, whose vertical derivative is the identity for the resulting translation. Its derivatives in the base do not affect the actions on the vertical space or on its quotient. The determinant thus multiplies the base factor $\prod_{j\neq i}(1 - Xz_i/z_j)$ and the factors $1 - X(2\tau(\alpha_a)r)$ over exactly the listed fiber roots, by the determinant of a block triangular matrix.

In Definition 243.2 these fiber roots occur exactly in the lists multiplied by X: outside v_0 the representative of Ψ is $\bar{\nu}_v$ by Lemma 236.1. At the base,

$$\frac{L_i(Xz_i)}{L_i(0)} = \prod_{j\neq i}(1 - Xz_i/z_j).$$

Here $L_i(0) \neq 0$ since y is invertible and its roots are distinct. Each constant factor from the unmodified roots is nonzero because $B_{\boldsymbol{\alpha},y,i}(1) > 0$ by Lemma 254.2. Consequently $B_{\boldsymbol{\alpha},y,i}(0) \neq 0$, and after division by this constant the factors give the asserted polynomial. At $X = 1$ the numerator is the same positive integer. □

Corollary 255.2. *Let H be regular, and use the finite set of correspondences and the level in Theorem 254.3. For $h \geq 0$, let $A_p^\vee$ denote the ordinary linear dual of A_p, acting by precomposition on covectors. Set $\bigwedge^h A_p^\vee = 0$ when h exceeds the tangent dimension. Then*

$$\frac{1}{[C : C_m]} \sum_{gC \in \operatorname{supp} H} H(g) \sum_{p:\, u_{g,2\boldsymbol{\alpha}}(p) = \pi_m(p)} \frac{(-1)^h \operatorname{tr}(\bigwedge^h A_p^\vee)}{\det(I - A_p)} = M_{\boldsymbol{\alpha},h}(H).$$

In particular for every F normalized as in Theorem 251.3 there is one regular left multiple H of $F(\tau)$ for which the displayed complex sum is nonzero at each specified h_j for some $\boldsymbol{\alpha} \in \mathcal{B}_n^\nu$.

Proof. For a complex endomorphism A, triangularize it by induction using an eigenvector as in Corollary 246.3. The diagonal entries of its dual are the same eigenvalues. On a basis of exterior monomials of degree h the diagonal entries are the products over h-element subsets. Thus $[X^h] \det(I - XA) = (-1)^h \operatorname{tr}(\bigwedge^h A^\vee)$, also at $h = 0$ with the empty product one. Theorem 255.1 therefore makes each fraction $[X^h] B_{\boldsymbol{\alpha},y,i}(X) / B_{\boldsymbol{\alpha},y,i}(1)$. On the fiber of a marked orbit the number of terms is precisely $[C : C_m] B_{\boldsymbol{\alpha},y,i}(1)$ by Theorem 254.3. The common Hecke value and finiteness in that theorem now give the equality after summation over the marked orbits, by Definition 251.2. Its nonvanishing in the stated applications is Theorem 251.3. □

256. Theta coordinates for the signed complex fibers

Lemma 256.1. *Put $g = fn$, $L = \Lambda_x$ and $E_0 = E_A$ from Proposition 253.3. There is a lattice basis $e_1, \dots, e_g, f_1, \dots, f_g$ with*

$$E_0(e_j, e_k) = E_0(f_j, f_k) = 0, \qquad E_0(e_j, f_k) = d_j \mathbf{1}_{j=k}, \qquad 0 < d_1 \mid d_2 \mid \cdots \mid d_g.$$

Each d_j is a multiple of R^2 from Theorem 236.2; $\prod_j d_j = h^n$.

Proof. The form is nonsingular by its positivity on $(v, I_J v)$. Among its positive integer values on pairs u, v in the lattice choose the smallest $d = E_0(u, v)$. For every w, both $E_0(u, w)$ and $E_0(w, v)$ are divisible by d: subtract integer multiples of v or u respectively and apply division with remainder and minimality. Thus the integral projection

$$w \longmapsto d^{-1} \big(E_0(w, v) u + E_0(u, w) v \big)$$

splits off $\mathbb{Z}u + \mathbb{Z}v$ orthogonally. The complementary lattice is free by Lemma 12.1 and its form remains nonsingular over $\mathbb{Q}$. On it each pairing $E_0(w, w')$ is divisible by d: replacing the second member of $E_0(u + w, w')$ by $w' + kv$ adds kd to that pairing. Its remainder cannot lie strictly between zero and d. Inducting gives all pairs and the stated divisibility chain. The form E_0 / R^2 is integral since $\xi / R^2 = u / \varpi$ with $u \in O_F$, by the local trace-dual argument of Proposition 253.3. This gives the common divisibility. In the paired basis the dual modulo the lattice is the product of two copies of $\mathbb{Z}/d_j\mathbb{Z}$ per pair. The order h^{2n} in that proposition then gives $\prod_j d_j = h^n$. □

Proposition 256.2. *Put $D = \operatorname{diag}(d_1, \dots, d_g)$ and use the basis of Lemma 256.1. At each point b of the period ball, in the complex basis $Q_b(e_1), \dots, Q_b(e_g)$ the lattice has the form*

$$\mathbb{Z}^g + \Omega(b) D \mathbb{Z}^g, \qquad \Omega(b)^{\mathsf{T}} = \Omega(b), \qquad \Im \Omega(b) > 0.$$

The matrix $\Omega(b)$ is holomorphic on the ball.

Proof. For the real span U of the e_j, if $v, I_J v \in U$, then $E_0(v, I_J v) = 0$ by isotropy, giving $v = 0$ by Proposition 253.3. Thus U and $I_J U$ span a direct sum of dimension $2g$. This proves the complex basis assertion through Q_b of Proposition 252.2. Define the matrix by $\Omega = (Q_b(e_1), \dots, Q_b(e_g))^{-1} (Q_b(f_1), \dots, Q_b(f_g)) D^{-1}$. The entries are holomorphic by the period formula and inverse matrix cofactors with nonvanishing determinant.

The Hermitian form in the complex structure of this fiber is $H(v,w) = E_0(v, I_J w) + iE_0(v,w)$. It is conjugate linear in the first variable and linear in the second by alternation and I_J-invariance, and is positive by the preceding proposition on the Riemann form. Its matrix M on the e_j is real symmetric positive definite. Write $\Omega = X + iY$. The equations $E_0(e_j, f_k/d_k) = \mathbf{1}_{j=k}$ give $MY = I$. Thus $Y = M^{-1}$ is positive definite and symmetric. The imaginary part of $\bar{\Omega}^{\mathsf{T}} M \Omega$ vanishes since all the f_k/d_k pair to zero. Consequently $X^{\mathsf{T}} MY - Y^{\mathsf{T}} MX = 0$, giving $X^{\mathsf{T}} = X$. □

Theorem 256.3. *For $k \geq 1$, define $\Theta_k(\Omega, D)$ to be the entire functions S on $\mathbb{C}^g$, periodic under $\mathbb{Z}^g$, with*

$$S(z + \Omega D v) = \exp\big(-\pi i k (Dv)^{\mathsf{T}} \Omega D v - 2\pi i k (Dv)^{\mathsf{T}} z\big) S(z) \qquad (v \in \mathbb{Z}^g).$$

These are the sections of a holomorphic line bundle, the k-th tensor power of the one defined by the $k = 1$ factors. A basis indexed by $r \in \mathbb{Z}^g / kD\mathbb{Z}^g$ is

$$\Theta_{k,r}(z) = \sum_{a \in r + kD\mathbb{Z}^g} \exp\big((\pi i/k) a^{\mathsf{T}} \Omega a + 2\pi i a^{\mathsf{T}} z\big).$$

In particular the dimension is $k^g h^n$. These sections vary holomorphically in b, z on the period ball.

Proof. The factors are multiplicative for two successive lattice translations: the cross term is determined by symmetry of Ω, and translations by $m \in \mathbb{Z}^g$ contribute $\exp(-2\pi i k (Dv)^{\mathsf{T}} m) = 1$. They thus give a lattice action on $\mathbb{C}^g \times \mathbb{C}$ by scaling the fiber, forming a line bundle by the disjoint local lattice charts of Theorem 252.3. Its sections lift to exactly the entire functions with these transformation laws. The factors in degree k are the k-th powers of the degree-one factors.

For a compact set of b, z, the least value on real unit vectors of $\Im\Omega(b)$ has a positive uniform lower bound by Proposition 256.2. The absolute values of the summands are bounded by $\exp(-c\|a\|^2 + C\|a\|)$, $c > 0$, summable by comparison with a product of one-dimensional geometric tails. The same bound with a polynomial factor applies to derivatives on smaller compact sets. Termwise differentiation, or the Cauchy formula on coordinate discs of Lemma A.3, proves holomorphy. Translation by $\Omega D v$ reindexes a by $a + kDv$ and gives the displayed factor. The sections for different classes r have disjoint Fourier supports and nonzero coefficients, so are linearly independent.

Every periodic entire function has a unique normally convergent Fourier expansion $\sum_{a \in \mathbb{Z}^g} c_a \exp(2\pi i a^{\mathsf{T}} z)$: the contour shift and product Fourier argument in Theorem 49.2 uses just periodicity and holomorphy. Substitution in the period equations gives

$$c_{a + kDv} = \exp\big(2\pi i a^{\mathsf{T}} \Omega D v + \pi i k (Dv)^{\mathsf{T}} \Omega D v\big) c_a.$$

Thus the coefficients on each class are the coefficients of $\Theta_{k,r}$ times one constant. The functions form a basis and $|\mathbb{Z}^g / kD\mathbb{Z}^g| = k^g \prod_j d_j = k^g h^n$ by Lemma 256.1. □

257. Separating points and tangents by translated theta products

Proposition 257.1. *Use any positive diagonal integral D and symmetric Ω with $\Im\Omega > 0$ in the construction of Theorem 256.3. For $k \geq 3$ the degree-k theta sections span a system without common zeros, separating points and nonzero tangent vectors on $\mathbb{C}^g / (\mathbb{Z}^g + \Omega D \mathbb{Z}^g)$. More precisely these properties already hold for the span of*

$$\prod_{j=1}^{k} S_j(z + t_j), \qquad S_j \in \Theta_1(\Omega, D), \quad t_j \in \mathbb{C}^g, \quad \sum_j t_j = 0.$$

Proof. Zero total shift makes the product's automorphy factors exactly the degree-k factors by Theorem 256.3. Choose one nonzero section S there. At fixed z each $S(z + t_j)$ is a nonzero entire function of the $k - 1$ independent shifts, also for the last shift $t_k = -\sum_{j<k} t_j$,

since each shift ranges over $\mathbb{C}^g$. Its nonzero set is open dense. Indeed vanishing on a ball gives vanishing along each complex line through the ball by the one-variable identity theorem of Lemma A.3. Finitely many such open dense sets have nonempty intersection. This gives a product nonzero at z.

Suppose $z, w \in \mathbb{C}^g$ have proportional evaluations on all these products. Both evaluation vectors are nonzero, so the constant of proportionality is nonzero. Taking all $S_j = S$ gives, wherever the ratios are defined,

$$\prod_{j=1}^{k} f(t_j) = c, \qquad f(t) = S(w+t)/S(z+t).$$

Differentiate with respect to each coordinate of t_1, holding $t_2, \dots, t_{k-1}$ fixed. Logarithmic differentiation gives $(\partial_a f/f)(t_1) = (\partial_a f/f)(t_k)$. The two arguments t_1, t_k vary independently because $k \geq 3$. This identity of meromorphic functions follows everywhere after clearing denominators from the open dense set on which the original products are nonzero. Consequently each $\partial_a f/f$ is constant: fix a generic value of its second argument and vary the first. It follows that $f(t) = C\exp(\ell(t))$ for a complex linear functional ℓ and $C \neq 0$. Indeed multiplication by $\exp(-\ell(t))$ gives zero derivative on a nonempty open ball of holomorphy; constancy there extends by clearing denominators and the identity theorem. Periodicity under $\mathbb{Z}^g$ gives $\ell(t) = 2\pi i m^{\mathsf{T}} t$ with $m \in \mathbb{Z}^g$.

Replacing only the first section in the proportionality relation by another $S' \in \Theta_1$ shows, by cancelling the other nonzero factors on an open set, that $S'(w+t)S(z+t) = S'(z+t)S(w+t)$. This entire identity holds for all t. Hence with $\eta = w - z$, there is a constant $C_1 \neq 0$, independent of S', such that

$$S'(v+\eta) = C_1 \exp(2\pi i m^{\mathsf{T}} v)\, S'(v) \qquad (v \in \mathbb{C}^g).$$

Use as S' each basis section in degree one. Its Fourier support is a whole class modulo $D\mathbb{Z}^g$ by Theorem 256.3. The shifted support on the right must agree with the original one, so $m \in D\mathbb{Z}^g$. Comparing the coefficient of frequency a for all such classes yields

$$\exp(2\pi i a^{\mathsf{T}} \eta) = C_1 \exp\bigl(\pi i (m^{\mathsf{T}} \Omega m - 2a^{\mathsf{T}} \Omega m)\bigr) \qquad (a \in \mathbb{Z}^g).$$

Taking $a = 0$ and then the coordinate vectors gives $\eta + \Omega m \in \mathbb{Z}^g$. Thus z, w are the same torus point, proving point separation.

If a vector v at z has zero projective derivative there is a constant c' with $\partial_v P(z) = c' P(z)$ for all products P. This follows by differentiating ratios to one nonvanishing product at z. With $S = \Theta_{1,0}$ in every factor, put $u(t) = (\partial_v S)(z+t)/S(z+t)$. The product rule gives $\sum_j u(t_j) = c'$ on the same open dense set. Differentiating in t_1 now gives $(\partial_a u)(t_1) = (\partial_a u)(t_k)$. By the preceding independent-arguments reasoning each partial derivative is constant, so u is affine linear. Its periodicity under $\mathbb{Z}^g$ makes it constant. In $\partial_v S = c'' S$, comparison of Fourier coefficients gives $2\pi i a^{\mathsf{T}} v = c''$ for all $a \in D\mathbb{Z}^g$. Taking zero and the columns of D gives $v = 0$. This proves tangent separation. □

Theorem 257.2. *For the actual $D, \Omega(b)$ of Proposition 256.2, the degree-one coordinates of Theorem 256.3 embed every fiber in $\mathbf{P}^{h^n-1}(\mathbb{C})$. Over a fixed period ball they give a closed holomorphic embedding of the family of one torus into $\mathbb{B} \times \mathbf{P}^{h^n-1}(\mathbb{C})$.*

Proof. By Lemma 256.1, $D' = D/R^2$ has positive integral diagonal. Put $\Omega' = R^2\Omega$, so $\Omega' D' = \Omega D$ and $\Im\Omega' > 0$. The automorphy factors for degree R^2 of (Ω', D') are exactly those for degree one of (Ω, D): substitute D', Ω' into the quadratic and linear terms of Theorem 256.3. Since $R^2 \geq 3$, Proposition 257.1 gives all three separation properties for these actual degree-one sections.

Their coordinates depend holomorphically on the base and on the complex torus parameter by Theorem 256.3. The coordinate change from $Q_b(w)$ to z in the basis $Q_b(e_j)$ is holomorphic in b by Proposition 256.2. The common scalar factor on a lattice shift leaves the projective

point unchanged. With the base recorded, fiber separation gives injectivity. A tangent vector in the kernel must have zero base component and is zero by tangent separation. The map is proper: over any compact set of the base the torus family is compact, since the base set times a fixed closed real fundamental parallelotope of Λ_x maps continuously onto that inverse image, as in Theorem 253.2. The inverse image of a compact set in the target is closed in such a compact set. This turns the continuous injection into a homeomorphism onto its closed image: for any convergent sequence of image points, its values and its limit form a compact set in the target; the preimages have a convergent subsequence, and injectivity fixes the possible limit whenever the target limit is in the image.

In complex coordinates select a square submatrix of the derivative of full rank; the corresponding target coordinates are holomorphic local coordinates on the source. Indeed after linear normalization their map is $x \mapsto x+e(x)$ with $e(0) = 0$ and derivative zero at zero. On a small closed convex ball its derivative norm is less than $1/2$, so $x \mapsto t - e(x)$ takes that ball into itself for small t and contracts distances by $1/2$. Iteration from zero gives a locally uniformly convergent sequence of holomorphic functions of t and hence a holomorphic inverse by the Cauchy formula of Lemma A.3. In these coordinates the remaining components give a holomorphic graph. This proves the embedding assertion. □

258. Finite polynomial projections of coordinate rings

Lemma 258.1. *Let $R = \mathbb{C}[r_1, \dots, r_N]$ be a unital domain and let $R_{\le m}$ be the linear span of products of at most m of the r_j, including one. There are linear combinations $t_1, \dots, t_d$ of the generators that are algebraically independent, meaning they satisfy no nonzero complex polynomial relation, such that R is a finite module over $A = \mathbb{C}[t_1, \dots, t_d]$. For some positive constants c, C one has*

$$c(m+1)^d \le \dim_{\mathbb{C}} R_{\le m} \le C(m+1)^d \qquad (m \ge 0).$$

Each r_j satisfies a monic polynomial over A.

Proof. If the current list is dependent, take a nonzero relation of total degree $e \ge 1$. Choose constants $c_1, \dots, c_{N-1}$ where its degree-e homogeneous part evaluated at $(c_1, \dots, c_{N-1}, 1)$ is nonzero. Such constants exist: dehomogenizing a nonzero homogeneous polynomial keeps distinct monomials distinct, and a nonzero polynomial over the infinite field $\mathbb{C}$ cannot vanish on every tuple, by induction using univariate division. Replace the first $N-1$ generators by $u_j = r_j - c_j r_N$. Substituting in the relation and dividing by that nonzero scalar gives a degree-e monic equation for r_N over the algebra of the u_j. Hence powers through $e-1$ span R over the smaller algebra. Repeat on its list until independent or empty. Finite module generation passes down the tower by multiplying the two finite lists of generators at each step. This constructs A. For any one endomorphism of the finite generating module given by multiplication by r_j, write its matrix on a generating list using coefficients in A. Multiplying the resulting equations by the adjugate of $r_j I$ minus that matrix shows its monic characteristic polynomial kills all generators, hence kills 1. This is the claimed equation.

The monomials of degree at most m in t are independent and lie in $R_{\le m}$. For the upper bound choose finite module generators including one, and express r_j times each generator in those generators with polynomial coefficients in t of degrees at most a common integer $b \ge 1$. Every word of length at most m applied to 1 then uses coefficients of degrees at most bm, by induction. There are at most $(bm+1)^d$ monomials in that range and at least $(\lfloor m/d \rfloor + 1)^d$ independent monomials in the lower bound if $d > 0$, by bounding each exponent separately. When $d = 0$ the entire ring is finite dimensional and contains one. This proves both inequalities. □

Lemma 258.2. *Let b be a nonzero polynomial on $\mathbb{C}^d$, $d \ge 1$. A holomorphic function on $U_b = \{t : b(t) \ne 0\}$ bounded there by $C(1 + \|t\|)^m$, m a nonnegative integer, extends uniquely to a polynomial of degree at most m. Also U_b is connected.*

Proof. Any two distinct points of U_b lie on an affine complex line where b has finitely many zeros, since its values at the endpoints are nonzero. In that line one can join the two points by a polygonal path avoiding the finite set, proving connectedness. At a point outside U_b choose one complex line direction where the polynomial on the line is not identically zero, by joining it to a point of U_b. Use coordinates (v, z) in that direction, with the given point at the origin. On some circle $|z| = \rho > 0$ at $v = 0$ the polynomial has no zero. The same holds uniformly for all sufficiently small v by continuity and compactness. Use the Cauchy integral of the function on this circle with kernel $1/(2\pi i(\zeta - z))$ for $|z| < \rho$. It is holomorphic in all the variables by the local Cauchy formula of Lemma A.3, applied successively in the coordinate variables on smaller closed polydiscs. For fixed v the original function is bounded near its finitely many missing points in the disc. A bounded holomorphic function at a puncture extends: multiply it by $(z - a)^2$, extend by zero at a, differentiate there by the difference quotient to obtain zero, and divide its Taylor series back. Thus the integral agrees with the original function wherever defined. These local extensions glue by their agreement on the dense open complement and continuity. The global polynomial growth bound also extends by continuity. On polydiscs of radius T around zero the repeated Cauchy coefficient bound is $C(1+\sqrt{d}\,T)^m T^{-|\beta|}$ for a coefficient of multi-index β. Letting T tend to infinity kills all coefficients of total degree greater than m, proving the conclusion and uniqueness. □

Proposition 258.3. *Use R, A, t from Lemma 258.1 with $d \geq 1$, and let $Z \subset \mathbb{C}^N$ be the common zero set of the polynomial relations among the r_j. Evaluation gives $Z = \mathrm{Hom}_{\mathbb{C}\text{-}alg}(R, \mathbb{C})$ in the coordinates r_j. The map $t : Z \to \mathbb{C}^d$ has finite fibers and bounded inverse image over each bounded subset. Given a particular fiber, there is a complex linear combination l of the r_j separating its distinct points, a monic $P(t, T) \in A[T]$ irreducible over $\mathbb{C}(t_1, \ldots, t_d)$, and a nonzero $b \in A$ such that*

$$Z|_{U_b} \longrightarrow \{(t, \lambda) : t \in U_b,\ P(t, \lambda) = 0\}, \qquad z \longmapsto (t(z), l(z))$$

is bijective with coordinates of its inverse polynomial in $\lambda, t, 1/b(t)$. The roots on U_b are all simple and give local holomorphic graphs. One has $P(t(z), l(z)) = 0$ on all of Z.

Proof. The evaluation description follows from the presentation by generators and relations. The monic equations of Lemma 258.1 give finitely many choices for every coordinate over fixed t. Roots of $T^s + \sum_{j<s} a_j T^j$ have magnitude at most $1 + \max_j |a_j|$, by dividing by $|T|^s$ and summing a geometric series if this bound is exceeded. Thus coordinates remain bounded on bounded t-sets, and individually satisfy a polynomial growth bound there and at infinity.

Put $k = \mathbb{C}(t_1, \ldots, t_d)$. Inverting nonzero elements of A in R gives a finite-dimensional k-domain, hence a field L: multiplication by a nonzero member is injective and therefore surjective. In characteristic zero its finite extensions are separable by the polynomial derivative. There are $[L : k]$ embeddings into an algebraic closure, by extending over each of the finitely many generators and counting the distinct roots of successive minimal polynomials, as in Lemma 82.2. Choose a constant linear combination l of the coordinate generators taking different values under each pair of embeddings and at each pair of distinct points of the given finite fiber. Each equality cuts out a proper linear condition: the generators separate the two homomorphisms in question. Finitely many such conditions do not exhaust the constant tuples, by the polynomial nonvanishing argument in the normalization lemma, also for polynomials with coefficients in a larger field. Then $L = k(l)$ since the minimal polynomial of l has at least $[L : k]$ roots. Denote that monic polynomial by P.

Express every r_j as a polynomial of degree below $[L : k]$ in l over k. Invert one nonzero $b \in A$ clearing the denominators of these expressions, of P, and of a Bezout identity between P and its derivative. The resulting presentation is $R[1/b] = A[1/b][T]/(P)$: evaluation at l is onto by the expressions, and monic division followed by independence of the lower powers

of l gives its kernel. Over U_b this is exactly the asserted bijection with simple roots. The holomorphic inverse function argument of Theorem 257.2, applied to $(t,T) \mapsto (t, P(t,T))$, makes each simple root a local holomorphic graph. All roots meet the polynomial growth bound in t from the coordinate bound, because l is their fixed linear combination of coordinates of points of Z. Thus the coefficients of P, the elementary symmetric functions of its roots, are polynomially bounded on U_b. They are regular there by construction. Applying Lemma 258.2 to each gives polynomials in A, equal as rational functions to their original values. The equation $P(t,l) = 0$ holds in L and hence in its subring R, so also at every point of Z. □

259. Projective algebraicity from the growth of sections

Theorem 259.1. *Let a connected compact complex manifold X of dimension $d \geq 1$ be holomorphically embedded in $\mathbf{P}^N(\mathbb{C})$ by sections $S_0, \ldots, S_N$ of a holomorphic line bundle. Suppose, for every $m \geq 1$, the span of the products of m of these sections has dimension at most $C(m+1)^d$ for a fixed C. Then the image is exactly the common zero set of a finite list of homogeneous complex polynomials. Its homogeneous ideal of relations is prime.*

Proof. In an affine chart $S_j \neq 0$ which meets X, use ratios $r_i = S_i/S_j$, $i \neq j$, and set $R = \mathbb{C}[(r_i)_i]$ as an algebra of functions there. A section of a holomorphic line bundle that vanishes on an open set of X vanishes everywhere: in trivializing coordinates use the identity theorem on complex lines as in Proposition 257.1, continuing over intersecting coordinate balls by connectedness. Thus the chart is dense in X, and R is a domain: if the product of two ratio polynomials is zero, multiply by a power of S_j to obtain an identity of products of two global sections. Where one is nonzero the other vanishes on an open set. Multiplication by S_j^m identifies $R_{\leq m}$ with the span of homogeneous degree-m coordinate sections, since every monomial can be padded by S_j. On the other hand, at a point in this chart some d of the ratios have independent differentials by the embedding. Their joint values contain a complex open set by the local inverse proof of Theorem 257.2. No nonzero polynomial can vanish there, so their monomials of degree at most m give a lower bound of order $(m+1)^d$. Lemma 258.1 therefore gives exactly d independent base coordinates t.

Let $Y \subseteq \mathbb{C}^N$ be this affine part of the image, and Z the common zero set of its affine polynomial relations. The set Y is closed in the affine chart since X is compact and projective space is Hausdorff. We prove $Z = Y$. Fix a point $z_0 \in Z$. Apply Proposition 258.3 with its fiber $t = t(z_0)$ distinguished. Use the resulting data l, P, b. Over U_b, Z is locally the union of disjoint graphs over an open ball, one for each simple root of P, with holomorphic coordinates in both directions. The intersection with Y is relatively closed and open there. Indeed Y is an embedded manifold of the same dimension as each graph, so at any common point its inclusion into the graph has full rank and is open by the same local inverse argument. There is a point of Y over U_b because $b(t) \neq 0$ in the algebra of functions R.

For each $t \in U_b$, multiply $T - \lambda$ over the sheets at t lying in Y, with λ their l-coordinates. The sheets admitted are locally constant in every evenly covered ball, by the preceding open and closed property. Their number is positive and constant since U_b is connected by Lemma 258.2. The coefficients of this monic polynomial are thus holomorphic on U_b. They have polynomial growth since all the roots have the bound in Proposition 258.3. The same two facts hold for the product over the omitted sheets, with an empty product equal to one. Both products therefore have polynomial coefficients in t by Lemma 258.2. Their product equals P on U_b and hence over $\mathbb{C}(t)$. Irreducibility of P forces the first, nonconstant factor in T, to have the entire degree. Every point of Z over U_b belongs to Y.

To include z_0, take $t_j \in U_b$ tending to $t(z_0)$, possible since the zero set of a nonzero polynomial has no interior. One can choose roots λ_j of $P(t_j, T)$ tending to $l(z_0)$. In fact this target is a root at the limit by the preceding proposition. If along a subsequence all roots stayed at positive distance from it, their ordered tuples have a further convergent

subsequence by boundedness of roots of a monic polynomial with bounded coefficients. Multiplication of the limiting linear factors would give $P(t(z_0), T)$ with no such root. Over U_b each chosen root gives a point $z_j \in Y$. Its coordinates are bounded by Proposition 258.3. Any subsequential limit is in Z by its polynomial equations and has $t = t(z_0)$, $l = l(z_0)$. The choice of l separates points of that fiber, so the limit is z_0. Closedness of Y in the affine chart gives $z_0 \in Y$.

Any affine relation on Y homogenizes using S_j to a relation on all of X by the identity theorem above. Conversely every homogeneous relation restricts to an affine one. Charts with S_j identically zero are excluded by that coordinate's own homogeneous relation. This proves the asserted description of the projective zero set. To see primeness, form the graded ring of the coordinate sections with their degrees. A product of nonzero homogeneous sections is nonzero by the identity theorem. The highest nonzero homogeneous terms of two finite sums therefore have nonzero product as well. The homogeneous relation ideal, the kernel of the map to this graded ring, is prime.

Finally order monomials by total degree and then lexicographically, taking the largest monomial of each nonzero homogeneous relation. The exponents so obtained are closed under addition by nonnegative vectors. By the finite generation argument of Lemma 237.1, finitely many of them generate this set; choose homogeneous relations with those leading monomials, of coefficient one. From any homogeneous relation subtract an appropriate scalar and monomial multiple of one of the chosen relations to cancel its leading term. Each step decreases the leading monomial among finitely many of that degree, so it ends at zero. Thus the finite list generates the homogeneous ideal. □

Corollary 259.2. *Each theta-embedded complex fiber of Theorem 257.2 is the zero set of finitely many homogeneous polynomials with complex coefficients and is irreducible. In each of its nonempty affine coordinate charts its coordinate ring is a finite algebra over a polynomial ring in $g = fn$ variables.*

Proof. The torus is connected as the continuous image of its complex vector space and is compact by Theorem 252.3. A product of m coordinate sections belongs to the degree-m theta space by multiplication of their automorphy factors. That space has dimension $m^g h^n$ by Theorem 256.3. Hence Theorem 259.1 applies and gives both the finite list and the finite projection in each chart. For irreducibility, if two proper projective algebraic subsets covered the image, choose a homogeneous polynomial for each subset which is nonzero at some point of the image and vanishes on that subset. Their product would vanish identically, contradicting primeness. □

260. Finite jets and the algebraic graphs of the torus operations

Lemma 260.1. *Let X be a compact complex manifold of dimension $d \geq 1$ and L a holomorphic line bundle on it. There are finitely many points $x_1, \ldots, x_s$, trivializations near those points, and $A \geq 1$, such that every holomorphic section of $L^{\otimes m}$, $m \geq 1$, is determined by its Taylor coefficients of degrees below $1 + \lfloor Am \rfloor$ at these points. In particular their space has dimension at most $C(m+1)^d$ for some C independent of m.*

Proof. Take polydisc coordinates and nowhere vanishing local sections of L whose domains contain the closed polydisc of radius three about each center, after rescaling. By compactness choose finitely many for which the open polydiscs of radius one cover. On each such domain take a continuous real function which is one on radius one and zero outside radius two, linear in the maximum coordinate radius between those radii. Extend it by zero to X. Use these functions to form a continuous positive squared norm on L: add the squared absolute values of a fiber vector's local coefficients multiplied by the corresponding functions wherever they are supported. At each point at least one term has positive weight. The norm of each chosen local basis vector on its closed radius-two polydisc has a positive minimum and a

finite maximum. Choose common bounds $0 < c \le C_0$ for all those basis norms. The induced norm of a local section $fe^{\otimes m}$ is $|f|\,\|e\|^m$.

If a nonzero section vanished to order at least k at every chosen center, scale it to have global maximum norm one. Its local coefficient has absolute value at most c^{-m} on radius two. For each point z of radius at most one, the function $\lambda \mapsto f(\lambda z)$ on the radius-two disc has a zero of order at least k at zero. Divide by λ^k and apply the maximum principle on its boundary to obtain $|f(z)| \le 2^{-k} c^{-m}$. The maximum principle here follows from the circle mean in the Cauchy formula of Lemma A.3: an interior maximum of a nonconstant holomorphic function is impossible since equality with its circle average in absolute value forces constancy on every small circle, and hence on a neighborhood. Thus on the covering inner discs the global norm is at most $2^{-k}(C_0/c)^m$. Choose $A \ge 1$ with $A \ge \log(C_0/c)/\log 2$, and $k = 1 + \lfloor Am \rfloor$. This would give a global bound strictly below one. Therefore simultaneous jets of orders below k separate sections. There are at most sk^d coefficients, bounding the number of monomials by allowing each exponent from zero to $k-1$. This proves the dimension bound. □

Corollary 260.2. *The image of every holomorphic embedding of a connected compact complex manifold of positive dimension in projective space is cut out by finitely many homogeneous complex polynomials. If X is such a manifold with a projective embedding and $f : X \to \mathbf{P}^M(\mathbb{C})$ is holomorphic, its graph in the product is cut out by finitely many bihomogeneous equations. This also holds for a product of finitely many projectively embedded connected compact manifolds as source.*

Proof. For a projective embedding, take the pulled-back line bundle whose local degree-one sections in a chart $x_j \neq 0$ are the functions x_i/x_j. On overlaps the changes of trivialization are the nonzero ratios of chart coordinates. The corresponding global sections give the original embedding. Their homogeneous products are sections in the indicated tensor powers. Lemma 260.1 therefore verifies the growth hypothesis of Theorem 259.1.

For two projective spaces use the map $([x],[y]) \mapsto [x_i y_j]_{i,j}$. Its image consists of the nonzero matrices up to scalar with all two-by-two minors zero: any nonzero column spans all columns by those equations. On a patch where $x_i y_j \neq 0$, column and row coordinate ratios recover the inverse holomorphically. Thus it is a holomorphic embedding, and iterations embed any finite product. The graph $x \mapsto (x, f(x))$ is a holomorphic embedding because the first projection recovers its inverse and identifies its differential with a graph. After the product embedding use the first assertion and pull each homogeneous equation back via the products $x_i y_j$. The resulting equations are bihomogeneous and cut out exactly the graph in the product. □

Corollary 260.3. *For any fiber of Theorem 257.2, the graphs of addition, inversion and multiplication by every element of O_E are complex projective algebraic sets. The same holds for the graph of every homomorphism between such fibers induced by a real-linear map intertwining their complex structures and taking their lattices into one another. This applies to the fiber legs of Proposition 254.1.*

Proof. In the period charts addition and inversion lift to vector addition and negation and preserve the lattice equivalence. The integral scalars act complex linearly and preserve the lattice by Proposition 252.2. An intertwining real-linear map is complex linear in the two period coordinates, so descends holomorphically precisely when it takes the source lattice into the target. Apply Corollary 260.2 to these maps and to the product torus as source for addition. For the indicated legs the lattice inclusion and intertwining were proved in Proposition 254.1. □

261. The canonical quadratic multiplier on a period fiber

Definition 261.1. Fix a finite frame x, its signed matrix A and a point of the ball. Use $E_0 = E_A$ and write $H(v,w) = E_0(v, I_J w) + iE_0(v,w)$, the positive Hermitian form of

Proposition 256.2. Put $\beta = (1+\sqrt{-d})/2 \in O_E$ as in Lemma 236.1. For $t \in V_{\mathbb{R}}$ regarded as a point of the complex vector space with structure I_J, define

$$\eta_\beta(t) = \exp\big(\pi i E_0(\beta t, t)\big),$$
$$(W_t S)(z) = \exp\big(-\pi H(t,z) - \tfrac{\pi}{2} H(t,t)\big)\, S(z+t), \qquad U_t = \eta_\beta(t)^{-1} W_t.$$

The canonical section space consists of the entire functions S with $U_\lambda S = S$ for every $\lambda \in \Lambda_x$. We use z, t either as vectors or in the complex basis of Proposition 256.2.

Lemma 261.2. *For $t, s \in V_{\mathbb{R}}$ and entire functions,*

$$W_t W_s = e^{\pi i E_0(t,s)} W_{t+s}, \qquad U_t U_s = e^{2\pi i E_0(\beta t, s)} U_{t+s}.$$

For t in the dual lattice $\Lambda_x^\# = \{t : E_0(t, \Lambda_x) \subseteq \mathbb{Z}\}$, U_t preserves the canonical section space and there depends only on $t \bmod \Lambda_x$. On that space $U_t U_s = e^{2\pi i E_0(t,s)} U_s U_t$ for dual s, t.

Proof. The alternating trace expression of Proposition 253.3 gives $E_0(\beta t, s) = E_0(t, \bar\beta s)$, and $\beta + \bar\beta = 1$. These imply $E_0(\beta t, s) + E_0(t, \beta s) = E_0(t, s)$. In the product of the two W-factors the excess exponent over the W_{t+s}-factor is $\pi(H(t,s) - H(s,t))/2 = \pi i E_0(t,s)$. For the U-factors this is increased by $\pi i(E_0(\beta t, s) + E_0(\beta s, t))$ on expanding $\eta_\beta(t+s)/(\eta_\beta(t)\eta_\beta(s))$. The adjoint identity gives the stated exponent. Interchanging s, t in this formula gives the commutation law. Since $\beta, \bar\beta$ preserve Λ_x, the numbers $E_0(\beta\lambda, t)$ and $E_0(\beta t, \lambda)$ are integers for every lattice λ and dual t. Thus $U_\lambda U_t = U_t U_\lambda = U_{t+\lambda}$, proving both assertions on sections. □

Theorem 261.3. *Write D, Ω for the data of Proposition 256.2, and $M = (\Im\Omega)^{-1}$. Choose real columns a, c with entries in $\{0, \frac{1}{2}\}$ by*

$$e^{2\pi i a_j} = \eta_\beta(e_j), \qquad e^{2\pi i c_j} = \eta_\beta(f_j)$$

where the expressions on the right mean the multiplier in Definition 261.1. The columns a, c are constant throughout the ball. Put $d_0 = \Omega a - D^{-1} c$. A basis of the canonical space in the complex coordinates z is

$$S_r(z) = \exp\big(\tfrac{\pi}{2} z^{\mathsf T} M z + 2\pi i a^{\mathsf T} z\big)\Theta_{1,r}(z + d_0), \qquad r \in \mathbb{Z}^g / D\mathbb{Z}^g.$$

In particular the canonical space has dimension h^n, and its projective evaluations embed the fibers holomorphically over the ball.

Proof. On a lattice vector λ, $E_0(\beta\lambda, \lambda)$ is an integer since $\beta\Lambda_x \subseteq \Lambda_x$. The multiplier is therefore a sign, determined by the fixed real form and lattice endomorphism independent of b. By Lemma 261.2, the U_λ on lattice vectors multiply by addition, so it suffices to impose the conditions on e_j, f_j. In the complex e-basis one has $H(t,z) = \bar t^{\mathsf T} M z$. For a section write $S(z) = G(z)h(z)$ with $G(z) = \exp(\pi z^{\mathsf T} M z/2 + 2\pi i a^{\mathsf T} z)$. For a general vector $t = u + \Omega v$ with $u, v \in \mathbb{R}^g$, expansion of the quadratic gives

$$\frac{(W_t S)(z)}{G(z)} = \exp\big(2\pi i v^{\mathsf T} z + \pi i v^{\mathsf T} t + 2\pi i a^{\mathsf T} t\big)\, h(z+t).$$

Indeed $(t - \bar t)^{\mathsf T} M = 2iv^{\mathsf T}$ by the formula for M. At e_j the equation $W_{e_j} S = \eta_\beta(e_j) S$ is just periodicity of h in coordinate j. At f_j one has $v = D\mathbf{e}_j$, $u = 0$, with $\mathbf{e}_j$ the j-th coordinate column. The multiplier of $h(z + \Omega v)$ relative to $h(z)$ becomes

$$\exp\big(-\pi i v^{\mathsf T}\Omega v - 2\pi i v^{\mathsf T}(z + d_0)\big),$$

since $v^{\mathsf T} d_0 = a^{\mathsf T}\Omega v - c_j$. Hence $h(z) = H_0(z + d_0)$ with H_0 satisfying exactly the standard degree-one theta equations of Theorem 256.3. The converse follows from the same equations. That theorem gives the basis and dimension. In projective coordinates the common nonzero factor $G(z)$ cancels. The translation by d_0 is holomorphic in b, and is a biholomorphic translation on each torus. The embedding result now follows from Theorem 257.2. □

262. Descent of the canonical theta coordinates to the level quotient

Theorem 262.1. *In the symplectic basis of Lemma 256.1 the dual lattice vectors have complex coordinates $t = u + \Omega v$ where $u \in D^{-1}\mathbb{Z}^g$, $v \in \mathbb{Z}^g$. With the canonical basis of Theorem 261.3 one has*

$$U_t S_r = \eta_\beta(t)^{-1} \exp\Big(\pi i v^{\mathsf{T}} u + 2\pi i\big(a^{\mathsf{T}} u + v^{\mathsf{T}} D^{-1} c + r^{\mathsf{T}} u\big)\Big)\, S_{r+v}.$$

The matrices in this basis of the translations indexed by $\Lambda_x^{\#}/\Lambda_x$ are independent of the period point and form a basis of the full endomorphism algebra.

Proof. A vector with real coordinates $\sum e_j u_j + \sum f_j s_j$ is dual exactly if Du, Ds are integral by the paired basis. Its complex coordinate is $u + \Omega D s$, proving the description with $v = Ds$. In particular t, as a real vector, and the value of its multiplier $\eta_\beta(t) = \exp(\pi i E_0(\beta t, t))$ are independent of the complex period when u, v are fixed. The Fourier summation in Theorem 256.3 gives

$$\Theta_{1,r}(z + d_0 + u + \Omega v) = e^{2\pi i r^{\mathsf{T}} u - \pi i v^{\mathsf{T}} \Omega v - 2\pi i v^{\mathsf{T}}(z + d_0)} \Theta_{1,r+v}(z + d_0).$$

Indeed for each frequency $m \in r + D\mathbb{Z}^g$ one has $e^{2\pi i m^{\mathsf{T}} u} = e^{2\pi i r^{\mathsf{T}} u}$, and completing the square replaces m by $m + v$. Multiply by the Gaussian translation factor in the proof of Theorem 261.3 and by $\eta_\beta(t)^{-1}$. Since $d_0 = \Omega a - D^{-1} c$, the z, Ω terms in the scalar cancel and give exactly the stated matrix. The formula respects changing r by $D\mathbb{Z}^g$, and the class of t suffices by Lemma 261.2.

For a nonzero class t there exists a class s with $E_0(t, s) \notin \mathbb{Z}$: the dual of the dual in the paired basis is the original lattice, as follows by testing at the vectors $e_j/d_j, f_j/d_j$. The commutator formula of Lemma 261.2 implies that conjugating U_t by U_s multiplies it by a scalar different from one. All these operators are invertible by the same formula with the negative shift. Trace invariance under conjugation gives $\operatorname{tr}(U_t) = 0$ off the zero class. In any linear relation among the class representatives multiply by U_{-t} and take the trace. Only the coefficient at t can contribute, with a nonzero scalar times the dimension. Thus every coefficient vanishes. The number of these operators is h^{2n} by Proposition 253.3, equal to the square of the dimension in Theorem 261.3. They are a basis. □

Corollary 262.2. *Fix a representative xC of one rational orbit of finite frames of the signed complex family. Use one symplectic basis on Λ_x throughout its period ball. The canonical projective evaluation*

$$(b, P) \longmapsto [S_r(P)]_r$$

descends to a holomorphic map from this component of $\mathcal{X}(C, 1)$ to $\mathbf{P}^{h^n - 1}(\mathbb{C})$, giving a projective embedding on each torus. If M is any positive integer with $\mathfrak{n} \in MO_E$, then the points of $M^{-1}\Lambda_x/\Lambda_x$ give individual holomorphic sections over that base component. The projective coordinates of each such section descend there as well.

Proof. Let $y \in \Gamma$ with $k = x^{-1} y x \in C$. Theorem 252.3 gives $y^* A y = A$. The rational map commutes with β, preserves E_0 by its trace formula, and carries the complex structure at b to the one at yb. It carries the positive Hermitian form H to the corresponding form, by the defining formula in Definition 261.1. Thus pullback of entire functions by this complex-linear map carries the canonical space at yb isomorphically to the one at b, and intertwines the translations at yt and t by their explicit Gaussian formulas.

Here y acts trivially on $\Lambda_x^{\#}/\Lambda_x$. Indeed $\Lambda_x^{\#} \subseteq h^{-1}\Lambda_x$ by Proposition 253.3, and $\mathfrak{n} \in hO_E$ by Proposition 237.2 and Corollary 239.4. Since $k - I \in hM_n(\widehat{O})$, local frame coordinates give $(y - I)h^{-1}\Lambda_x \subseteq \Lambda_x$. In the chosen canonical bases, the pullback matrix from yb to b commutes with every translation matrix by Theorem 262.1. Its commutant is scalar: commuting with the full matrix algebra entails commuting with each diagonal coordinate projection and then with each elementary matrix between two coordinates. Thus the evaluation lists at (yb, yP) and (b, P) differ by a common nonzero factor. Changing the

torus representative likewise multiplies all sections by the same factor by Definition 261.1. Their projective evaluation is invariant and holomorphic by Theorem 261.3 and the local quotient charts in Theorem 252.3.

For the second assertion replace h by M in the local inclusion. This shows y fixes every labelled point mod Λ_x. In the ball charts the section corresponding to a fixed real vector v is the holomorphic section $b \mapsto Q_b(v)$ modulo its periods, by Proposition 252.2. Hence it descends, as do its projective coordinates. □

263. Torsion evaluations as coordinates on a compact period set

Definition 263.1. Fix x and the symplectic basis of Corollary 262.2. Before taking the rational quotient let $T = V_{\mathbb{R}}/\Lambda_x$ with its fixed real differentiable structure and let $F_b : T \to \mathbf{P}^{h^n-1}(\mathbb{C})$ be its canonical projective theta embedding at $b \in \mathbb{B}$. Set $T[m] = m^{-1}\Lambda_x/\Lambda_x$. Use any ordered integer basis to identify $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$, with the distance given by the least Euclidean length of a lift of the difference. On projective space use the distance $\|P(z) - P(z')\|$ with the Euclidean matrix norm, where $P([u]) = uu^*/(u^*u)$. This is independent of the nonzero vector representative u. The matrix recovers each affine coordinate u_i/u_j where $u_j \neq 0$ by dividing its ij-entry by its jj-entry. In particular this gives a smooth embedding of projective space into a real matrix space.

Proposition 263.2. *For a compact $K \subseteq \mathbb{B}$, there is $m \geq 1$ such that*

$$b \longmapsto (F_b(t))_{t \in T[m]}$$

is injective on K and its derivative is injective at all points of K. It is holomorphic on $\mathbb{B}$. Also there are $0 < c_K \leq C_K < \infty$ with

$$c_K \operatorname{dist}(v, w) \leq \operatorname{dist}(F_b(v), F_b(w)) \leq C_K \operatorname{dist}(v, w) \qquad (b \in K,\ v, w \in T).$$

Proof. The period coordinates of Proposition 252.2 and the theta functions of Theorem 261.3 show that $(b, v) \mapsto F_b(v)$ is smooth in real coordinates and holomorphic in b for fixed v. Indeed fixed real vectors have holomorphic periods, and in projective coordinates only the translated Fourier sections remain. In the real torus $\bigcup_m T[m]$ is dense, by approximating real basis coordinates by rational numbers. If $F_b(t) = F_{b'}(t)$ at all torsion points, continuity gives equality at all v. As both maps are holomorphic embeddings for their respective complex structures by Theorem 261.3, the identity of the real torus is biholomorphic from one structure to the other. Differentiating gives $I_b = I_{b'}$. Thus the positive lines, and hence their coordinates b, b', are identical by Proposition 252.2.

Suppose a real tangent vector w at b gives zero derivative of $F_b(t)$ for every torsion t. Continuity of the derivative gives $\partial_w F_b(v) = 0$ at all v. In a local complex target chart the differential in v satisfies

$$d_v F_b \, I_b = i \, d_v F_b,$$

since the embedding of that torus is holomorphic. In these coordinates $\partial_w(d_v F_b) = d_v(\partial_w F_b) = 0$ at the chosen b since the two-variable function is smooth and its w-derivative vanishes identically in v. Differentiating the equation now gives $d_v F_b \, \partial_w I_b = 0$. Fiber immersion implies $\partial_w I_b = 0$. Thus $\partial_w J_b = 0$ at the distinguished embedding. Differentiate $J_b(1, \bar{b}) = (1, \bar{b})$ along w. This puts $(0, \bar{w})$ in the positive line of $(1, \bar{b})$, forcing $w = 0$. For $n = 1$ the parameter space has zero-dimensional tangent space and a single point, giving the same two assertions.

At any b a finite number of torsion evaluations suffice for an injective derivative: while a common kernel of the chosen real differentials has positive dimension, the preceding assertion supplies a further differential reducing that dimension. By the holomorphic inverse argument of Theorem 257.2, these evaluations are locally injective on a neighborhood where the derivative keeps full rank. Finitely many such neighborhoods U_a cover K. Outside $\bigcup_a U_a \times U_a$, the pairs of $K \times K$ form a compact set of distinct points. Each pair is distinguished by some torsion evaluation, and that same evaluation distinguishes pairs on a

neighborhood by continuity. A finite subcover supplies a combined finite list of tests also separating those pairs. Take m a common multiple of the orders of the tests.

For the distance bounds compose F_b with the real matrix map P of Definition 263.1. The derivative in the torus directions is injective, and on the compact set of $b \in K$, torus points and unit tangent vectors its norm has positive minimum and finite maximum. Integration along a least-length lifted segment gives the upper bound. The derivative is uniformly continuous in the torus variable on this compact set. Thus for sufficiently short segments, integration of its difference from the initial derivative gives an error at most half of that positive minimum times the segment length. Subtracting the error gives the lower bound at short distances. At distances bounded below, the compact set of triples has a positive minimum image distance by fiber injectivity; divide this by the finite diameter of the real torus. Taking the smaller lower bound proves the assertion. □

264. A finite level embedding by labelled torsion points

Lemma 264.1. *For a component representative xC let $\Gamma_x = \{y \in \Gamma : x^{-1}yx \in C\}$, acting on Λ_x and the ball. In an ordered basis of the lattice write $D_1 = 2g$ for its rank. Fix a compact $K \subset \mathbb{B}$. There exist a finite subset $S_0 \subset \Gamma_x$ and positive integer m_0 such that whenever m is a multiple of m_0 and*

$$F_b(t) = F_{b'}(\gamma t) \quad (t \in T[m]), \qquad b, b' \in K, \quad \gamma \in \Gamma_x,$$

the action of γ on $T[m]$ is that of a member of S_0.

Proof. Write C_1, c_1 for the distance constants in Proposition 263.2. Compare the equality at zero and at each $t = \mathbf{e}_j/m$, where the $\mathbf{e}_j$ are the coordinate lattice basis elements. Their images at b have distance at most C_1/m. Thus $\gamma\mathbf{e}_j/m$ has torus distance at most $C_1/(c_1 m)$ from zero, by the lower bound at b'. Choose a shortest lift in $\mathbb{R}^{D_1}$ and multiply it by m. This gives an integral matrix A_0, congruent to γ modulo m, whose columns have Euclidean lengths at most C_1/c_1. Since γ permutes $T[m]$, the same argument with b, b' interchanged gives a matrix $B_0 \equiv \gamma^{-1} \bmod m$ with the same bound. There are finitely many such integral pairs.

Each γ commutes with multiplication by the chosen integral basis of O_E and preserves the integral Gram matrix of the rational trace form B_A, as proved in Theorem 252.3 and Lemma 253.1. For every possible pair A_0, B_0, take all entries of $A_0B_0 - I, B_0A_0 - I$, the commutators of A_0 with these multiplication matrices, and $A_0^{\mathsf{T}} B_A A_0 - B_A$ in the chosen basis. Their absolute values have a common finite bound independent of m. For m larger than it all these entries vanish whenever the hypotheses hold, because reduction modulo m gave exactly zero. Thus A_0 is an invertible integral matrix commuting with O_E and preserving B_A. Rational extension then gives an E-linear isometry of the Hermitian A: varying the second argument through all E-multiples and taking traces recovers its values, as in Proposition 246.1.

Choose m_0 also divisible by the positive integer $|N_{E/\mathbb{Q}}\mathfrak{n}|$ and larger than the bound. Multiplication by this integer maps O_E into $\mathfrak{n}O_E$ by the adjugate of multiplication by $\mathfrak{n}$, whose determinant is its norm. Hence $m\Lambda_x \subseteq \mathfrak{n}\Lambda_x$. One has $(\gamma - I)\Lambda_x \subseteq \mathfrak{n}\Lambda_x$ by completion in the frame x. Consequently the same inclusion holds for $A_0 - I$ since $A_0 \equiv \gamma \bmod m$. In frame coordinates $x^{-1}A_0x$ is integral invertible by its action on local lattices, unitary by $A_0^* A A_0 = A$ and $x^* A x = \varepsilon I$, and congruent to I mod $\mathfrak{n}$. It lies in C, so $A_0 \in \Gamma_x$. The finite collection of members of Γ_x with the specified bound on columns gives S_0. □

Theorem 264.2. *Let $\Gamma_x(m)$ be the kernel of the action of Γ_x on $T[m]$. For every multiple of some positive integer m_x, the map*

$$\Gamma_x(m)\backslash\mathbb{B} \longrightarrow \left(\mathbf{P}^{h^n-1}(\mathbb{C})\right)^{T[m]}, \qquad [b] \longmapsto (F_b(t))_{t \in T[m]}$$

is a closed holomorphic embedding. The group $\Gamma_x/\Gamma_x(m)$ permutes the projective factors by its action on the labelled points.

Proof. The map is well defined and equivariant for the indicated permutations because $F_{\gamma b}(\gamma t) = F_b(t)$ by Corollary 262.2. It is holomorphic by Proposition 263.2. The coarse quotient $\Gamma_x\backslash\mathbb{B}$ is the component at xC in Theorem 252.3 with no marked torus coordinates. It is compact by Theorem 253.2, since these components form a disjoint open union and are therefore closed. It has a compact set $K \subset \mathbb{B}$ meeting all orbits: images of relatively compact ball neighborhoods cover the quotient and a finite subcover suffices. Use Lemma 264.1 for this K. Apply Proposition 263.2 to the compact set $K' = K \cup \bigcup_{s\in S_0} s^{-1}K$. Choose m_x a common multiple of the resulting testing integer and m_0 in the preceding lemma. The whole list at m then separates all points of K' and is immersive there. Equivariance and the coverage by K give immersion everywhere.

Suppose b_1, b_2 give the same labelled evaluations. Choose $u_1, u_2 \in \Gamma_x$ with $b = u_1 b_1,\ b' = u_2 b_2$ in K. Equivariance yields $F_b(t) = F_{b'}(\gamma t)$ for all $t \in T[m]$ and $\gamma = u_2 u_1^{-1}$. The preceding lemma gives $s \in S_0$ with the same action as γ on these labels. Therefore $F_b(t) = F_{s^{-1}b'}(t)$ for all labels. Both base points belong to K', so $b = s^{-1}b'$. It follows that $b_2 = u_2^{-1} s u_1 b_1$, with $u_2^{-1} s u_1 \in \Gamma_x(m)$ by its label action. This proves injectivity.

The finer quotient is compact because $\Gamma_x(m)$ has finite index, being the kernel of a permutation action on a finite set. The translates of K by coset representatives therefore exhaust it after projection. Both groups act freely with locally disjoint charts by Theorem 252.3. A holomorphic injective immersion of this compact quotient into the Hausdorff target is a homeomorphism onto a closed image, and is a local holomorphic graph by the inverse argument of Theorem 257.2. This proves the result. □

265. Projective equations for the complex level correspondences

Lemma 265.1. *Let a finite group D_2 act freely on a compact complex manifold Y. Suppose $Y \to \mathbf{P}(V)$ is a holomorphic embedding intertwining its action with projective linear transformations. Then Y/D_2 admits a closed holomorphic embedding into a projective space.*

Proof. Identify $v \in V$ with the linear polynomial $\lambda \mapsto \lambda(v)$ on its dual. Choose linear representatives for the projective transformations. At the point represented by $[v]$ send its orbit to the line of the polynomial $\prod_{s\in D_2}(sv)$ in the homogeneous polynomials of degree $|D_2|$ on $V^\vee$. Nonzero factors have nonzero product; rescaling any lifts changes only the overall scalar. Thus this is a well-defined holomorphic projective map on Y, constant on orbits. Equality of two products up to scalar forces equality of their sets of linear factor lines with multiplicities. Indeed a nonzero linear polynomial generates a prime ideal, since after an invertible coordinate change its quotient ring is the polynomial ring in the remaining variables. Comparing factors and cancelling proves the assertion by induction. Hence equality identifies one orbit.

Because the action is free the factor lines at a point are distinct. Differentiate the product in a local representative v of the projective embedding. If the derivative is a scalar multiple of the product, reduce the differentiated equation modulo the ideal of the identity factor v. All terms except its derivative vanish, and none of the other factors vanishes in this integral domain. Therefore the derivative of v is a scalar multiple of v; projective immersion on Y makes the original tangent zero. A free finite action has local quotient charts by shrinking a neighborhood away from its finitely many nonidentity translates. The quotient map is locally biholomorphic, and the descended map is an injective immersion. Compactness and the local inverse argument of Theorem 257.2 prove it is a closed embedding. □

Theorem 265.2. *The complex spaces $\mathcal{X}(C, \nu)$ of Definition 252.1 and $\mathcal{X}(C', \nu)$ for every open finite-index $C' \subset C$ admit closed holomorphic embeddings in projective spaces whose images are cut out by finitely many homogeneous complex polynomials. For all levels and finite supports in Proposition 254.1, the cover map and each of its indicated Hecke legs have graphs cut out by finitely many bihomogeneous complex polynomials.*

Proof. First take the base without the marked torus coordinates and fix a frame xC representing one of its components. Write $\Gamma_{x,C'} = \{z \in \Gamma : x^{-1}zx \in C'\}$ and $\Gamma_x = \Gamma_{x,C}$. Over this component every fine frame can be represented by xkC', $k \in C$, by undoing a rational change; finitely many suffice by the finite index. For each such representative one may substitute xk for x below. The open subgroup C' contains $\{k \in C : k - I \in m_1 M_n(\widehat{O})\}$ for an integer m_1 by the neighborhood argument of Proposition 254.1. For m divisible by m_1, the subgroup $\Gamma_x(m)$ of Theorem 264.2 is contained in $\Gamma_{x,C'}$: its congruence on $T[m]$ means $(z-I)\Lambda_x \subseteq m\Lambda_x$, the displayed matrix congruence upon using the local lattice frames. Take m also divisible by the integer m_x of that theorem. The group $\Gamma_x(m)$ is normal in $\Gamma_{x,C'}$ and the quotient group is finite.

Embed the deep quotient by the labelled theta evaluations of Theorem 264.2 followed by the product embedding of Corollary 260.2. Permuting these factors acts linearly by permuting coordinates in the ambient tensor product. The finite quotient group acts freely on the deep quotient: if an element fixes a deep class, multiplication by an element of $\Gamma_x(m)$ makes it fix a ball point, whose stabilizer in Γ_x is trivial by Theorem 252.3. Lemma 265.1 therefore gives a projective embedding of the base at C', component by component, including $C' = C$.

For ν marked points use the joint map of the base embedding just constructed and the ν corresponding fiber evaluations in Corollary 262.2. They descend for the subgroup $\Gamma_{x,C'}$ and are holomorphic in the family charts. If two points have the same image, the base embedding makes the two bases agree in the quotient; after choosing the same frame and base representative, fiber injectivity identifies each of their marked points. Similarly a tangent vector killed by the joint derivative is vertical by the base embedding, and then each of its vertical components is zero by fiber immersion. The domain is compact. For example, above any fixed component use a compact set of ball representatives for the finite-index subgroup, by the compact-set argument of Theorem 264.2. For each torus factor add a closed real fundamental parallelotope as in Theorem 253.2. There are finitely many components, since the coarse compact base is a disjoint union of open components and the finer frames over each have the finite list given above. Therefore the joint injection is a closed immersion in complex analytic charts by the local inverse argument already used. Compose with the product embedding. If desired a single projective space for all components is obtained by placing their spaces in disjoint projective coordinate subspaces of their direct sum.

Corollary 260.2 applies on each positive-dimensional component and to each holomorphic map in the statement; the zero-dimensional components are individual projective points, cut out by linear equations. Finite unions of the resulting closed sets are cut out by the finite lists of products formed by taking one equation from each list, since any point outside all the sets fails some equation from each list. The same argument applies in the product of the source and target projective spaces to the graphs, giving the bihomogeneous assertion. □

266. Simple algebraic coordinates at a chosen period point

Lemma 266.1. *In Proposition 258.3 let $a \neq 0$ in R. For any point $z_0 \in Z$ and any additional prescribed nonzero polynomial on the base, there exist points $z_j \to z_0$ with $a(z_j) \neq 0$, lying over the complement of that polynomial's zero set and over the simple-sheet locus for a choice of l separating the fiber of z_0. For a finite-dimensional unital complex domain R, every nonzero element is nonzero at its unique complex algebra homomorphism to $\mathbb{C}$.*

Proof. Make the indicated choice of l, P, b in Proposition 258.3. Its fraction field calculation expresses a^{-1} in powers of l with coefficients rational in t. Enlarge b by their nonzero denominators and the extra polynomial. Then a is invertible in $R[1/b]$, hence nonzero on every sheet above U_b. Choose values $t_j \in U_b$ tending to $t(z_0)$. As in the root-limit argument of Theorem 259.1, roots λ_j of the monic $P(t_j, T)$ can be chosen to converge to $l(z_0)$. The sheets yield z_j with these values. All their coordinates are bounded by the same proposition,

and any convergent subsequence has the same unique limit z_0 by separation of the fiber. This gives the claimed approximating points. In dimension zero a finite-dimensional domain over $\mathbb{C}$ is a field by injectivity of multiplication, and its elements are algebraic over $\mathbb{C}$. Hence it is $\mathbb{C}$ by the complex root property in Lemma 16.2. □

Theorem 266.2. *Let Y be a nonempty affine coordinate chart of a connected compact complex manifold holomorphically embedded in projective space, of positive complex dimension d. Use its algebraic coordinate ring R and the identification of Y with its affine zero set in Theorem 259.1 and Corollary 260.2. For every $y_0 \in Y$ there are a finite linear projection $t : Y \to \mathbb{C}^d$, a linear function l and a monic $P(t,T) \in \mathbb{C}[t,T]$ for which $P(t,l) = 0$, $s(y_0) \neq 0$ with $s = P_T(t,l)$ the derivative in T, and*

$$R[1/s] = (\mathbb{C}[t,T]/(P))[1/P_T], \qquad T \mapsto l.$$

Here brackets denote localization, adjoining the reciprocal of the indicated element. In particular the points have algebraically nonsingular charts of dimension d.

Proof. Begin the generator elimination of Lemma 258.1 with the affine coordinate ratios of Y, whose differentials have combined rank d at y_0 by its analytic embedding. At each dependent stage, the number N' of current generators is greater than d: at rank d any d with independent differentials are algebraically independent by the local inverse theorem proved in Theorem 257.2. In the choice of projection along $(c_1, \ldots, c_{N'-1}, 1)$ impose also that this vector lie outside the d-dimensional image of the tangent space under the current differential. There is such a vector satisfying the leading polynomial condition: this tangent image is a proper subspace, so a nonzero linear functional annihilates it, and the product of that functional with the nonzero top homogeneous polynomial is nonzero. The same dehomogenization argument as in the lemma applies. The linear projection with this kernel preserves the tangent rank. At termination the number of coordinates is d by the growth bounds in Theorem 259.1. Thus t is locally biholomorphic at y_0.

Apply Proposition 258.3 with l separating the fiber of $t(y_0)$. The root $l(y_0)$ is simple at this base value. Indeed if it had multiplicity at least two, choose t_j on the simple-sheet locus tending to $t(y_0)$. At least two distinct roots there tend along a subsequence to $l(y_0)$: take a convergent subsequence of all ordered root lists and multiply their limiting linear factors. Points of both sheets are bounded, and their only possible limit is y_0 by separation of the fiber. They would belong eventually to the same neighborhood on which t is injective, contradicting the distinctness of the roots. Hence $P_T(t(y_0), l(y_0)) \neq 0$.

For each affine coordinate r of Y, on the simple-sheet locus U_b of that proposition form the polynomial

$$Q_r(t,T) = \sum_{z\colon t(z)=t} r(z) \prod_{\substack{z'\colon t(z')=t \\ z' \neq z}} (T - l(z')).$$

The coefficients are holomorphic since permutation of the locally holomorphic sheets does not change the sum. They have polynomial growth on U_b since all coordinates and l satisfy such bounds by the proposition. They are therefore polynomials in t by Lemma 258.2. Substitution at $T = l$ gives $Q_r(t,l) = rP_T(t,l)$ above U_b, and on all of Y by continuity and Lemma 266.1 (with $a = 1$ and the original b as additional polynomial). These equalities put every coordinate in the stated localization of $\mathbb{C}[t,l]$. Monic division and independence of the lower powers of l over $\mathbb{C}(t)$ identify $\mathbb{C}[t,l]$ with $\mathbb{C}[t,T]/(P)$, and $t, l \in R$. This proves the ring equality. On this chart there is one equation whose derivative in T is invertible, giving the asserted nonsingular hypersurface coordinates. Explicitly the linearized equation determines the increment of T uniquely from those of t; even over a square-zero ideal the same holds for a correction of a tentative lift by Taylor expansion modulo the square of that ideal. □

267. Algebraic regularity of the holomorphic maps

Lemma 267.1. *Let $r \in \mathbb{C}(t_1, \ldots, t_d)$ be bounded on the complement of finitely many polynomial zero sets in some open ball about t_0, with all excluded polynomials nonzero. It is a quotient of polynomials with denominator nonzero at t_0.*

Proof. The polynomial ring is factorial. For the needed argument, if B is a unique factorization domain, call a polynomial in $B[T]$ primitive when its coefficients have no common prime divisor. Products of primitive polynomials are primitive, since reduction modulo any prime is a product of nonzero polynomials over a domain. Clearing denominators and removing the coefficient gcd therefore identifies factorizations of primitive polynomials with those over the fraction field up to scalar units: the product of the primitive representatives has coefficient gcd one, and a fraction relating two primitive polynomials must have numerator and denominator units by prime divisibility. Factorization with uniqueness over the fraction field follows by Euclidean division and Bezout. Combining with factorization of the coefficients gives unique factorization in $B[T]$. Induction from $\mathbb{C}$ proves the assertion, and unique factorization makes irreducible factors prime.

Write $r = N/D$ with relatively prime numerator and denominator, taking $D = 1$ if $r = 0$. If an irreducible factor g_0 of D vanished at t_0, the domain $\mathbb{C}[t_1, \ldots, t_d]/(g_0)$ would contain a nonzero residue of N. Its complex points are exactly the zeros of g_0 by its single generator relation. By Lemmas 258.1 and 266.1, arbitrarily near t_0 on this hypersurface there is a point with $N \neq 0$, including the zero-dimensional case. At such a point in the ball the denominator vanishes. Approaching it from the dense complement of the excluded sets and of $D = 0$ makes the absolute value of N/D unbounded, a contradiction. Thus $D(t_0) \neq 0$. $\square$

Theorem 267.2. *For a connected compact complex manifold X with projective embedding, every holomorphic map $f : X \to \mathbf{P}^M(\mathbb{C})$ is an algebraic morphism for the reduced algebraic structure of Theorem 259.1. Here algebraic regularity means that locally in affine source and target charts each target coordinate is a quotient of polynomials in the source coordinates with denominator nonvanishing at the given point.*

Proof. For dimension zero X is a point. Otherwise fix y_0 and an affine source chart Y with domain R as in Theorem 266.2. Use the t, l, P chosen there and their fraction fields $k = \mathbb{C}(t) \subseteq K = \operatorname{Frac} R = k(l)$. The fiber over $t_0 = t(y_0)$ is finite by Proposition 258.3. Choose affine target coordinates $h_1, \ldots, h_M$ using a linear denominator nonzero at the images of all points in that fiber. Such a linear form exists by avoiding finitely many proper linear subspaces. Regard the h_j as functions pulled back by f. They are bounded on the entire inverse image in Y of a sufficiently small ball about t_0. Indeed they are holomorphic and bounded on some neighborhood of each fiber point after shrinking it. If points outside these neighborhoods projected to a sequence tending to t_0, the coordinate bounds of Proposition 258.3 and closedness of Y in its affine chart would give a subsequential limit in that fiber and outside the neighborhoods.

First all h_j belong to K. On the affine product chart of the graph let $S = R[h_1, \ldots, h_M]$ be its coordinate ring, viewing the functions on their common domain. Its complex points are exactly that part of the graph by Corollary 260.2. It is a domain by the identity theorem in Theorem 259.1. In the product embedding of the graph all source and target affine coordinates occur as ratios of degree-one sections with a common denominator, by holding the other factor's chart coordinate fixed. The filtered span of words of length at most m in them has dimension $O((m+1)^d)$ by Lemma 260.1, with $d = \dim_{\mathbb{C}} X$, after multiplication by the denominator to power m. Any h_j transcendental over K would give $d+1$ algebraically independent coordinates $t_1, \ldots, t_d, h_j$; their monomials would contradict this bound by the counting argument of Lemma 258.1. Thus $\operatorname{Frac} S/K$ is a finite separable extension. The primitive-element and monic-division proof of Proposition 258.3 applies with R in place of its base algebra: take a linear combination of the h_j separating embeddings, express each h_j in powers of it, and invert one nonzero element of R clearing the coefficients, its minimal

polynomial and the derivative Bezout identity. Over the complement in Y of that element's zero set the fibers of the affine graph then have as many distinct points as the extension degree. The complement is nonempty because R is the algebra of functions on Y. Each fiber of a graph has at most one point, forcing degree one.

Now take $h = h_j \in K$ and form over k the polynomial

$$Q_h(t,T) = \mathrm{Tr}_{K/k}\Big(h\,\frac{P(t,T)-P(t,l)}{T-l}\Big),$$

with trace taken on coefficients. Over the simple sheets, after excluding finitely many polynomial zero sets in the base to clear the denominators in h, this is $\sum_{z:\,t(z)=t} h(z)\prod_{z'\neq z}(T-l(z'))$. In detail, evaluation at the distinct roots of P in its splitting field diagonalizes multiplication on $k[l]$ by the Vandermonde argument of Lemma 82.2. The rational expression for h agrees with the target coordinate wherever its denominator and the target chart denominator are nonzero, by the graph identity. Near t_0 the latter holds for all sheets by the first paragraph, and denominators in R can be inverted on all sheets after inverting a nonzero base polynomial as in Lemma 266.1. Consequently every coefficient of Q_h, a rational function of t, is bounded near t_0 off those zero sets. This uses the uniform bound for h on all these sheets and the bound for l from Proposition 258.3. By Lemma 267.1, these coefficients have polynomial denominators nonzero at t_0. Evaluating the interpolation identity in the field K gives $Q_h(t,l) = hP_T(t,l)$, since exactly the sheet for l contributes there. As $P_T(t,l)$ is nonzero at y_0 by Theorem 266.2, this represents h by an algebraically regular fraction near y_0. It agrees there with the given holomorphic function by continuity from the dense open where the two expressions agree. Applying this to all target coordinates proves the statement. □

Corollary 267.3. *At all levels in Theorem 265.2 the complex spaces are nonsingular projective algebraic varieties component by component. The covers and the Hecke legs are algebraic morphisms over* $\mathbb{C}$. *Each projective period fiber is an abelian variety over* $\mathbb{C}$, *meaning here a connected nonsingular projective variety with commutative algebraic group operations. Its* O_E*-action and the homomorphisms of Corollary 260.3 are algebraic.*

Proof. Use the projective embeddings of Theorem 265.2 and the nonsingular charts of Theorem 266.2, with points as charts in dimension zero. Every map in question is holomorphic by Proposition 254.1 or Corollary 260.3. Apply the preceding theorem on each source component, using the product embedding for addition. The regular maps take values in their specified algebraic targets since the target equations vanish on their images. The group identities hold on the torus by its vector-space quotient construction. □

268. Relative smoothness and finiteness at complex levels

Proposition 268.1. *Let* $f : X \to Y$ *be an algebraic morphism of the nonsingular complex projective varieties in Corollary 267.3. If its complex derivative at a point is onto, it has a smooth algebraic presentation there of relative dimension* $\dim X - \dim Y$. *This means a presentation over an affine chart of* Y *by* k *variables and* r *equations after inverting functions nonzero at the point, with an invertible* r*-rowed square minor of their Jacobian in those variables and* $k - r = \dim X - \dim Y$. *At equal dimensions this is an étale presentation.*

Proof. Use Theorem 266.2 on source and target. Write the source in coordinates $x_1, \dots, x_a, w$ with $G(x,w) = 0,\ G_w \neq 0$, and target in $t_1, \dots, t_b, l$ with $P(t,l) = 0,\ P_l \neq 0$. Denote the regular coordinate expressions of f by $F_i(x,w)$, $L(x,w)$, inverting their denominators on the source. The graph presentation over the target has equations

$$G = 0, \qquad t_i = F_i(x,w)\ (1 \le i \le b), \qquad l = L(x,w),$$

with all the chart denominators inverted. Substituting $t_i = F_i$ makes $P(t,L) = 0$ modulo G, since the regular map takes values in the target. The difference $P(t,l) - P(t,L)$ is $(l - L)$

times a polynomial in t, l, L whose value at the given point is $P_l \neq 0$. Localize also at this polynomial. There $l = L$ follows from the first $b + 1$ equations and the target relation. The Jacobian rows $dG, dF_1, \ldots, dF_b$ in the source coordinates are independent: $G_w \neq 0$ and the last b rows restricted to the tangent kernel of dG are independent by surjectivity of the derivative and the local target coordinates t. One minor is consequently nonzero, hence a unit after a further localization. Clearing unit denominators of the equations preserves this minor's invertibility on their common zero set. This gives the claimed presentation with $k = a + 1, r = b + 1$. If a chart is a point the same proof omits its variables and equations; a map from a point is submersive only to a point. □

Lemma 268.2. *A projective algebraic morphism over $\mathbb{C}$ whose complex fibers are finite is finite. Here projective means that its source is closed in a product of the base with some projective space, followed by projection, and finite means that over an affine neighborhood of every base point its preimage is affine with coordinate ring a finite module over the base ring.*

Proof. We give the homogeneous elimination argument. First, every proper ideal in a finitely generated complex algebra has a complex point annihilating it. Extend it to a maximal ideal by the maximal-element principle of Definition 16.1. The residue field is a finite-type complex domain; apply Lemma 258.1. A field finite as a module over a polynomial subring forces that subring to be a field: for any nonzero a in the subring, a^{-1} is integral by the same finite-module determinant argument. Multiplying its monic equation by the necessary power of a puts a^{-1} in the subring. The polynomial ring must therefore have zero variables. The residue field is then finite over $\mathbb{C}$ and equals $\mathbb{C}$ by Lemma 16.2. Furthermore, if a polynomial p vanishes at all common zeros of an ideal I, adjoining an extra variable v gives $1 \in (I, 1 - vp)$ by this fact. Substituting $v = 1/p$ in a finite expression for 1 and clearing denominators gives $p^a \in I$ for some $a \geq 1$, with the zero polynomial immediate.

Work in an affine chart of the base near y_0, and use fiber coordinates $z_0, \ldots, z_N$ in the projective space, choosing z_0 nonzero at each point of the finite fiber by avoiding finitely many hyperplanes in the dual. The closed source has homogeneous equations in these variables with coefficients in the affine base ring; for the graphs considered here they are the dehomogenized equations of Theorem 265.2. At y_0 and $z_0 = 0$ these equations have no projective common zero. The power containment just proved supplies a positive power of each of $z_1, \ldots, z_N$ in the specialized ideal. In sufficiently large degree m every monomial in these variables is divisible by one of these powers, by the sum of their exponents. Thus homogeneous multiples of finitely many of the specialized equations span the entire space of degree m polynomials in these variables. One can take homogeneous multiples because collecting the degree-m part of any ideal expression preserves its degree-m target.

The coefficient matrix for those multiples has full row rank at y_0. Invert a nonzero maximal minor on the base. Solving by this minor now expresses every such monomial over the base as a combination of the equations evaluated at $z_0 = 0$, in that same degree. It follows that throughout this affine base neighborhood there are no projective points of the source with $z_0 = 0$. It also holds scheme-theoretically: on each chart $z_j = 1,\ j > 0$, the degree-m expression for z_j^m makes the ideal with $z_0 = 0$ the unit ideal. The source over the neighborhood therefore lies in the affine chart $z_0 = 1$, closed there and hence affine. Lifting the same degree-m expressions to the original homogeneous equations, the differences all contain z_0. After dehomogenizing, every degree-m monomial is therefore a linear combination of monomials of smaller total degree in the affine coordinates over the base. Multiplying these reductions and inducting on degree expresses all monomials in those of degree below m. Their finite list generates the coordinate algebra as a module. The case $N = 0$ is a closed immersion and its algebra is a quotient of the base ring, generated as a module by one. This proves finiteness. □

Theorem 268.3. *The projections $\mathcal{X}(C', \nu) \to \mathcal{X}(C', 0)$ are smooth projective algebraic morphisms over $\mathbb{C}$ of relative dimension $\nu f n$. The cover $\pi_m : \mathcal{X}(C_m, \nu) \to \mathcal{X}(C, \nu)$ of*

Proposition 254.1 is finite étale, of degree $[C : C_m]$ on every complex fiber. Each of its unscaled Hecke legs $u_{g,\mathbf{1}}$ is finite étale as well.

Proof. In the charts of Theorem 252.3 the first map is projection from the ball and its torus coordinates to the ball, with surjective derivative and the stated relative dimension. The cover is locally the identity with the fiber count of Proposition 254.1. The unscaled leg is locally the identity in the base and on its vector-space fiber coordinates, with a change from the period lattice to a larger lattice. Thus it too has invertible derivative at each point. A local isomorphism from a compact manifold has finite fibers: each fiber is closed and compact, and is discrete by its local charts, hence finite. All these maps are algebraic by Theorem 267.2. They are projective since their graphs are closed in products with a projective source embedding, by Corollary 260.2. Apply Proposition 268.1 for the smooth and étale presentations, and Lemma 268.2 for finiteness of the last two maps. □

269. Subspace kernels and the pointwise Frobenius polynomial

Lemma 269.1. *Fix a safe place w in Theorem 241.2. Write $O = O_w$, $\pi = \pi_w$, $\kappa = O/\pi O$, $q = |\kappa|$. The summands in $\tau_{w,r}$, $0 \le r \le n$, can be labelled $g_A C$ by the r-dimensional subspaces $A \subseteq \kappa^n$. In finite frame coordinates the w-torsion on a complex torus is naturally the module $\pi^{-1}O^n/O^n$, identified with κ^n by the chosen π. The unscaled leg from x to xg_A kills precisely A on this torsion.*

Proof. Let $d_r = \operatorname{diag}(\pi I_r, I_{n-r})$. Choose $k_A \in \mathrm{GL}_n(O)$ with $\ker(\bar{d}_r \bar{k}_A) = A$, by taking a residue basis adapted to A and lifting an invertible matrix entrywise; its determinant remains a unit. Set $m_A = d_r k_A$ and $a_A = m_A^{\mathsf{T}}$. The lattice $a_A O^n$ lies between πO^n and O^n, with residue subspace $\operatorname{im}(\bar{m}_A^{\mathsf{T}}) = A^{\perp}$ for the standard dot product. This identity follows by pairing against the kernel and comparing dimensions by matrix rank. The lattice is the full preimage of that subspace since it contains πO^n. Every matrix in the double coset of d_r similarly gives the full preimage of a codimension-r subspace, and two matrices give the same lattice exactly when they differ on the right by $\mathrm{GL}_n(O)$. The orthogonal complement labels all such subspaces exactly once, proving the parametrization of the local sum. Put $g_A = s(z_w^{-r}) b_w(a_A)$. The central scalar is supported over the conjugate quadratic place, so at w the inverse of g_A is $a_A^{\mathsf{T}} = m_A$. Distinct labels also give distinct global cosets: equality would give $a_A^{-1} a_{A'} \in \mathrm{GL}_n(O)$ on applying d_w, by Definition 245.1 and integrality at w.

The torsion of $V_{\mathbb{R}}/\Lambda_x$ is E^n/Λ_x, since the lattice spans E^n over $\mathbb{Q}$. Its map $t \mapsto x^{-1}t \bmod \widehat{O}^n$ identifies it with the finite adelic quotient $\mathbb{A}_{E,f}^n/\widehat{O}^n$. The kernel is zero by definition of the lattice. For surjectivity, any target tuple has bounded denominators. After multiplication by x, its class modulo $x\widehat{O}^n$ lies in $N^{-1}x\widehat{O}^n/x\widehat{O}^n$ for an integer $N \ge 1$. This is the quotient of the completions of $N^{-1}\Lambda_x, \Lambda_x$, hence equals their rational quotient by the finite Chinese remainder argument in Lemma 13.4. The quotient splits as the direct sum of its prime components. Killing it by all elements of w leaves just $\pi^{-1}O^n/O^n$ at w: locally w generates (π) there and the unit ideal at every other prime by Proposition 82.3. The map on torsion for the inclusion $\Lambda_x \subseteq \Lambda_{xg_A}$ multiplies these finite coordinates by g_A^{-1}. The class of $\tilde{v}/\pi$ supported at w dies exactly when $m_A \tilde{v} \equiv 0 \bmod \pi$, the defining condition $\bar{v} \in A$. □

Proposition 269.2. *Let V be an n-dimensional vector space over a field of q elements, $n \ge 1$, and fix a line $W \subseteq V$. Let $Z(A)$ be an element of an abelian group B for each subspace $A \subseteq V$. If $Z(A) = Z(A + W)$ for every A, then*

$$\sum_{A \subseteq V} (-1)^{\dim A} q^{\binom{\dim A}{2}} Z(A) = 0.$$

The same implication holds when the value depends only on $A + W'$ for a specified nonzero subspace $W' \subseteq V$.

Proof. Group the sum by $D = A + W$. For fixed D of dimension $r \geq 1$ the unique member containing W is $A = D$. The others are the complements of W in D, of dimension $r - 1$. There are precisely q^{r-1}: relative to one complement U each other one is the graph of a linear map $U \to W$, with independent values on a basis. All the values in this group equal $Z(D)$. Its coefficient is

$$(-1)^r q^{\binom{r}{2}} + q^{r-1}(-1)^{r-1} q^{\binom{r-1}{2}} = 0$$

since $\binom{r}{2} = \binom{r-1}{2} + r - 1$. If $W' \neq 0$, choose any line inside it; adjoining that line leaves the sum with W' unchanged. □

Corollary 269.3. *In the labeling of Lemma 269.1, let X_0, Z_0 be sets, $\phi : Z_0 \to Z_0$ a map and $u_A : X_0 \to Z_0$ maps. If for each $x \in X_0$ there is a line $W_x \subseteq \kappa^n$ such that*

$$\phi^{\,n-\dim A} u_A(x) = \phi^{\,n-\dim(A+W_x)} u_{A+W_x}(x) \qquad (A \subseteq \kappa^n),$$

then, in the free abelian group on Z_0,

$$\sum_{r=0}^{n} (-1)^r q^{\binom{r}{2}} \sum_{\dim A = r} [\phi^{\,n-r} u_A(x)] = 0 \qquad (x \in X_0).$$

In particular this is the pointwise Frobenius congruence for a reduction of the actual legs whenever the displayed hypothesis is established there with ϕ the q-power Frobenius on geometric point coordinates.

Proof. The free abelian group on a set is the group of finite integer-valued formal sums of its points, with $[z]$ denoting its basis element at z. Apply Proposition 269.2 separately at each x to $Z(A) = [\phi^{n-\dim A} u_A(x)]$, and collect summands by dimension. The inner indexing agrees with all summands of the safe spherical operator by Lemma 269.1. The coefficients and exponents are those of the operator polynomial in Corollary 235.5. □

270. Fields of definition for the projective Hecke diagrams

Lemma 270.1. *Fix ordered homogeneous coordinates on a projective space over $\mathbb{C}$, a closed algebraic set Z, and a subfield $k \subseteq \mathbb{C}$. Take the homogeneous vanishing ideal of Z, that is, the direct sum of the spaces of homogeneous forms vanishing on it. The following conditions are equivalent.*

(1) *The ideal is generated over $\mathbb{C}$ by homogeneous polynomials with coefficients in k.*
(2) *Coordinatewise application of every $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ preserves Z as a set.*

In the chosen coordinates there is a smallest subfield of $\mathbb{C}$ containing k over which this ideal is generated. It is finitely generated as a field over k.

Proof. The fixed field of $\mathrm{Aut}(\mathbb{C}/k)$ is k. Indeed, for an element algebraic over k outside k, its minimal polynomial in characteristic zero has a distinct conjugate. Sending the element to that conjugate extends to an automorphism of the algebraic closure of k in $\mathbb{C}$ by the embedding argument of Lemma 16.2. Fix a transcendence basis of $\mathbb{C}$ over this closure and extend over its algebraic closure by the same argument. Here a maximal algebraically independent set exists by Definition 16.1, and maximality makes the remaining extension algebraic. For an element transcendental over k, include it in such a basis over k, replace it by its sum with one, fix the other basis members and extend the resulting rational function field automorphism to $\mathbb{C}$. This moves the element as required.

In each degree m, list the monomials in a fixed order. The degree-m part of the ideal is a vector subspace I_m of the corresponding finite complex coefficient space. A subspace has a unique matrix of nonzero reduced row echelon rows spanning it. Successively using the first nonzero column on the subspace with all preceding pivot coordinates zero determines its pivot positions. Projection to the pivot coordinates is an isomorphism, and the inverse images of the coordinate basis are the rows. These facts also show that entrywise field automorphisms carry the matrix to the corresponding matrix for the conjugate subspace. Invariance of

Z gives invariance of I_m, since evaluating a conjugated polynomial at a conjugated point conjugates its value. The entries of its reduced matrix then belong to k by the fixed-field assertion.

There is d such that the homogeneous ideal is generated in degrees at most d. For completeness, the exponents of the leading monomials of its nonzero homogeneous elements in degree-lexicographic order form an upward closed set. By Lemma 237.1 it is generated by finitely many exponents. Choose monic homogeneous elements attaining them. Subtracting suitable monomial multiples cancels the greatest monomial of any homogeneous element, and terminates since there are only finitely many monomials of its degree. This proves the assertion, with $d = 0$ for the zero ideal. The echelon rows through degree d thus suffice. Conversely, equations over k are carried to themselves as equations by every such automorphism, giving setwise invariance in both directions. Finally adjoin to k the finitely many entries of the indicated rows. They give generators over the resulting field. Any field of definition containing k fixes the set under its own automorphisms, so its fixed-field argument puts every entry in that field. This proves minimality. □

Proposition 270.2. *Let a finite diagram over $\mathbb{C}$ consist of projective algebraic sets with their reduced structures, regular maps, and finitely many identities between composites of these maps. Let $R_0 \subseteq \mathbb{C}$ be a subring. There is a subring $R \subseteq \mathbb{C}$ generated by finitely many elements over R_0 on which projective schemes and morphisms with all those identities are defined, recovering the given diagram by extending coefficients to $\mathbb{C}$. One may preserve any finite list of the relative smooth or étale presentations of Proposition 268.1 covering specified maps, as well as finiteness for maps certified by the projective elimination of Lemma 268.2.*

Proof. Use finitely many homogeneous equations for each projective source and target, as in Lemma 270.1. A standard affine chart sets a nonzero projective coordinate equal to one and uses the corresponding dehomogenized equations. Its algebra is the quotient by those equations. Within such a chart, an open set $D(a)$ means the locus where the regular function a is invertible. Its algebra is presented by adjoining v and $av - 1 = 0$. Such opens form a basis for the polynomial topology since an open complement of common zeros is a union of the opens of the individual equations. Near each point a regular map into a target chart is given by regular coordinate functions on some such open of a source chart. On shrinking, write these functions as polynomials in the coordinates and v. The domains may be chosen from a finite list. Indeed, in each affine chart the functions specifying any covering by principal opens generate the unit ideal by the maximal ideal argument in Lemma 268.2, and an expression for one uses finitely many of the functions. Refine the covers for the finitely many maps, composites and identities by intersections, using that inverse images of principal opens in an affine target chart are again principal on each of these domains. A finite affine cover of each necessary overlap suffices by the same argument. Transitions between the standard projective charts invert one additional coordinate and divide the other coordinates by it.

All data are now finite polynomial lists. Include the coefficients of the equations, of the maps and of the covering functions. For each cover include coefficients of an expression for one witnessing the cover. For each map include a witness of ideal membership for every pulled-back target relation in the localized source presentation. Include the same kind of witnesses for equality of coordinate formulas on overlaps, for transition identities and for the specified composite identities. They exist since equality in the quotient polynomial ring is exactly ideal membership. Agreement of two maps to projective space can also be tested by the cross products of their homogeneous coordinates on their target chart domains, since each list there has a unit coordinate. Pointwise identities on the reduced complex domains are identities of regular functions by the power-containment argument of Lemma 268.2. Take R to contain all these coefficients. The very same polynomial identities give covers on

the affine schemes over R, maps into the designated target charts, and agreement on their overlaps. Gluing gives the projective diagram, with the stated base change and identities.

A specified relative Jacobian chart is an isomorphism of localizations with a finite polynomial presentation in which a specified determinant is a unit. Give the formulas for that isomorphism and its inverse, including the inverse of the determinant, and give the finite witnesses for both composites and relations as above. This preserves the charts and their covering property over R, with exactly the same relative dimensions. For a specified finite map use the closed projective graph over the target. Include its equations and the isomorphism from the original source to the graph and back. Over the finitely many target opens in the elimination proof of Lemma 268.2, retain the coordinate changes, the unit minor witnesses and the homogeneous reductions of all monomials in its specified degree. These identities again establish that the graph over each open lies in one affine chart and that lower degree monomials generate its coordinate algebra as a module. Enlarging R for this finite list gives finiteness on the same covering. All choices involve only finitely many actual complex coefficients or their indicated inverses, so adjoining them produces a subring of the claimed form. □

Corollary 270.3. *For every finite selection of levels, powers, covers and Hecke legs of Theorem 268.3, together with any finite selection of their algebraic operations and diagram identities, there is a simultaneous projective model over a finitely generated O_E-subalgebra of $\mathbb{C}$. Its indicated projections are smooth, and the indicated covers and unscaled legs are finite étale. Here O_E is embedded by the chosen inclusion $E \subseteq \mathbb{C}$. In fixed projective coordinates, the equations of the varieties and their graphs have field of definition contained in E if and only if each of those embedded point sets is invariant under* $\mathrm{Aut}(\mathbb{C}/E)$.

Proof. The projective equations are supplied by Theorem 265.2, the algebraic maps by Corollary 267.3, and the covering Jacobian and finite presentations by Theorem 268.3. Select finite subcovers using the principal-open argument of Proposition 270.2 and apply it with $R_0 = O_E$. For the last assertion place each graph in a projective space by the product embedding of Corollary 260.2. That embedding has coordinates $x_i y_j$, and hence commutes with field automorphisms. Apply Lemma 270.1 with $k = E$ to each of the finitely many embedded sets. □

271. Specialization of the labelled torsion subgroups

Lemma 271.1. *Let k be an algebraically closed field, in any characteristic, and D a nonzero finite-dimensional commutative k-algebra with identity. Its points are the k-algebra maps $x : D \to k$. The algebra is the product of finitely many local algebras D_x, one at each point, whose maximal ideals $\mathfrak{m}_x$ are nilpotent. The tangent space at x, consisting of lifts of x to $k[\varepsilon]/(\varepsilon^2)$, identifies with the k-dual of $\mathfrak{m}_x/\mathfrak{m}_x^2$. If its dimension is at most one and $e_x = \dim_k D_x$, then*

$$D_x \simeq k[T]/(T^{e_x}).$$

When $e_x > 1$ one may use any element of $\mathfrak{m}_x$ nonzero modulo its square as T. Any quotient of this local algebra of dimension j, $0 \le j \le e_x$, has kernel (T^j).

Proof. Every maximal ideal gives a finite-dimensional residue field over k, hence exactly k, since every minimal polynomial splits. Distinct maximal ideals are comaximal. For two comaximal ideals with $a + b = 1$ in their sum, the elements a, b lift the two coordinate idempotents in the residues, and their intersection is their product by multiplying an element of both by $a + b$. Induction gives the Chinese remainder map onto the product for any finite set of pairwise comaximal ideals. In this induction an ideal comaximal to each of a list is comaximal to their product by multiplying elements congruent to one modulo that ideal. The number of distinct maximal ideals is thus at most $\dim_k D$.

Write J for their product. The descending powers of J stabilize by dimension. If $N = J^a = JN$, on a finite generating list for N one obtains a matrix with entries in J

expressing the generators in terms of themselves. As in Lemma 16.2, the determinant of the identity minus this matrix annihilates N. It is one modulo every maximal ideal, hence a unit since a nonunit lies in a maximal ideal by Definition 16.1. Thus $J^a = 0$ for sufficiently large a. The a-th powers of distinct maximal ideals are still comaximal by expanding $(a_1 + a_2)^{2a-1} = 1$ for $a_1 + a_2 = 1$ in the two ideals. Their product is zero. The Chinese remainder map therefore identifies D with the product of the quotients by these powers. Each quotient has nilpotent maximal ideal and residue k, and all elements outside that ideal are units by a finite geometric sum.

The ring of dual numbers is local and its only idempotents are zero and one, as is seen by reducing modulo its nilpotent ideal. A lift of x therefore factors through the x-factor. Writing its value at $c+v$ as $c+\lambda(v)\varepsilon$ for $c \in k$, $v \in \mathfrak{m}_x$, multiplicativity is exactly $\lambda(\mathfrak{m}_x^2) = 0$ for the linear functional λ. If $\mathfrak{m}_x/\mathfrak{m}_x^2 = 0$, nilpotence gives $\mathfrak{m}_x = 0$. Otherwise choose the stipulated T. In $D_x/(T)$ the image of the maximal ideal equals its square, so nilpotence again makes it zero. Consequently $\mathfrak{m}_x = (T)$. Successive subtraction of a residue and division in this principal ideal express every element as a polynomial in T. If $T^b = 0$ with b minimal, a nonzero polynomial of degree less than b has value T^j times a unit for some $j < b$, by its first nonzero coefficient. It cannot vanish. Thus $b = e_x$, and each nonzero ideal is generated by the lowest power occurring among its nonzero elements. The zero ideal is (T^{e_x}). This proves the quotient assertion, including $e_x = 1$ with $T = 0$. □

Proposition 271.2. *Let V be a valuation ring with fraction field K, maximal ideal $\mathfrak{m}$, and algebraically closed residue field k. A valuation ring here is a domain in which, for nonzero $a, b \in K$, at least one of $a/b, b/a$ belongs to V. Let S be a finite free commutative V-algebra such that $K \otimes_V S = K^\Omega$ with a specified finite set Ω. Every projection indexed by $\omega \in \Omega$ restricts on S to a V-valued point. Let $\bar{\omega}$ be its reduction. For $A \subseteq \Omega$ put $S_A = \operatorname{im}(S \to K^A)$, allowing the zero algebra when A is empty. Then S_A is finite free of rank $|A|$, and for each point x of the special algebra $D = S/\mathfrak{m}S$ the local factor at x of $S_A/\mathfrak{m}S_A$ has dimension*

$$|\{\omega \in A : \bar{\omega} = x\}|.$$

An absent factor is counted as zero. In particular reduction from Ω onto the points of D is surjective.

Proof. Multiplication on the finite free module gives a monic equation for every element of S by the adjugate argument of Lemma 258.1. An element of K integral over V belongs to V. Otherwise its inverse would be a nonunit of V, and division of its monic equation by the highest power would put one in $\mathfrak{m}$. Evaluation therefore takes values in V. The finite V-module S_A is torsion free as a submodule of K^A. Any finite generating list in a torsion-free module can be reduced to a basis here. A nontrivial K-linear relation after inverting nonzero elements of V can be divided by one of its nonzero coefficients so all resulting coefficients lie in V and one equals one, by total divisibility in a valuation ring. The relation already holds in the module by torsion freeness, so that generator can be removed. Iteration leaves an independent spanning list. Inverting nonzero elements of V in S_A gives all of K^A, by the given generic product and coordinate projection. This proves its rank.

By Lemma 271.1, choose $\bar{t} \in D$ with distinct residue values $c_x \in k$ at all its points, by taking these constants in the product factors. The field k is infinite, since on a finite field the polynomial one plus the product of $X - c$ over all c has no root. Lift $\bar{t}$ to $t \in S$. On S_A the characteristic polynomial of multiplication by t becomes $\prod_{\omega \in A}(Y - t(\omega))$ over K. Both sides already have coefficients in V, so they are equal there. The special algebra of S_A is a quotient of D. It is a product of the corresponding quotients of the local factors of D, by the images of their idempotents. In the factor at x multiplication by $\bar{t} - c_x$ is nilpotent. A nilpotent map has a basis in which its matrix is strictly triangular, by extending bases along its increasing sequence of kernels. The characteristic polynomial on this factor is $(Y - c_x)^{d_x}$, where d_x is its dimension. Reducing the characteristic polynomial over V and comparing

multiplicities of the pairwise distinct roots gives the formula. Take $A = \Omega$ for surjectivity, using the injectivity of $S \to K \otimes_V S$. □

Theorem 271.3. *Assume the hypotheses of Proposition 271.2. Suppose* $\mathrm{Spec}(S)$ *is a commutative group over* V *with a scalar action of a finite field* κ*, and identify its generic points as a* κ*-space* $\Omega \simeq \kappa^n$*. A group over* V *here means that multiplication, inverse and unit are morphisms over* V *satisfying the group identities, with the scalar maps satisfying the module identities. If the special tangent space at the identity has dimension one over* k*, then the kernel* W *of reduction on generic labels is a nonzero* κ*-subspace. For every point* x *of the special group and every subspace* $A \subseteq \Omega$ *the ideal of the closure of* A *on the special local factor at* x *is*

$$(T_x^{j_A(x)}), \qquad j_A(x) = \begin{cases} |A \cap W|, & x \in \bar{A}, \\ 0, & x \notin \bar{A}, \end{cases}$$

where $\bar{A}$ *is the reduction image, and* T_x *is any generator of the local maximal ideal. Each such local algebra is* $k[T_x]/(T_x^{|W|})$*.*

Proof. Composition of the V-points and reduction commute with the group operations and scalar maps since these morphisms are defined over V. Thus reduction is linear on labels, and each nonempty fiber on A is a translate of $A \cap W$. By Proposition 271.2 all special points occur, and each full local factor has dimension $|W|$. Translation by a special point, with inverse translation by its negative, identifies its tangent space with that at the identity. By Lemma 271.1 each local factor is the asserted truncated polynomial algebra. At the identity its maximal ideal has a nonzero class modulo its square, so its length is at least two. Hence W contains a nonzero vector, and therefore a κ-line. The closure of the generic labels A is cut out by the elements vanishing on all those labels, giving exactly the quotient S_A. Its local quotient dimension is $j_A(x)$ by Proposition 271.2. Lemma 271.1 determines its ideal from that dimension, including the unit ideal when the local factor is absent. □

272. Frobenius powers on a torsion component

Lemma 272.1. *Let* k *be an algebraically closed field of characteristic* $p > 0$*,* $P = p^a$ *with* $a \geq 0$*,* Y *a scheme over* k*, and* $y \in Y(k)$ *a closed point. Write* $Y^{(P)}$ *for change of scalars by* $c \mapsto c^P$ *on* k*. The relative Frobenius* $F_{Y,P} : Y \to Y^{(P)}$ *is defined on affine coordinate rings by*

$$R \otimes_{k,\, c \mapsto c^P} k \longrightarrow R, \qquad r \otimes c \longmapsto r^P c.$$

For a k*-map* $v : H \to Y$ *from a finite* k*-scheme and a point* $x \in H(k)$*, let* I_x *be the ideal in the local factor* D_x *of* H *cutting out* $v^{-1}(y)$*. The ideal in* D_x *cutting out* $(F_{Y,P} v)^{-1}(F_{Y,P}(y))$ *is generated by the* P*-th powers of the elements of* I_x*. In particular, if* $I_x = (T^j)$*, this ideal is* (T^{Pj})*, also when* $j = 0$*.*

Proof. The formula respects the tensor relations, addition and products by the characteristic-p binomial identity, iterated a times. It commutes with localization since inverting a function is equivalent to inverting its P-th power, so defines the indicated morphism. The field map $c \mapsto c^P$ is bijective by algebraic closedness and injectivity of a field homomorphism. The map on the underlying point spaces is bijective. In an affine chart the scalar twist is isomorphic as a ring to R, by $r \otimes c \mapsto r c^{1/P}$. In that identification the displayed map raises elements to their P-th powers, and r^P belongs to a prime ideal precisely when r does. The argument respects the affine opens.

If $v(x) \neq y$, both local ideals are the unit ideal. Otherwise an affine neighborhood of y contains the whole image of $\mathrm{Spec}(D_x)$, which has just one point by Lemma 271.1. If $\mathfrak{n}$ is the ideal of y in this chart, the ideal of $F_{Y,P}(y)$ in its twist is generated by $r \otimes 1$ for $r \in \mathfrak{n}$. The quotient is $k \otimes_{k,\, c \mapsto c^P} k \simeq k$ by the displayed evaluation. The new pullback ideal is generated by $v^*(r)^P$ for these r. This also contains the P-th power of any element of I_x, by

raising its expression as a linear combination of the $v^*(r)$ to that power. Conversely each generator is already of this form. A principal ideal gives the asserted generator. □

Proposition 272.2. *Use V, K, k as in Proposition 271.2. Let G be a commutative group scheme over V and let $H \subseteq G$ be a finite free closed subgroup whose generic algebra is a product of copies of K. Let $\psi : G \to G'$ be a homomorphism. If its kernel on H is finite flat over V and its generic points in H form a subset A, then that kernel on H is the closure of A. On the special fiber its ideal at $x \in H(k)$ is therefore determined by the cluster length in Proposition 271.2 whenever the tangent dimension at x is at most one.*

Proof. Write S for the algebra of H. The kernel is assumed a closed subgroup, with coordinate quotient S/J. Flatness means that tensoring with this module preserves injections. Applying this to multiplication by a nonzero element of V on V proves torsion freeness, hence S/J injects into its generic algebra. An ideal in a product of finitely many fields is the product of its coordinate ideals, by multiplying by the coordinate idempotents. Its quotient here is exactly K^A. Thus $J = \ker(S \to K^A)$, which is the stated closure ideal. Reduction gives its local quotient lengths by Proposition 271.2. The assertion about the ideal follows from Lemma 271.1. □

Theorem 272.3. *Suppose k has characteristic $p > 0$. Let H be a finite free commutative group scheme over V with split generic algebra as in Proposition 271.2, and special tangent dimension at the identity at most one. Inside its generic point group fix a subgroup $L \simeq \kappa^n$ for a finite field κ of cardinality $q = p^e$, $e \geq 1$. Assume reduction $\nu : L \to H(k)$ respects the κ-scalar action on its image. For each subspace $A \subseteq L$ let $\psi_A : H \to Y_A$ be a homomorphism to a group scheme over V with closed identity, having finite flat kernel and generic kernel labels A. Put $r = \dim_\kappa A$, let $\bar{\psi}_A$ denote the special map, and set*

$$f_A = F_{(Y_A)_k,\, q^{n-r}} \bar{\psi}_A.$$

Its kernel on the special fiber of H has, at $x \in H(k)$, local ideal

$$\begin{cases} (T_x^{q^n/|\nu(A)|}), & x \in \nu(A), \\ (1), & x \notin \nu(A), \end{cases}$$

where T_x generates the maximal ideal in the local algebra at x. In particular the special kernels of f_A, f_B agree as closed subschemes whenever $\nu(A) = \nu(B)$. If $W = \ker \nu$ contains a line W_0, this gives kernel equality for the pair $A, A + W_0$ occurring in Corollary 269.3.

Proof. Translations by k-points identify the special tangent spaces at all points with the one at the identity. Fix a generator T_x in the resulting truncated polynomial algebra of Lemma 271.1, taking zero when the algebra is k. Proposition 272.2 gives the ideal (T_x^j) of $\bar{\psi}_A^{-1}(0)$ with $j = |\{a \in A : \nu(a) = x\}|$. For a nonempty fiber, translation by a preimage identifies it with $A \cap W$. Partitioning A into its fibers therefore gives $j = |A|/|\nu(A)|$. Since $|A| = q^r$, application of Lemma 272.1 gives the exponent $q^{n-r} j = q^n/|\nu(A)|$. A power exceeding the nilpotence index simply gives the zero ideal. An empty fiber gives the unit ideal before and after Frobenius. Equality of these ideals in all the factors is equality in their product, the entire finite coordinate algebra by Lemma 271.1. Finally $\nu(A + W_0) = \nu(A)$ for any $W_0 \subseteq \ker \nu$. □

273. The signature and the split factors of the isogenies

Lemma 273.1. *In the notation of Lemma 236.1, identify E with its image under σ, and write $g_\tau = \sigma^{-1}\tau$. Choose an integral primitive element θ of $E/\mathbb{Q}$. Outside a finite set of rational primes, if w is a degree-one prime of E over p, then p splits completely in E, that is, pO_E is a product of $2f$ distinct primes with residue field $\mathbf{F}_p$. Write $\rho : O_E \to \mathbf{F}_p \subseteq k$ for reduction at w into an algebraically closed field k of characteristic p. For $m \geq 1$, the factors of $O_E/p^m O_E$ are copies of $\mathbb{Z}/p^m\mathbb{Z}$, indexed by $v_g = \ker(\rho \circ g)$, $g \in \mathrm{Gal}(E/\mathbb{Q})$. On*

the tangent space at the identity of every complex torus of Proposition 252.2, the polynomial of θ is the image in $\mathbb{C}[T]$ of

$$P_\theta(T) = (T-\theta)(T-\bar{\theta})^{n-1} \prod_{\tau \in \Psi \setminus \{\sigma\}} (T - g_\tau \theta)^n \in O_E[T].$$

If $u = w|_{K_0}$ for the imaginary quadratic subfield, the roots indexed by v_g above u have multiplicity one at w and zero at all other such primes. Above $\bar{u}$ the multiplicities are $n-1$ at $\bar{w}$ and n at each remaining prime.

Proof. The generators $\sqrt{-d}, \zeta_{N_1}$ have all their conjugates in E. Every embedding of this finite field preserves it and the $2f$ embeddings are the automorphisms, as follows by adjoining successively the roots of their separable polynomials as in Lemma 16.2. They commute since on the first generator each changes at most its sign and on the second each raises it to a unit power. An integral primitive element and the finite index of its power basis are provided by Lemma 82.2. Exclude 2, the primes dividing that index and the nonzero discriminant of the minimal polynomial. Its $2f$ conjugate roots then have distinct residues under ρ, since the product of squared differences is the discriminant. All residues belong to $\mathbf{F}_p$ by the degree-one assumption. For every m these roots lift uniquely to roots modulo p^m by the digit correction in Lemma 113.1. Their pairwise differences are units, so the minimal polynomial modulo p^m is the product of the corresponding linear factors. Indeed division by each monic factor preserves the other roots by invertibility of the differences. Since the power basis index is prime to p, it is a basis modulo p^m by its invertible determinant. The Chinese remainder argument of Lemma 271.1 gives the stated product with its residue maps indexed by g. In particular p is a uniformizer in each completion at these primes by the valuation factorization of Proposition 82.3.

Proposition 252.2 computes the polynomial on the complex tangent space. The normality just proved makes every g_τ an automorphism of E; it preserves integrality by transporting monic integral equations. For $\tau \neq \sigma$ in Ψ, g_τ acts by conjugation on K_0 by Lemma 236.1. Together with complex conjugation these are all the f automorphisms with that restriction, since Ψ has one member from each conjugate pair. The maps ρ and $\rho \circ \mathfrak{c}$ are distinct on O_{K_0} outside the finite set of primes ramifying there, which we also exclude. This identifies $u, \bar{u}$ and proves the multiplicity assertions. $\square$

Lemma 273.2. *Let V be as in Proposition 271.2, with $K = \mathbb{C}$, equipped with $O_E \to V$ inducing the chosen inclusion into $\mathbb{C}$ and the map ρ above in its residue field k. Let G be a smooth commutative group scheme over V with an O_E-action, whose generic group with action is one of the complex tori of Lemma 273.1. Suppose the kernel of multiplication by p^m is finite flat over V. Then*

$$G[p^m] = \prod_{v \mid p} H_{v,m}$$

with finite free closed factors cut out by the respective idempotents of $O_E/p^m O_E$. The special tangent dimension of $H_{v,m}$ equals the multiplicity attached to v in Lemma 273.1. Here smoothness is used in the relative Jacobian presentation of Proposition 268.1.

Proof. At the identity point of the special fiber take a smooth chart. The identity section factors through it over V: the inverse image is an open containing the closed point of a local spectrum, hence contains a principal neighborhood of it defined by a unit. Tangent vectors along that section are solutions of the linearized chart equations over V. The unit Jacobian minor uniquely solves for the corresponding coordinates in terms of the others. Thus the tangent module is finite free, and its formation commutes with passage to both K and k. The derivative of θ on it has characteristic polynomial P_θ over V, since both sides agree over K by Lemma 273.1 and $V \subset K$. In particular the special polynomial is its reduction.

On the kernel the scalar action factors through $O_E/p^m O_E$. For each ring idempotent e_v of the product in Lemma 273.1 let $H_{v,m}$ be the equalizer of that endomorphism and the

identity on the kernel. The kernel is affine finite, so the equalizer is closed, defined on its algebra by all differences $s - e_v^* s$. This quotient identifies with the image of the idempotent algebra map e_v^*: the kernel of that map consists of differences with zero image, and every difference is killed. Hence its algebra is a finite torsion-free V-module, using flatness of the original kernel as in Proposition 272.2. It is free by the generating-list argument of Proposition 271.2. The maps $z \mapsto (e_v z)_v$ and $(z_v)_v \mapsto \sum_v z_v$ are mutual inverse morphisms by the orthogonal idempotent identities. This proves the product on the whole group scheme, after any base change as well.

The derivative of the group operation at the identity is addition. Its restrictions to both coordinate axes are the identity, and its derivative is linear by the linearized equations on dual numbers. Hence multiplication by p^m has derivative zero on the special tangent space. All tangent vectors there belong to the kernel, by evaluating that map on the dual numbers. The product idempotents split the tangent into their corresponding images. They act through O_E/pO_E on it, since it is a k-vector space and the derivative of an integer scalar is that integer times the identity. On the v_g image the derivative of θ is scalar $\rho(g\theta)$ by the product description. These scalars are pairwise distinct. Their multiplicities in the reduced characteristic polynomial are therefore exactly the dimensions of the summands, proving the claim. □

Theorem 273.3. *Take the data of Lemma 273.2 with G separated, w safe, and finite flat multiplication kernels for $m = 1, n$. Suppose all the unscaled complex fiber isogenies labelled $A \subseteq \mathbf{F}_p^n$ in Lemma 269.1 extend over V to O_E-linear homomorphisms $\psi_A : G \to G_A$ with finite flat kernels and separated target groups. Put $r = \dim_{\mathbf{F}_p} A$ and $f_A = F_{(G_A)_k, p^{n-r}}(\psi_A)_k$. There exists a line $W_0 \subseteq \mathbf{F}_p^n$ such that the kernels of f_A, f_{A+W_0} agree on $\prod_{v|u}(H_{v,n})_k$ for every A, where $u = w|_{K_0}$.*

Proof. Use uniformizer p at w in Lemma 269.1. The generic w-part killed by the isogeny is exactly A even among all p^n-torsion labels. In the finite coordinates of that lemma its matrix is $d_r k_A$, whose inverse has denominators dividing p. Thus any killed class is already killed by p, and the computation modulo p applies. On the other components above u the matrix is the identity, as follows from Lemma 245.2: the local $b_w(a_A)$ is supported on $w, \bar{w}$ and the central scalar on the places above $\bar{u}$. At $\bar{w}$ the inverse frame matrix is $p^r \bar{a}_A^{-1}$. Its own inverse has denominators dividing p^r since a_A is integral. At the other places above $\bar{u}$ the inverse frame is scalar p^r, and it is a unit at all remaining places. These formulas use that the local norm of the chosen uniformizer is p in the degree-one completion. They show that p^n kills the entire generic kernel on points, including $r = 0$. This holds on the generic kernel scheme: its local tangent spaces vanish because the complex isogeny is locally invertible by Theorem 268.3, and its finite complex algebra is reduced by Lemma 271.1.

Multiplication by p^n on the integral kernel of ψ_A is consequently zero. Separatedness of G makes the identity section closed, by pulling back its closed diagonal along the identity map and the constant identity-section map. Thus $G[p^n]$ is closed. Pulling back the ideal of $G[p^n]$ to the finite flat kernel gives zero by injection of its algebra into its generic fiber. Decomposing this kernel by the same $O_E/p^n O_E$ idempotents as in Lemma 273.2 gives finite free factors. Their intersections with $H_{v,n}$ are exactly these factors, by the fixed-point description for the idempotents. Their generic labels above u were just computed.

The special tangent dimension of $H_{w,1}$ and of $H_{w,n}$ is one by Lemma 273.2. The generic points of $H_{w,1}$ are $\mathbf{F}_p^n$ by Lemma 269.1; their algebra splits since $[p]$ on the generic group has invertible derivative and $\mathbb{C}$ is algebraically closed. Now Theorem 271.3 supplies a line W_0 reducing to zero. Use Theorem 272.3 on $H_{w,n}$ with $L = H_{w,1}(\mathbb{C})$, the generic labels just determined and their finite flat kernels on this component. This identifies the kernels for $A, A + W_0$ there. At the other $v \mid u$ each generic component kernel is the identity. Its closure is the identity section, by Proposition 272.2. The special tangent dimension there is zero by Lemma 273.2, so all local algebras are copies of k by Lemma 271.1. Applying Lemma 272.1

preserves both the zero and the unit ideals in these fields. Finally the kernel of a linear map on this product splits into its component kernels: apply each idempotent to an element in the kernel over any k-algebra, using the O_E-linearity of the reduced map and the naturality of relative Frobenius for the scalar maps. This proves the asserted equality. □

274. The local algebras of the Frobenius kernels

Lemma 274.1. *Let G be a separated smooth commutative group scheme of finite type and dimension g over an algebraically closed field k of characteristic $p > 0$. For $a \geq 1$ put $P = p^a$ and $J_a = \ker F_{G,P}$ for the relative Frobenius of Lemma 272.1. Then J_a is supported at the identity and has coordinate algebra*

$$k[t_1, \ldots, t_g]/(t_1^P, \ldots, t_g^P).$$

Multiplication by P is zero on J_a as a morphism. In particular $J_a \subseteq G[P]$ as a closed subgroup.

Proof. Take a relative Jacobian chart as in Proposition 268.1, now with base k. Subtract the coordinates of the identity and write the remaining variables as t, z, with g free variables t and a square invertible derivative in z for the defining equations. In each quotient by the b-th power of the maximal ideal, for $b \geq 1$, the equations have a unique solution $z = z_b(t)$ with zero constant term in $k[t]/(t)^b$. Induct from $b = 1$. A correction of degree b multiplies the constant derivative matrix, so its inverse gives the unique correction killing the homogeneous degree-b error. The same inductive identities in the original chart modulo its maximal ideal powers show that its variables z equal these polynomials in t to each order. Denominators inverted in the chart have nonzero constant and so have inverses modulo each such power by finite geometric sums. The inverse maps on truncated rings identify their inverse limits. Thus at the identity the formal functions are $k[[t_1, \ldots, t_g]]$. The same construction for products gives independent tuples of formal parameters for each factor.

The Frobenius has a single point above the identity by Lemma 272.1. Its ideal in an affine neighborhood of that point is generated by the P-th powers of the elements vanishing there. This already cuts out a finite local algebra: the maximal ideal in an affine chart is generated by its finitely many affine coordinates minus their values, with any presentation denominators treated as additional affine coordinates. If there are h such generators, every product of $h(P-1)+1$ generators contains a P-th power. Thus one can compute the quotient in the truncated rings just described. In formal coordinates its ideal is $(t_1^P, \ldots, t_g^P)$, since P-powering is additive and all parameter multiples have their P-th powers in this ideal. This proves the algebra formula and shows that J_a is finite of length P^g.

In the parameters at the identity, express the sum of p points by g formal series in p independent tuples $t^{(i)}$. The constant terms are zero and the series are invariant under cyclic permutation of the tuples, by associativity and commutativity of the group law. Identify all the tuples with t. Monomials in a nonconstant orbit of the cyclic permutation occur in orbits of size p with equal coefficients and hence cancel in characteristic p. In a fixed monomial the full multi-index is the same on every tuple, giving a monomial in $t_1^p, \ldots, t_g^p$. This argument is degree by degree with finite sums. Each coordinate of $[p]$ therefore lies in $k[[t_1^p, \ldots, t_g^p]]$ with zero constant term. Iterating the substitution a times puts each coordinate of $[P]$ in the series ring in $t_1^P, \ldots, t_g^P$ with zero constant term. It vanishes in the algebra calculated above. Thus the morphism $[P]$ restricted to the entire Frobenius kernel is zero. Since G is separated its identity is closed, and both kernels are closed subgroups. □

Theorem 274.2. *Let G/V satisfy Lemma 273.2 with $m = a \geq 1$ and G separated. Put $P = p^a$. For $v \mid p$ let d_v be the tangent multiplicity in Lemma 273.1, and $J_{v,a} = J_a \cap (H_{v,a})_k$ in G_k. Then*

$$J_a = \prod_{v \mid p} J_{v,a}, \qquad \mathcal{O}(J_{v,a}) \simeq k[t_1, \ldots, t_{d_v}]/(t_1^P, \ldots, t_{d_v}^P).$$

The full special factor $(H_{v,a})_k$ has length P^n. For $d_v = n$ it equals $J_{v,a}$; for $d_v = 0$ the Frobenius kernel there is the identity section.

Proof. The group J_a in the statement is the kernel for G_k. It is killed by P by Lemma 274.1 and is stable under each scalar endomorphism by the affine formula for relative Frobenius in Lemma 272.1. Apply the idempotents of O_E/PO_E on this subgroup, using the equalizer construction and the sum inverse in Lemma 273.2. Its factors are exactly the indicated intersections. Each coordinate algebra B_v is a quotient of the local algebra of J_a and is local with residue k. Every element of its maximal ideal has P-th power zero by the explicit algebra of Lemma 274.1. The tangent of J_a is the full tangent space of G_k, since $P \geq 2$ and all the defining powers vanish on dual numbers. Thus the tangent of its v-factor has dimension d_v, by the idempotent decomposition and the proof of Lemma 273.2.

Choose d_v lifts of a basis of the maximal ideal modulo its square in B_v. They generate the ideal: in the quotient by them the remaining maximal ideal equals its square and is nilpotent by Lemma 271.1. They also generate B_v as a k-algebra. Subtracting residues expresses an element successively as a polynomial of degree less than j in the lifts plus an element of the j-th power of the maximal ideal. Each next power is generated by products of j lifts, so subtracting the residue of each coefficient supplies the next step. Nilpotence terminates the process. Sending the variables of the displayed truncated algebra to the lifts is now a surjection, which gives $\dim_k B_v \leq P^{d_v}$. The products of all these dimensions equal P^{fn}: coordinate algebras of the product of finite affine schemes tensor over k, as follows by specifying the algebra maps from the factors to any commutative k-algebra; dimensions of tensor products multiply by taking bases. On the other hand $\sum_v d_v = fn$ by Lemma 273.1. Equality in every factor bound follows, so every stated surjection is an isomorphism.

The generic v-factor in $G[P]$ consists of the classes $p^{-a}\mathbb{Z}_p^n/\mathbb{Z}_p^n$ in the finite frame coordinates of Lemma 269.1. It is reduced by invertibility of the derivative PI and Lemma 271.1, hence its coordinate dimension equals its P^n points. Finite freeness in Lemma 273.2 gives the same special dimension. For $d_v = n$ the surjection from its algebra to B_v has equal source and target dimensions and is an isomorphism. For $d_v = 0$ the displayed algebra is k at the identity. □

Corollary 274.3. *Under the hypotheses of Theorem 273.3, assume also that the targets G_A are smooth groups of finite type. Then every full* $\ker f_A$ *is a closed subgroup of $G_k[p^n]$, with the same p^n for every subspace A.*

Proof. The generic matrix computation in Theorem 273.3 gives that p^r kills the entire generic kernel of ψ_A when $r = \dim A \geq 1$: at w a single p suffices and at the conjugate factors p^r suffices. For $r = 0$ the generic kernel is already the identity. Finite flatness and the closed identity of G give this same annihilation on the integral kernel by injection into its generic coordinate algebra, as in that theorem. For any special point over any k-algebra killed by f_A, its image under $(\psi_A)_k$ belongs to the Frobenius kernel of exponent p^{n-r} on the target. It is killed by p^{n-r} by Lemma 274.1; for $r = n$ this kernel is the identity. Consequently p^{n-r} times the original point is in the kernel of $(\psi_A)_k$, and multiplication by p^r kills it. This gives the assertion as morphisms of schemes, by applying the functorial identity on affine charts of the kernel itself. Closedness follows from separatedness of the targets and the source. □

275. Flatness of Frobenius and descent through an isogeny

Proposition 275.1. *Let k be a perfect field of characteristic $p > 0$, so p-powering is an automorphism of k. On a smooth k-scheme X of pure dimension g, relative Frobenius $F_{X,P}$ for $P = p^a$, $a \geq 0$, is finite locally free of rank P^g and surjective. Finite locally free of positive rank means that on a cover by affine target opens the source is affine with coordinate algebra a finite free module of positive rank.*

Proof. Use a smooth chart

$$R = (B[z_1, \dots, z_h]/(f_1, \dots, f_h))[1/d], \qquad B = k[t_1, \dots, t_g],$$

with $J = \det(\partial f_i/\partial z_j)$ a unit in R, as in Proposition 268.1. Inverting several denominators is equivalent to inverting their product. Write $R^{(P)}, B^{(P)}$ for the scalar twists of Lemma 272.1. Use variables t_i', z_j' in the twists. Their equations are those of the original presentation with all coefficients raised to P, since $c \otimes 1 = 1 \otimes c^P$ for a scalar. Form

$$T = R^{(P)} \otimes_{B^{(P)}} B, \qquad t_i' \longmapsto t_i^P.$$

There is a homomorphism $T \to R$ sending t_i to t_i and z_j' to z_j^P. We prove it is an isomorphism.

Adjoin to T variables v_j with $v_j^P = z_j'$, obtaining a finite free algebra S with basis $\prod_j v_j^{e_j}$, $0 \le e_j < P$, by division by the successive monic equations. The basis contains one (including the case $P = 1$). The elements $f_i(t, v)$ have P-th power zero by the equations in T. The ideal I they generate is nilpotent by the pigeonhole argument in Lemma 274.1. Every element of I has P-th power zero by additivity of P-powering. Both $d(t, v)$ and $J(t, v)$ are units since their P-th powers are the corresponding units in the twisted presentation.

There exists a tuple $b \equiv v \bmod I$ with $f(t, b) = 0$. Indeed, at an approximate solution with error in I^j multiply the negative error column by the inverse Jacobian matrix and add the resulting correction. Polynomial expansion makes the new error belong to I^{2j}. The determinant remains a unit under a nilpotent correction. Iteration terminates, and $b_j^P = z_j'$. We use also the following uniqueness of such lifts. For two exact solutions with invertible Jacobian whose differences generate a nilpotent ideal N, the linear Taylor term gives their differences in N^2 after multiplication by the inverse Jacobian. Thus $N = N^2$, forcing $N = 0$.

In $S \otimes_T S$ the two images of b_j have equal P-th powers. Their differences generate a nilpotent ideal, again by the pigeonhole argument. The uniqueness just proved gives $b_j \otimes 1 = 1 \otimes b_j$. The T-linear functional selecting the coefficient of one in the free basis of S takes one to one. Applying it on the first factor gives $b_j \in T$. The assignment $z_j \mapsto b_j$ defines $R \to T$: the equations hold in S, which contains T, and $d(t, b)^P$ is the corresponding unit in T. The composite on T fixes each z_j' by $b_j^P = z_j'$, and fixes t and the scalars. The two tuples on the R-side also solve the original equations with the same P-th powers and unit Jacobians, so uniqueness proves the other composite is the identity.

Finally B is free over $B^{(P)}$ with basis $\prod_i t_i^{e_i}$, $0 \le e_i < P$, by partitioning the monomials by their exponent residues. Thus $R \simeq T$ is free of rank P^g over $R^{(P)}$. These affine charts and their twists correspond under Frobenius on underlying opens, by the prime-ideal argument of Lemma 272.1, valid over any perfect field. It gives surjectivity as well. □

Lemma 275.2. *Let $b : X \to Y$ be a surjective finite locally free morphism of schemes and $h : X \to Z$ a morphism. If h has the same composite with both projections $X \times_Y X \rightrightarrows X$, it factors uniquely through b. If b, h are homomorphisms of commutative group schemes over the same base with closed identity sections, the same conclusion holds whenever* $\ker b \subseteq \ker h$ *as closed subgroup schemes, and the factor is a homomorphism.*

Proof. Finite maps are closed. In fact over an affine target $\operatorname{Spec}(A)$ a closed part of the finite source has coordinate ring M which is a finite A-module. Its image is the common zero set of its annihilator. To check this at a prime $\mathfrak{p}$, the localization $M_{\mathfrak{p}}$ is zero exactly if an element outside $\mathfrak{p}$ kills its finite generating list, by taking the product of individual denominators. When the localization is nonzero, its residue algebra is nonzero: otherwise the determinant argument of Lemma 271.1 applied to the local maximal ideal would kill that module by a unit. A nonzero commutative ring admits a maximal ideal by Definition 16.1, giving a point of the fiber. This proves closedness.

Two points over the same point y of Y have a common lift to their fiber product over y. Indeed the tensor product of their residue fields over $k(y)$ is nonzero by vector space bases,

and a prime ideal of it gives such a lift. The hypothesis therefore gives all points of $b^{-1}(y)$ the same image in Z. Choose an affine neighborhood of that image, remove from Y the closed image under b of its inverse complement under h, and take an affine neighborhood of y in the remaining open. Shrink it so that the finite algebra S over its ring A is free. Because the fiber at y is nonempty, the vector for one in a module basis of S has a coordinate that is a unit after shrinking again: otherwise its residue at y would be zero in the nonzero fiber algebra. There is consequently an A-linear $\lambda : S \to A$ with $\lambda(1) = 1$. The ring map $A \to S$ is injective, and $s \otimes 1 = 1 \otimes s$ in $S \otimes_A S$ implies $s = \lambda(s)1$ on applying λ to the first factor. Every coordinate function of h on the chosen target chart thus belongs to A, and its relations hold there by injectivity. This defines the unique local factor. Such factors agree on overlaps: the common value on underlying points is fixed by surjectivity, and in any common affine chart equality of pulled-back functions implies equality by the same injection. Gluing gives the first assertion.

For group maps the map $(x, x') \mapsto (x, x' - x)$ identifies $X \times_Y X$ with $X \times \ker b$ over the original base, with inverse $(x, z) \mapsto (x, x + z)$. The kernel containment thus gives the required equality. Base change of a surjective finite locally free map is again so, by tensoring its free basis locally and by nonzero residue fiber algebras as above. Its square on products has the same property by composing two base changes. Uniqueness of descent for this square verifies addition for the factor, and the identity section and inverse satisfy the corresponding identities by surjectivity and uniqueness. □

Corollary 275.3. *Let U, Y be smooth commutative groups of dimension g over an algebraically closed field k of characteristic $p > 0$, and $\phi : U \to Y$ a surjective finite locally free homomorphism. If $\ker \phi = \ker F_{U,p}$ as closed subgroups of U, there is a unique isomorphism $i : U^{(p)} \to Y$ of groups over k satisfying $\phi = iF_{U,p}$. If ϕ is linear for an action of a commutative ring, then i is linear for the twisted action on $U^{(p)}$. Any marked points carried by ϕ are carried by i after Frobenius on their source coordinates.*

Proof. Proposition 275.1 makes the Frobenius map surjective finite locally free. Apply Lemma 275.2 in the two directions to obtain factorizations of ϕ by Frobenius and of Frobenius by ϕ. Uniqueness applied to the identity factorizations shows the two resulting homomorphisms are inverse. The affine formula of Lemma 272.1 commutes with all scalar morphisms after twisting the target action. Equality after composition with $F_{U,p}$ therefore gives the scalar compatibility by the same uniqueness. The statement on marked points is evaluation of $\phi = iF_{U,p}$. □

276. Consecutive Hecke neighbours along a specializing line

Lemma 276.1. *Use a safe degree-one place as in Lemma 273.1, with uniformizer p. For $A \subset B \subseteq \mathbf{F}_p^n$ with $\dim B = \dim A + 1$, let K_A, K_B denote the generic complex fiber kernels of the unscaled Hecke maps with labels A, B. Then*

$$K_A \subseteq K_B, \qquad pK_B \subseteq K_A.$$

The orders on the factors of the quotient K_B/K_A at $w, \bar{w}$ are p, p^{n-1}, respectively. At the other primes above $\bar{u}$ they are p^n, and at all remaining primes they are one. Its w-points are the image of B/A in the target of the map with label A.

Proof. Put $r = \dim A$, $L = \mathbb{Z}_p^n$, and for a residue subspace D let $L(D)$ be its full inverse image in L. At w the two kernels are A, B in $p^{-1}L/L$ by Lemma 269.1 and the inverse denominator calculation in Theorem 273.3. At $\bar{w}$, identify the coordinates using conjugation. The kernel for A is $p^{-r}L(A^\perp)/L$, since its inverse frame is $p^r \bar{a}_A^{-1}$ and the lattice $a_A L$ is $L(A^\perp)$ by Lemma 269.1. For $r = 0$ this formula gives the zero group. For B replace r, A by $r + 1, B$. We have $pL(A^\perp) \subseteq pL \subseteq L(B^\perp)$ and $L(B^\perp) \subseteq L(A^\perp)$. These prove both inclusions at this place. The index of $L(B^\perp)$ in $L(A^\perp)$ is p by their residue dimensions, so

the quotient of the two enlarged lattices has order $p^n/p = p^{n-1}$. At a remaining place above $\bar{u}$ the two lattices are $p^{-r}L, p^{-(r+1)}L$, giving the inclusions and order p^n. Elsewhere the kernels are zero by Theorem 273.3. The final assertion follows from the map with kernel A on the w-labels. □

Proposition 276.2. *In the notation of Theorem 273.3 suppose each $\psi_A : G \to G_A$ is surjective finite locally free. Suppose G_A is smooth and separated, with the given generic complex group and scalar action, and $G_A[p]$ is finite flat over V. For every pair of Lemma 276.1 there is a unique O_E-linear homomorphism $\phi_{A,B} : G_A \to G_B$ with $\phi_{A,B}\psi_A = \psi_B$. Its kernel $K_{A,B}$ is a finite free closed subgroup of $G_A[p]$. In each prime factor it has rank equal to the order given by Lemma 276.1.*

Proof. The inclusion of the generic kernels extends to the integral kernels: the ideal of the closed identity of G_B pulled back along the restriction of ψ_B to $\ker \psi_A$ vanishes generically. The coordinate algebra of that kernel injects into its generic algebra by flatness, as in Proposition 272.2. The same argument applied to $[p]$ on $\ker \psi_B$ followed by ψ_A gives $[p] \ker \psi_B \subseteq \ker \psi_A$. In both arguments the generic algebras are products of fields by Theorem 273.3. Now use Lemma 275.2 on ψ_A and ψ_B. It gives the homomorphism and its uniqueness. Scalar compatibility follows by equality after composition with ψ_A.

Pulling back ψ_A along the closed immersion $K_{A,B} \hookrightarrow G_A$ gives $\ker \psi_B \to K_{A,B}$, surjective finite locally free by base change. Multiplication by p on $K_{A,B}$ vanishes after this cover by the second kernel inclusion. It vanishes on $K_{A,B}$ by uniqueness of descent. Thus $K_{A,B}$ is a closed subgroup of the finite affine $G_A[p]$, hence finite over V. The coordinate map of the cover is injective by the local functional argument of Lemma 275.2. Torsion freeness of the coordinate algebra of $\ker \psi_B$ over V therefore implies torsion freeness for the finite coordinate algebra of $K_{A,B}$; it is free by Proposition 271.2. The idempotent factors are also finite free by the image argument of Lemma 273.2. Its generic subgroup is reduced since it is closed in the split reduced group $G_A[p]_{\mathbb{C}}$, by the product-of-fields calculation in Proposition 272.2. Every generic point lifts through the displayed cover over $\mathbb{C}$, since a nonzero finite-dimensional complex fiber algebra has a complex point by Lemma 271.1. Thus its point group is K_B/K_A with the indicated factors. □

Theorem 276.3. *Take the hypotheses of Proposition 276.2. There is a line $W_0 \subseteq \mathbf{F}_p^n$, independent of A, with the following property. For $B = A + W_0$ and $W_0 \not\subseteq A$, put $U = (G_A)_k$. On all factors of $U[p]$ except possibly the factor at $\bar{w}$, the special subgroup $(K_{A,B})_k$ equals $\ker F_{U,p}$. On the $\bar{w}$-factor the two subgroups both have length p^{n-1}. If $n = 1$, equality also holds on this factor.*

Proof. Choose a line of generic w-labels in $G[w]$ reducing to the identity, as in Theorem 273.3. By Lemma 276.1, the p generic points on the w-factor of $K_{A,B}$ are the images of W_0 under ψ_A, since W_0 maps bijectively to B/A. Integral sections and their reductions commute with ψ_A, so every one of them reduces to zero. The special subgroup on that factor is therefore supported at zero and has length p, by Proposition 271.2 and Proposition 276.2. The w-factor of $U[p]$ has one-dimensional tangent at zero by Lemma 273.2. Its local algebra there has a unique quotient ideal of colength p by Lemma 271.1. The w-factor of the Frobenius kernel has precisely that support and length by Theorem 274.2, so the two are equal.

At a remaining place over u both have length one and contain the identity, by Lemma 276.1 and Theorem 274.2. At each remaining place over $\bar{u}$, both are the full p-torsion factor, since both are closed subgroups of its length p^n and already have that length. At $\bar{w}$ the rank and dimension formulas in the same results give the common length p^{n-1}. This is one when $n = 1$, proving equality there in that case. □

277. Characters of finite groups and the first Frobenius kernels

Lemma 277.1. *Let H be a finite commutative group scheme over a perfect field k of characteristic $p > 0$. There is a finite commutative group $H^\vee$ over k whose points over*

a k-algebra S are the group characters $H_S \to \mathbb{G}_{m,S}$, with pointwise multiplication. Here $\mathbb{G}_m(S) = S^\times$. The natural evaluation identifies H with its double dual. There is a natural map $V_H : H^{(p)} \to H$ satisfying

$$V_H F_{H,p} = [p]_H, \qquad \langle V_H t, \chi \rangle = \langle t, F_{H^\vee,p}\chi \rangle_{(p)}.$$

The brackets on the right evaluate the twisted perfect duality between $H^{(p)}$ and $(H^\vee)^{(p)}$.

Proof. Let D be the finite-dimensional coordinate algebra of H. Addition gives a coassociative cocommutative algebra map $\Delta : D \to D \otimes D$, the identity gives evaluation $\epsilon : D \to k$, and inverse gives the map on functions for negation. These assertions are the group identities under pullback. On the linear dual $D^* = \mathrm{Hom}_k(D, k)$ define multiplication by $(bc)(x) = (b \otimes c)\Delta(x)$ and unit ϵ. Use the transpose of ordinary multiplication of D as comultiplication of D^*, evaluation at $1 \in D$ as counit and the transpose of negation as inverse. The ring, multiplicativity and group identities on these maps follow by transposing the corresponding identities on D, using the natural tensor dual identification in finite dimensions. Thus they define a finite commutative group on $\mathrm{Spec}(D^*)$.

A linear map $D^* \to S$ is a tensor $v \in S \otimes D$. It is a unital algebra map exactly if $\Delta(v) = v \otimes_S v$ after extending scalars to S, and $\epsilon(v) = 1$. This follows by evaluating on all pairs of dual basis elements and on the dual unit. Such v is a unit, its inverse being the pullback by negation by the group inverse identity. Sending the coordinate on $\mathbb{G}_m$ to v is therefore exactly a character. This proves the representation assertion. Dualizing twice restores D with all its operations, and under the described identification the resulting map is evaluation. A map of finite groups dualizes by precomposition of characters, equivalently by transposing its coordinate ring map.

There is a natural identification $(D^*)^{(p)} = (D^{(p)})^*$ from the perfect pairing

$$(b \otimes c,\ x \otimes d) \longmapsto cd\, b(x)^p.$$

Twisted dual bases pair as dual bases. Define V_H to be the dual of $F_{H^\vee,p}$ in this identification. The evaluation identity and naturality in H follow from precomposition of characters and naturality of relative Frobenius in Lemma 272.1.

For the remaining identity take a basis u_i of D and dual basis b_i. The transpose defining V_H sends $x \in D$ to $\sum_i b_i^p(x)(u_i \otimes 1) \in D^{(p)}$. Here b_i^p is the p-th power in the dual multiplication. After pulling back by Frobenius this is $\sum_i b_i^p(x) u_i^p$. On the other hand the pullback of $[p]$ is the iterated comultiplication into p copies of D followed by multiplying all p factors. The tensor coefficients in the basis u_i are invariant under cyclic rotation by commutativity of addition on H. Nonfixed cyclic orbits of indices have length p, so their terms cancel after multiplication. The tuple $(i, \ldots, i)$ has coefficient $b_i^p(x)$, by the dual multiplication formula. This gives exactly the previous sum and proves the identity. □

Lemma 277.2. *Take the hypotheses of Lemma 273.2 for $m = 1, 2$ and fix a prime $v \mid p$. On special fibers put $H_1 = (H_{v,1})_k$, $H_2 = (H_{v,2})_k$. Then $[p] : H_2 \to H_1$ is surjective finite locally free of rank p^n, with kernel H_1.*

Proof. The idempotents modulo p^2 reduce to those at the same prime modulo p by the residue labels of Lemma 273.1. The functor of points therefore identifies the kernel of $[p]$ on this factor with H_1. Every k-point of H_1 specializes from an integral point labelled by one of its complex points by Proposition 271.2. Each label lifts under $[p]$ to the degree-two group of labels in $p^{-2}\mathbb{Z}_p^n/\mathbb{Z}_p^n$, using the finite coordinates of Lemma 269.1. Evaluating at this generic point gives an integral point by the same valuation result. Its reduction lifts the chosen k-point. Each fiber is thus isomorphic to H_1 by translation by such a lift, including the scheme structure since the group map commutes with translations.

The ranks over k of the two finite coordinate algebras are $d = p^n$ and d^2, respectively, by Theorem 274.2. Decompose the first algebra as $\prod_x B_x$ by Lemma 271.1; the second decomposes as a module over it into factors M_x by the idempotents. Each $M_x/\mathfrak{m}_x M_x$ has

dimension d by the fiber isomorphism. Lifting a basis gives a surjection $B_x^d \to M_x$, since its cokernel equals its product by the nilpotent ideal $\mathfrak{m}_x$ and must be zero by iteration. Summing the dimensions gives equal total dimensions d^2 for the sources and targets of these surjections. Thus every one is an isomorphism. This gives the asserted local freeness and surjectivity. □

Proposition 277.3. *Let H_1, H_2 be as in Lemma 277.2, let H' be a finite commutative group over k, and let $\gamma : H_1 \times H' \to \mathbb{G}_m$ be bilinear as a morphism of group functors in its two variables. Then γ is identically one on* $\ker F_{H_1,p} \times \ker F_{H',p}$.

Proof. By Lemma 277.1, γ is equivalently a homomorphism $\lambda : H' \to H_1^\vee$; on an affine test scheme the first variable gives the character for each value of the second variable. Naturality gives a morphism by evaluating at the universal point of the finite affine scheme H'. Thus a point $y \in \ker F_{H',p}$ maps to a character in $\ker F_{H_1^\vee,p}$. For any test scheme and $x \in \ker F_{H_1,p}$, pull back the finite locally free cover in Lemma 277.2 along x. On this cover choose $t \in H_2$ with $pt = x$. Naturality of Frobenius gives $pF_{H_2,p}(t) = F_{H_1,p}(x) = 0$, viewing $H_1^{(p)}$ in $H_2^{(p)}$. Thus $F_{H_2,p}(t)$ belongs to $H_1^{(p)}$, since formation of the kernel in that lemma commutes with twisting the scalars. Call it z. Naturality of V for $H_1 \hookrightarrow H_2$ and Lemma 277.1 give $x = V_{H_1} z$. Its value with $\lambda(y)$ is one by the evaluation identity and $F_{H_1^\vee,p}\lambda(y) = 0$. Equality to one descends across this cover by uniqueness in Lemma 275.2. Applying the assertion to the universal pair of points proves the morphism identity. □

278. The annihilator at the conjugate prime

Lemma 278.1. *Let H be a finite commutative group over an algebraically closed field k of characteristic $p > 0$, with tangent dimension one at its identity. Put $J = \ker F_{H,p}$ and write $d = \dim_k \mathcal{O}(H)$. Suppose J has length p. The subgroup $J^\perp \subseteq H^\vee$ of characters trivial on J is closed of length d/p.*

Proof. Write $D = \mathcal{O}(H)$, $C = \mathcal{O}(J)$, and $\pi : D \to C$ for restriction. In the dual algebras of Lemma 277.1, the restriction map on characters $H^\vee \to J^\vee$ corresponds to the inclusion $C^* \to D^*$. Its kernel at the identity is defined by the ideal $D^* C_+^*$, where $C_+^* = \{\lambda : \lambda(1) = 0\}$. Its quotient dimension equals the dimension of the orthogonal vector subspace of D to this ideal. By dual multiplication, this is

$$D^J = \{b \in D : (\mathrm{id} \otimes \pi)\Delta(b) = b \otimes 1\}.$$

Indeed pairing the second tensor component with all functionals vanishing at $1 \in C$ forces that component into $k\,1$, as is seen by extending 1 to a basis. The counit then gives b for the first component. These are exactly the functions invariant under translation by J.

By Lemma 271.1 and translation each local factor at $x \in H(k)$ is $D_x = k[t]/(t^e)$, where e is independent of x. The subgroup J is supported at zero by Lemma 272.1; its local algebra is $k[t]/(t^p)$, since it has length p and is a quotient of D_0. Every p-th power of an element $b \in D$ is invariant. In fact the difference of the coaction on b and $b \otimes 1$ lies in $D \otimes \mathfrak{m}_C$, by the counit. Its p-th power vanishes using the algebra just calculated.

Choose a nonzero tangent vector of J at zero and evaluate the coaction on that dual-number point in its second factor. The resulting map is $b \mapsto b + \partial(b)\varepsilon$, with ∂ a k-derivation which kills every invariant. It preserves each local factor, since a derivation kills an idempotent a: $(1-2a)\partial a = 0$ and $(1-2a)^2 = 1$. On D_x the value $\partial(t)$ is a unit. Indeed the chosen tangent is nonzero in H, by injectivity of dual-number points for a closed immersion, and translation by x is an isomorphism taking it to the tangent at x. The parameter t detects any nonzero vector there by Lemma 271.1. Applying ∂ to $t^e = 0$ gives $et^{e-1}\partial(t) = 0$, so $p \mid e$. Written on the basis $1, t, \ldots, t^{e-1}$, its kernel is exactly the span of t^j with $p \mid j$, by the power rule and invertibility of $\partial(t)$. These are precisely the p-th powers in D_x since k is perfect. Thus on the product the invariants are all the p-th powers and have dimension d/p. This computes the length of $J^\perp$. □

Proposition 278.2. *At the safe place of Lemma 276.1, omit also the rational primes dividing h of Proposition 253.3. For a label A denote the period lattice of its target by Λ_A and its integral Riemann form by E_A. On $p^{-1}\Lambda_A/\Lambda_A$ the formula*

$$b_A(v,t) = \exp(2\pi i\, pE_A(v,t))$$

defines a perfect alternating pairing with values in the p-th roots of unity. It pairs the factor at w perfectly with the one at $\bar{w}$. For $A \subset B$, $\dim B = \dim A + 1$, it is trivial on pairs in the generic kernel $K_{A,B}(\mathbb{C})$ of Proposition 276.2.

Proof. Changing a lift by a lattice element changes $pE_A(v,t)$ by an integer. Its product by p is also integral, and alternation and bilinearity follow from Proposition 253.3. In a lattice basis, the Gram determinant of E_A has absolute value h^{2n} by the dual quotient in that proposition and the determinant index of Lemma 20.2. Thus its reduction modulo p is nonsingular. Writing lifts as $v = v_0/p$, $t = t_0/p$, gives $b_A(v,t) = \exp(2\pi i E_A(v_0,t_0)/p)$. The exponential identifies the additive group of $\mathbf{F}_p$ with the p-th roots of unity, so this pairing identifies the full space with its character dual. The trace formula defining E_A gives $E_A(av,t) = E_A(v,\bar{a}t)$ for $a \in O_E$. Passing to the idempotents modulo p in Lemma 273.1 shows the w-factor pairs trivially with all except $\bar{w}$; global perfection and their equal dimensions imply perfection on the pair.

Let the initial signed form be E_0. The normalization of the signed frame in Proposition 254.1 gives $E_A = p^{\dim A}E_0$. Indeed the two conjugate inverse frame matrices in Theorem 273.3 have scalar Gram product of valuation $\dim A$ at p, and unit Gram product away from p. Multiplying the initial form by the positive rational $p^{\dim A}$ therefore gives a unit Gram multiplier for the new frame at every finite prime, preserving the line and signatures. The sign and the scalar are unique by the frame argument of Lemma 247.2. Consequently $E_B = pE_A$. The complex intermediate isogeny is the identity on the common covering vector space modulo the enlarged lattice Λ_B, by the construction in Proposition 254.1 and the generic inclusions of Lemma 276.1. Represent its kernel points by elements of Λ_B. Their pairing exponent $pE_A(v,t) = E_B(v,t)$ is an integer by integrality on this lattice. □

Theorem 278.3. *In Theorem 276.3, suppose also that $G_A[p^2]$ is finite flat for every A. At each A assume a bilinear morphism over V*

$$(G_A[p])_w \times (G_A[p])_{\bar{w}} \longrightarrow \mathbb{G}_m$$

whose generic fiber is b_A on those factors and whose special fiber induces an isomorphism from the $\bar{w}$-factor onto the character dual of the w-factor. Then for every $B = A + W_0$ with $W_0 \not\subseteq A$ there is a unique O_E-linear special group isomorphism

$$i_{A,B} : (G_A)_k^{(p)} \longrightarrow (G_B)_k, \qquad (\psi_B)_k = i_{A,B}F_{(G_A)_k,p}(\psi_A)_k.$$

In particular if sections of G_A, G_B are the images of common sections of G, $i_{A,B}$ carries the Frobenius of the former reductions to the latter reductions.

Proof. Write H, H' for the two special torsion factors at $w, \bar{w}$ of $U = (G_A)_k$ and $J = \ker F_{H,p}$. This kernel is the intersection of $\ker F_{U,p}$ with H, since the twisted inclusion is a closed immersion and relative Frobenius commutes with the inclusion. Its length is p by Theorem 274.2. It is already the w-factor of the special $K_{A,B}$ by Theorem 276.3. Under the assumed special duality the annihilator of J in H' has length p^{n-1} by Lemma 278.1. The special pairing is trivial on the two factors of $K_{A,B}$. Indeed their product over V is finite free by Proposition 276.2, so its coordinate algebra injects into its generic algebra. On that algebra the pairing equals one by Proposition 278.2, the generic factors being split reduced. The $\bar{w}$-factor of the special kernel is therefore contained in the annihilator. So is $\ker F_{H',p}$, by Proposition 277.3. Both have length p^{n-1} by Theorem 276.3; the surjections from the coordinate algebra of the annihilator are isomorphisms by dimension. Combining the factors proves $\ker(\phi_{A,B})_k = \ker F_{U,p}$.

Thus $F_{U,p}(\psi_A)_k$ and $(\psi_B)_k$ have the same kernel, by Proposition 276.2. Both are surjective finite locally free. Indeed Proposition 275.1 gives such a presentation for the Frobenius. Its map on underlying open sets is a bijection by Lemma 272.1. Near each point of its target we may consequently shrink a free affine Frobenius chart to a principal open whose inverse image lies inside a free affine chart for $(\psi_A)_k$. This inverse image is affine since Frobenius is finite. Restricting the two free bases to these opens and multiplying their elements gives a free basis for the composite.

Apply Lemma 275.2 in both directions to the two maps with equal kernels. Their factorizations are inverse by uniqueness, as in Corollary 275.3. Their scalar compatibility and statement on sections follow by that same uniqueness and by evaluation. □

279. Constructing the torsion pairing from a theta line bundle

Definition 279.1. For a commutative group scheme G over a base and a line bundle L on G, write $\mathrm{Pic}(T)$ for the group of isomorphism classes of line bundles on T, with tensor product denoted by addition. A line bundle is a locally free sheaf of rank one, with inverse its dual. For T-points x, y of G put

$$l(x) = [x^*L] - [0^*L], \qquad d(x,y) = l(x+y) - l(x) - l(y).$$

Say that L satisfies the parallelogram law if $l(x+y) + l(x-y) = 2l(x) + 2l(y)$ for every T, x, y. Equivalently it suffices to test the two universal points on $G \times G$ over the base, since every other pair gives a pullback of that identity. For a T-point a, write $\pi : G_T \to T$ for projection, t_a for translation by a, and define the line bundle

$$D_a = t_a^* L_T \otimes L_T^{-1} \otimes \pi^*((a^*L)^{-1} \otimes 0^*L).$$

Its pullback by a section z has class $d(a,z)$. Its fiber at the identity is canonically trivial by contracting each line with its dual.

Proposition 279.2. *On every complex torus G of Proposition 253.3, let L be the pullback of the hyperplane line bundle by the canonical theta embedding of Theorem 261.3. It satisfies the parallelogram law of Definition 279.1 as an algebraic line bundle. Its coordinate sections are a basis of $H^0(G, L)$, the space of global algebraic sections.*

Proof. Use the period lattice and H, E_0, η_β of Definition 261.1. The line on the vector space descends to the torus by the factors

$$j_\lambda(z) = \eta_\beta(\lambda) \exp\left(\pi H(\lambda, z) + \tfrac{\pi}{2} H(\lambda, \lambda)\right), \qquad (z,u) \longmapsto (z+\lambda, j_\lambda(z)u).$$

They satisfy the translation rule by Lemma 261.2. The canonical theta sections are precisely the holomorphic sections of this line, by their defining translation law. They have no common zero by Theorem 261.3. On each chart where one is nonzero their ratios give the embedding coordinates, so the line is holomorphically the pullback of the hyperplane line. Its algebraic coordinate sections identify in this way with the theta basis. Any algebraic section is holomorphic and hence is a complex linear combination of this basis by that theorem. Since all its members are algebraic, the basis assertion follows.

On the universal cover of $G \times G$ one has

$$j_{\lambda+\mu}(z+w) j_{\lambda-\mu}(z-w) = j_\lambda(z)^2 j_\mu(w)^2.$$

The H-terms agree by expansion and the two η_β-products agree by expanding the quadratic expression $E_0(\beta v, v)$. Thus there is a holomorphic isomorphism between the product of the pullbacks of L under sum and difference and the product of the squares of its pullbacks under the two projections. The fixed fiber factors in l are constant complex lines and can be trivialized.

This identity of line bundles is algebraic. Indeed both sides, M, N, are generated by finite lists of algebraic sections, obtained by pullback and multiplication of the coordinates, since the group operations are algebraic by Corollary 267.3. Given the holomorphic isomorphism,

regard the two lists as one list of holomorphic sections of M, using the inverse isomorphism on the list of N. It has no common zero and defines a holomorphic map from the compact projectively embedded $G \times G$ to a projective space. This map is algebraic by Theorem 267.2. On an open where the chosen original sections of both bundles are nonzero, the isomorphism is given, in these two frames, by an invertible ratio of coordinates of this map. The ratio and its inverse are regular there. The opens cover, proving algebraicity and hence the universal Picard identity. □

Lemma 279.3. *If L satisfies the parallelogram law, then $d(ma, z) = md(a, z) = d(a, mz)$ in $\mathrm{Pic}(T)$ for all $m \geq 0$ and T-points a, z. In particular, if $ma = 0$, the pullback of D_a by $[m]$ on G_T is a trivial line bundle.*

Proof. Expanding by Definition 279.1 and applying the parallelogram identity at $x + z, y$ and at x, y gives

$$d(x + y, z) + d(x - y, z) = 2d(x, z).$$

Taking $x = ja, y = a$ gives a two-step recurrence for $d(ja, z)$. Its values at zero and one are $0, d(a, z)$, so induction gives the identity in the first variable. The formula is symmetric and hence gives the other variable as well. Apply it on G_T itself with the universal variable point z and the pulled-back section a. The pullback has class $d(a, mz) = d(ma, z) = 0$, which by definition of Pic gives a trivialization. □

Theorem 279.4. *Let V be a valuation ring of $\mathbb{C}$ with algebraically closed residue field of characteristic p. Let G be a flat projective commutative group over V with generic fiber one of the complex tori in Proposition 279.2. Suppose $Q = G[p]$ is finite flat. Suppose a line bundle L over G satisfies the parallelogram law and on the generic fiber is the canonical theta line of that proposition. There is a unique morphism $b : Q \times Q \to \mathbb{G}_m$ extending*

$$b(v, t) = \exp(2\pi i\, pE_0(v, t))$$

on the generic torus torsion, in the lift convention of Proposition 278.2. It is alternating and bilinear over V.

Proof. First $\Gamma(G, \mathcal{O}_G) = V$. A regular function restricts to a holomorphic function on the compact connected complex torus, hence is constant there by the maximum principle (apply the one-variable principle of Lemma A.3 to complex lines in a chart of a maximum). Its value on the identity section belongs to V, and subtracting it gives zero on every affine chart by flatness over the domain V. Indeed a flat module over V injects into its localization at the fraction field by the torsion-free argument of Proposition 272.2. For each flat V-algebra S the same equality gives $\Gamma(G_S, \mathcal{O}) = S$. Take a finite affine cover of the projective scheme G. Intersections of its affines are affine since they are obtained by pulling back its closed diagonal to the affine product over V. The ring of global functions is the kernel of the difference of restrictions from the finite product of chart rings to the finite product on intersections. After tensoring with the flat algebra S this remains the kernel, giving the claim. In particular it holds for finite products of Q over V, whose coordinate algebras are finite free by Proposition 271.2's torsion-free module argument.

Let a be the universal point on Q. On G_Q, choose a nowhere-zero section s of $[p]^* D_a$ using Lemma 279.3. Over $Q \times Q$ use the two universal torsion points a, t. Since $[p]t = 0$, translation by t preserves the map $[p]$, so identifies the translated pullback of D_a canonically with the same pullback. In this identification the quotient $t_t^* s / s$ is an invertible regular function on $G_{Q \times Q}$, hence an invertible function on $Q \times Q$ by the preceding paragraph. It defines b. Any change of the chosen section multiplies it by a unit from Q, which cancels in the quotient. This construction commutes with pulling back the section and its scalar identity to any base scheme.

On the complex fiber let v represent a modulo the period lattice and let w represent the point t. In the factors of Proposition 279.2 the translation difference D_a has lattice

multiplier $\exp(\pi H(\lambda, v))$, by dividing $j_\lambda(z+v)$ by $j_\lambda(z)$. The constant fiber normalization does not affect this multiplier. After pullback by $[p]$, a nowhere-zero holomorphic section is $s(z) = \exp(\pi H(pv, z))$. In fact $H(pv, \lambda) - H(p\lambda, v) = 2iE_0(pv, \lambda)$ and the latter pairing is integral before multiplication by $2i$. Any other nowhere-zero section differs by a constant by compactness. In translating z by w we must identify the fibers of D_a at $pz + pw$ and pz, using the inverse of the multiplier at the lattice vector pw. The quotient is therefore

$$\frac{s(z+w)}{s(z)} \exp(-\pi H(pw, v)) = \exp(2\pi i\, pE_0(v, w)).$$

The coordinate algebra of every finite product of Q is flat over V, hence injects into its generic algebra. The generic torsion is split and reduced since $[p]$ has invertible derivative there, by Lemma 271.1. Its point formula proves uniqueness, and verifies bilinearity and alternation on Q^3 and Q by that same injection. □

280. Detecting the radical in the complete theta coordinates

Lemma 280.1. *Let U be a connected reduced projective scheme over an algebraically closed field k, with a k-point. Then $\Gamma(U, \mathcal{O}_U) = k$. For a line bundle M on U and any k-algebra S, the natural map*

$$H^0(U, M) \otimes_k S \longrightarrow H^0(U_S, M_S)$$

is an isomorphism. In particular all global units of U_S come from units of S.

Proof. We recall the closed-image argument for the constants. The affine point and power-containment argument of Lemma 268.2 holds over every algebraically closed field. Indeed the only field properties used for finite polynomial normalization in Lemma 258.1 are infinitude, to choose a nonzero leading value, and the field operations. For the residue at a maximal ideal one uses algebraic closedness. Such a field is infinite by the polynomial argument of Proposition 271.2. Thus in each affine chart vanishing at all k-points means membership in the radical of the ideal.

For a global function, consider its graph in the product of $\mathbf{P}^1$ with the projective space containing U, viewing its values in the affine chart of $\mathbf{P}^1$. The graph is closed since the projective line has closed diagonal, defined by the homogeneous cross product. Its image on k-points is closed in $\mathbf{P}^1(k)$. At any k-point outside the image, the homogeneous fiber equations of the graph generate an ideal containing powers of every homogeneous variable of the source by the preceding power-containment argument. Their homogeneous multiples consequently span every monomial of a sufficiently large common degree. A nonzero minor witnessing this spanning at the chosen point remains invertible on an affine neighborhood of it, so the graph has empty fibers there, as in Lemma 268.2. The closed image avoids infinity and is therefore finite, by the univariate root bound on an affine chart. The product of the corresponding linear polynomials in the global function vanishes in every reduced affine chart by the radical assertion above. The closed inverse images of its finitely many distinct values are disjoint and cover U, so each is open as well. Connectedness gives just one value, and reducedness gives equality to that constant as a regular function. Evaluation at the assumed point identifies the constants with k.

Choose finitely many affine trivializing charts for M, refining the projective affine cover. They can be chosen from a finite collection since a covering by principal opens of an affine scheme has a finite subcover by the unit ideal argument in Proposition 270.2. Their intersections are affine by separatedness, and the transition functions describe global sections as the kernel of the differences of the restrictions there. Tensoring with S preserves the kernel since vector spaces over a field have bases and injections split on a complementary basis. Sections on each trivializing chart and its intersections tensor to the sections there after base change. This proves the displayed isomorphism; applying it to $M = \mathcal{O}_U$ gives the last assertion. □

Lemma 280.2. *A homomorphism $j : H \to H'$ between finite commutative groups over an algebraically closed field k is a closed immersion if it is injective on k-points and on tangent spaces at the identity. If the two coordinate algebras also have equal dimension over k, it is an isomorphism.*

Proof. At each point of H the differential is injective by translation to the identity. In the local factors of Lemma 271.1, the induced cotangent maps are therefore surjective by duality of finite-dimensional vector spaces. For a local source point x, choose images of elements of the target maximal ideal spanning the source maximal ideal modulo its square. These elements generate the local source algebra over k, by the nilpotence argument of Theorem 274.2. Thus the map of local algebras at $j(x)$ and x is surjective. The source points have distinct target points on underlying sets, and their local factors correspond to the images of the respective target idempotents. The coordinate ring map on the products is surjective, which gives a closed immersion. Equal dimensions make it an isomorphism. □

Theorem 280.3. *Use V, G, Q, L, b of Theorem 279.4. Let $U = G_k$ be connected and reduced, $M = L_k$. Assume $[p] : U \to U$ is surjective finite locally free. Suppose the complete space $D = H^0(U, M)$ gives a projective embedding of U. Suppose there are translations by k-points c_i killed by a positive integer h prime to p, with isomorphisms $t_{c_i}^* M \simeq M$ whose matrices P_i on D span $\mathrm{End}_k(D)$. Then $a \mapsto b(a, -)$ identifies $U[p]$ with its character dual over k. With the action and good split conditions of Proposition 278.2, the induced pairing between w and $\bar{w}$ in Theorem 278.3 is perfect as a pairing of finite group schemes.*

Proof. Write $H = U[p]$ and let $j : H \to H^\vee$ be the homomorphism defined by the special pairing using Lemma 277.1. Take either $S = k$ or $S = k[\varepsilon]/(\varepsilon^2)$, and any point $a \in H(S)$ in its kernel. By the construction of Theorem 279.4, there is a nowhere-zero section s of $[p]^* D_a$ on U_S for which $t_y^* s / s = b(a, y)$ after the natural identification for any kernel point y over any S-algebra. This follows by pulling back the universal section used there. The right side is one for all such y. Thus the two pullbacks of s to $U_S \times_{[p], U_S, [p]} U_S$ coincide, using the identification with $U_S \times H_S$ by subtraction in Lemma 275.2. It descends to a nowhere-zero section of D_a. In fact, on a trivializing open of D_a it is a unit in the covering algebra satisfying the equalizer identity for that finite locally free map. The proof of Lemma 275.2 gives a descended function there, with inverse obtained by the same argument. The sections agree on overlaps by injectivity.

The lines from $\mathrm{Spec}(S)$ in Definition 279.1 are trivial, since any open containing the closed point of the local spectrum is the whole spectrum. Thus there is an isomorphism $t_a^* M_S \simeq M_S$. Pulling back sections and applying this isomorphism gives an invertible matrix P on $S \otimes_k D$ by Lemma 280.1. Its projective action expresses translation on the embedded U_S, with the usual dual convention for point coordinates. An automorphism of a line bundle over the identity is multiplication by a global unit: in each rank-one frame it is a unit and these units agree on overlaps. Such units on U_S belong to S by the same lemma. It follows that P^p is scalar, each P_i^h is scalar, and $P P_i = \lambda_i P_i P$ for some $\lambda_i \in S^\times$, since the underlying translations commute. Iterating this last identity with the two scalar powers gives $\lambda_i^p = \lambda_i^h = 1$. Their exponents are coprime, so $\lambda_i = 1$. Hence P commutes with the full matrix algebra over S by the spanning hypothesis. Commuting with all diagonal coordinate projections makes it diagonal, and commuting with the elementary off-diagonal matrices makes all diagonal entries equal. Thus P is scalar. The projective embedding gives $t_a = \mathrm{id}$, and evaluating at zero gives $a = 0$.

We have proved injectivity on k-points and on dual-number tangents, the latter by the kernel assertion for the second S. Both coordinate algebras have the same dimension since they are linear duals by Lemma 277.1. Apply Lemma 280.2. For the split assertion the adjoint identity $b(ca, y) = b(a, \bar{c}y)$ holds over V by its complex formula in Proposition 278.2 and injection of the finite flat product algebra into its generic algebra. Reduce this identity and apply it to lifts in O_E of the idempotents modulo p. Thus the isomorphism j pairs each

prime summand only with its conjugate. The dual of a product of groups is the product of their duals, since a character is the product of its restrictions to the summands on every test scheme. The corresponding individual maps are isomorphisms since their product is. Alternation allows either order of the two factors. □

281. Theta coordinates indexed by a fixed finite group

Lemma 281.1. *Let T be a finite abelian group of odd order m, k an algebraically closed field whose characteristic does not divide m, and $e : T \times T \to k^\times$ a perfect alternating bicharacter. Perfect means that the induced map to the group of characters of T is bijective. Let $D_0 = D_0^\perp$ be a subgroup, where the perpendicular uses e. Then $|D_0|^2 = m$. Put $I = T/D_0$, $c = -(m-1)/2$, and choose a section $j : I \to T$ with $j(0) = 0$, $j(-s) = -j(s)$. Suppose a nonzero vector space W of dimension $\sqrt{m}$ has operators U_t with $U_0 = 1$ and*

$$U_t U_v = e(t,v)^c U_{t+v}.$$

There is a basis s_b, $b \in I$, unique up to one common scalar, with s_0 fixed by D_0 and $s_b = U_{j(b)} s_0$. In this basis, for $b' = b + t + D_0$ and $l = t + j(b) - j(b')$,

$$U_t s_b = e(t, j(b))^c e(j(b'), l)^{-c}\, s_{b'}.$$

The matrices U_t are a basis of $\mathrm{End}_k(W)$. Any invertible J satisfying $JU_tJ^{-1} = U_{-t}$ acts projectively as the permutation $b \mapsto -b$.

Proof. Characters on a subgroup extend one element at a time, choosing a root of the order of the cyclic extension, as in Lemma 58.1. These orders divide m and are prime to the characteristic, so the root polynomials have exactly their degrees in distinct roots. Hence the dual of each subgroup has its order. Restricting the perfect pairing to D_0 gives a surjection $T \to D_0^\vee$ with kernel $D_0^\perp = D_0$; here the dual denotes ordinary characters of the finite abstract group. This gives the order and identifies I with that dual. No nonzero b equals $-b$ in the odd group I, so the required section exists by choosing representatives in opposite pairs.

The U_l for $l \in D_0$ form an ordinary group representation because e is trivial on these pairs. They commute and diagonalize simultaneously: each is killed by $X^m - 1$ with simple roots, and the eigenspaces of one are stable under all the others. Since $2c = 1 - m$, the commutator of U_t, U_v is $e(t,v)$. The invertible U_t with inverse U_{-t} therefore carry the D_0-eigenspaces amongst themselves by multiplication of their characters by the restrictions of $e(\cdot, t)$. These run through every character of D_0. Since $W \neq 0$, all eigenspaces are nonzero with a common dimension. Their number equals $\dim W$, so each is a line. This proves the asserted uniqueness and the basis property. Applying the product rule twice and using $U_l s_0 = s_0$ gives the displayed action. For nonzero t some U_v conjugates U_t to a nontrivial scalar multiple of itself, so its trace is zero. Multiply a proposed relation among the U_t by U_{-v} and take the trace. The trace of the identity is $\sqrt{m}$, nonzero in k, leaving only the coefficient of U_v. All vanish, and there are $(\dim W)^2$ matrices. Finally J preserves the invariant line and carries $U_{j(b)}$ to $U_{-j(b)} = U_{j(-b)}$, proving the permutation statement. □

Proposition 281.2. *Let $T = L^\perp/L$ and $D_0 = \Lambda/L$ be the finite group and its self-perpendicular subgroup in Theorem 236.2. Fix one section j as in Lemma 281.1. For every framed complex torus, identify T with $\Lambda_x^\#/\Lambda_x$ using the finite frame. On T write $e_x(t,v) = \exp(2\pi i E_x(\tilde t, \tilde v))$, where E_x is its integral Riemann form and the tildes are lifts to its dual lattice. The theta embedding has a basis indexed by this same $I = T/D_0$ in every frame, well defined up to a scalar. The projective coordinates are holomorphic on each period family and are respected by rational changes of frame and by the level group. Translations have the matrices of Lemma 281.1 for $e = e_x$; inversion is its indicated permutation in projective coordinates.*

Proof. The local dual lattices are computed in Proposition 253.3. Transport by the finite frame and reduction of rational lifts as in Lemma 269.1 give exactly T. Changing a lift changes E_x by an integer, so e_x is well defined. It has trivial radical by the dual-of-the-dual calculation of Theorem 262.1, and hence is perfect by the finite character count of Lemma 58.1. The subgroup D_0 is self-perpendicular also for e_x. Indeed every subgroup of the finite abelian T is the product of its primary parts: the corresponding primary projections are integer multiplications by the Chinese remainder identity on the exponent. Pairings between different primary parts are one since their orders are coprime. In local frame coordinates each alternating trace pairing is scaled by a unit $\varepsilon_p \in \mathbb{Z}_p^\times$ by Proposition 253.3. A local fractional trace is integral exactly when its product by this unit is integral. Thus its annihilator on each primary part is precisely that of Theorem 236.2, giving $D_0 = D_0^\perp$. Here the agreement of the rational and local integrality tests follows by taking all completions of the rational trace, as in Lemma 13.4. The order $m = h^{2n}$ is odd since $2 \mid P$ in Lemma 236.1 and h is prime to P.

Replace β in Definition 261.1 by $\beta' = m\beta - (m-1)/2$. It still satisfies $\beta' + \bar{\beta}' = 1$. On a lattice vector the multiplier $\eta_{\beta'}$ equals $\eta_\beta^m = \eta_\beta$: $E_x(\lambda, \lambda) = 0$ and $\eta_\beta(\lambda)$ is a sign. Hence the section space and its embedding are the same. On the dual lattice take $U'_t = \eta_{\beta'}(\tilde{t})^{-1} W_{\tilde{t}}$. The proof of Lemma 261.2 applies to β', which preserves the lattice and its dual. Since $m\beta\tilde{t} \in \Lambda_x$, it gives $U'_t U'_v = e_x(t, v)^c U'_{t+v}$. The operators depend only on the indicated classes. Their matrices are constant over the period ball by Theorem 262.1: the further scalar adjustment uses only E_x, β' and fixed real lifts. The invariant line and the other new basis vectors of Lemma 281.1 can therefore be chosen by a fixed constant change of the old basis on that ball. Projective evaluations are then holomorphic by Theorem 261.3. For inversion, $S(z) \mapsto S(-z)$ acts on the section space: the multiplier at $-\lambda$ is the same sign as at λ and the two linear arguments change sign together. It conjugates U'_t to U'_{-t} by the defining formula. Apply the last assertion of Lemma 281.1.

Rational equivalences carry the trace form, the lattice, the complex structure and the labelled torsion to those of the other representative. Pullback therefore intertwines the U'_t with the same label by their defining formula. A level change fixes the T-labels by the divisibility argument in Corollary 262.2. Such an intertwiner carries the invariant line and the basis specified from it to their counterparts, up to a common scalar. This proves the descent assertions. □

Corollary 281.3. *For an odd prime $p \nmid h$, and a valuation ring of $\mathbb{C}$ of residue characteristic p, the matrices in Proposition 281.2 are integral and invertible over the valuation ring. They specialize to a basis of the full matrix algebra over the residue field. When the finite pairing e_x is replaced by e_x^p, all entries of the new matrices are the p-th powers of the previous entries.*

Proof. Each nonzero entry is a product of integral powers of values of e_x, hence a root of unity of order dividing $m = h^{2n}$. Roots of unity are units in a valuation ring by integrality of them and their inverses, as in Proposition 271.2. The matrices permute basis vectors up to these units. If a nonidentity root ξ of order $s \mid m$ reduced to one, reducing $1 + \xi + \cdots + \xi^{s-1} = 0$ would give $s = 0$ in the residue field, impossible. Thus the reduced pairing remains perfect and the commutator trace argument of Lemma 281.1 still gives linear independence and spanning, also directly over the residue field. The matrix formula of that same lemma commutes termwise with raising the nonzero entries to p. □

282. Equality of the marked projective coordinates after Frobenius

Lemma 282.1. *Let $b : X \to Y$ be a surjective finite locally free homomorphism of connected reduced projective commutative groups over an algebraically closed field k. Suppose* $\ker b$ *is killed by an odd positive integer d. If a symmetric line bundle M on Y, meaning $[-1]^* M \simeq M$, has trivial pullback under b, then M is trivial.*

Proof. Fix a nowhere-zero section s of b^*M. Translation by a kernel point z identifies its pullback with b^*M, because $b\,t_z = b$. The ratio $t_z^* s/s$ is thus a scalar $\lambda(z)$ on the test scheme by Lemma 280.1, applied to the universal point of the affine finite kernel. Composition of translations gives a character λ, independent of multiplying s by a constant. The line $[-1]^*M$ gives the inverse character by pulling the section back under inversion, since that map conjugates t_z to t_{-z}. Isomorphic lines give the same character by transporting the trivializing section. Thus $\lambda^2 = 1$, while $\lambda^d = 1$ since the kernel is killed by d. A Bezout identity gives $\lambda = 1$. As in the proof of Theorem 280.3, the section now has identical pullbacks to $X \times_Y X$ and descends to a nowhere-zero section of M by Lemma 275.2. □

Lemma 282.2. *Use the prime-to-characteristic data and basis of Lemma 281.1. The only projective linear transformation commuting projectively with all U_t and with the inversion permutation J is the identity. Here $Js_b = s_{-b}$.*

Proof. Take a linear representative P. The scalars in $PU_tP^{-1} = \lambda_t U_t$ form a character of T, by multiplying the displayed equations and using the product law of Lemma 281.1. By perfection $\lambda_t = e(v,t)$ for some $v \in T$. Thus $U_{-v}P$ commutes linearly with all U_t and is scalar by their spanning the full endomorphism algebra. The basis formula, $j(-b) = -j(b)$, and bilinearity give $JU_vJ^{-1} = U_{-v}$, since simultaneously negating both arguments of the bicharacter preserves it. Projective commutation with J now gives that U_{2v} is scalar. Its commutators and perfection imply $2v = 0$. The odd order of T gives $v = 0$, as required. □

Theorem 282.3. *Assume the integral group and isogeny hypotheses of Theorem 278.3, including $p \nmid 2h$. Its special perfect pairing condition can be supplied by the following theta data. Let L, L_A on G, G_A be symmetric line bundles with the parallelogram identity and the stipulated generic theta lines. Assume on the special fibers that*

1. *the groups are connected, reduced and projective, $[p]$ is surjective finite locally free, and the labelled homogeneous coordinates of Proposition 281.2 give the complete linear embeddings for these lines, with their indicated matrices for translations and inversion,*
2. *$(\psi_A)_k^*(L_A)_k \simeq (L_k)^{\otimes p^{\dim A}}$ for every A, and the integral groups are projective and flat over V.*

Mark any finite list of sections of G/V and use their images under each ψ_A. Include the labelled T-translations as common sections with the specified labels. Write z_A for the tuple of projective coordinates of the reduced marked sections on G_A. Writing Fr *for coordinatewise p-powering, there is a line $W_0 \subseteq \mathbf{F}_p^n$ such that*

$$\mathrm{Fr}^{\,n-\dim A} z_A = \mathrm{Fr}^{\,n-\dim(A+W_0)} z_{A+W_0} \qquad (A \subseteq \mathbf{F}_p^n).$$

In particular the sum in Corollary 269.3 is zero for these tuples.

Proof. By Corollary 281.3 the special translation matrices span the endomorphism algebra. Their points have order dividing h. Theorems 279.4 and 280.3 provide the integral pairing and special perfection required in Theorem 278.3. Take its line W_0. When $W_0 \subseteq A$ the asserted identity is immediate. Otherwise put $B = A + W_0$, $r = \dim A$, $U = (G_A)_k$, and take the isomorphism $i : U^{(p)} \to (G_B)_k$ of that theorem. Let M_A, M_B denote the two special theta lines. Then

$$i^* M_B \simeq M_A^{(p)}.$$

Indeed pull back their ratio by the surjective finite locally free map $b = F_{U,p}(\psi_A)_k$ in that theorem. For any line bundle N, $F_{U,p}^* N^{(p)} = N^{\otimes p}$ as classes, since its transition functions are raised to p. Thus both pullbacks are $L_k^{\otimes p^{r+1}}$ by the assumed identities. The ratio is symmetric since both lines are symmetric and i is a group map. The kernel of b equals that of $(\psi_B)_k$, killed by p^{r+1} in the proof of Corollary 274.3. Apply Lemma 282.1.

For any two labels of T the finite alternating pairings on the generic fibers satisfy $e_B = e_A^p$. Indeed the two points on the B-fiber are the images of those on the A-fiber, and their lifts in the common vector space may be kept. Their Riemann form is multiplied by p by Proposition 278.2. The two labels on reduction are still matched by i after Frobenius by the section assertion in Theorem 278.3. In the chosen section bases the translation matrices for the twisted A-fiber and the B-fiber therefore agree by Corollary 281.3. Their inversion permutations agree as well. An isomorphism of the two lines above gives a linear isomorphism on their complete spaces of sections. It intertwines all these matrices projectively: the actions lift the same translated points under i, and two isomorphisms of the same lines above the same map differ by a global unit, a scalar by Lemma 280.1. The same applies to inversion. Consequently the intertwiner is projectively the identity by Lemma 282.2. Evaluation at each marked section and its matched image gives $z_B = \mathrm{Fr}(z_A)$. Here the twist of a projective coordinate list on a Frobenius image is its coordinatewise p-th power by Lemma 272.1. Applying Fr^{n-r-1} proves the displayed identity. The formal cancellation now follows from Corollary 269.3. □

283. One projective coordinate space for all level components

Lemma 283.1. *Let $\mathcal{T} = \mathbb{A}^n_{E,f}/L$, and for a frame x write $P_x(t)$ for its torus torsion point of label $t \in \mathcal{T}$ via Lemma 269.1. Write $F_{x,b}$ for its theta embedding in the common coordinates I of Proposition 281.2, with period parameter b. Suppose x, x' are two frames with parameters b, b' and $k \in C$, and*

$$F_{x,b}(P_x(t)) = F_{x',b'}(P_{x'}(kt)) \qquad (t \in \mathcal{T}).$$

Then there is $y \in \mathrm{GL}_n(E)$ with $yx = x'k$ and $yb = b'$, where y acts on the period structures.

Proof. The torsion points of each real torus are dense by rational approximation in a lattice basis, as in Proposition 263.2. The two images of the entire tori are compact and equal, since they are the closures of the same set of labelled point images and k permutes all labels. The composite $Y = F^{-1}_{x',b'}F_{x,b}$ is a biholomorphic map of tori matching those labels. The label maps $P_x, P_{x'}$ are O_E-linear by their finite frame formula, and k is O_E-linear. On the dense torsion subgroup Y is therefore additive and respects every element of O_E. Continuity gives these identities on the whole torus, using also density of the product of the two copies of the torsion.

Let y be its real derivative at zero in the covering space coordinates. The curve $s \mapsto Y(sv + \Lambda_x)$ has constant velocity yv in translated target coordinates, by additivity and differentiation at zero. Subtracting the curve $s \mapsto syv + \Lambda_{x'}$ has derivative zero and initial value zero, so the two agree for every real s, by local vector-space charts. The inverse map gives $y\Lambda_x = \Lambda_{x'}$. It follows that y preserves their common rational span E^n. Its O_E-linearity there implies E-linearity by taking fractions, so $y \in \mathrm{GL}_n(E)$. Biholomorphy at zero gives $yI_b = I_{b'}y$, hence $yb = b'$ by Proposition 252.2.

In finite frame coordinates the action on torsion says that $x'^{-1}yx - k$ carries $m^{-1}L$ into L for every positive integer m. Thus each matrix entry lies in $m\widehat{O}_E$ for every m. At a prime over p, taking $m = p^a$ and letting a increase forces it to be zero by the local valuation. This proves $yx = x'k$. □

Theorem 283.2. *Fix an open subgroup C' of finite index in C. For each $M \geq 1$, let D_M be the permutation group on $\mathcal{T}[M]$ induced by C'. For a framed base point put*

$$\mathbf{v}_M(x,b) = \bigotimes_{t \in \mathcal{T}[M]} F_{x,b}(P_x(t)), \qquad \mathbf{z}_M(x,b) = \prod_{\gamma \in D_M} \gamma \mathbf{v}_M(x,b).$$

The first expression is the line of a simple tensor of point representatives in the fixed ordered coordinates. The second is the line of the product of the indicated linear forms on the dual of that tensor space. There is an M_0 such that for every multiple M of M_0 the map $\mathbf{z}_M$

is a closed holomorphic embedding of the entire base $\mathcal{X}(C',0)$. For $\nu \geq 0$ the joint map with the ν individual $F_{x,b}(P_a)$ is a closed holomorphic embedding of $\mathcal{X}(C',\nu)$ in a product of projective spaces. Its image is algebraic over $\mathbb{C}$. The tensor, permutation and product operations above are given by homogeneous polynomials with integer coefficients.

Proof. Use any fixed order for the finite labels and bases. The tensor product is the projective product embedding of Corollary 260.2. A rational change preserves the evaluations with their labels by Proposition 281.2. Replacing x by xk for $k \in C'$ changes $P_x(t)$ to $P_x(kt)$ in the same underlying torus, by its finite-coordinate definition. It preserves the preferred projective embedding since it fixes the smaller finite group of theta translations and its labelled basis. Thus the orbit product descends. It is holomorphic on the period charts, since for each fixed torsion label one is evaluating at the periods of a fixed real vector; see Proposition 263.2. Equality of two orbit products means equality of the two ordered lists up to an element of D_M, by linear factor uniqueness in Lemma 265.1 and injectivity of the projective product embedding. The factor lines of a single such product are distinct. Indeed the points $P_x(t)$ for these labels are distinct, and $F_{x,b}$ embeds the torus, so no nonidentity effective permutation fixes the ordered list.

There are finitely many component representatives x and a compact set K_x in each period ball meeting every orbit for that component by Theorem 265.2 and the compact representative argument of Theorem 264.2. On each K_x, finitely many torsion evaluations have jointly injective derivative and suffice for local injection near every point of K_x, by Proposition 263.2. Its proof applies in the present coordinates since the change of section basis in Proposition 281.2 is constant on this ball. For multiples of a common integer the tensors therefore immerse each base chart. For the orbit product, reduce its differentiated expression modulo the linear factor of the identity as in Lemma 265.1. Distinctness of those factors shows that a vanishing projective derivative of the product forces that of the ordered tensor to vanish. Thus the orbit map is immersive at every representative.

To prove separation, suppose injectivity fails for arbitrarily large factorial levels $M = j!$. Choose pairs of distinct base classes with equal product. After taking subsequences, their representatives have fixed frames x, x' and parameters $b_j \in K_x,\ b'_j \in K_{x'}$ converging to b, b'. Take $k_j \in C'$ carrying one ordered list to the other in the direction

$$F_{x,b_j}(P_x(t)) = F_{x',b'_j}(P_{x'}(k_j t)) \quad (t \in \mathcal{T}[j!]).$$

After a further subsequence $k_j \to k \in C'$. For clarity, extract successively subsequences constant modulo each integer factorial in the integral matrix coordinates. The compatible limits of the matrices and their inverses define an invertible integral matrix by the finite local residue description of Lemma 225.1. The limit lies in C' since C is compact closed and an open finite-index subgroup is closed by its open cosets. The labels of any fixed order are eventually fixed under $k^{-1}k_j$. Passing to the limit hence gives the hypothesis of Lemma 283.1. Take y supplied by that lemma.

Pulling the second parameter back by y^{-1} now puts it in the first ball, tending to b. In this frame the displayed equality of the lists replaces k_j by $k^{-1}k_j$, since $yx = x'k$. Choose fixed torsion labels giving a local injection near b as above. Their orders divide $j!$ and $k^{-1}k_j$ fixes them for all sufficiently large indices. The two parameters must then be identical in that neighborhood: $b_j = y^{-1}b'_j$. Together with $yx = x'k$, this says their original classes coincide at level C', a contradiction. Hence some factorial level large enough for immersion also separates all points. An equality of ordered lists up to C' at any multiple restricts to one at the chosen level, proving the assertion for all its multiples.

The space is compact and a holomorphic injective immersion there is a closed embedding by the local inverse argument of Theorem 257.2. For the marked total space the individual evaluations descend by Proposition 281.2. After the bases are matched, the fiber embeddings separate the marked points and detect their vertical tangents. The same compactness and immersion proof as in Theorem 265.2 now applies. Algebraicity of the image follows from

Corollary 260.2 and the projective product embedding. Finally each coordinate operation takes products of entries, permutes factors, or takes coefficients of a product of linear forms. These are the stated integral polynomials. □

284. Complex periods under conjugation of the coefficients

Lemma 284.1. *Let Y be a smooth irreducible projective commutative group over $\mathbb{C}$, of dimension g. Put $W = T_0 Y(\mathbb{C})$. There is a unique holomorphic additive map $e : W \to Y(\mathbb{C})$ with derivative the identity at zero. It induces an isomorphism $W/\Lambda \simeq Y(\mathbb{C})$ of complex Lie groups, where Λ has a basis over $\mathbb{Z}$ which is a real basis of W. Every algebraic homomorphism between two such groups lifts along these maps to its complex derivative at the identity.*

Proof. The invertible Jacobian minor in a smooth chart gives complex manifold coordinates by the local inverse argument of Theorem 267.2. Projective space is compact as the quotient of the unit sphere; hence its closed algebraic subset $Y(\mathbb{C})$ is compact. Its manifold components are open and closed and there are finitely many by compactness. Each component is a closed embedded complex submanifold of projective space and is algebraic by Corollary 260.2. Irreducibility forces a single component.

Take a holomorphic coordinate z at zero with identity derivative. Write $F(z, w)$ for the local sum, and set $M(z) = \partial_2 F(z, 0)$. It is invertible near zero. The columns $X_j(z) = M(z)e_j$ are carried to themselves by local translations, by differentiating associativity in the last variable. They commute: the mixed derivatives in s, t of $F(F(z, se_i), te_j) = F(F(z, te_j), se_i)$ give $DX_j(z)X_i(z) = DX_i(z)X_j(z)$. Thus the rows of $\omega = M(z)^{-1}dz$ are closed. Indeed their values on the X_j are constant, and differentiating these values and subtracting the two orders gives $d\omega(X_i, X_j) = 0$. On a ball put $h(z) = \int_0^1 \omega(tz)z\,dt$. Closedness and differentiation under this integral give $dh = \omega$ by integrating $d(t\omega(tz))/dt$. Translation invariance gives $h(F(z, w)) = h(z) + h(w)$ near $(0, 0)$, by differentiating in w and setting $w = 0$. The local inverse of h therefore supplies a local additive map e_0 of derivative one. Shrink to a convex ball around zero on which the local sum rule holds whenever both summands and their sum lie in that ball.

For $v \in W$ put $e(v) = me_0(v/m)$ for a sufficiently large positive integer m. Subdividing the segment from 0 to v/m shows that this does not change on replacing m by a multiple; a common multiple proves independence. The local sum rule at $v/m, w/m, (v + w)/m$ gives global additivity. Translations of e_0 give holomorphy everywhere. Its image is open, hence closed by the openness of its cosets, and connectedness gives surjectivity. Its kernel is discrete and the quotient has the local charts of e_0. Any two additive maps with the same derivative have a difference with derivative zero at every point by translation. Their difference is constant, and is zero at the origin. The same argument with a homomorphism composed with e proves functoriality.

The discrete kernel is closed. Its intersection with a bounded set is finite, since two members of any convergent subsequence would have arbitrarily small nonzero difference. Its real span is W, as the compact quotient maps onto the real vector space modulo that span. Choose a real basis from the kernel; its integral span has finite index in the kernel by translating into the bounded basis parallelepiped. Thus the kernel lies between this basis lattice and its multiple by $1/d$, where d is the finite index (multiplication by d kills the quotient, by partition into translates of a cyclic subgroup). A subgroup of $\mathbb{Z}^r$ is free: induct on r by projecting to the last coordinate, adjoining a lift of the positive generator of the image when it is nonzero. Applied in the coordinates of the enlarged basis lattice this gives a basis of rank $2g$, since the kernel spans W. □

Proposition 284.2. *Suppose Y is as in Lemma 284.1 and an algebraic line bundle B satisfies the parallelogram law of Definition 279.1. For $m \geq 1$ and $a, t \in Y[m](\mathbb{C})$ the scalar*

$$b_{m,B}(a, t) = t_t^* s/s, \qquad 0 \neq s \text{ a nowhere-zero section of } [m]^* D_a,$$

is well defined, using the equality $[m]t_t = [m]$ to identify fibers. If B is symmetric, let $q_B(a)$ for $2a = 0$ be the scalar on B_a of the isomorphism $[-1]^*B \to B$ normalized to be one on B_0. These scalars commute with conjugation by every field automorphism of $\mathbb{C}$, acting on all the algebraic data. For the actual period tori and lines of Proposition 279.2, with $a = v + \Lambda$, $t = w + \Lambda$, one has

$$b_{m,B}(a,t) = \exp(2\pi i\, mE_0(v,w)), \qquad q_B(a) = b_{2,B}(\beta a, a) \quad (2a = 0).$$

Proof. Lemma 279.3 gives the section. Regular functions on Y are constant by Lemma 280.1, so its ratio with a translated section is scalar, independent of the section. Two isomorphisms of the two symmetric lines differ by a global unit, giving the uniqueness of the normalization. Double inversion then gives the identity on the whole line, in particular $q_B(a)^2 = 1$. Transport of local defining equations, algebraic sections, and their pullbacks by a field automorphism preserves these ratios and the origin normalization. This proves the asserted naturality.

For the first evaluation use the multiplier of D_a from Theorem 279.4. On the vector cover a trivializing section after $[m]$ is $\exp(\pi H(mv, z))$, since $mv \in \Lambda$ and $E_0(mv, \lambda) \in \mathbb{Z}$ for $\lambda \in \Lambda$. Dividing by the multiplier for mw on translation by w gives the displayed exponential, exactly as in that theorem. A holomorphic nowhere-zero section differs from the algebraic one by a constant by compactness.

The canonical factors satisfy $j_{-\lambda}(-z) = j_\lambda(z)$; the normalized inversion lift in the vector cover is $(z,u) \mapsto (-z,u)$. It is algebraic by the joint section argument of Proposition 279.2, applied to B and $[-1]^*B$. At $v = \lambda/2$ the fiber scalar is $j_\lambda(-\lambda/2) = \eta_\beta(\lambda)$ (the opposite direction gives the same sign). By Definition 261.1 this is $\exp(\pi i E_0(\beta\lambda, \lambda))$, which equals $b_{2,B}(\beta a, a)$ by the first formula. □

285. A positive curvature form with constant coefficients

Definition 285.1. On a complex vector space with $z_j = x_j + iy_j$, let $\partial f = \sum_j (f_{x_j} - if_{y_j})\, dz_j/2$ and $\bar\partial f = \sum_j (f_{x_j} + if_{y_j})\, d\bar z_j/2$. On forms extend by the signed product rule, with the coordinate differentials constant. Thus $d = \partial + \bar\partial$, and commuting real partial derivatives gives $\partial^2 = \bar\partial^2 = 0$ and $\bar\partial\partial = -\partial\bar\partial$. A real $(1,1)$-form ω is positive when $\omega(v, iv) > 0$ for all nonzero real tangent vectors. The curvature of a smooth Hermitian metric on a holomorphic line, in a nowhere-zero holomorphic frame of squared norm h, is $-i\partial\bar\partial \log h/(2\pi)$. Changing the frame multiplies h by $|u|^2$ for a holomorphic unit u. The logarithm of this factor is locally the real part of a holomorphic function (use the power series for the logarithm at a nonzero value). Consequently the formula gives a well-defined real closed $(1,1)$-form.

Lemma 285.2. *Let W be a finite-dimensional complex vector space, Λ a full real lattice, and ω a smooth real closed $(1,1)$-form on W/Λ. Average its coefficients over a lattice parallelepiped with normalized volume and denote the constant form by E_0. There is a real smooth periodic function ϕ such that*

$$\omega - E_0 = \frac{i}{2\pi}\partial\bar\partial\phi.$$

If ω is positive, then E_0 is a positive i-invariant alternating real form.

Proof. Write $\Lambda^\vee = \{k \in W^*_{\mathbb{R}} : k(\Lambda) \subseteq \mathbb{Z}\}$. In a real lattice basis the modes are $u_k(v) = \exp(2\pi i k(v))$. The Fourier coefficients ω_k in fixed constant differentials decay faster than each negative power of $1 + |k|$. In fact for a nonzero integral frequency one coordinate has size at least a fixed positive multiple of $|k|$, and repeated integration by parts in that coordinate moves every derivative to the smooth periodic coefficient with no boundary term. The sum of $(1 + |k|)^{-s}$ over the lattice converges for $s > 2\dim_{\mathbb{C}} W$ by counting successive dyadic cubes. The Fourier series and all its derivatives converge absolutely and represent the stated coefficients: continuous periodic functions with zero Fourier coefficients vanish by the product kernel argument of Lemma 53.2.

Write $k = \alpha + \bar{\alpha}$ for its parts of type $(1,0),(0,1)$. Closedness at each nonzero frequency gives separately $\alpha \wedge \omega_k = 0$ and $\bar{\alpha} \wedge \omega_k = 0$, by their distinct types. Extend α to a complex covector basis; comparison of exterior basis coefficients in the two identities makes ω_k a scalar multiple of $\alpha \wedge \bar{\alpha}$. The latter has norm proportional to $|k|^2$ in a Hermitian orthonormal basis, by taking the first covector in the direction of α. Consequently the equation for this individual frequency is solved uniquely by a coefficient ϕ_k, since

$$\partial\bar{\partial} u_k = -4\pi^2 u_k \, \alpha \wedge \bar{\alpha}.$$

The solution satisfies $|\phi_k| \le C|\omega_k|/|k|^2$. Give ϕ mean zero. Its series is smooth by the same derivative estimates. Complex conjugation of $i\partial\bar{\partial}\phi$ gives $i\partial\bar{\partial}\bar{\phi}$, so reality follows by uniqueness in the nonzero modes. The zero mode of ω is its stated average. A real $(1,1)$-form is invariant under simultaneous multiplication by i in its two arguments, as each elementary $dz_j \wedge d\bar{z}_l$ is. Averaging preserves this identity and gives a strictly positive value on each fixed nonzero pair v, iv if the integrand has that property. □

Proposition 285.3. *Let $F : W/\Lambda \to \mathbb{P}^{d-1}(\mathbb{C})$ be a holomorphic embedding. The line $B = F^*\mathcal{O}(1)$ has a smooth Hermitian metric whose curvature is a constant positive real alternating form E_0. This constant is independent of the original metric used to compute its average curvature.*

Proof. The line $\mathcal{O}(1)$ is the dual of the tautological line in $\mathbb{C}^d$. Its pullback has the metric dual to the Euclidean metric. For a local nonzero holomorphic lift f of F the dual frame has squared norm $1/\sum |f_j|^2$. The curvature of Definition 285.1 is positive. Indeed in real tangent directions v, iv its value is

$$\frac{\|f\|^2 \|df(v)\|^2 - |\langle f, df(v)\rangle|^2}{\pi \|f\|^4}.$$

This follows by differentiating the logarithm of the squared norm in the complex line of direction v. The numerator is the squared length of the component of $df(v)$ perpendicular to f, times $\|f\|^2$. It is strictly positive for $v \neq 0$, since the zero derivative of the chart ratios f_j/f_l would mean that $df(v)$ is a multiple of f, contrary to immersion. Lemma 285.2 now provides ϕ for this curvature. Multiplying the squared norm by $\exp(\phi)$ subtracts $i\partial\bar{\partial}\phi/(2\pi)$, as required. Two smooth metrics on a line differ by a globally defined positive smooth function. The second derivatives of its logarithm have zero average by periodicity in lattice coordinates, proving independence. □

286. Period factors and the signs at half-periods

Lemma 286.1. *A holomorphic line bundle on a complex vector space with a smooth Hermitian metric of zero curvature has a global holomorphic frame of constant unit norm. Any two such frames differ by a constant of modulus one.*

Proof. In a holomorphic frame with squared norm h, write $u = \log h$. By Definition 285.1, $\bar{\partial}\partial u = 0$, so the closed form ∂u has holomorphic coefficients (the Cauchy–Riemann equations in each variable, or the one-variable Cauchy formula successively in a polydisc). On a ball it has a holomorphic primitive v, given by the radial integration formula in Lemma 284.1. As u is real, $d(u - v - \bar{v}) = 0$. The original frame times $\exp(-v)$ therefore has constant norm, which can be scaled to one. Two unit frames differ by a holomorphic function of modulus one, constant on each connected overlap by the maximum principle of Lemma A.3. Starting with a frame value at zero, continue it along a polygonal path in these unit frames, matching values at each subdivision point. Refinements do not change the result because transition constants agree on each connected overlap used by the path. Two paths to the same point are homotopic with fixed endpoints by linear interpolation. Subdivide their compact parameter square into sufficiently small rectangles that each rectangle maps into one frame patch: a finite covering by smaller open sets with closures in the patches gives

a uniform radius by compactness and uniform continuity. Transport around each such boundary is the identity in its one frame; cancellation of paired edges gives equality along the outside paths. Continuation thus gives a single global unit frame, holomorphic by the local construction. The constant ratio argument proves uniqueness up to a unit complex scalar. □

Theorem 286.2. *Let F, B, W, Λ, E_0 be as in Proposition 285.3. Define the Hermitian form, conjugate linear in its first argument, by*

$$H(v,w) = E_0(v,iw) + iE_0(v,w).$$

It is positive definite, $E_0(\Lambda,\Lambda) \subseteq \mathbb{Z}$, and B on W/Λ is the quotient of the trivial line on W by

$$(z,u) \longmapsto (z+\lambda,\ \eta(\lambda)\exp(\pi H(\lambda,z) + \tfrac{\pi}{2}H(\lambda,\lambda))u),$$

where $|\eta(\lambda)| = 1$ and $\eta(\lambda+\mu) = (-1)^{E_0(\lambda,\mu)}\eta(\lambda)\eta(\mu)$. If B is symmetric then $\eta(\lambda) = q_B(\lambda/2+\Lambda)$ with q_B as in Proposition 284.2. If B also satisfies the parallelogram law, then

$$b_{m,B}(\lambda/m+\Lambda, \mu/m+\Lambda) = \exp\left(2\pi i E_0(\lambda,\mu)/m\right) \qquad (\lambda,\mu \in \Lambda,\ m \geq 1).$$

Proof. The invariance $E_0(iv,iw) = E_0(v,w)$ and alternation give Hermitian symmetry and complex linearity in w. Its squared norm $E_0(v,iv)$ is strictly positive off zero by Lemma 285.2. The trivial line on W with squared norm $\exp(-\pi H(z,z))$ in its constant frame has curvature E_0: in a complex orthonormal basis the formula of Definition 285.1 gives $i\sum dz_j \wedge d\bar{z}_j/2 = \sum dx_j \wedge dy_j$. Pull back the metric of Proposition 285.3 to W and tensor by the inverse of this model line. Curvatures subtract and vanish, so Lemma 286.1 gives a holomorphic isometric identification of the pullback of B with the model.

In this trivialization the deck transformation for λ multiplies the fiber by a holomorphic unit $j_\lambda(z)$. Its absolute value is $\exp(\pi\Re H(\lambda,z) + \pi H(\lambda,\lambda)/2)$ since the deck map preserves the metric. Dividing by the holomorphic exponential in the theorem gives a constant $\eta(\lambda)$ of modulus one. Applying the transformations for λ,μ in the two possible orders gives $\exp(2\pi i E_0(\lambda,\mu)) = 1$. Thus this real number $E_0(\lambda,\mu)$ is an integer, and composition in either order now yields the asserted sign in the formula for $\eta(\lambda+\mu)$.

For a symmetric line, compare its normalized inversion lift with the metric pulled back by inversion. The logarithm of the ratio of the squared norms on the torus has $\partial\bar{\partial} = 0$, since inversion preserves the constant curvature. It is constant by the nonzero-frequency calculation in Lemma 285.2; its value at zero is zero. Thus the normalized lift is isometric. Upstairs inversion preserves $H(z,z)$, so its holomorphic fiber multiplier is a constant of modulus one and equals one by normalization. At the half-period $\lambda/2$, translation from $-\lambda/2$ back to $\lambda/2$ multiplies the fiber by $j_\lambda(-\lambda/2) = \eta(\lambda)$. The square of the normalized involution is one, so these constants are signs. Finally, the first calculation in the proof of Proposition 284.2 uses only the displayed transformation factors and the integrality of their imaginary form. In detail the constant multiplier of $D_{\lambda/m}$ at $\gamma \in \Lambda$ is $\exp(\pi H(\gamma,\lambda/m))$. The section $\exp(\pi H(\lambda,z))$ after $[m]$ has this multiplier at $m\gamma$ because $E_0(\lambda,\gamma)$ is integral. Translate by μ/m and identify the resulting fiber by the inverse multiplier at μ. The resulting ratio is the displayed pairing. □

287. The rational form and the finite frame after conjugation

Lemma 287.1. *Fix a framed torus Y of Proposition 281.2 with lattice $D = \Lambda_x$, hyperplane line B, and alternating form E_A. Let $\iota \in \mathrm{Aut}(\mathbb{C}/E)$; apply it to projective coordinates, algebraic maps and equations to form Y', B'. Denote the lattice and curvature form of Y', B' by D', E' using Lemma 284.1 and Theorem 286.2. There is an O_E-linear isomorphism*

$$c : \widehat{D} \longrightarrow \widehat{D'}, \qquad E'_p(cu,cv) = \epsilon_p E_{A,p}(u,v) \quad (u,v \in D \otimes \mathbb{Z}_p)$$

at every prime p. Here $\widehat{D} = \prod_p (D \otimes \mathbb{Z}_p)$, the subscript p denotes scalar extension, and $\epsilon = (\epsilon_p)_p \in \widehat{\mathbb{Z}}^\times$ is uniquely specified by $\iota(\exp(2\pi i/m)) = \exp(2\pi i\epsilon_m/m)$ for each $m \geq 1$,

with $\epsilon_m = \epsilon \bmod m$. On m-torsion the map on the classes of u/m is exactly reduction of c modulo m in the two lattice coordinates.

Proof. The original torus is a compact connected complex manifold algebraic by Corollary 260.2. A proper closed algebraic subset has empty analytic interior: a homogeneous equation nonzero somewhere cannot vanish on an open set, by the identity theorem in connected holomorphic coordinate charts (apply the one-variable identity theorem successively on a polydisc). Two proper closed subsets therefore cannot cover it. Its homogeneous vanishing ideal is prime, as the zero set of a product is the union of the zero sets of its factors. Conjugation preserves primality, the smooth chart Jacobian identities, the algebraic group and O_E-action of Corollary 267.3. It gives a conjugate smooth irreducible projective group, and transports the symmetric isomorphism and the parallelogram law by the algebraicity proved in Propositions 279.2 and 284.2. The cited uniformization and period factors thus apply. Functoriality of uniformization gives the action on the new lattice by derivatives.

Via the two period maps, the ι-image of each point of order dividing m gives an additive O_E-linear bijection $c_m : D/mD \to D'/mD'$, because division of a lattice vector by m identifies each quotient with the whole corresponding torsion group. For $m \mid l$, multiplication of points by l/m corresponds to reduction from l to m and commutes with conjugation. The integral lattices have finite bases; in these bases the compatible matrices and their inverses give mutually inverse matrices over $\widehat{\mathbb{Z}}$. Indeed at p the entries are compatible residues at all p^j, hence p-adic integers, and the Chinese remainder identity recovers the original entries modulo each m. This gives c with the stated action. The powers ϵ_m exist and are units modulo m since the automorphism takes a primitive m-th root to a primitive m-th root. Raising roots to l/m proves their compatibility and the same residue argument gives ϵ. Proposition 284.2 and Theorem 286.2 now give, on all lattice classes modulo m,

$$E'(c_m u, c_m v) \equiv \epsilon_m E_A(u, v) \pmod{m}.$$

Taking $m = p^j$ for every j gives the claimed identity over $\mathbb{Z}_p$, since the intersection of its ideals $p^j\mathbb{Z}_p$ is zero. □

Proposition 287.2. *With the preceding notation, choose any E-basis of $D'_{\mathbb{Q}} = D' \otimes_{\mathbb{Z}} \mathbb{Q}$. It identifies $D'_{\mathbb{Q}}$ with E^n. There are a frame $x' \in \mathrm{GL}_n(\mathbb{A}_{E,f})$ and a nonsingular Hermitian matrix A' over E with*

$$\widehat{D'} = x'L, \qquad D' = E^n \cap x'L, \qquad E'(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}(\xi\, w^* A' v), \qquad x'^* A' x' = \epsilon\, x^* A x.$$

Here the middle form identity is on $D'_{\mathbb{Q}}$, the last is over the finite adeles, and $L = \widehat{O}_E^n$. The labels $t \in \mathbb{A}_{E,f}^n / L$ are carried by ι from the frame x to the identical labels for x'.

Proof. The ring action transported from Y is unital and additive, so the action on $D'_{\mathbb{Q}}$ extends from O_E to $O_E \otimes_{\mathbb{Z}} \mathbb{Q} = E$. The last equality follows by multiplying the rational coordinates in an integral basis by a common denominator. Since the $\mathbb{Z}$-rank of D' is $2fn$ and $[E : \mathbb{Q}] = 2f$, the E-dimension is n. Extend c of Lemma 287.1 by rational scalars in each completion and put $x' = cx$. The chosen rational lattices, including O_E^n, are commensurable by clearing the finitely many basis denominators in both directions. Thus c is expressed by an $E \otimes \mathbb{Q}_p$-linear matrix at p, integral with integral inverse away from finitely many primes. It defines an adelic matrix. The completion of D is xL by the lattice approximation in Lemma 269.1 (equivalently use rational approximation modulo each fixed multiple of xL). Hence the completion and intersection identities follow, the latter since a rational number integral in all $\mathbb{Z}_p$ is integral in a lattice basis. On torsion of order dividing m, t corresponds to the class of mxt modulo mD in completed lattice coordinates; replacing t by any lift changes the class by $m\widehat{D}$. Applying c_m replaces x by x', proving the label assertion.

The adjoint relation $E_A(av, w) = E_A(v, \bar{a}w)$ holds for $a \in O_E$ by the trace formula of Proposition 253.3. It passes through c and ϵ_p to each completion of D', hence holds rationally

for E' and all $a \in E$ by linearity. For $v, w \in D'_{\mathbb{Q}}$ define $H_0(v,w) \in E$ by

$$E'(av, w) = \mathrm{Tr}_{E/\mathbb{Q}}(\xi a H_0(v,w)) \qquad (a \in E).$$

The trace map from H_0 to such functionals is injective: if $H_0 \neq 0$, take $a = 1/(\xi H_0)$ and the trace is $2f$. Equality of the finite rational dimensions gives existence. Uniqueness and the adjoint relation make H_0 linear in v and conjugate linear in w. Alternation and $\bar{\xi} = -\xi$ give $H_0(w,v) = \overline{H_0(v,w)}$ by testing against all a and using invariance of trace under conjugation. It thus equals $w^* A' v$ for a Hermitian matrix. Degeneracy would contradict the strict positivity of the curvature on v, iv in the real scalar extension. Finally the equality of alternating forms in Lemma 287.1, extended to the full rational p-adic spaces, gives equality of the two corresponding Hermitian matrices in frame coordinates. Indeed one may still test with every a; the trace coefficient matrix was invertible over $\mathbb{Q}$ and remains invertible over $\mathbb{Q}_p$. This proves the last identity and in particular its scalar is a unit at every finite place. □

288. Descent of the embedded marked level varieties

Proposition 288.1. *With x', A' as in Proposition 287.2, the conjugate Y', with its O_E-action, labelled torsion and hyperplane line B', is holomorphically isomorphic with these structures to the actual framed torus for x' and a reflection J' of Definition 252.1.*

Proof. In the chosen E-basis extend the identification of the rational periods to $D'_{\mathbb{Q}} \otimes \mathbb{R} \simeq V_{\mathbb{R}}$; this is the real tangent space by Lemma 284.1. The complex structure I' on it commutes with $E \otimes \mathbb{R} = \prod_{\tau \in \Psi} \mathbb{C}$ by functoriality for the O_E-action. The product decomposition here is given by field embeddings: for a primitive element the distinct roots of its minimal polynomial give the coordinate decomposition over $\mathbb{C}$ by polynomial interpolation; the real vectors are the conjugation-fixed pairs. Thus I' is a complex-linear matrix on each ordinary embedding factor, with eigenvalues $i, -i$. On its i-eigenspace the complex Lie action of a has eigenvalue $\tau(a)$, and on its $-i$-eigenspace it has eigenvalue $\bar{\tau}(a)$ using the new complex structure. The characteristic polynomial for an integral primitive θ is P_θ of Lemma 273.1: the differential of an algebraic map in a smooth chart is given by its linear terms and therefore conjugates with the coefficients. All coefficients of P_θ belong to E, fixed by ι. The $2f$ embeddings take distinct values on θ; comparison of multiplicities therefore makes $I' = i$ at $\tau \neq \sigma$ in Ψ, and $I' = iJ'$ at σ with positive-one multiplicity one and negative-one multiplicity $n-1$ for J'.

Write $J_\sigma = J'$ and $J_\tau = 1$ at the other places. The real form of the reconstructed trace formula is $2\Re \sum_\tau \tau(\xi)\, w_\tau^* A'_\tau v_\tau$. Invariance under I' in both arguments implies $J_\tau^* A'_\tau J_\tau = A'_\tau$: test vectors supported at that place and replace one vector by its ordinary i-multiple to recover both real and imaginary parts. Consequently $A'_\tau J_\tau$ is Hermitian. Evaluating at $v, I'v$ gives $2\Im(\tau(\xi)) v_\tau^* A'_\tau J_\tau v_\tau$ for vectors supported there, as in Proposition 253.3. Both the curvature value and $\Im(\tau(\xi))$ are strictly positive for nonzero v, by Theorems 286.2 and 236.2. This verifies the required sign at every place. The finite equation of Proposition 287.2 verifies the unit similitude condition there. We have the signed frame and its full period lattice and torsion.

It remains to identify the line, since the alternating form alone leaves its semicharacter undetermined. For $\lambda \in D'$, let $v = \lambda/2 + D'$. The algebraic identity $q_B(a) = b_{2,B}(\beta a, a)$ and its naturality in Proposition 284.2 give $q_{B'}(v) = b_{2,B'}(\beta v, v)$, as the conjugated endomorphism acts as the same β in the new period coordinates. By Theorem 286.2 this is $\exp(\pi i E'(\beta\lambda, \lambda))$ and is exactly the multiplier sign of Definition 261.1. The Hermitian form in that theorem is uniquely specified by E', I'. All period factors therefore agree with those of the actual framed theta line. Identifying the vector-cover trivializations gives the required line isomorphism. □

Theorem 288.2. *Fix an open finite-index level $C' \subseteq C$ and $\nu \geq 0$. For each admissible multiple M of the cutoff in Theorem 283.2, the homogeneous ideal of its embedded $\mathcal{X}(C', \nu)$*

is generated over E, in the common projective coordinates there (using the projective product embedding if necessary). For each individual framed torus and each $\iota \in \mathrm{Aut}(\mathbb{C}/E)$, the conjugated evaluation of every point is the canonical theta evaluation on the reconstructed frame of Proposition 288.1. This holds simultaneously for all torsion labels and for arbitrary marked points.

Proof. Use $T = L^{\perp}/L$, of odd order $m = h^{2n}$, and the labels, D_0, and section j fixed in Proposition 281.2. The conjugation of the finite labels is identified in Proposition 287.2. The equality of alternating forms up to a local unit sends the integral dual lattice to the integral dual at every prime. Consequently on the new frame precisely the same T-labels are the theta translations.

For $t \in T$, the operator U_t on sections from Proposition 281.2 is induced by a lift of translation on the line. It satisfies $U_t^m = 1$ by the product rule and alternation, and inversion conjugates it to $U_{-t} = U_t^{-1}$. These properties characterize the lift operator uniquely: two lifts differ by an automorphism of the line over the identity, hence by a nonzero scalar on the connected compact torus. Its m-th power is one, and conjugation by inversion gives that it equals its inverse. Since m is odd the scalar is one. The lifts used here are algebraic: their holomorphic line isomorphisms between a line and its translate are algebraic by the joint globally generated sections argument of Proposition 279.2. This also applies to the line comparison of the preceding proposition after identifying the compact projective tori by their biholomorphic map; that map is algebraic by Theorem 267.2. Changing the scalar of a line identification leaves its conjugation of operators unchanged.

Thus ι carries the original U_t to the new canonical U_t, by the two characterizing identities. The D_0-fixed line of sections is unique and the whole basis is given from it by $s_b = U_{j(b)} s_0$, by Lemma 281.1. Conjugating the old basis gives the new one up to a common scalar. For evaluations the scalar and the choice of frame in each one-dimensional value line disappear on passing to projective coordinates. This proves the claimed identity pointwise on each torus, with the same correspondence for all its labels. The orbit product over D_M and the projective product with the individual marked points are defined by integer polynomials by Theorem 283.2, so they commute with ι. Each conjugated point belongs to the image, and applying ι^{-1} proves set equality. The closed algebraic image therefore has its homogeneous ideal generated over E, by Lemma 270.1. □

289. The field of definition of the Hecke maps

Lemma 289.1. *Let X, Y be reduced complex projective algebraic sets in specified coordinates with their homogeneous ideals generated over a subfield $k \subseteq \mathbb{C}$. If a regular map $f : X \to Y$ on complex points commutes with all $\mathrm{Aut}(\mathbb{C}/k)$ acting on both sets, it is obtained by scalar extension from a unique regular map of their models over k. Its projective graph has homogeneous ideal generated over k.*

Proof. Use the ideal of all homogeneous equations over k in each case. Its extension to $\mathbb{C}$ is the given ideal. The graph is closed since the diagonal in a product of projective spaces is closed by the coordinate cross products; pull this diagonal back along (f, id). Equivariance makes the graph invariant, so its ideal descends by Lemma 270.1 using the projective product embedding.

For each target coordinate y_i, the closed set $Z_i = f^{-1}(y_i = 0)$ in X has the same invariance. Its homogeneous ideal therefore has finite generators over k by Lemma 270.1. On any standard source chart, dehomogenize these generators. The principal opens U of their nonvanishing cover that chart outside Z_i, with U defined over k. Take all i; these opens cover over k also. Indeed their defining functions generate the unit ideal in each complex affine chart by the cover argument of Proposition 270.2; ideal membership for elements of a k-algebra descends from $\mathbb{C}$ by taking the quotient and observing that any k-vector space injects into its scalar extension (use a basis). Let A be the localized affine chart ring over k

on one such U. Each function $f^*(y_j/y_i)$ is a member of $A \otimes_k \mathbb{C}$ on this open and is fixed there by the coefficient action. Here equality as functions implies equality in the reduced affine chart ring by Lemma 268.2's power-containment argument. In a k-basis of A the finitely many coefficients of this member are fixed. They lie in k by Lemma 270.1. The resulting elements of A satisfy the target relations and the projective transition identities on overlaps, by the same injection into scalar extension. These formulas glue and are unique by that injection. □

Theorem 289.2. *In the common coordinates of Theorem 288.2, forgetting level or marked slots, the group operations on the fibers, scalar endomorphisms by O_E, and every map $u_{g,\mathbf{a}}$ of Proposition 254.1 are defined over E. This also holds for right translations by $k \in C$ between levels for which the translation is well defined. All identities between their complex composites hold over E.*

Proof. For a representative x, conjugation by $\iota \in \mathrm{Aut}(\mathbb{C}/E)$ on all its marked evaluations is the single reconstructed group with frame $\tilde{x}$ of Theorem 288.2. This description already commutes with forgetting labels, with group operations, and with the specified scalars, since these were transported as algebraic morphisms on the individual complex tori in Lemma 287.1. The same rational period basis may be used at each level on forgetting level. Changing x to xk, for an integral level translation k preserving the underlying lattice, changes the associated labelling by $t \mapsto kt$. The reconstructed frame then changes to $\tilde{x}k$ using the same period basis, by the finite formula in Proposition 287.2.

For a Hecke leg first omit the scalars $\mathbf{a}$. Denote by $\tilde{u}$ a frame reconstructed for the target xg in any rational period basis. The complex isogeny sends $P_x(t)$ to $P_{xg}(g^{-1}t)$ for every finite torus label t, by the lattice quotient formula in Lemma 269.1; the underlying vector is unchanged. The conjugated isogeny therefore sends $P_{\tilde{x}}(t)$ to $P_{\tilde{u}}(g^{-1}t)$. Its complex derivative sends source periods into target periods by Lemma 284.1. In rational period coordinates this derivative is a matrix y over E, since it intertwines the O_E-actions. Thus $\tilde{u}^{-1}y\tilde{x} - g^{-1}$ takes $l^{-1}L$ into L at every integer $l \geq 1$, by the label identity. Taking $l = p^j$ at each prime forces every entry to vanish there. Hence

$$y\,\tilde{x}g = \tilde{u}.$$

In particular y is invertible, since it is invertible in every finite completion. Its complex-linearity in the period structures carries the reflection at $\tilde{x}$ to that at $\tilde{u}$, by Proposition 252.2. Thus y^{-1} identifies the conjugated target with the actual frame $\tilde{x}g$, and in these coordinates the conjugated isogeny is the vector identity modulo the larger lattice. It agrees on all marked points by uniformization. Composing with the scalars a_j gives equivariance of the complete map, including the case $a_j = 0$.

All the maps are regular over $\mathbb{C}$ by Corollary 267.3 and Theorem 267.2, using the product embedding for their domains where needed. Apply Lemma 289.1 with $k = E$ to obtain the maps and their graphs. An equality of composites can be checked on chart coordinate expressions, which inject into their scalar extensions as in that lemma. □

290. Integral models of fixed diagrams

Lemma 290.1. *Let $E \subset \mathbb{C}$ be a number field. A finite system of polynomial equalities and nonvanishing conditions over E having a complex solution has a solution in a number field $K \subset \mathbb{C}$ finite over E. All entries of that solution and all required inverse entries belong to $O_K[1/N]$ for some positive integer N.*

Proof. Replace each $b \neq 0$ by $bv = 1$ with a new variable. Over the algebraic closure of E in $\mathbb{C}$ the ideal of these equations is proper, as evaluation at the given complex solution takes one to one. This closure is algebraically closed: roots in $\mathbb{C}$ of polynomials with algebraic coefficients are algebraic over a finite extension of E, hence over E. The affine point argument of Lemma 268.2 in the generality checked in Lemma 280.1 therefore supplies

a solution in that closure. The field generated by its finitely many entries is finite over E, by successively using their minimal polynomials. For any algebraic number z, choose a positive integer d clearing the denominators in a monic rational equation for z; substituting dz into the equation multiplied by the corresponding power of d shows that dz is an algebraic integer. Taking a common multiple for all the entries, including the inverse variables, proves the assertion. □

Theorem 290.2. *Fix a finite list of the levels, marked powers and maps in Theorem 289.2. One may include identifications of powers as fiber products over their bases, together with finitely many group or diagram identities. There is a common positive integer N such that the diagram admits projective models and regular maps over $O_E[1/N]$ satisfying those identities. For any valuation ring V with fraction field $\mathbb{C}$ and residue characteristic $p \nmid N$, the base changes to V admit the smooth and finite étale presentations of Theorem 268.3, for the maps specified there. The same holds in their geometric fibers at every such prime of O_E. In particular the base and its marked powers are smooth there, and the one-point family is a smooth projective commutative group over the base, of relative dimension fn.*

Proof. For the last assertion take the diagram enlarged by the zero, inversion, addition, projections, and the identities expressing the group laws on at most three factors. The fiber product of two powers embeds in their product by equating their maps to the same base, a closed condition given by projective cross products. Over $\mathbb{C}$ the forgetful projections identify the combined power with this fiber product. Indeed the source and target have the same local smooth period charts by Theorem 252.3; the chosen level has trivial stabilizer there by the neatness in Proposition 254.1. The fiber product is reduced: in smooth chart coordinates for the two projections from Proposition 268.1, their equations after base change use disjoint variable lists over the smooth base, with invertible minors in diagonal blocks. This gives a smooth chart, hence a reduced chart by the local calculation in Theorem 266.2. The inverse, which concatenates the two point lists on these charts, is holomorphic, hence algebraic by Theorem 267.2. Both maps commute with coefficient automorphisms by Theorem 288.2. Lemma 289.1 descends the inverse as well. The product presentation itself extends from E, since its explicit diagonal equations and their chart rings commute with extension of scalars. We may retain that presentation and both inverse identities.

All projective equations and map formulas can now be taken over E, by Theorem 289.2 and the chart proof of Lemma 289.1. Take their covers, relations, and transition and composite identities as in Proposition 270.2. One may take all the ideal membership witnesses for these formulas over E: in a complex polynomial identity with coefficients to be found, keep the finitely many monomials of the multiplier polynomials in one witness. The linear system for their coefficients has entries in E; elimination over E gives a solution if one exists over $\mathbb{C}$. This applies to cover witnesses for one as well. Clearing the finitely many number denominators as in Lemma 290.1 gives the models with their identities over $O_E[1/N]$.

For a projection or leg retain the additional certificates in the last paragraph of Proposition 270.2. These may use complex coefficients for auxiliary chart changes and the nonzero minors. Fix all the finite lists of monomials in one collection of these certificates, including the maps between charts, both inverse identities, covering unit expressions and all ideal membership witnesses. With the already chosen equations and maps held fixed, equality of coefficients in those polynomial identities gives a finite polynomial system over E for the additional entries. Unit conditions are expressed by their displayed inverse identities. This includes the relative dimensions and the monomial degree used for a finite morphism, since the certificate fixes those integer sizes. Apply Lemma 290.1. Over a single $O_K[1/N]$, increasing N, the same identities yield the smooth charts, finite presentations as modules, and étale charts, exactly as in Proposition 270.2. The smooth charts for the bases over a point are included by Theorem 266.2. The projective models and maps to which the certificates apply still have their original $O_E[1/N]$ equations.

The given V contains O_K: a root of a monic integer polynomial cannot have negative valuation, since then its leading power has strictly smaller valuation than every lower term. Also N is a unit of V. The certificates thus remain valid over V. To verify the statement in a geometric fiber at any prime over $p \nmid N$, extend its residue map to $O_K[1/N]$ in an algebraic closure of the residue field. For completeness, localize O_E at that prime. The localized O_K is a nonzero finite module there: an integral basis over $\mathbb{Z}$ generates already over O_E, and the inclusion is injective upon localization. Its reduction at the maximal ideal is nonzero, by the finite generator determinant argument $M = \mathfrak{m}M \Rightarrow M = 0$ in Lemma 271.1. The nonzero finite residue algebra has a maximal ideal, and its residue field, finite over that of O_E, embeds in the chosen algebraic closure by adjoining roots successively. This extends the residue map, and base change of the same certificates proves the fiber statement. The commutative group identities, including the indicated fiber products, were kept over $O_E[1/N]$ in the first construction. □

291. Graded presentations on finitely many base strata

Lemma 291.1. *Let A be a noetherian domain, that is, a domain whose ideals are finitely generated. Set $P = A[X_1, \ldots, X_d]$ with $\deg X_i = 1$. Let M be a finitely generated graded P-module. Then after inverting one nonzero element of A, M is free as an A-module with a basis given by homogeneous monomials times specified generators. In each degree the list of basis elements is finite. This basis remains a basis degree by degree after every further scalar extension of the localized base.*

Proof. The polynomial ring over a noetherian ring is noetherian. For one variable and an ideal, consider the ideals of coefficients of X^j in members of degree at most j. They increase by multiplication by X and stabilize: the union is finitely generated and its generators occur in finitely many stages. Through a stabilization index choose polynomials witnessing finite coefficient generators at each stage. Subtract multiples of these, shifted by a power of X in higher degrees, to lower the degree of any member of the ideal. This generates the ideal. Induct on the number of variables. A submodule of a finite free module is finitely generated by projecting to one coordinate and inducting on the number of remaining coordinates. Thus a presentation by finite homogeneous generators has a finitely generated homogeneous relation module D; take the homogeneous parts of any finite generating list.

Use a free module $F = \bigoplus_j P(-a_j)$ mapping onto M, where $P(-a)_t = P_{t-a}$. Order its terms $X^\alpha e_j$ by total degree of the monomial, then lexicographically by exponents, then by j; multiplication by a fixed monomial preserves this order and every strictly decreasing sequence terminates. Over $K = \operatorname{Frac}(A)$, the leading monomials of the relations in D_K are closed under multiplication by monomials, separately in each coordinate. By the first argument of Lemma 237.1, finitely many of them generate each such set. Realize the chosen terms by monic homogeneous relations: the homogeneous component of a relation containing its leading term has the same leading term. After inverting one nonzero element of A, these finitely many monic relations belong to D, by clearing the denominators of their expressions in the original generators. Include 1 if the list is empty.

Subtract multiples of the chosen relations to eliminate divisible terms, successively taking the largest remaining term and putting it in the remainder when it is indivisible. Monicity makes all subtractions possible over the localized ring. Each step strictly decreases the largest unprocessed term. The residue classes of indivisible terms thus span M. No nonzero combination of them can belong to D, since its leading term over K would then be divisible by a chosen leading monomial, and A injects into K. They give a homogeneous free basis, finite in each degree since the variable and generator lists are finite. Tensoring a free basis preserves it. □

Proposition 291.2. *Let S be a scheme covered by finitely many affine schemes of finite type over $O_E[1/N]$. On each specified affine chart $\operatorname{Spec}(A)$ take finitely generated graded*

modules over $P = A[X_1, \dots, X_d]$, with $d \geq 1$. For each fixed positive integer l there is a finite cover by locally closed reduced affine strata such that on each stratum every indicated module M, restricted by scalar extension, has l exact sequences

$$0 \longrightarrow K_{i+1} \longrightarrow F_i \longrightarrow K_i \longrightarrow 0 \quad (0 \leq i < l), \qquad K_0 = M.$$

Here the F_i are finite direct sums of shifts of the graded polynomial ring, and the K_i, including K_l, are free over the base with finite homogeneous bases in each degree. The sequences stay exact after arbitrary scalar extension of that stratum. Strata from different original affine charts may overlap.

Proof. The coefficient rings are noetherian. Indeed O_E is a free abelian group of finite rank by Theorem 18.2; its ideals, as subgroups of a finite free abelian group, are finitely generated by the subgroup basis argument of Lemma 284.1. Finite generation persists under forming polynomial rings by Lemma 291.1, quotients by taking images of generators, and localization by contracting an ideal and extending it again.

Work on a reduced irreducible closed subset of one affine chart with its domain as base ring, restricting the module presentations by scalar extension. Choose finite homogeneous generators to obtain $F_0 \to K_0 = M$. The kernel is a finite graded module by the polynomial and submodule argument of Lemma 291.1. Repeating gives all l sequences. Apply that lemma simultaneously to all the K_i, for all indicated modules, multiplying the finitely many nonzero denominators. On the resulting nonempty principal open, each surjection $F_i \to K_i$ splits as a map of graded modules over the coefficient ring, by choosing homogeneous preimages of its free basis. Tensoring by any algebra over the coefficient ring therefore preserves exactness, including the injections.

This construction covers a dense open in each irreducible closed subset. In a noetherian affine chart descending chains of closed subsets stabilize by the ascending chain condition on their radical ideals. Every closed subset is a finite union of irreducible ones: otherwise a minimal counterexample is reducible and the two proper closed parts have finite decompositions. The coordinate ring of an irreducible reduced affine closed set is a domain, since the zero set of a product is the union of the factor zero sets. Apply induction on proper closed subsets to the complements of the exhibited opens in the irreducible pieces. This gives finitely many locally closed affine strata covering every chart, with the stated properties. Nilpotents of the original base vanish under every map to a field and do not change this covering of its points. □

292. A finite kernel for projective sections

Definition 292.1. For $P = A[X_1, \dots, X_d]$, a graded P-module M, and $J \subseteq \{1, \dots, d\}$, let M_J be the localization inverting X_j for $j \in J$. Fix a degree $t \in \mathbb{Z}$. Form the augmented cochain complex with one summand $(M_J)_t$ in degree $|J| - 1$ for each J, including M_t in degree -1. Its differential on the summand of the result indexed by $J = \{j_0 < \cdots < j_r\}$ is the sum of the restrictions from $J \setminus \{j_a\}$, with signs $(-1)^a$. Two deletions in either order have opposite signs, so the composite is zero. Cohomology means the kernel of the next differential modulo the image of the previous one. Write $H^0_{\mathrm{proj}}(M, t)$ for the kernel at degree zero *before* division by the augmented image. It is precisely the tuples of local expressions which agree on overlaps of the standard projective charts. For $M = P/I$ with I homogeneous, these are the global sections of the hyperplane twist $\mathcal{O}(t)$ on the projective scheme cut out by I: divide the degree-t expression by X_i^t to read it in the usual frame on $X_i \neq 0$. This convention also describes sections of the sheaf given by a graded module by its localized chart modules.

Lemma 292.2. *In Proposition 291.2 take $l = d + 1$. There is one integer t_0 for its finite strata such that on each stratum and after any scalar extension of its ring,*

$$M_t \simeq H^0_{\mathrm{proj}}(M, t) \qquad (t \geq t_0)$$

by restriction to the charts. In fact the entire augmented complex of Definition 292.1 is exact in these degrees.

Proof. For a free summand $P(-a)$ the complex splits into subcomplexes indexed by Laurent monomials X^α with $\sum_j \alpha_j = t - a$. Each term is a finite sum of Laurent monomials and the number of charts is finite, so this is a direct sum. Put $D = \{j : \alpha_j < 0\}$. A coefficient in that subcomplex is allowed exactly on subsets J containing D. When D is proper, choose $v \notin D$. On a coefficient array define the homotopy at J by zero for $v \in J$, and otherwise by taking the coefficient at $J \cup \{v\}$ times $(-1)^{|\{j \in J : j < v\}|}$. It acts between successive degrees and includes the augmented term when J is empty and D empty. In the sum of differential then homotopy and homotopy then differential, the terms deleting an index other than v cancel in pairs, while deleting v returns the original coefficient once. Terms outside $J \supseteq D$ are read as zero; adjoining $v \notin D$ preserves this convention. Thus the complex is exact there. When $t \geq a$ there can be no monomial with all exponents negative. The augmented complex for this free summand is then exact.

Choose t_0 at least every shift of every F_i in the finite collection. On this range the complexes for all F_i are exact, including after scalar extension. The short exact module sequences in Proposition 291.2 give short exact sequences on each cochain term. Localization preserves exactness: multiply a fraction with zero image by a common product of the inverted elements to get a numerator with zero image, then lift using exactness before localization. Taking homogeneous parts preserves the sequences. Each short exact sequence with exact middle complex identifies the cohomology in degree j for K_i with that in degree $j+1$ for K_{i+1}. To see this, lift a closed array to the middle complex and take its differential, which is an array in the subcomplex. Changing the lift or adding a boundary changes its class by a boundary. A zero class allows adjustment of the lift by a subcomplex array to make it closed, hence a boundary in the middle complex. Conversely a closed subcomplex array is a boundary in the middle one, and its primitive projects to a closed array in the quotient. Iterating $d+1$ times from any $j \geq -1$ ends in degree at least d, where there is no term since J has size at most d. This proves exactness and identifies M_t with the compatible chart sections. □

Proposition 292.3. *With these strata and any integer u, choose $b \geq 1$ with $u + b \geq t_0$. On each stratum the sections $H^0_{\mathrm{proj}}(M, u)$, after any scalar extension there, are naturally the kernel of*

$$(M_{u+b})^d \longrightarrow (M_{u+2b})^{\binom{d}{2}}, \qquad (f_i)_i \longmapsto (X_i^b f_j - X_j^b f_i)_{i<j}.$$

These are matrices between finite free modules over the ring of the stratum. The fiber dimension of the section space at field-valued points is constant on a further finite cover by locally closed strata.

Proof. Multiply a global degree-u section s by X_i^b to obtain the global higher-degree section f_i, read as a polynomial module element by Lemma 292.2. The indicated differences vanish. Conversely on $X_i \neq 0$ use the local fraction f_i/X_i^b. The kernel identities give equality on all overlaps, thus a section s. On every $X_j \neq 0$, $X_i^b s = X_i^b f_j / X_j^b = f_i$, so the recovered higher sections are precisely the given ones. These constructions are inverse, including when $d = 1$. The finite homogeneous bases and their base-change properties are supplied by Proposition 291.2. The rank of a matrix over a field is the largest order of a nonzero minor, by row elimination. Vanishing of all larger minors and nonvanishing of a chosen minor therefore give finitely many locally closed conditions on which the kernel dimension is fixed, including rank zero. □

293. Dimensions of sections on the fixed integral level

Lemma 293.1. *Let finitely many locally closed affine strata of a finite-type $O_E[1/N]$-scheme be given. After increasing N, each stratum with a geometric point has a complex point for*

the specified embedding $E \hookrightarrow \mathbb{C}$. Consequently any collection of numerical data constant on field-valued points of each stratum, and having the same prescribed value at all complex points, has that value on every remaining geometric fiber. The data on a stratum may form an infinite sequence if the whole sequence is constant there.

Proof. If the affine ring A of a stratum has $A \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$, this is a nonzero finite-type E-algebra since $O_E \otimes_{\mathbb{Z}} \mathbb{Q} = E$. Its scalar extension to $\mathbb{C}$ is nonzero by the basis injection in Lemma 289.1. It has a complex point by Lemma 268.2's affine point argument. Conversely, if this localization is zero, then 1 is killed in A by a nonzero integer, by the definition of localization. Inverting that integer removes the stratum. There are only finitely many integers to invert. A single complex point on each retained stratum compares all the entries that are constant there. □

Theorem 293.2. *Fix $C' \subseteq C$ as in Theorem 288.2, put $g = fn$, $d = h^n$, and use the fixed one-point family over $S = \mathcal{X}(C', 0)$. Choose its integral model in Theorem 290.2 inside $S \times \mathbb{P}^{d-1}$ by the common marked embedding. Write $B = \mathcal{O}(1)|_Y$ on a geometric fiber Y over an algebraically closed field k. After increasing the single integer N of this model, simultaneously for all such fibers one has*

$$\dim_k H^0(Y, \mathcal{O}_Y) = 1, \qquad \dim_k H^0(Y, B^{\otimes m}) = d\, m^g \qquad (m \geq 1).$$

The d coordinates are a basis for $H^0(Y, B)$. Moreover, on each of a fixed finite affine cover of S, write P/I for the homogeneous presentation in the marked coordinates. There is an integer $m_0 \geq 1$ such that on every geometric fiber the restriction $(P_k/I_k)_m \to H^0(Y, B^{\otimes m})$ is an isomorphism for all $m \geq m_0$. Here I_k is the ideal generated by the specialized equations.

Proof. The embedding is the product of the common base coordinates with the individual marked projective coordinates in Theorem 283.2; take its closed image over E by Theorem 288.2. In constructing the model one can retain the graph of the base projection inside this product by the chart identities of Theorem 290.2. On a standard affine of the base, dehomogenizing only the base variables gives the required homogeneous presentation in the marked coordinates. Equivalently one can take equations in the projective product embedding and substitute the pairwise products of its coordinates; the usual charts with one nonzero coordinate in each factor recover the same closed subscheme by division. Take the finite affine cover by these base charts.

At a complex point the constant functions form $\mathbb{C}$ by Lemma 280.1. For $m \geq 1$ the theta functions for the m-th power of the canonical factors are obtained by multiplying the gauge $G(z)$ of Theorem 261.3 to the power m and translating the standard degree-m theta functions by the same d_0. In fact raising the multiplier identity in its proof to the m-th power gives exactly the degree-m transformation equations of Theorem 256.3; division by $G(z)^m$ and reverse translation give the inverse on all sections. That theorem gives dimension dm^g. All these holomorphic sections are algebraic. For any one section adjoin it to the list of all degree-m products of the d original coordinates; the latter generate $B^{\otimes m}$ at every point. The resulting holomorphic map to projective space is algebraic by Theorem 267.2. The ratios against a nonvanishing original product give regular coefficients in those local frames, proving the assertion. The degree-one coordinates themselves form a basis by Proposition 281.2.

Apply Proposition 291.2 to $M = P/I$ with $l = d + 1$ on the chosen affine cover. Let $m_0 = \max(1, t_0)$ using Lemma 292.2. On a stratum the numbers of its standard monomials in degrees $m \geq m_0$ specify simultaneously the entire sequence of dimensions in these degrees. They are unchanged at any field-valued point there, and the chart section comparison is an isomorphism. For the finitely many degrees $0 \leq m < m_0$, refine using Proposition 292.3. Include a finite rank test for the list of degree-one coordinates: under the matrix for $u = 1$ each X_j has column $(X_i^b X_j)_i$ in the finite free domain. The dimension of their span is the rank of this list of columns, again determined by minors. Each refinement is given by

vanishing of finitely many minors and inverting a specified minor, so consists of finitely many locally closed affine strata after taking their reduced structures. Now apply Lemma 293.1 to all the data. The complex fiber calculation gives exactly the indicated sequence and the rank d for the degree-one coordinate list. This proves every assertion with one finite exclusion. □

294. The section lattices along a good valuation arc

Lemma 294.1. *Let Y be a nonempty projective scheme of finite type over an algebraically closed field k, with smooth Jacobian charts as in Proposition 268.1. If $H^0(Y, \mathcal{O}_Y) = k$, then Y is reduced and irreducible.*

Proof. At a k-point the completed local ring is a formal power series ring over k in the free tangent parameters. Indeed translate the point to zero, split the variables according to the unit Jacobian minor, and solve for the dependent variables to each order in the remaining ones. At each successive degree the linear coefficient matrix in those variables is the same invertible minor evaluated at zero. The corrections make all equations vanish modulo the next power and are unique there. Inverses of the inverted chart functions are given to each order by the finite geometric sum after removing the nonzero constant. This yields inverse coordinate homomorphisms modulo each power, as in the proof of Lemma 274.1, and hence the assertion on taking compatible sequences. The power series ring is a domain: the first homogeneous parts of two nonzero series have a nonzero polynomial product.

The noetherian local ring D at the point injects into its completion. Put $\mathfrak{m}$ for its maximal ideal and $I = \bigcap_{r \geq 0} \mathfrak{m}^r$. The graded ring $\bigoplus_{r \geq 0} \mathfrak{m}^r t^r$ is generated over D by t times finite generators of $\mathfrak{m}$; it is noetherian by Lemma 291.1. The homogeneous ideal $\bigoplus_r (I \cap \mathfrak{m}^r) t^r$ has finitely many homogeneous generators. In any degree above their degrees every coefficient in $I \cap \mathfrak{m}^r = I$ then belongs to $\mathfrak{m} I$. Thus $I = \mathfrak{m} I$ and $I = 0$ by the determinant argument for finite modules in Lemma 271.1. This proves the injection and shows that the local ring is a domain.

Every nonempty closed subset of an affine chart has a k-point by the affine point argument of Lemma 280.1. If a nilpotent in such a chart were nonzero, a maximal ideal containing its proper annihilator would give a localization where it was still nonzero, contradicting the domain assertion. There are finitely many irreducible components by the noetherian argument of Proposition 291.2. If two components met, a k-point in their intersection would have in an affine neighborhood two distinct minimal primes contained in its maximal ideal. Indeed the nonempty open part of each component is dense in it by irreducibility; the ideals of these two maximal irreducible closed subsets are the distinct minimal primes, since primality is exactly irreducibility of the zero set for a radical ideal. They remain distinct minimal primes on localization, by the bijection between primes disjoint from the inverted elements and primes of the localization, obtained by extending and contracting ideals. A domain cannot have two minimal primes. The components are thus disjoint, hence open as well as closed. More than one would give a nonconstant idempotent section by assigning zero on one and one on the remaining components. The hypothesis excludes this, proving the lemma. □

Theorem 294.2. *Let V be a valuation ring with fraction field $\mathbb{C}$, and let $\operatorname{Spec}(V) \to S$ be a point of the fixed model in Theorem 293.2. Let G/V be the pulled-back group, with relative hyperplane line B. Every geometric fiber is integral. There is a homogeneous coordinate presentation R for G with the following properties for $t \geq m_0$. The module R_t is finite free over V. Its elements give global sections of $B^t = B^{\otimes t}$, and in every geometric fiber R_t after scalar extension maps isomorphically onto the complete space of these sections. The scheme G is flat over V. The same assertions hold for each finite positive product of such groups over V, embedded by the projective product and the tensor product of their hyperplane lines, with the largest of their degree thresholds.*

Proof. Select a base chart containing the image of the closed point of Spec(V); its inverse image is all of Spec(V) since an open containing the maximal ideal of a local ring contains every prime. Take its presentation $R = V[X_1, \ldots, X_d]/I$ by pullback from Theorem 293.2. In each degree it is a quotient of a finite free module by a finite coefficient matrix, using the fixed homogeneous generators of I. Over V any such matrix diagonalizes: choose a nonzero entry of least valuation from the finite list, move it to the first position, and subtract multiples of its column and row to clear them since it divides all entries. Repeat on the smaller rectangle until it is zero. The rank over the fraction field equals the number of nonzero diagonal entries, and over the residue field it equals the number of unit diagonal entries. For $t \geq m_0$ these ranks coincide because the two cokernel dimensions are the same dt^g by Theorem 293.2 and do not change under extension of a field. Thus there are only unit or zero diagonal entries. The quotient is free of the stated dimension. Restriction of homogeneous elements to local charts gives global sections, and the same theorem gives the comparison after base change to every geometric fiber. Smoothness there is supplied by Theorem 290.2, also in characteristic zero by the same Jacobian certificates over a finite extension. With $H^0(\mathcal{O}) = k$, Lemma 294.1 gives integrality.

Over $X_i \neq 0$ the chart ring is the direct limit of the R_t, $t \geq m_0$, by successive multiplication by X_i: the elements are precisely fractions r/X_i^t, with equality detected after increasing the common power. A directed limit of flat modules is flat. Indeed tensor products commute with such limits by their bilinear presentation, and an element killed in the target of a directed limit of injections is already killed at some stage and hence becomes zero in the source. Free modules are flat by taking direct sums of any given injection. This proves flatness on the covering charts.

For a product take the graded ring with degree t part the tensor product over V of the constituent degree t parts, in nonnegative degrees. It is generated in degree one: in each factor lift a homogeneous term to a sum of products of degree-one entries and collect one entry per factor at each of the t positions. Its standard charts at pure coordinate products are exactly the affine product charts. Indeed fractions in the different factors can be brought to one common power of their denominators; the same common-power rule tests equality, or equivalently the tensor product of the directed limits above is the limit along simultaneous powers. Thus this ring presents the projective product, with free high pieces and flat charts. Over a field, for line bundles on a finite product the global sections of their external tensor product are the tensor product of the individual section spaces. For two factors take finite standard affine covers with affine intersections. On a product of two affines the external line has the tensor product of the two section modules, by their local trivializations and gluing. First imposing agreement on overlaps in the second variable gives the tensor product with the kernel of its restriction differences, since tensoring over a field is exact. Then impose the same equalities in the first variable. These two sets of conditions imply agreement on all product overlaps, which proves the assertion. Induction gives the finite product. It applies both in degree $t \geq m_0$ and to the trivial line, giving $H^0(\mathcal{O}) = k$. The products have smooth charts by using all the equations with diagonal Jacobian blocks. The preceding lemma now gives integral geometric fibers for the products as well. □

295. Extending the generic homomorphism by translations

Lemma 295.1. *Let $W \to X$ be flat and surjective, with the inverse image of each member of an affine open cover of X a union of finitely many affine opens. Let Z be closed in a projective space over a common affine base. A morphism $W \to Z$ whose two pullbacks to $W \times_X W$ coincide descends to a unique morphism $X \to Z$.*

Proof. Over a member Spec(A) of the affine cover replace its inverse image by the disjoint union of a finite affine cover, with coordinate ring B the finite product of these affine rings. It is flat over A and its spectrum surjects onto Spec(A). Here flatness on affine charts can

equivalently be checked after localization at every point: the kernel of a tensor injection localizes to its local kernels, and a nonzero element of a module survives at a maximal ideal containing its annihilator. This also shows that the finite product is flat. It is faithfully flat: a nonzero cyclic submodule A/I of any nonzero module remains nonzero upon tensoring, since a maximal ideal above I has a point above it and $IB \neq B$; its inclusion remains injective by flatness.

The equalizer of the two maps $B \rightrightarrows B \otimes_A B$ is A. Indeed the sequence $0 \to A \to B \to B \otimes_A B$ with final difference map becomes exact after tensoring by B: the new first map is $b \mapsto b \otimes 1$, split by multiplication, and if $\sum b_i \otimes c_i \otimes 1 = \sum b_i \otimes 1 \otimes c_i$, multiplying the first two factors gives $\sum b_i c_i \otimes 1 = \sum b_i \otimes c_i$. Flatness preserves kernels and images; faithfulness detects that their quotients vanish. The same proof holds after localization in A.

On $\operatorname{Spec}(B)$, take the ideal J of the inverse image of the closed projective hyperplane where a target coordinate vanishes. The equality on the product says that its two extended ideals in $B \otimes_A B$ are equal. If I is the inverse image of J in A, then $IB = J$. In fact tensor the kernel of $A \to B/J$ with B. For $b \in J$, $1 \otimes b$ lies in the extended ideal from the second factor, hence in the one from the first factor, and is zero in $(B/J) \otimes_A B$. Thus $b \in IB$; the reverse containment follows from the definition of I. Consequently the inverse image of each standard target chart is the pullback of the open complement of $V(I)$ in $\operatorname{Spec}(A)$. These opens cover by surjectivity. Over each principal open inside one of them, the affine coordinate functions of the map belong to the localized equalizer and so descend to the source. They satisfy the target relations and agree on overlaps, by the injection into the flat cover. They define the desired map, uniquely by the same argument on smaller affines. Its pullback agrees on the original W since it agrees on its covering affines. This uniqueness also glues the maps obtained over the different affines of X. □

Theorem 295.2. *Let G, G' be two group schemes along the same good valuation ring V as in Theorem 294.2, allowing distinct points of the fixed base. Every algebraic group homomorphism $f_{\mathbb{C}} : G_{\mathbb{C}} \to G'_{\mathbb{C}}$ extends uniquely to a homomorphism $G \to G'$. Equalities of composites of such homomorphisms and their compatibility with O_E-actions extend from the generic fibers.*

Proof. Choose a standard affine chart of the integral generic source meeting its generic point. The target homogeneous coordinates of $f_{\mathbb{C}}$ there give a nonzero projective tuple of rational functions: divide by one coordinate nonzero at that generic point. In the fraction field of the affine source chart, clear a common denominator; its affine coordinate functions are polynomials in the source coordinate ratios. Homogenizing to one degree and multiplying further by powers of the nonzero chart coordinate gives a tuple in $R_t \otimes_V \mathbb{C}$, with $t \geq m_0$, representing $f_{\mathbb{C}}$ on a dense open. At least one entry is nonzero there. Write all entries in a free V-basis of R_t from Theorem 294.2. Divide all coefficients by one nonzero coefficient of smallest valuation. The resulting entries lie in R_t, with at least one unit coefficient in their collected list. They define a map $f_U : U \to \mathbb{P}^{d'-1}_V$ on the open U where these homogeneous sections do not all vanish; d' is the target coordinate count. Every geometric fiber U_s is nonempty and dense: a unit coefficient survives in every residue field and the high-degree comparison of Theorem 294.2 makes the corresponding section nonzero on that integral fiber. On the generic part of U this map equals $f_{\mathbb{C}}$ by the closed diagonal and irreducibility. It maps into G' over V as well, since all pulled-back defining equations vanish generically and chart rings of U are flat over V. Indeed multiplication by each nonzero scalar of V is injective on a flat module, so every such ring injects into its fraction-field base change.

Let

$$W = \{(z, u) \in G \times_V G : u \in U,\ z + u \in U\}, \qquad h(z, u) = f_U(z + u) - f_U(u).$$

The projection $W \to G$ is flat, as restriction of the base change of the flat map $G \to \operatorname{Spec}(V)$. Flatness is preserved under scalar extension by the associativity of tensor products and on localizing by exactness of localization. This projection is surjective: over an algebraically

closed extension of the residue field of z, the two opens U_s and $-z + U_s$ are dense and nonempty in the integral G_s. Their intersection has a point by the affine point argument of Lemma 280.1. Over each standard affine in the first factor, W has a finite affine cover. The product with the second projective factor has its finitely many standard affine charts. On it the conditions defining W are nonvanishing of at least one of each of two finite lists of sections of lines, pulled back under projection and sum. On each affine both lines have finite trivializing principal covers (principal opens are a basis, and an affine has a finite principal subcover by the unit ideal argument of Proposition 270.2). On these the conditions are finite unions of principal opens, giving the assertion.

The two induced maps h on $W \times_G W$ agree generically, since the generic homomorphism gives $h(z, u) = f_{\mathbb{C}}(z)$. This product is open in G^3; it is flat over V by Theorem 294.2. The equalizer is a closed subscheme by the closed diagonal of G'; its ideal vanishes because it vanishes after fraction-field extension in flat charts. Apply Lemma 295.1. The resulting map agrees with the original homomorphism generically by surjectivity and flat descent. It is unique by the same flat-chart and closed-diagonal argument on G. Compare the two group-law maps on G^2 by that argument to see that the extension is a homomorphism. Identities of composites and identities with the extended scalar endomorphisms follow in precisely this way on their common flat source. □

296. Extension of identities of line bundles

Lemma 296.1. *Let X be a closed subscheme of projective space of finite dimension over an affine ringed base* $\operatorname{Spec}(A)$*. Write $B = \mathcal{O}_X(1)$. For any line bundle D on X, all sufficiently large t give finitely many sections generating $D \otimes B^t$ at every point. In particular D^{-1} embeds into a finite direct sum of copies of B^t with locally free quotient.*

Proof. Use the finitely many standard affines U_i where the coordinate X_i does not vanish. On an affine $U = \operatorname{Spec}(C)$ a line bundle has a finite principal cover where it is free. Indeed shrink each trivializing open to principal opens and choose a finite subcover as in Proposition 270.2. Its module of sections, described by compatible tuples in free frames on this cover, localizes at $a \in C$ to the module on $D(a)$. This follows by localizing the kernel of the finite array of differences on the pairwise intersections; these intersections are again principal, the transition functions localize, and localization is exact as in Lemma 292.2. In particular, clearing powers of each covering function in the unit frame on its principal open gives a section on U which generates on that open. These finitely many sections generate on all of U.

Take any such generator s for the restriction of D to a standard chart U_i, meaning one member of that chart's finite generating list. On U_j the overlap with U_i is the principal open of X_i/X_j. By the localization just proved, a suitable power $(X_i/X_j)^a$ times the restriction of s extends to a section s_j on U_j. Choose one exponent large enough for all charts. These sections, interpreted as local sections of $D \otimes B^a$ in the frames X_j^a of B^a, agree on overlaps after restriction to U_i. On each full overlap $U_j \cap U_k$, which is affine, their difference consequently vanishes after inverting X_i/X_j. Some power of this function kills the difference there. Increase the common twist by the largest of these finitely many powers, multiplying all local sections by that power of the global coordinate X_i. They now glue to a global section agreeing with sX_i^t on U_i. Repeating for the finite generator lists and multiplying any section by further powers of its own X_i makes all their twists equal to any sufficiently large t. They still generate on their respective charts since X_i is nonzero there.

The resulting surjection from a finite trivial bundle onto $D \otimes B^t$ splits locally, since near each point some generator is a frame. Dualizing and tensoring by B^t gives the asserted embedding with locally free quotient, directly in these split charts. □

Theorem 296.2. *Let X/V be a group or a finite nonempty product of groups in Theorem 294.2. The restriction* $\operatorname{Pic}(X) \to \operatorname{Pic}(X_{\mathbb{C}})$ *is injective. More precisely, a nonzero*

global generic section of any line on X has a scalar multiple extending over V with nonzero restriction to the closed fiber.

Proof. For the section assertion use Lemma 296.1 with the inverse of the given line D, obtaining $D \hookrightarrow (B^t)^{\oplus r}$ with locally free quotient. Here B is the projective hyperplane line, and we choose t above the threshold in Theorem 294.2. The generic section is a nonzero tuple of generic sections of B^t. Write each entry in the generic basis from the free homogeneous piece R_t of that theorem. Dividing by a coefficient of smallest valuation gives a tuple in $R_t^{\oplus r}$ with at least one unit coefficient. It defines global sections of $(B^t)^{\oplus r}$ over V. Their image in the quotient bundle is zero, since it is zero generically and locally the free quotient over a flat V-chart injects into the fraction-field base change. In the locally split embeddings the tuple therefore defines a section of D, agreeing on all overlaps by injectivity. Its special restriction is nonzero: the tuple reduces to a nonzero section of a coordinate line B^t in at least one position, by the closed-fiber comparison in Theorem 294.2. The same proof permits an algebraically closed extension of the residue field for that test.

If D is trivial on $X_{\mathbb{C}}$, take a generic trivializing section and its inverse. Apply the section assertion to each of them. The product of the resulting integral sections is a regular function on X which is generically a nonzero constant $c \in \mathbb{C}$. Evaluating on the identity section of the product shows $c \in V$. Subtracting this constant gives zero everywhere by flatness. Over an algebraic closure of the residue field the two reduced sections are nonzero and their common nonvanishing locus is nonempty, since this special fiber is integral by Theorem 294.2. Their product therefore does not vanish identically there and c is a unit in V. The two integral sections multiplied by an inverse unit give mutual inverses. Hence D is trivial over V. Taking the quotient of any two lines with isomorphic generic restrictions proves injectivity. □

297. The polarized maps on good valuation arcs

Proposition 297.1. *Fix the integral level family of Theorem 293.2 and a valuation ring $V \subset \mathbb{C}$ with fraction field $\mathbb{C}$ and residue characteristic $p \nmid 2hN$. Every complex point of its base, and every chosen point of its corresponding torus, extends uniquely over V. Write G, B for the resulting integral group and relative hyperplane line. Then B is symmetric, satisfies the parallelogram law of Definition 279.1, and $[m]^*B \simeq B^{\otimes m^2}$ for every $m \geq 1$. The action of each O_E-scalar and the projective translations by all T-labels extend; on the reduced fiber their coordinates use the matrices of Corollary 281.3. Inversion has its prescribed permutation matrix. These coordinates give the complete linear embedding on that fiber.*

Proof. For a complex point of a closed projective model, divide all homogeneous coordinates by one nonzero coordinate of smallest valuation. The entries then lie in V and one is a unit. They define a section over V, since every defining polynomial vanishes by its generic value and $V \subset \mathbb{C}$. Two such sections agree by the closed diagonal and injection into the fraction field. Apply this first to the base and then to the projective fiber over V; this works for any finite list of points, with the group-law equalities preserved by the same uniqueness.

The generic line is symmetric by Proposition 284.2 and satisfies the algebraic parallelogram law by Proposition 279.2. Apply Theorem 296.2 to G and to $G \times_V G$ to extend the line isomorphisms for symmetry and for the two universal points in the parallelogram law. The origin fibers contributing to the normalized law are pulled back from a line on V, trivial over the local base since a frame at its closed point trivializes on an open containing that point. The class formula $[m]^*B \simeq B^{\otimes m^2}$ follows by applying the law with $x = jz,\ y = z$ at the universal z: $l((j+1)z) + l((j-1)z) = 2l(jz) + 2l(z)$. Induction with $l(0) = 0$ gives $l(jz) = j^2 l(z)$.

Each scalar action is obtained from the generic endomorphism by Theorem 295.2. Their ring identities follow there by uniqueness, including addition by comparison with the group sum. Each labelled translation is translation by its extended section. The equalities between

its embedded coordinates and the projective action of the generic translation matrix hold over V, since these matrices have unit entries in their nonzero positions and unit determinant by Corollary 281.3. Here the matrix on evaluated homogeneous coordinates is the action dual to the specified action on sections, with pullback following the convention of Proposition 281.2. Both sides define maps into projective space; equality on the generic fiber extends by the closed diagonal and the flatness of Theorem 294.2. The inversion permutation extends in the same way. Degree-one completeness on the fiber is Theorem 293.2; the reduced translation matrices span the full endomorphism algebra by Corollary 281.3. □

Proposition 297.2. *Let w be a safe degree-one place over a completely split prime p as in Lemma 273.1, with $p \nmid 2hN$. Take a valuation ring as in Proposition 297.1 inducing this place of E. For every framed complex base point and residue subspace $A \subseteq \mathbf{F}_p^n$, form the integral groups G, G_A by extending that point and its complex neighbour xg_A on the fixed base. The unscaled map extends to a unique O_E-linear homomorphism $\psi_A : G \to G_A$, and on their hyperplane lines*

$$\psi_A^* B_A \simeq B^{\otimes p^{\dim A}}.$$

For adjacent $A \subset D$ the complex intermediate isogeny also extends over V, composing with ψ_A to ψ_D.

Proof. Use Theorem 295.2 on the algebraic complex homomorphisms of Corollary 267.3. The nested lattices and their intermediate complex map are described in Lemma 276.1. The scalar identities and the composite identity persist by uniqueness on the flat domains. By Proposition 278.2 the alternating form of the target in the common covering space is $p^{\dim A}$ times the source form. Its complex structure is the same, so its Hermitian form has the same multiple, and its canonical η_β is the $p^{\dim A}$-th power of the initial one by Definition 261.1. Restricted to the source period lattice its automorphy factors in Proposition 279.2 are therefore those of the claimed tensor power. This identifies the generic lines holomorphically and algebraically by the joint section argument of that proposition: both lines are generated by pullbacks or products of the homogeneous coordinates. Theorem 296.2 extends their equality of classes over V. □

298. Finiteness and the image of a polarized map

Lemma 298.1. *Suppose G, G' are as in Theorem 295.2, and $f : G \to G'$ satisfies $f^* B' \simeq B^{\otimes \mu}$ for their relative hyperplane lines and some integer $\mu \geq 1$. Then f is finite.*

Proof. In any geometric fiber of f, the restriction of B^μ is trivial by the displayed identity. Take a reduced irreducible component of that fiber, projective over its algebraically closed field. Its global functions are constant by Lemma 280.1. In a frame of the trivial restriction, all degree-μ monomials in the projective coordinates are constants. These monomials distinguish the original points: on a chart with $X_j \neq 0$ recover X_i/X_j as $X_i X_j^{\mu-1}/X_j^\mu$. Thus each component has one point, and there are finitely many components by Proposition 291.2's noetherian argument over the geometric field. Nilpotent elements do not change the set of points.

The graph makes G projective over G'. We give the finite-presentation argument also for this possibly nonnoetherian valuation base. Over an affine chart of G' it is closed in projective space over its ring. One can use homogeneous equations for this closed subscheme: on each standard chart take elements of its ideal. Each such local element extends, after multiplying by a sufficient power of that chart coordinate, to a homogeneous polynomial vanishing on the closed subscheme. Indeed homogenize a polynomial representing the local element; its restriction to the quotient on each other chart is zero after inverting the first coordinate. Some power kills it there, and a common power works on the finitely many charts. The homogeneous vanishing ideal therefore gives exactly the original chart ideals.

At a point y of the target, its residue field K_y is infinite: it contains the residue field of the localization of V at the corresponding prime. That localization is a valuation ring of the algebraically closed field $\mathbb{C}$, and its residue field is algebraically closed because a lift of a monic polynomial splits into linear factors with integral roots, by the valuation argument of Theorem 290.2. An algebraically closed field is infinite by the argument of Proposition 271.2. A linear coordinate over K_y can therefore be chosen nonzero at all points of the finite geometric fiber: the product of the finitely many nonzero linear conditions is a nonzero polynomial on the dual space over an algebraic closure, and cannot vanish on the whole grid over an infinite subfield, by induction using the one-variable root bound. Lift its change of coordinates over an affine neighborhood of y, shrinking to make the determinant and all denominators units. Write X_0 for this first coordinate.

Modulo X_0, over the algebraic closure of K_y, the remaining homogeneous equations span all forms of a sufficiently high degree by homogeneous multiples, by the empty-projective-fiber argument of Lemma 268.2 and its algebraically closed field version in Lemma 280.1. Only finitely many multiples are needed. The resulting full-size minor is nonzero already over K_y, and lifts to a unit on a smaller affine neighborhood. Consequently $X_0 \neq 0$ everywhere on the graph there, and the same span relations express every monomial of that degree on $X_0 = 1$ by terms of lower degree over the base. The graph is closed in this affine chart and its ring is generated as a module by the finitely many monomials of smaller degree, exactly as in the cited finite-fiber lemma. The argument also applies to an empty fiber, or a projective space of dimension zero as in that lemma. These neighborhoods prove finiteness. □

Lemma 298.2. *Let Y be a geometric fiber of the one-point family in Theorem 293.2, over k. Its field of rational functions $k(Y)$, the fraction field of a nonempty affine open, has transcendence degree g over k. The rational function field of any nonempty proper irreducible closed subset has transcendence degree at most $g-1$. Consequently a finite map between two such geometric fibers over k is surjective.*

Proof. For a nonempty integral closed projective set Z write D_Z for its homogeneous coordinate domain in its chosen projective coordinates, the quotient by all homogeneous vanishing equations. It is finite as a graded module over a polynomial ring in some $s \geq 1$ algebraically independent linear forms. To see this, if the current degree-one generators are dependent, take a nonzero homogeneous relation, obtained by collecting homogeneous parts. Make it monic in the last variable by a linear change: since k is infinite its highest homogeneous part takes a nonzero value in some direction. Taking this direction as the last basis vector and dividing by that value gives the required monic equation. Its bounded powers generate over the subalgebra in the remaining variables. Repeat on that subalgebra, stopping at independence as in Lemma 258.1. The module generators may all be homogeneous by this construction. At least one independent generator remains, since a nonzero degree-one coordinate is transcendental over k by the grading in a domain. Counting monomials in the s independent forms and through the finitely many shifted generators gives positive lower and upper constant multiples of m^{s-1} for $\dim_k (D_Z)_m$ as m grows. Here the count for degree m is $\binom{m+s-1}{s-1}$, obtained by separating m identical units into s ordered parts. The fraction field of D_Z has transcendence degree s by the finite module extension. For a nonzero linear coordinate L, $D_Z[1/L]$ is the Laurent polynomial ring in L over its degree-zero part: divide each homogeneous element by the power of L of its degree, and distinct degrees cannot cancel. Its degree-zero part is the affine coordinate ring on $L \neq 0$. Hence $k(Z)$ has transcendence degree $s-1$.

For $Z = Y$ the high homogeneous pieces have precisely dimension dm^g by Theorem 293.2. Indeed the given ideal and the full homogeneous vanishing ideal have equal pieces there: a polynomial vanishing pointwise gives zero as a section on the reduced fiber by Lemma 294.1 and the affine point argument of Lemma 280.1; the high-degree restriction is injective. Thus $s - 1 = g$. For a proper closed integral subset choose a homogeneous equation P of positive

degree e vanishing on it but nonzero on Y, from its closed defining equations. Multiplication by P in the domain D_Y is injective, so in all sufficiently large degrees its quotient has dimension $d(m^g - (m-e)^g) = O(m^{g-1})$. The coordinate ring of the subset is a further quotient, giving the claimed upper bound by the same monomial count.

The image of the finite map on geometric points is closed: the graph is projective, and the empty-fiber minor argument of Lemma 280.1 gives an open neighborhood of each point outside its image. The closed image is irreducible since the source is. Over a nonempty affine chart of that image the source has a finite coordinate algebra, a domain, over the image coordinate domain. Indeed the finite presentation as a module restricts from the target, and equations vanishing on the image are zero on the reduced source. The ring map is injective by density of the image there. The source fraction field is therefore finite over the image fraction field: localize at all nonzero elements of the base domain, and a finite-dimensional domain over a field is a field by injectivity of multiplication by any nonzero element. Both fields have the same transcendence degree, so the image cannot be a proper closed subset. This gives a preimage for every closed point, and for arbitrary scheme points by the same argument after algebraic closure of the residue field of the point, since both geometric fibers have the same section dimensions after that extension. □

299. The degree of an isogeny along the arc

Theorem 299.1. *Let G, G' be two groups obtained from the same fixed level model along V, and let $u : G \to G'$ be a homomorphism with $u^*B' \simeq B^{\otimes \mu}$, where μ is a positive integer. Then u is surjective and finite locally free of rank μ^g, with $g = n[F : \mathbb{Q}]$ the common relative dimension. Its kernel is finite free of that rank over V.*

Proof. By Lemma 298.1 the map is finite, and by Lemma 298.2 it is surjective on each geometric fiber and hence on the whole scheme. On a fixed geometric base fiber over an algebraically closed field k, write $u : Y \to Z$ for the map. Over a nonempty standard affine of the integral Z, its finite direct image algebra $D = u_*\mathcal{O}_Y$ is free of some positive rank r on a nonempty open. In detail choose among finite module generators a basis of the generic vector space and clear denominators in expressions for all other generators. Equality can be checked in the fraction field because the coordinate algebra of the preimage is a domain containing the target ring, as in Lemma 298.2. Shrinking to that denominator open gives the claimed basis. Choose a k-point of this open, using the affine point argument of Lemma 280.1. Every other k-point is its translate by $u(y)$ for some $y \in Y(k)$, since the nonempty finite fibers contain k-points. Translating in Y above the corresponding translation in Z is an isomorphism of the finite morphisms since u is a group map. The same freeness and rank therefore hold on neighborhoods of every closed point, which cover Z since any nonempty closed complement has a closed point. Thus D is a vector bundle of rank r.

There is an integer a and injections of sheaves on Z with $L = B'|_Z$,

$$(L^{-a})^{\oplus r} \hookrightarrow D \hookrightarrow (L^a)^{\oplus r}.$$

Indeed the proof of Lemma 296.1 applies to a finite locally free sheaf of any rank: use a finite set of basis sections on each trivializing principal open, clear powers of the covering function and extend with the same powers of the chart coordinates. It gives finite global generators of both $D \otimes L^a$ and $D^\vee \otimes L^a$, increasing a as needed. In each list choose r sections independent at the generic point. The first list gives the left map, the second gives the right one by evaluating the linear forms. Both are generically isomorphisms, hence injective in the integral charts since the sources are locally free. Taking sections after tensoring with L^m gives, for $m > a$,

$$r\, d(m-a)^g \;\le\; \dim_k H^0(Y, (u^*L)^m) \;\le\; r\, d(m+a)^g.$$

Here $H^0(Z, D \otimes L^m) = H^0(Y, (u^*L)^m)$: on an affine open trivializing L these sections are the functions on its preimage in the pulled-back frame, and the transition functions are the same. The two outside dimensions follow from Theorem 293.2. The middle dimension is $d(\mu m)^g$ by the pullback identity and that same theorem. Divide by dm^g and let m increase to get $r = \mu^g$. This argument works over each geometric residue field and over $\mathbb{C}$.

Over an affine chart of G' with ring A, let M be the finite algebra of the preimage. At each chosen prime of A the dimension of its fiber algebra is μ^g, since extension of the residue field to an algebraic closure preserves that dimension. Lift a basis to module generators on an affine neighborhood: first clear the denominators of residue representatives in the local ring, and use the finite-module determinant argument of Lemma 271.1 to get generation in that local ring. The finitely many original module generators are then expressed by the lifts after inverting a common element outside the prime. Denote the new affine ring again by A. The surjection $A^{\mu^g} \to M$ is an isomorphism after extending V to $\mathbb{C}$, since there it is a surjection between locally free modules of equal rank. In local bases its matrix modulo each maximal ideal is invertible, so its determinant is a unit locally. The kernel over A is therefore zero, since A^{μ^g} injects into this generic localization by the flatness of G' over V in Theorem 294.2. This proves local freeness over the entire integral target without any noetherian hypothesis on V. Pulling back by the identity section gives a finite locally free kernel over the local ring V; the open trivializing its closed point covers its spectrum. □

Corollary 299.2. *For the groups of Proposition 297.2 each ψ_A is surjective finite locally free of rank $p^{g \dim A}$. On every such group, multiplication by $m \geq 1$ is surjective finite locally free of rank m^{2g}, also when m is divisible by p. The finite flat multiplication and neighbour kernel hypotheses of Theorem 278.3 are thus satisfied on these valuation arcs.*

Proof. Use Theorem 299.1 with $\mu = p^{\dim A}$ for ψ_A by Proposition 297.2. For $[m]$ use $\mu = m^2$ from Proposition 297.1. In particular this supplies the kernels of $[p]$ and $[p^2]$ over V for every neighbour. The nested kernel and quotient statements follow by Proposition 276.2 using exactly these maps, the extended scalar action and the smooth separated models of Theorem 290.2. □

300. Frobenius on reductions of the actual marked neighbours

Theorem 300.1. *Fix the coarse level C, the number $\nu \geq 0$ of marked points, and a single admissible M for the common embedding of Theorem 283.2. There is a finite set of rational primes such that the following holds at every safe degree-one w outside it, of residue characteristic p. Suppose $V \subset \mathbb{C}$ is a valuation ring with fraction field $\mathbb{C}$ inducing w on E, and write k for its residue field. For a framed complex point $(x, J, (P_j))$, let $\bar{z}_A \in \mathcal{X}(C, \nu)(k)$ be the reduction, in the fixed projective model, of its neighbour $(xg_A, J, (\psi_A P_j))$, for each $A \subseteq \mathbf{F}_p^n$. With* Fr *acting on the projective coordinates by p-th powers, there exists a line $W \subseteq \mathbf{F}_p^n$ such that*

$$\mathrm{Fr}^{\,n-\dim A}\bar{z}_A = \mathrm{Fr}^{\,n-\dim(A+W)}\bar{z}_{A+W} \qquad (A \subseteq \mathbf{F}_p^n).$$

Consequently in the free abelian group on the geometric points of this special fiber one has

$$\sum_{r=0}^{n}(-1)^r p^{r(r-1)/2} \sum_{\dim A = r} [\mathrm{Fr}^{\,n-r}\bar{z}_A] = 0.$$

The finite exclusion and the model in these statements are independent of A, w, V and the initial framed point.

Proof. Use the projective models for the fixed level with its base, one-point family, marked power and group operations given in Theorem 290.2. Exclude their denominators, the primes in Theorem 293.2, the divisors of $2hM$, and the finite splitting exceptions of Lemma 273.1. Each remaining degree-one prime is completely split over its rational prime. The valuation residue field is algebraically closed by the monic lifting argument of Lemma 298.1. The

fixed model and coordinates reduce over $\mathbf{F}_p$, so coordinate p-powering preserves the reduced equations and sends their k-points to k-points.

Propositions 297.1 and 297.2 and Corollary 299.2 supply the integral groups, scalar action, signed theta lines and polarized isogenies on this same V. Their special fibers are connected reduced and projective by Theorem 294.2. In particular the complete matrix and section hypotheses of Theorem 282.3 hold by Proposition 297.1. Its pairing and the opposite-prime kernel comparison therefore apply to these very neighbours.

Add to the chosen sections P_j all source torsion sections of labels in $\mathcal{T}[M]$, as well as the T-translations. They are defined over V by Proposition 297.1. At every place away from p the chosen g_A is the identity, since $b_w(a_A)$ is supported on the one conjugate pair and z_w has its only nonunit entry at $w|_{K_0}$ and equals one elsewhere, as in Theorem 241.2 and Lemma 245.2. The component of a prime-to-p torsion label at a place over p is zero, since multiplication by its order is a local unit. Thus g_A^{-1} preserves each of these labels individually, as $p \nmid hM$. The label action of the generic isogeny in Lemma 269.1 and uniqueness of integral point extension identify its images with the indicated target labels. Theorem 282.3 now gives a single line W and the displayed Frobenius identities for the individual torus evaluations of all these sections.

To pass to the actual coarse base coordinate take their ordered tensor over $\mathcal{T}[M]$ and the product of the corresponding linear forms over the finite permutation group D_M, as in Theorem 283.2. This procedure commutes with reduction and with Frobenius. Indeed each evaluation has homogeneous representatives in V with a unit coordinate. The reduced tensor is nonzero by selecting one such coordinate in each factor. Each permuted linear form is nonzero after reduction and their product is nonzero since a polynomial ring over a field is a domain. The resulting coefficient tuple thus also has a unit entry; its reduction is the projective specialization, with every operation given by integer polynomials. This argument is valid even if some factors coincide after reduction. Keeping the individual marked coordinates and taking the product embedding gives exactly the $\bar{z}_A$ by uniqueness of projective specialization. The equality for these points follows. Finally apply Corollary 269.3 with $q = p$. All exclusions were made for the fixed objects and coordinates before choosing the place. □

301. All points of the reduced correspondence graph

Lemma 301.1. *Let $R = O_E[1/N]$, and let H_E be a reduced closed subscheme of a projective space over E, or in a finite projective product over E. Take its reduced closure H over R, that is, the reduced closed subscheme on the closure of the underlying generic subset in the corresponding projective model. For every point $z \in H$ there exist a valuation ring V with fraction field $\mathbb{C}$ and a map $\operatorname{Spec}(V) \to H$ whose closed point maps to z and induces an embedding of residue fields. Its generic point gives a complex point of H_E for the prescribed embedding of E.*

Proof. The generic scheme has finitely many irreducible components by the noetherian argument of Proposition 291.2. The union of their closures is the closure of their union, so choose a component whose closure contains z. The reduced structure on that closure is integral: the closure of an irreducible set is irreducible, and the radical ideal on each nonempty affine chart is prime by irreducibility. Its local ring A at z is a domain whose fraction field K is finitely generated over E, because its generic component has a chart of finite type over E. The field K embeds in $\mathbb{C}$ over the given E: choose a maximal algebraically independent subset of finitely many field generators, choose the same number of independent complex numbers over E, then extend by roots of the remaining algebraic generators. Such independent numbers exist successively since the algebraic closure in $\mathbb{C}$ of a field generated over the countable E by finitely many elements is countable (there are only countably many polynomials), whereas the reals are uncountable by the diagonal argument on binary sequences embedded using the ternary digits zero and two.

Inside $\mathbb{C}$, extend A to a maximal local subring V dominating it. Domination means inclusion with contraction of the larger maximal ideal equal to the smaller one. Local subrings so ordered admit as upper bound of a chain their union with the union of its maximal ideals, which is local since every element outside that union has its inverse at some stage. The maximal-element principle of Definition 16.1 applies. For $x \in \mathbb{C}^\times$ at least one of x, x^{-1} belongs to V. Indeed if $\mathfrak{m}V[x]$ is proper, take a maximal ideal containing it and localize $V[x]$ there. Its contraction is $\mathfrak{m}$, the maximal ideal of V, so maximality gives $x \in V$. Otherwise an identity $1 = a_0 + \cdots + a_j x^j$ with $a_i \in \mathfrak{m}$ makes x^{-1} integral over V by multiplying by x^{-j} and dividing by the unit $1 - a_0$. Thus $V[x^{-1}]$ is a nonzero finite V-module and $\mathfrak{m}V[x^{-1}]$ is proper by the finite-module determinant argument of Lemma 271.1. The same localization now gives $x^{-1} \in V$. This proves the valuation property and also that its fraction field is $\mathbb{C}$. The map from the local ring and domination give the stated center and injection $\kappa(z) \hookrightarrow V/\mathfrak{m}$. The generic map factors through the chosen generic component by construction of the embedding of K. □

Theorem 301.2. *Use the fixed level, ν, M and finite exceptional set of Theorem 300.1. At each indicated w, choose a fine complex level C_m as in Proposition 254.1 on which all the maps with labels $A \subseteq \mathbf{F}_p^n$ are defined. Let H_w be the reduced closure in the product of fixed integral marked targets of the joint generic image of these maps. For every geometric point z of its fiber at w, with projections z_A to those targets, there is a line $W \subseteq \mathbf{F}_p^n$ such that*

$$\mathrm{Fr}^{\,n-\dim A} z_A = \mathrm{Fr}^{\,n-\dim(A+W)} z_{A+W} \qquad (A \subseteq \mathbf{F}_p^n).$$

In particular the sum in Theorem 300.1 vanishes on this entire graph fiber. The result also holds on projecting a joint graph that records any further finite list of coarse-target coordinates defined over E on a common refined level.

Proof. The complex joint map is algebraic and defined over E by Theorem 289.2. Its complex image is closed: take its projective graph and project to the target product. Outside the image the homogeneous equations in the source variables have empty fiber, and the maximal minor witnessing this in one degree remains invertible on a target neighborhood, as in Lemma 280.1. The image is invariant under $\mathrm{Aut}(\mathbb{C}/E)$ by equivariance, hence its ideal descends by Lemma 270.1. Use this closed reduced image over E in the closure; its complex points are exactly the tuples arising from the complex fine level. The same statements hold for the additional maps in the last assertion, taking the joint map on the common fine level.

Let z_0 be the underlying scheme point of z. Apply Lemma 301.1 to z_0 and the joint image, using the product embedding of Corollary 260.2. The generic tuple is an actual complex tuple, so choose one complex fine-level preimage and a framed representative. Projection of the V-point to each fixed projective target is the unique extension of its generic value by Proposition 297.1's coordinate scaling argument. Its reductions in $V/\mathfrak{m}$ thus satisfy the line identities of Theorem 300.1. These identities already hold in $\kappa(z_0)$ for the same finite-indexed line W: in residue coordinates each equality is tested by homogeneous cross products of coordinate powers, and $\kappa(z_0) \hookrightarrow V/\mathfrak{m}$ is injective. They persist under the given field embedding from $\kappa(z_0)$ defining the geometric point z. This gives the claimed equalities, and the sum vanishes by Corollary 269.3. The argument treats the whole joint graph at once and therefore compares its projected coordinates in the same valuation ring also when extra maps are present. □

302. Projective models of finite diagrams of correspondences

Theorem 302.1. *Fix the coarse marked model $\mathcal{Z} = \mathcal{X}(C, \nu)$ over $R = O_E[1/N]$ of Theorem 300.1. Suppose finitely many coarse-target Hecke maps u_j are defined on the fine generic level $X_m = \mathcal{X}(C_m, \nu)_E$ of Proposition 254.1. Include its forgetful map π_m in the list. There is a reduced projective R-scheme $\mathcal{W}$ with generic fiber X_m, an action of the finite group $D = C/C_m$, and maps to $\mathcal{Z}$ extending all u_j and their translates by D. In characteristic*

zero the map π_m is a finite étale cover with D acting simply transitively on each geometric fiber. Any finitely many such models admit a common projective refinement, surjecting onto each of them and respecting all the specified coarse maps from their respective generic levels. The ring R can be fixed independently of their supports.

Proof. Use any of the closed projective embeddings $i : X_m \hookrightarrow \mathbb{P}^b_E$ of Theorem 288.2. On the generic level send v to the tuple

$$\big((i(vk))_{k\in D},\ (u_j(vk))_{(j,k)}\big),$$

where in the second list j runs over the prescribed maps and k over D. All entries are morphisms over E by Theorem 289.2. The factor $i(v)$, together with the closed graphs of the other maps, makes this a closed immersion. Explicitly, on its first factor the point must lie in $i(X_m)$, and equating the other factors with their prescribed images is a closed diagonal condition. Take its reduced closure in the product of copies of $\mathbb{P}^b_R$ and $\mathcal{Z}$. It is projective under the projective product embedding. It recovers the original reduced generic image since that image is already closed there. Right translation $v \mapsto va$ permutes the factors by sending the coordinate at k to the value originally at ak. This permutation and its inverse preserve the closed generic image, hence also its reduced closure, and their compositions give the right group action. Projection gives all the desired maps. The equalities between the $\pi_m(vk)$ extend to the whole model: on a reduced closure, a regular function vanishing on the dense generic subset vanishes on every affine chart, by restriction to each of its integral components as in Lemma 301.1.

The generic cover over $\mathbb{C}$ has the asserted degree, fibers and transitive free action by Theorem 268.3 and Proposition 254.1. These properties hold after passage to any algebraically closed extension of E. Indeed finite étale charts can be certified over a finite extension of E as in Theorem 290.2, and hence over each such algebraically closed field. The assertion on the action is equivalently that $(v, k) \mapsto (v, vk)$ from the finite disjoint union of the X_m is an isomorphism onto $X_m \times_{\mathcal{Z}_E} X_m$. Over $\mathbb{C}$ this map is a bijection and is locally the identity on the analytic covering charts. Its inverse is holomorphic and thus algebraic by Theorem 267.2. The schemes in that fiber product are reduced by the étale charts over the smooth base. The inverse commutes with field automorphisms by uniqueness, so Lemma 289.1 descends it to E. Cover surjectivity persists after scalar extension: for any point its nonempty finite algebra fiber stays nonzero over a field extension. Nonemptiness at each point over E itself can be tested after embedding its finitely generated residue field in $\mathbb{C}$ by Lemma 301.1. Finiteness over E follows from the homogeneous finite-fiber argument of Lemma 298.1: each residue field contains the infinite field E, and the geometric fibers are finite. On an affine target chart its finite algebra is locally free of rank $|D|$. Lift a basis at a point to generators on a neighborhood by the argument of Theorem 299.1; the resulting surjection becomes an isomorphism after extending to $\mathbb{C}$, by the rank and freeness there. Its kernel is already zero by faithful scalar extension. In the residue algebra at any point choose an element whose values at all the $|D|$ geometric fiber points are distinct. Such an element is a linear combination of algebra generators: for each pair at least one generator distinguishes them, and finitely many proper linear conditions can be avoided over the infinite residue field. Its powers of degrees $0, \dots, |D| - 1$ form a basis by the Vandermonde determinant over the split reduced geometric fiber. Lift the element on an affine neighborhood. Its powers give a basis on inverting their coordinate determinant. The algebra is the quotient by one monic polynomial of degree $|D|$, from expressing the next power in that basis. Its derivative is a unit after shrinking again: multiplication by the derivative has nonzero determinant at the point since the geometric roots are distinct. This is an étale chart over E.

For a finite list of models choose a finer factorial level mapping to all their generic levels, using the nested subgroups in Proposition 254.1. Include their generic forgetful maps, now with targets the already constructed projective models, and all their deck translates as extra factors in the same construction. The generic forgetful maps are the level projections of

the period quotients and are surjective: each point has a framed representative giving a lift. Projection of the new reduced closure to each earlier model is projective (the graph is closed in a projective product) and has closed image. At an empty geometric fiber the homogeneous equations in the projected-away variables span all monomials of a fixed degree after base change to an algebraic closure, as in Lemma 298.1; one minor keeps that fiber empty in an open neighborhood. This proves the closed-image assertion at arbitrary residue fields. The image contains the dense generic fiber, so it is the whole earlier model. The specified composite identities persist by closed diagonals and density on the reduced source. All new ambient projective spaces and products were defined over the original R. □

303. Cancellation on irreducible components

Proposition 303.1. *Fix w as in Theorem 301.2. Take any cover $\mathcal{W}$ of Theorem 302.1 recording all of its coarse maps u_A and the source projection c. Over an algebraically closed extension k of $\mathbf{F}_p$ let Z be the coarse special target and let D be any reduced irreducible component of $\mathcal{W}_k$. Write $\Phi : Z \to Z$ for the map over k obtained by base change of the coordinatewise p-power Frobenius of the $\mathbf{F}_p$-model. There is a line $W_D \subseteq \mathbf{F}_p^n$ such that, as morphisms on D,*

$$\Phi^{\,n-\dim A} u_A|_D = \Phi^{\,n-\dim(A+W_D)} u_{A+W_D}|_D \qquad (A \subseteq \mathbf{F}_p^n).$$

Proof. The projection of $\mathcal{W}$ to the product of the coarse targets factors through H_w of Theorem 301.2. Indeed it does so on the generic fiber by definition of the joint image, and the defining homogeneous equations vanish on the reduced closure by density, as in Theorem 302.1. Take the point of D at the algebraic closure of its function field induced by its generic point. Apply Theorem 301.2 to its image in H_w. Its resulting line W_D works for all A simultaneously. Equality of the morphisms already holds over the function field by injectivity into its algebraic closure. Their equalizer is closed since the target is projective, and every rational function in any integral affine chart that vanishes generically vanishes there. Thus its ideal pulls back to zero on all charts of D. This proves equality as morphisms on the entire component. The coordinate power map used here is over k: it raises the homogeneous coordinate variables to p-th powers and fixes the scalars in k; the defining equations over $\mathbf{F}_p$ ensure that it maps into Z. □

Definition 303.2. For a projective scheme X over an algebraically closed field, a cycle is an element of the free abelian group on its integral closed subvarieties. For an integral projective D and a morphism $b : D \to X$, let C be its reduced closed image. Define the image cycle of $[D]$ to be zero if $k(D)/k(C)$ has positive transcendence degree, and otherwise put

$$b_*[D] = [k(D) : k(C)]\,[C].$$

The image is closed by the projective empty-fiber argument of Theorem 302.1, and is irreducible by continuity. Pullback of rational functions injects $k(C)$ into $k(D)$ by density. The fields have finite type over the base field, so an algebraic extension here has finite degree, by adjoining finitely many generators successively.

Corollary 303.3. *In Proposition 303.1 assign an integer m_D to each reduced irreducible component D of the special cover. There is an identity of cycles on $Z \times_k Z$,*

$$\sum_D m_D \sum_{A \subseteq \mathbf{F}_p^n} (-1)^{\dim A} p^{\binom{\dim A}{2}}\, (c, \Phi^{\,n-\dim A} u_A)_*[D] = 0.$$

Before taking image cycles, the same inner sum of the corresponding maps is zero in the free abelian group on morphisms $D \to Z \times_k Z$, for each single component.

Proof. On D the two maps paired by adjoining W_D are identical by Proposition 303.1; their first component $c|_D$ is fixed throughout. Proposition 269.2 applied to the free group on these morphisms gives the second assertion. Sending each generator to its image cycle in

Definition 303.2 defines a homomorphism of abelian groups and preserves the zero relation. There are finitely many components by the noetherian argument of Proposition 291.2; multiplying their relations by m_D and summing gives the formula, including for the lengths at the generic points if those multiplicities are chosen. □

304. Elliptic curves with quadratic multiplication and finite level

Definition 304.1. Let $K = \mathbb{Q}(\sqrt{-d})$ with its fixed embedding of positive imaginary square root, let $O = O_K$, and put $\widehat{O} = \prod_v O_v$, $\mathcal{T}_K = \mathbb{A}_{K,f}/\widehat{O}$. The product is over finite places. For $x \in \mathbb{A}_{K,f}^{\times}$ let $\mathfrak{a}_x = K \cap x\widehat{O}$, a fractional O-ideal, and put $Y_x = \mathbb{C}/\mathfrak{a}_x$. Its torsion label $P_x(t)$ for $t \in \mathcal{T}_K$ is the unique class of $z \in K/\mathfrak{a}_x$ with $x^{-1}z = t \bmod \widehat{O}$. For an integer $m \geq 1$ set $H_m = \{b \in \widehat{O}^{\times} : b - 1 \in m\widehat{O}\}$ and $\mathcal{C}_m = \mathbb{A}_{K,f}^{\times}/(K^{\times}H_m)$. A label of level m means an element of $\mathcal{T}_K[m]$. The symbol H_m here denotes the quadratic idele subgroup used in Proposition 241.1.

Lemma 304.2. *For any full lattice $\mathfrak{a} \subset \mathbb{C}$, the functions $(\wp_{\mathfrak{a}}, \wp'_{\mathfrak{a}})$ identify $\mathbb{C}/\mathfrak{a}$ biholomorphically with a smooth projective plane cubic, taking zero to its point at infinity. The torus law and every holomorphic homomorphism between two such tori are algebraic in their cubic coordinates.*

Proof. Use the series and its termwise computations in Proposition 31.3. Before the choice of a scale in that proof they apply to every lattice by taking an oriented basis $\lambda, \lambda\tau$. Write $x = \wp_{\mathfrak{a}}, y = \wp'_{\mathfrak{a}}$, and e_1, e_2, e_3 for the values at the nonzero half-periods. Then $y^2 = 4\prod_{j=1}^{3}(x - e_j)$ and the roots are distinct by the double-zero argument of Proposition 32.2. The homogeneous cubic $Y^2Z = 4\prod(X - e_jZ)$ has no affine singular point: both partial derivatives can vanish on it only at a repeated root. Its only point at infinity is $[0 : 1 : 0]$, where the derivative in Z is nonzero. The poles of orders two and three at the origin make x/y a local parameter there, with linear term $-z/2$, and $1/y$ holomorphic and vanishing. Thus the map extends to the indicated point with nonzero derivative. Elsewhere its derivative is nonzero, since away from half-periods $x' = y \neq 0$ and at a half-period y has a simple zero by the same period computation.

For each finite $a \in \mathbb{C}$ the function $x - a$ has two zeros counted with multiplicity by the pole count of Lemma 31.1. When a is not a branch value these are $z, -z$ with opposite nonzero y; at a branch value there is the double zero of the indicated half-period. This accounts bijectively for every affine point of the curve. The local holomorphic inverse argument of Theorem 257.2 then gives the claimed isomorphism. Now use Theorem 267.2 on these compact projective manifolds, and their projective product for the sum map, to obtain the algebraic morphisms. □

Proposition 304.3. *The definitions of P_x give additive O-linear bijections from $\mathcal{T}_K$ onto the torsion of Y_x. At level m, the isomorphism classes of the curves Y_x equipped with their O-action and the individual labels $P_x(t)$ for $t \in \mathcal{T}_K[m]$ are exactly $\mathcal{C}_m$, which is finite. More generally every smooth connected complex projective elliptic group with unital O-action by the given embedding on its tangent line and an O-linear bijection from $\mathcal{T}_K[m]$ to its m-torsion represents one of these classes. The same classification for all labels simultaneously has equivalence $x \sim ax$, $a \in K^{\times}$.*

Proof. In the valuation factorization of Proposition 82.3, $\mathfrak{a}_x$ has at each finite place the local valuation of x, so its completion is $x\widehat{O}$. It is a full lattice since it lies between two rational integer multiples or reciprocal multiples of O. Modulo $\mathfrak{a}_x$, the torsion points are exactly $K/\mathfrak{a}_x$. The assignment $z \mapsto x^{-1}z \bmod \widehat{O}$ is injective there. For surjectivity multiply a proposed lift in $\mathbb{A}_{K,f}$ after applying x by an integer to clear its denominators and those of $x\widehat{O}$. Approximation of an integral tuple modulo a fixed integer multiple of $\widehat{O}$ by a member of O holds by the Chinese remainder identity in an integral basis, as in Lemma 269.1. Choose

that multiple inside the cleared lattice and undo the denominator. This gives all labels and proves additivity and O-linearity.

For an elliptic group as in the statement, connectedness and smoothness imply irreducibility by Lemma 294.1 and constancy of global functions on a connected projective manifold by the maximum principle. Use Lemma 284.1: the periods in its tangent line have rational rank two. Their rational span is a one-dimensional K-space because the action extends rationally to K and is ordinary multiplication by the given embedding on that line. Dividing by one nonzero rational period therefore identifies the lattice with a fractional O-ideal in K. Choose an idele x of its local generators; existence follows from the ideal factorization and the local uniformizers in Proposition 82.3. The finite labels may differ from P_x by an automorphism of $\mathcal{T}_K[m]$. This module is O/mO via multiplication of representatives by m and reduction in $\widehat{O}/m\widehat{O}$. An automorphism is multiplication by a unit of that finite ring. Lift the unit locally at primes dividing m and choose one at the other primes. This gives $b \in \widehat{O}^\times$, and $P_{xb}(t) = P_x(bt)$, realizing the given labels. For compatible labellings on all torsion, the corresponding units are compatible under reduction, by multiplication of torsion points by l/m for $m \mid l$. They give one $b \in \widehat{O}^\times$ by the same residue limit argument as in Lemma 287.1.

A pointed O-group isomorphism lifts to its complex derivative by the period uniformization. It sends one fractional ideal onto the other, hence is multiplication by an element $a \in K^\times$. The induced finite unit $x'^{-1}ax$ fixes the level labels exactly when it lies in H_m, and it fixes all labels exactly when it is one, by testing modulo p^j at every prime. Conversely any such scalar gives the required isomorphism, algebraic by Lemma 304.2. Finiteness of $\mathcal{C}_m$ is the ideal-class and finite unit-residue calculation in Proposition 241.1. □

305. Rigid coordinates for the quadratic classes

Lemma 305.1. *Fix $6 \mid m$, two distinct nonzero labels of order two $e_1, e_2 \in \mathcal{T}_K[m]$, their sum e_3, and a label t_3 of order three. On a labelled complex curve of Proposition 304.3 there are unique rational functions X, Y with the following properties. They are regular off the origin, X has pole order at most two and takes the values zero and one at the first two labels, and Y has pole order at most three, is odd under inversion, and takes the value one at t_3. They embed the affine complement of the origin as $bY^2 = X(X-1)(X-\lambda)$, where, writing $u = X(P_x(t_3))$, one has*

$$\lambda = X(P_x(e_3)), \qquad b = u(u-1)(u-\lambda).$$

Here $b\lambda(\lambda-1) \neq 0$. Over any algebraically closed field of characteristic different from two, on a smooth cubic of this displayed form marked by the same coordinate normalizations, with origin at infinity and inversion $(X, Y) \mapsto (X, -Y)$, the same uniqueness holds.

Proof. In Lemma 304.2 let $a_i = \wp(P_x(e_i))$. Take $X = (\wp - a_1)/(a_2 - a_1)$ and $Y = \wp'/\wp'(P_x(t_3))$. The denominator in Y is nonzero since the finite zeros of $\wp'$ occur exactly at the nonzero points of order two. The cubic equation in that lemma, evaluated at t_3 as well, gives the displayed relation and the nonvanishing. The linear affine coordinate changes extend invertibly to the projective plane and put the origin at $[0:1:0]$.

In homogeneous coordinates $[X_h : Y_h : Z_h]$, put $t = X_h/Y_h$, $v = Z_h/Y_h$ near that point. The equation is $bv = t(t-v)(t-\lambda v)$. Rewriting it gives $v(b + (1+\lambda)t^2 - \lambda tv) = t^3$, whose parenthesis is a unit there. The local maximal ideal is generated by t, and v/t^3 is a unit. The pole orders of $X = t/v$ and $Y = 1/v$ are thus two and three. On the affine complement every regular function is an element of its coordinate ring and has a unique expression $A(X) + B(X)Y$ for polynomials A, B, by monic division in Y after dividing by the nonzero b. The highest poles in the two parts have even and odd orders, respectively, hence cannot cancel. It follows that all functions of pole order at most two there are spanned by $1, X$, and the odd ones of pole order at most three by Y, since the characteristic is different from two.

The imposed values pick one element in each case. This proof applies in all the indicated fields. □

Theorem 305.2. *For the chosen labels of Lemma 305.1, the tuple consisting of b, λ and the coordinates $(X(P_x(t)), Y(P_x(t)))$ for all nonzero $t \in \mathcal{T}_K[m]$ distinguishes precisely the elements of $\mathcal{C}_m$. All its entries are algebraic over K. The permutations of these tuples by coefficient automorphisms fixing K factor through a finite Galois extension of K. The group laws and the O-action on each normalized cubic are defined over the algebraic closure of K in $\mathbb{C}$.*

Proof. For clarity, every pointed holomorphic map between compact complex tori is a group map. In their translation coordinates the matrix of its derivative is globally holomorphic, since coordinate translations have identity derivative and holomorphic maps have holomorphic derivatives by the Cauchy formula of Lemma A.3. Each entry is constant by compactness and the maximum principle. On the vector source cover subtract from the map the target period map applied to this constant linear derivative. The difference has zero derivative and is zero at zero, hence vanishes. This proves the assertion also when the two group structures on the same underlying pointed compact curve were given separately, by their uniformizations in Lemma 284.1.

An isomorphism matching the group and labels preserves the normalized functions by Lemma 305.1. Conversely equal tuples identify the same cubic and all the labelled points, including the origin. Under this pointed identification the group laws agree by the first paragraph. Their endomorphisms for $a \in O$ agree since both have derivative multiplication by the fixed complex number a, and an endomorphism is determined by its derivative by the same argument. This proves separation by Proposition 304.3.

Conjugate a normalized cubic, its algebraic group law, endomorphisms and marked points by any $\iota \in \mathrm{Aut}(\mathbb{C}/K)$. Algebraicity of the group and maps before conjugation is Lemma 304.2. The result is a smooth irreducible projective group with the same tangent scalars $a \in O$, since the derivative on a one-dimensional tangent space is a scalar and its linear chart terms conjugate. All the finite torsion labels are transported bijectively, since multiplication and its point equations are transported. Thus this is a curve from the same finite class list by Proposition 304.3. The normalized coordinates conjugate by their uniqueness or directly by the conjugated cubic equation and values. Each entry therefore has finite orbit under these automorphisms. The monic polynomial with the distinct orbit elements as roots has coefficients fixed by all of them and hence in K, by the fixed-field computation of Lemma 270.1. The finite set of tuples therefore lies in one finite Galois extension, taking the splitting fields of the finitely many orbit polynomials. This proves the assertion on permutations.

Finally an automorphism fixing the algebraic closure of K fixes each normalized cubic and its origin as embedded data. The conjugate group law must be its same law by the first paragraph. The conjugate endomorphisms by a have the original tangent scalars and hence are the same maps. Apply Lemma 289.1 over that algebraic closure to their regular maps, also to the sum on the projective product. This proves the assertion on fields of definition. □

306. Conjugation and translation on the quadratic levels

Lemma 306.1. *Give $\mathcal{C} = \mathbb{A}_{K,f}^{\times}/K^{\times}$ the quotient of the restricted product topology. It is compact, and reduction identifies it as a topological group with the inverse limit of the finite groups $\mathcal{C}_m$. The multiples of six suffice as levels.*

Proof. Projection to $\mathcal{C}_1$, the finite ideal class group, has kernel $\widehat{O}^{\times}/O^{\times}$, by the valuation map of Proposition 241.1. The subgroup $K^{\times} \cap \widehat{O}^{\times} = O^{\times}$ is finite: an integral quadratic unit has complex absolute value one by its norm and there are finitely many points of the lattice

O in a bounded disk. The compactness of $\widehat{O}^\times$ can be seen from its finite residues modulo $j!$. Compatible residues give unique local integers and inverses by the local completeness and residue description of Lemma 225.1. The kernels of the residue maps are a neighborhood basis in this product, since finitely many prescribed local powers divide some $j!$. If an open cover had no finite subcover, one could choose recursively a nested sequence of factorial residue classes with the same failure, using the finite subdivision at the next stage. The compatible residues would define a point with a whole neighborhood in one covering open, a contradiction. The quotient of the unit group and each of its finitely many translates are compact, giving compactness of $\mathcal{C}$.

The maps to finite levels are continuous and surjective. A class trivial at every level has a representative $b \in \widehat{O}^\times$. For each j it equals some unit in $O^\times$ modulo $H_{j!}$. Finiteness of the units makes a single one occur for unbounded j; equality at all powers gives equality in every completion. Thus the limit map is injective. For a compatible sequence of finite classes the nonempty preimages of its factorial terms are nested closed sets in a compact space, so their intersection supplies a representative. The finite level residues of that representative give the whole sequence. The limit is Hausdorff since distinct sequences differ in some finite discrete coordinate; the continuous bijection from the compact group has continuous inverse as closed subsets are compact and compact subsets of Hausdorff spaces are closed. Factorials divisible by six are cofinal, proving the final assertion. □

Proposition 306.2. *Let $\bar{K}$ be the algebraic closure of K in $\mathbb{C}$ and $G_K = \mathrm{Gal}(\bar{K}/K)$, with the finite Galois quotient topology. There exists a continuous homomorphism*

$$r_K : G_K \longrightarrow \mathcal{C}$$

such that for each $6 \mid m$ a Galois element takes the rigid tuple of $[x] \in \mathcal{C}_m$ to the rigid tuple of $r_K(\gamma)[x]$ at that level.

Proof. Lift γ to an automorphism of $\mathbb{C}$ by choosing a transcendence basis and extending over the algebraic closure as in Lemma 270.1. Conjugating all the O-action and torsion labels of any Y_x gives a full labelled curve of the same tangent type, by Proposition 304.3. Write $\widetilde{x}$ for the class of its reconstructed frame in $\mathcal{C}$. At each level $6 \mid m$ its class is exactly the conjugated tuple of Theorem 305.2, so it depends only on γ. By Lemma 306.1 this holds for the full class as well.

Let y be any finite idele of K. Multiplying its inverse by a suitable positive integer clears all its denominators. Dividing y by that integer leaves its class in $\mathcal{C}$ unchanged, so we may assume $y^{-1} \in \widehat{O}$. Then $\mathfrak{a}_x \subseteq \mathfrak{a}_{xy}$; the vector identity gives a nonzero algebraic homomorphism $Y_x \to Y_{xy}$ by Lemma 304.2. Its effect on labels is $t \mapsto y^{-1}t$ by Definition 304.1. Conjugating both curves and this map gives a nonzero group homomorphism between the reconstructed curves with exactly this same label formula. Its derivative in the two rational period coordinates is multiplication by some $a \in K^\times$: it carries one full fractional ideal into the other and a nonzero complex homomorphism cannot have zero derivative by Lemma 284.1. Use arbitrary representatives of the reconstructed classes. Then $\widetilde{xy}^{-1}a\widetilde{x} - y^{-1}$ maps $l^{-1}\widehat{O}$ into $\widehat{O}$ for every positive integer l, since its action on all finite torsion classes vanishes. Taking $l = p^j$ above each rational prime proves this adelic difference is zero. Thus $[\widetilde{xy}] = [\widetilde{x}]\,[y]$ in $\mathcal{C}$, including the original y since rational scalings identify the full labelled groups.

The action is therefore translation by its value on the identity class; call this value $r_K(\gamma)$. Composition of conjugations on the finite tuples gives composition of their translations, hence the homomorphism identity in $\mathcal{C}$. Its projection to each finite level $6 \mid m$ is continuous because the tuple entries are in the finite Galois extension of Theorem 305.2. The inverse limit topology of the preceding lemma gives continuity. □

307. Good reduction on one fixed quadratic level

Lemma 307.1. *Fix $6 \mid m_0$ and the normalizing labels of Lemma 305.1. Choose one normalized cubic and its full ordered list of m_0-labels for each element of $\mathcal{C}_{m_0}$. There are a finite Galois extension L/K and an integer $N > 0$, divisible by $6m_0$, with the following properties over $O_L[1/N]$. Each equation*

$$bY^2Z = X(X - Z)(X - \lambda Z)$$

defines a smooth projective commutative group of relative dimension one with the prescribed origin, inversion and O-action. Its geometric fibers are integral. The affine coordinates of its nonzero marked points are integral in this ring, and the marked tuples for distinct classes have distinct reductions at every remaining prime.

For every valuation ring $V \subset \mathbb{C}$ containing this ring and every such cubic over V, put $B = \mathcal{O}(1)$. In every geometric fiber, for every $t \geq 1$, the complete section space of B^t is obtained by reducing a free degree-t coordinate module of rank $3t$. The section space of the trivial line on a geometric fiber has dimension one. The corresponding flatness, integrality and high-degree section assertions of Theorem 294.2 hold for any finite product of the listed cubics over V.

Proof. Theorem 305.2 gives algebraic coordinates, group laws and endomorphisms. Choose the integral basis $1, \beta$ of O from Lemma 11.3. It suffices to retain the single endomorphism $[\beta]$, its integral quadratic relation and its additivity. All other scalars are integral combinations of it and the identity. Include the origin, inversion, addition, their group equations on products of at most three copies and the finite label identities. The finite affine presentations of these morphisms and their identities in Proposition 270.2 require only finitely many algebraic coefficients. Take one finite Galois extension of K containing them and all the tuples. The denominator-clearing argument in Theorem 290.2 gives all the maps and identities over the indicated ring after increasing N. Invert also the finitely many denominators of the tuples and the norms of the nonzero elements $b, \lambda, 1 - \lambda$ after those denominators are cleared. For each pair of distinct tuples invert the norm of one nonzero coordinate difference in the same way. These norms are nonzero rational integers after clearing denominators. This proves both assertions on the marks. The inversion formula is $(X, Y) \mapsto (X, -Y)$ since it is so over the complex numbers.

On the affine chart $Z = 1$ the equation has a unit partial derivative at every geometric point. If its Y-derivative vanishes there, then $Y = 0$ and the three distinct roots give a nonzero X-derivative. At $[0 : 1 : 0]$ the derivative in Z/Y is b. These charts give smoothness of relative dimension one. Over the function field of the affine X-line the cubic polynomial $X(X - 1)(X - \lambda)/b$ cannot be a square, by its simple zero at $X = 0$. Monic division in Y thus gives a domain on the affine chart over every algebraically closed residue field. Its homogeneous closure is the entire cubic: the homogeneous polynomial has no factor Z, and a homogeneous factor becoming constant at $Z = 1$ would be a power of Z times a scalar. This gives geometric integrality.

For $P = V[X, Y, Z]$ the homogeneous equation is monic up to sign in X of degree three. Division gives a free basis of its degree-t quotient consisting of monomials with X-exponent at most two. In positive degree their number is $3t$, by summing the counts at exponents zero, one and two that occur. For the section comparison use the graded sequence $0 \to P(-3) \to P \to R \to 0$, with R the cubic ring. It stays exact after every scalar extension by the same monic division. In the augmented chart complex of Definition 292.1, the monomial contraction in Lemma 292.2 contracts each monomial unless all three exponents are negative. The latter monomials occur only in cochain degree two, since all three charts must be used. Thus both augmented complexes for P_t and $P(-3)_t$ are exact for $t \geq 1$, since $t - 3 \geq -2$. For $t = 0$ the latter complex still has zero cohomology in degrees zero and one. The lifting argument for the short exact sequence in the same lemma identifies R_t

with the compatible sections for every $t \geq 0$. In degree zero $R_0 = V$. All computations apply over each geometric residue field. The direct-limit description of the affine charts and the tensor-product argument in Theorem 294.2 now apply with these free coordinate pieces. They give flatness, complete product sections and $H^0(\mathcal{O}) = k$ on products over an algebraically closed field k. The product Jacobian charts and Lemma 294.1 give their geometric integrality. □

Lemma 307.2. *Let V in Lemma 307.1 have fraction field $\mathbb{C}$ and residue characteristic $p > 0$. On each listed cubic, specialization of complex torsion points of order prime to p is injective.*

Proof. Normalize a finite list of homogeneous coordinates of a complex point by dividing by one of smallest valuation. They give a V-point of the same cubic, unique since two such normalized tuples for the same point differ by a unit. Group operations commute with this specialization by Lemma 307.1. It suffices to show that a point Q reducing to the origin and killed by n prime to p is the origin.

On the chart at the origin let $t = X/Y$, $v = Z/Y$. The equality $v(b+(1+\lambda)t^2-\lambda tv) = t^3$ from Lemma 305.1 makes the ideal of the identity section (t) on an open neighborhood of the closed identity. Its conormal line, the identity ideal modulo its square, has basis t over V: reducing the displayed relation to first order sets $v = 0$ and imposes no linear relation on t. On that line the pullback by multiplication by n is n. Indeed at the pair of origins the two first-order coefficients of the sum map are both one by its restriction to each axis; compose with the diagonal and induct on n. Thus, on a neighborhood of the closed identity on which the source and image coordinates are defined, $[n]^*t = nt + t^2h$ with h regular there. The section Q lies in this neighborhood, since its inverse image is an open containing the closed point of the local spectrum. If $[n]Q = 0$, then $t(Q)(n + t(Q)h(Q)) = 0$ in V; the parenthesis is a unit since $t(Q)$ belongs to the maximal ideal. Consequently $t(Q) = 0$, and the cubic equation gives $v(Q) = 0$. This is precisely $[0 : 1 : 0]$. □

308. Integral isogenies of the quadratic cubics

Proposition 308.1. *Use any two cubic models G, G' and a valuation ring V with fraction field $\mathbb{C}$ as in Lemma 307.1. Every algebraic group homomorphism of their generic fibers extends uniquely over V. Suppose the generic map f is induced by a nonzero complex linear map of the period lines and its finite kernel has odd cardinality d. Then its extension is surjective finite locally free of rank d, and its kernel is free of length d over V, meaning that its coordinate algebra is a free V-module of that rank.*

Proof. For the extension apply the argument of Theorem 295.2 with the free high cubic coordinate pieces and integral fibers of Lemma 307.1. In detail, clear denominators in a homogeneous tuple for the generic map, normalize its coefficients in a high free piece by one of least valuation, and take the open where the reduced tuple is defined. This open is dense in each geometric fiber because at least one coefficient is a unit and the fiber comparison is injective there. The sum of this open with itself covers the group fiberwise, since two dense opens on the same integral geometric fiber meet. The flat surjection from pairs of points of this open by their sum therefore descends the sum of their images, by Lemma 295.1. On its overlap the two formulas agree generically and hence everywhere by the flat chart and product calculation in Theorem 295.2. The same calculation gives uniqueness and the homomorphism identity. In particular it extends equalities of composites.

Write O_0 for the identity point and B, B' for the respective hyperplane lines. The line at infinity cuts out $3[O_0]$, by the equation for Z/Y in Lemma 305.1. The generic map is locally a biholomorphism since its linear coefficient is nonzero. Its inverse image of the target origin is thus the reduced set $H = \ker f$, with multiplicity one at each point. Every nonzero element of H belongs to a two-element pair $\{Q, -Q\}$ since d is odd. For such a pair the function $X - X(Q)$ has a simple zero at both points and a double pole at O_0, with no further

zeros or poles. Indeed $Y(Q) \neq 0$ so $X - X(Q)$ is a parameter there by the cubic equation, and the two points of that X-coordinate have opposite Y-coordinates. The product over the pairs has divisor $\sum_{Q \in H}[Q] - d[O_0]$, also for $d = 1$ by the empty product. Cubing this identity gives the generic line isomorphism $f^*B' \simeq B^d$. Apply the proof of Theorem 296.2 to the quotient of the two lines. The free high pieces, their comparison on the special fiber and flatness are those of Lemma 307.1. Normalizing the generic trivialization and its inverse gives regular sections whose product is a unit, by integrality of the closed fiber as in that theorem. This proves the pullback identity over V.

The ample pullback gives finiteness by the affine chart argument of Lemma 298.1. On each geometric fiber the image is the entire target. Indeed its function field and that of the source both have transcendence degree one, directly from the monic quadratic equation over $k(X)$ on the affine chart. For a finite map the image and source have equal transcendence degrees and a proper closed subset has smaller transcendence degree, by Lemma 298.2. On an integral target fiber the direct image of the structure sheaf is then locally free by translating a generic basis, as in Theorem 299.1. If its rank is r, sandwich it between sums of r negative and positive twists by B' as in the same proof. By Lemma 307.1 the two outside bounds on the dimension of the sections after a twist by t are $3r(t - c)$ and $3r(t + c)$ for some fixed c on this fiber and all $t > c$. The pullback has $3dt$ sections, so $r = d$. The final flat-chart argument of Theorem 299.1 now gives local freeness on the full target: a lifted basis at any point gives a surjection from the free module of rank d locally; the map is an isomorphism on the generic base fiber, and the source module has no V-torsion. Pull back at the identity to obtain the asserted free kernel algebra. □

Corollary 308.2. *For a split rational prime $p \nmid N$ write $pO = \mathfrak{u}\overline{\mathfrak{u}}$, and put $t_{\mathfrak{u}} = p$ at $\mathfrak{u}$ and 1 at every other finite place, as an idele of K. For each finite idele x, the identity on period lines gives a degree-p isogeny*

$$\psi_{\mathfrak{u}} : Y_x \longrightarrow Y_{x t_{\mathfrak{u}}^{-1}}.$$

On normalized models from the fixed level m_0 it extends surjectively and finite locally freely of rank p over each good V. It carries the label s to the label $t_{\mathfrak{u}} s$; in particular it preserves every label of order prime to p.

Proof. The corresponding period lattices are $\mathfrak{a}_x$ and $\mathfrak{u}^{-1}\mathfrak{a}_x$ by Definition 304.1 and Proposition 82.3. Their inclusion has index p, since at $\mathfrak{u}$ the quotient is its one-dimensional residue field of size p, and it is zero at every other completion, by the ideal residue calculation in Lemma 225.1. The map is algebraic by Lemma 304.2. In the target coordinates of Definition 304.1 its effect on the label is multiplication by $t_{\mathfrak{u}}$. A label of order prime to p has zero component at $\mathfrak{u}$, proving the last assertion. Identifying each complex group with the model for its m_0-tuple by Theorem 305.2, the extension and rank now follow from Proposition 308.1, since p is odd. □

309. Frobenius on the split quadratic isogeny

Theorem 309.1. *In Corollary 308.2 assume that the maximal ideal of V contracts to $\mathfrak{u}$ in O. Write G_k, G'_k for the reductions of the source and target over an algebraic closure k of the residue field. There is a unique isomorphism of pointed groups*

$$i : G_k^{(p)} \longrightarrow G'_k, \qquad (\psi_{\mathfrak{u}})_k = i F_{G_k,p}.$$

Here $F_{G_k,p}$ is the relative Frobenius. For every label of order prime to p this isomorphism takes the Frobenius of its source reduction to the target reduction with the same label.

Proof. Put $J = \ker F_{G_k,p}$. By Lemma 274.1, it is killed by $[p]$, has one geometric point, and has coordinate algebra $k[t]/(t^p)$. Every scalar of O preserves J, since relative Frobenius commutes with the corresponding twisted endomorphism by the coordinate formula in Lemma 272.1. Choose $a \in O$ with residues zero modulo $\mathfrak{u}$ and one modulo $\overline{\mathfrak{u}}$. Such a exists

since these distinct maximal ideals have sum O: write one as a sum of elements of the two ideals and take the first summand. Their intersection is their product pO, since any member of the intersection times this sum belongs to the product. Thus $a^2 - a \in pO$, so $[a]|_J$ is idempotent under composition.

The tangent scalar of $[a]$ on G_k is the image of a in k, hence is zero. Indeed the tangent line at the origin over V has a basis dual to the parameter of Lemma 307.2. In that basis the derivative is a scalar of V; over the fraction field it equals a by Proposition 304.3, so its reduction has the asserted value. On the augmentation ideal $I = (t)$ of J the pullback by $[a]$ therefore maps I into I^2. Indeed t is the image of a tangent parameter of G_k. The j-fold iterate maps I into I^{2^j}, which vanishes when $2^j \geq p$. Idempotence identifies that iterate with the first map. Thus $[a]|_J$ is the zero group map and $[1-a]|_J$ is the identity.

Put $e = 1 - a$. The linear map on the period line given by e/p carries $\mathfrak{a}_x$ into $\mathfrak{u}^{-1}\mathfrak{a}_x$, since $e \in \overline{\mathfrak{u}} = p\mathfrak{u}^{-1}$. It gives an algebraic homomorphism $\phi : Y_x \to Y_{xt_{\mathfrak{u}}^{-1}}$ by Lemma 304.2. On the generic groups $\psi_{\mathfrak{u}}[e] = \phi[p]$ by the equality of the two linear maps. By Proposition 308.1 these homomorphisms extend with their equality over V. Restrict the equality to J on the special fiber. Its left side is $(\psi_{\mathfrak{u}})_k|_J$ and its right side is zero. Hence J is a closed subgroup of the kernel of $(\psi_{\mathfrak{u}})_k$. Both have length p, by the first paragraph and Corollary 308.2. The induced surjection of their coordinate algebras is therefore an isomorphism. Apply Corollary 275.3. The final assertion follows from the label formula in Corollary 308.2 and specialization of its extended map. □

Corollary 309.2. *Under Theorem 309.1, write (b_x, λ_x) for the two coefficients of the normalized model, and $Q_x(s)$ for the pair of affine coordinates of its point of nonzero label s. For every s of order prime to p these coordinates are integral at V. If a tilde denotes reduction and $y = xt_{\mathfrak{u}}^{-1}$, then*

$$(\widetilde{b}_y, \widetilde{\lambda}_y) = (\widetilde{b}_x^{\,p}, \widetilde{\lambda}_x^{\,p}), \qquad \widetilde{Q}_y(s) = \widetilde{Q}_x(s)^{\,p},$$

with entrywise powering on the right.

Proof. By Lemma 307.2 the point of any such label reduces away from the origin. It thus lies in the affine chart over V, since its reduction does, proving integrality. On $G_k^{(p)}$ the cubic has coefficients $\widetilde{b}_x^{\,p}, \widetilde{\lambda}_x^{\,p}$, and the Frobenius images of the chosen normalizing points have the same values zero, one and $Y = 1$ at their respective labels. All of those labels have order prime to p. The isomorphism of Theorem 309.1 preserves them, the origin and group inversion. By the uniqueness in Lemma 305.1 the pullbacks of the target functions X, Y are the coordinate functions on that twisted cubic. Evaluating X at the third nonzero two-torsion point gives the coefficient λ, and evaluating the cubic relation at the chosen order-three point gives b. These evaluations prove the first equality. The coordinate formula for relative Frobenius and the assertion on all prime-to-p labels in the theorem prove the second one. □

310. The quadratic Galois action away from the residue prime

Lemma 310.1. *Let $h : O_{\overline{K}} \to \overline{\mathbf{F}}_p$ be a residue map at a place of K above p, where $\overline{K}$ is its algebraic closure in $\mathbb{C}$. There is a valuation ring V of $\mathbb{C}$ such that reduction on the elements of $O_{\overline{K}}$ is h followed by an embedding of its residue image. If $g \in G_K$ satisfies $h(gz) = h(z)^b$ on $O_{\overline{K}}$, for $b = 1$ or p, this identity holds on $\overline{K} \cap V$ with its extended residue map, and g preserves $\overline{K} \cap V$.*

Proof. Localize $O_{\overline{K}}$ at $\ker h$. The resulting ring A is the directed union of the localizations of O_D at the contracted primes, as D runs through the finite subextensions of $\overline{K}/K$. Indeed an integral numerator and denominator lie together in one such D, and a denominator outside the kernel remains outside it on every larger level. Each finite localization is a valuation ring of D, by the local ideal factorization in Proposition 82.3. Thus A is a valuation ring

of $\overline{K}$, by testing an element and its inverse in one finite field. Extend to a local ring V of $\mathbb{C}$ dominating A by the maximal-local-subring argument of Lemma 301.1. That argument gives the valuation property in $\mathbb{C}$. The contraction is A, since if $x \in \overline{K}$ is outside A, then x^{-1} is in the maximal ideal of A, hence in that of V. Its residue embedding agrees with h on $O_{\overline{K}}$ upon identifying the residue of A with the fraction field of the image of h. If $h(gz) = h(z)^b$, the prime is preserved in both directions since a power is zero exactly at zero and g is invertible. The formula extends to fractions whose denominator is outside the prime. This proves the last assertions. □

Theorem 310.2. *For the imaginary quadratic field K, take $m_0 = 6$ and $N_K = N$ of Lemma 307.1 at this level. Let $p \nmid N_K$ split in K, choose a place $\mathfrak{u} \mid p$, and let h extend its residue map. Write $U_p \subset \mathcal{C}$ for the image of $\prod_{v|p} O_v^\times$ supported over p. Let $I_h \subset G_K$ consist of the automorphisms s with $h(sz) = h(z)$ for every algebraic integer z. For r_K of Proposition 306.2 one has*

$$r_K(I_h) \subseteq U_p, \qquad r_K(\gamma) \in [t_{\mathfrak{u}}]\, U_p$$

for every geometric Frobenius datum $(\mathfrak{u}, h, \gamma)$ of Definition 221.1. These assertions hold with the same N_K at all levels.

Proof. Use the valuation of Lemma 310.1; it contains the localized coefficient ring of Lemma 307.1. Reduce the normalized model Y_1 and its labels there. All labels of finite order have algebraic coordinates by Theorem 305.2 applied to a multiple of their order and six, using the same normalizing points. The affine coordinates of every nonzero prime-to-p label are in the valuation ring by Corollary 309.2. The map γ^{-1} acts on their residues and on the model coefficients by p-th powering by Definition 221.1 and the preceding lemma.

Put $y = t_{\mathfrak{u}}^{-1}$. By Corollary 309.2 the conjugate of the level-six tuple by γ^{-1} and the level-six tuple of Y_y have identical reductions. Both are members of the fixed finite list by Theorem 305.2. Lemma 307.1 separates that list on reduction. Hence they have identical tuples in characteristic zero and are the very same normalized cubic with the same group law, as proved in Theorem 305.2. For every n prime to p, the conjugate of each n-label and the corresponding n-label of Y_y are now points on that one curve killed by n. Their reductions agree by the same Frobenius coordinate formula. Lemma 307.2 identifies the points themselves. The zero label is preserved by their group maps. At every n divisible by six and prime to p, the classification in Proposition 304.3 therefore gives $r_K(\gamma^{-1}) = [t_{\mathfrak{u}}^{-1}]$ in $\mathcal{C}_n$. Inverting gives the positive uniformizer for geometric Frobenius. For $s \in I_h$ compare the s-conjugated tuple with Y_1 itself. The residues are unchanged, and the identical fixed-list and torsion injection arguments give $r_K(s) = 1$ in every such $\mathcal{C}_n$.

It remains to identify the common kernel of these finite projections. Its projection to the ideal class quotient $\mathcal{C}_1$ is trivial, so represent any member by $u \in \widehat{O}^\times$, by the calculation in Proposition 241.1. At each indicated n triviality means $u \in O^\times H_n$, since the principal factor relating two everywhere integral units must be a global unit. Choose increasingly divisible n_j cofinal among those integers, for instance six times the product of all positive integers up to j prime to p. The global unit group is finite by Lemma 306.1. One unit c therefore occurs for an unbounded subsequence in $u \in cH_{n_j}$. At each finite $v \nmid p$ the valuations of these moduli tend to infinity, giving $u_v = c$. Thus $c^{-1}u$ is supported above p. Conversely every such integral unit lies in all these kernels by Definition 304.1. This proves both assertions in $\mathcal{C}$. A unit supported above p representing any given class is unique, since a principal idele equal to one at another finite place is the identity. □

311. The scalar character of the approximating form

Proposition 311.1. *Let ω be a character from Proposition 241.1, and choose integers A, B with $\omega(z) = z^A \bar{z}^B$ on $K^\times$. Fix the auxiliary prime ℓ and the isomorphism $\iota : \mathbb{C} \to K_\ell$*

of Lemma 229.2. Denote by $u_+, u_- \mid \ell$ *the places of* K *induced by* $z \mapsto \iota(z)$, $z \mapsto \iota(\bar{z})$, *respectively, using the corresponding identifications of their completions with* $\mathbb{Q}_\ell$. *The formula*

$$\omega_\ell([x]) = \frac{\iota(\omega(x))}{x_{u_+}^A x_{u_-}^B} \qquad (x \in \mathbb{A}_{K,f}^\times)$$

defines a continuous character of $\mathcal{C}$ *taking values in the unit group of one finite extension of* $\mathbb{Q}_\ell$. *Choose* $m \geq 1$ *with* $H_m \subseteq \ker \omega$. *For every* $p \nmid m\ell$ *it is trivial on* U_p.

Proof. The two identifications with $\mathbb{Q}_\ell$ and their distinct induced primes follow from Lemma 229.2. The numerator is a locally constant character on finite ideles, since the kernel of ω is open. By the proof of Proposition 241.1 there is such an m; select representatives x_i of the finite quotient $\mathcal{C}_m$. Writing $x_i^{h_i} = z_i v_i$ for its class order h_i, where $z_i \in K^\times$, $v_i \in H_m$, gives $\omega(x_i)^{h_i} = z_i^A \bar{z}_i^B \in K^\times$. Thus all values of ω are in a single number field $D = K((\omega(x_i))_i)$: each idele is a representative times a principal idele and a member of H_m. The numerator now takes values in $D_\ell = \mathbb{Q}_\ell(\iota(D))$, which is finite over $\mathbb{Q}_\ell$ by the finite algebraic tower argument in Lemma 16.2. The denominator is a continuous multiplicative function into $\mathbb{Q}_\ell^\times$, and on a principal idele the numerator equals that denominator by the choice of the two local embeddings. The displayed quotient therefore descends continuously for the quotient topology. Its image is a compact subgroup by Lemma 306.1. The absolute values on a compact set are bounded, and both positive and negative powers of any element in this subgroup still belong to it. Each element therefore has absolute value one. Finally, for units supported above $p \nmid m\ell$ the numerator is one by membership in H_m, and both local components in the denominator are one. □

Theorem 311.2. *For the form of Theorem 241.2 put*

$$a_\omega(g) = \omega_\ell(r_K(g)), \qquad g \in G_E \subseteq G_K.$$

Use the common algebraic closure of the indicated fields in $\mathbb{C}$. *This is a continuous character into* $K_\ell^\times$. *Outside a finite set of rational primes, at every degree-one place* w *of* E *and every geometric Frobenius datum* (w, h, γ),

$$a_\omega(\gamma) = \iota(\omega(z_w)), \qquad a_\omega(I_h \cap G_E) = \{1\},$$

where z_w *is the uniformizer norm idele of Theorem 241.2. In particular this supplies the scalar character required in Corollary 238.4 for each approximating form.*

Proof. Continuity and multiplicativity follow from Proposition 306.2 and Proposition 311.1. Restriction to G_E is continuous: a prescribed finite normal restriction over K pulls back to a finite normal restriction after compositing with E. Exclude the finitely many primes from the chosen level and safe conditions, $N_K m\ell$, and the ramified rational primes of E and K. At each remaining rational prime p under a degree-one w, its contraction $\mathfrak{u}$ to K also has residue field $\mathbf{F}_p$. It therefore splits in the quadratic field in the sense that there are two distinct places over p, by the degree formula in Lemma 219.3. Both local fields E_w and $K_\mathfrak{u}$ are $\mathbb{Q}_p$: their ramification and residue degrees are one by the same formula and the completion calculation of Lemma 227.2. Thus z_w and $t_\mathfrak{u}$ differ only by a unit at $\mathfrak{u}$. This factor belongs to H_m, since $p \nmid m$.

A datum over E is a datum over K at $\mathfrak{u}$ under the inclusion, since the exponent in Definition 221.1 is p for both and the algebraic integers in the closure are the same. By Theorem 310.2, $r_K(\gamma)$ differs from $[t_\mathfrak{u}]$ by an element of U_p, which Proposition 311.1 kills. Both ℓ-components of $t_\mathfrak{u}$ are one, so the value is $\iota(\omega(t_\mathfrak{u})) = \iota(\omega(z_w))$. The same two results give the triviality on $I_h \cap G_E$. The finite exclusions satisfy the quantifier for most data in Definition 221.3. Consequently the exact scalar in Theorem 238.3 is available for the indicated spherical polynomial. □

312. The order of choice of the coarse level and the prime

Proposition 312.1. *Fix E and a finite list of definite forms as in Theorem 229.3. For each of their ranks n and each distinguished embedding $\sigma : E \to \mathbb{C}$, the polarization data of Theorem 236.2 and an integer $b_\sigma > 0$ can be chosen before the coefficient prime ℓ; the dependence of b_σ on n is understood. Put $C_b = K(b_\sigma)$ as in Definition 239.1. All stabilizers at this level are trivial. One may choose ℓ, w_0, w_1 as in Theorem 229.3 so that at every $w \mid \ell$ each coarse complex base $\mathcal{X}(C_b, 0)$ and its one-point group have the smooth projective models of Theorem 290.2. Their equations, group maps and O_E-action are defined over a localization of O_E good at every such w. Over the completion $O_{E_w} = \mathbb{Z}_\ell$ these models admit their smooth Jacobian charts. Every finite marked power of the coarse group is smooth projective over $\mathbb{Z}_\ell$ as well.*

For each resulting approximant, the element $\mathfrak{n}$ of Corollary 239.4 may be chosen in $b_\sigma O_E$, with w_0 the only prime dividing $\mathfrak{n}$ above ℓ. The auxiliary place w_* there can be chosen over a rational prime coprime to $b_\sigma \ell$.

Proof. There are finitely many ranks and $2[E^+ : \mathbb{Q}]$ possible embeddings at each rank. For each pair apply Lemma 236.1, Theorem 236.2 and Proposition 237.2 with a fixed finite set of primes in the padding lemma, for example the empty set. Call the resulting multiple for the level N_σ. For an odd rational prime q_b split completely in E outside this finite list of integer divisors, put $b_\sigma = q_b N_\sigma$. Such primes exist by Lemma 229.2. The principal level with this integer is admissible by Proposition 237.2; Proposition 239.2, applied at a place above q_b, gives trivial stabilizers. All these choices concern only the ranks, the distinguished embeddings and the arithmetic field.

For each coarse level take the base, one-point family, projections, group operations and the actions of integral generators of O_E, including their ring relations. One may use an integral basis, with its finite table of sums of basis multiples for the products, by Theorem 18.2. Theorem 290.2 gives models over O_E with finitely many rational primes inverted. Its proof gives the charts for relative smoothness over a finite number field extension with finitely many more primes inverted. Keep all these fields and primes for the finite list, including the exclusions in Theorem 293.2 and Lemma 273.1. The marked ν-point family on the coarse base is the ν-fold fiber product of the one-point family for all ν, by the period coordinates in Theorem 252.3 and the descent of marked slots in Theorem 289.2. The products are projective by the product embedding. Their relative Jacobian charts use disjoint lists of fiber variables with block diagonal relative minors. Thus ν adds no excluded primes.

Choose a finite normal extension $D/\mathbb{Q}$ containing E and all fields needed for the charts. By Theorem 220.2, applied to the identity in its Galois group, one can choose ℓ split completely in D, outside the finite exceptions above, the primes dividing all b_σ, and those excluded for the original forms in Theorem 229.3. One can include any further fixed finite list of rational primes. Complete splitting outside the power-basis indices gives completions $\mathbb{Q}_\ell$ by the root-lifting proof of Lemma 229.2, applied also to D. Any embedding of E at $w \mid \ell$ extends to one of D by Lemma 16.2. The necessary integral chart coefficients and their inverted denominators map to $\mathbb{Z}_\ell$ with the denominators units. The retained presentations hold there. Composing a relative smooth chart with a smooth base chart multiplies their unit minors in a block triangular matrix. This proves the assertion on smoothness over $\mathbb{Z}_\ell$.

The proof of Theorem 229.3 uses, after the exclusions for the given forms, only the split completions and the two distinct indicated places above the same quadratic prime. It applies to this choice of ℓ. For any resulting approximant choose w_* by Lemma 229.2 away also from the finitely many divisors of b_σ and the exceptional primes of the form. Use the support and depths in the proof of Corollary 239.4, with b_σ in place of N in Lemma 239.3. Every resulting $\mathfrak{n}$ is still divisible by N_σ. The extra congruence at q_b only restricts the subgroup fixing the form, and q_b differs from both ℓ and the prime under w_*. The support identity of

that lemma gives exactly the stated support above ℓ. All coprimalities used for the safe places in Corollary 239.4 hold, since the chosen primes avoid N_σ. □

313. The torsion fixed by the finer congruence level

Lemma 313.1. *Use the choice in Proposition 312.1, with the identification of E by its distinguished embedding as in Lemma 273.1. Write S for the coarse base and G/S for its one-point group after passage to $O_{E_{w_1}} = \mathbb{Z}_\ell$. One may impose finitely many further exclusions before choosing ℓ, independent of the approximants. For each $\alpha \in O_E \setminus \{0\}$ divisible at places above ℓ only possibly by w_0, the relative differential of $[\alpha]$ is invertible on the geometric fibers of G/S at every point. This applies in particular to $\alpha = \mathfrak{n}$ for each of the fine levels in that proposition.*

Proof. Fix the integral primitive element θ of Lemma 273.1. Include its characteristic polynomial on the Lie space at the identity section in the finite data spread at the coarse level. More explicitly, on each relative Jacobian chart containing part of the identity section the differentials of the free parameters form a basis of the relative cotangent there: differentiating the chart equations expresses the remaining differentials in this basis by inverting the chosen minor. Pullback by $[\theta]$ has a matrix of regular functions in these frames on the base. The characteristic polynomial of that matrix equals $P_\theta(T)$ of Lemma 273.1 on every complex point, since the transpose acts on the tangent space. Its coefficient identities therefore hold on the reduced generic base, by the affine point argument of Lemma 280.1. They extend after a finite exclusion: an element of the integral chart ring zero after passage to the number field is killed by a nonzero integer after clearing number denominators. Use the finitely many chart identities and covers of Theorem 290.2 for all the coarse data in Proposition 312.1. This imposes the indicated exclusions before the coefficient prime is chosen.

On any geometric fiber in residue characteristic ℓ the tangent representation of O_E factors through $O_E/\ell O_E$, since the differential of multiplication by an integer is that integer times the identity, by the first-order group-law calculation of Lemma 307.2, in each free coordinate. The same calculation makes this action additive, as well as multiplicative by the chain rule. The split algebra here is a product of $2f$ copies of $\mathbf{F}_\ell$; its orthogonal idempotents split the tangent space into the corresponding subspaces. By Lemma 273.1 the residues of θ at all those factors are distinct. Their multiplicities in $P_\theta(T)$ are thus the dimensions of those subspaces, also on the special fiber by the preceding coefficient identity. In particular $w_0 \neq w_1$ lies over the same prime of K as w_1 and has multiplicity zero. On every other factor the scalar α is a unit at its place. This proves invertibility at the identity. In characteristic zero, α divides the nonzero integer $N_{E/\mathbb{Q}}(\alpha)$ in O_E, by its monic characteristic equation. The corresponding two endomorphisms multiply on tangent spaces to this integer times the identity, proving invertibility there also. Translations in a geometric group identify the differential at any point with that at the identity, since $[\alpha]$ intertwines translation by a point with translation by its image. □

Proposition 313.2. *Under Lemma 313.1 let $H_\alpha = \ker([\alpha] : G \to G)$, the closed inverse image of the identity section. It is projective over S and has relative Jacobian charts of dimension zero over S. Its geometric fibers are finite and reduced. Each finite positive product of H_α over S has the same properties and admits smooth Jacobian charts over $\mathbb{Z}_\ell$.*

Proof. The identity section is closed since the relative diagonal of the projective group is closed. Thus H_α is closed in the projective G/S. Consider a relative smooth chart at the identity over a point of S. Denote the free coordinates by t and the remaining coordinates by z, so that $F(t, z) = 0$ with invertible square z-Jacobian. Shrink the base so that the identity section lies in that chart and write its coordinates as t_0, z_0. Near the identity the ideal of this section is generated by $t - t_0$. Indeed in $F(t, z) - F(t_0, z_0)$ telescope the differences one variable at a time. The coefficient matrix of $z - z_0$ at the identity is $F_z(t_0, z_0)$, hence is

invertible on a neighborhood of it. The equations then express $z - z_0$ in the ideal of $t - t_0$, and setting all coordinates equal to their section values gives precisely the section on its chart.

At a point of H_α add the pullbacks of those $g = fn$ equations $[\alpha]^*(t - t_0) = 0$ to a relative smooth chart of the source G. They give exactly the kernel there by the preceding ideal identity. Their differentials restrict on the g-dimensional fiber tangent to independent forms by Lemma 313.1. Together with the independent old Jacobian rows they therefore give an invertible square matrix in the source variables over the residue field. Invert its determinant locally, clearing the unit denominators of the functions as in Proposition 268.1. This is the asserted relative dimension-zero presentation.

On each geometric fiber the local maximal ideal at a closed point is generated by coordinate differences. Modulo its square the Jacobian relations just obtained kill all of them. The finite-module determinant argument of Lemma 271.1 in the local ring therefore kills the maximal ideal itself. The local ring is its residue field. Every prime in an affine chart belongs to a maximal ideal there; the prime correspondence on localization and the preceding field calculation show that the prime was already maximal. The noetherian fiber has only finitely many irreducible components by Proposition 291.2; each is now a point. A nilpotent nonzero on any chart would remain nonzero on localization at a maximal ideal containing its annihilator. The field calculation rules this out, proving reducedness. For a product take the disjoint variable lists of the factor charts. The relative Jacobian is block diagonal, and its composition with the smooth base charts has a block triangular unit minor as in Proposition 312.1. The projectivity assertion for products follows from the projective product embedding. □

314. Components in a smooth chart over the local integers

Lemma 314.1. *Let Y be of finite type over $\mathbb{Z}_p$ with smooth Jacobian charts in the sense of Proposition 268.1 over this base ring. All its local rings are domains and multiplication by p is injective on them.*

Proof. First consider a closed special point y with finite residue field k. There is a complete discrete valuation ring D, finite free over $\mathbb{Z}_p$, with uniformizer p and residue k. For example adjoin successive lifts of monic irreducibles for a finite list of field generators of $k/\mathbf{F}_p$; apply Lemma 18.1 at each step using the irreducible residue polynomial, which is separable over a finite field. Separability follows since $x \mapsto x^p$ is bijective on that field, so an irreducible with zero derivative would have all coefficients and monomials p-th powers. Finite freeness follows from the monic power bases. Over D choose the point y' with these same residue coordinates in k. Its local ring is flat over $\mathcal{O}_{Y,y}$, since scalar extension by the free D and localization preserve injections after tensoring. A flat local homomorphism of local rings $A \to A'$ is injective: for $0 \neq c \in A$, the inclusion $Ac \hookrightarrow A$ stays injective, and $Ac \otimes A' = A'/\operatorname{Ann}(c)A'$ is nonzero since that annihilator maps into the maximal ideal. Its generator maps to the image of c. It therefore suffices to work at y'.

In its chart write t, z for the d free and r dependent variables with r equations $F = 0$; the matrix F_z is a unit matrix in determinant at the point. Choose lifts t_0, z_0 of the residue values in D. Put $P = D[[T_1, \ldots, T_d]]$ and $I = (p, T_1, \ldots, T_d)$. This ring is complete and separated for the powers of I: in degree b modulo I^b, each coefficient of a monomial T^μ with $|\mu| < b$ is specified modulo $p^{b-|\mu|}$, and all these specifications have unique limits in D. Solve $F(t_0 + T, z_0 + Z) = 0$ with $Z \in I^r$. Modulo I this holds with $Z = 0$. Given a solution modulo I^b, the correction in $(I^b/I^{b+1})^r$ is uniquely obtained by applying the inverse of the residue matrix F_z to the negative error, since all terms of order two in the correction vanish there. Passing to the limit gives the solution in P. Any inverted chart functions have nonzero residues and therefore are units in P, by the inverse series after factoring out a unit of D.

Write A' for the local ring at y', with maximal ideal $\mathfrak{m}'$ generated by $p, t - t_0, z - z_0$. Evaluation at the solution identifies $A'/(\mathfrak{m}')^b$ with P/I^b for each b. Indeed substitution $T = t - t_0$ defines an inverse on the truncated polynomial representatives. The two choices of z in the resulting quotient satisfy the equations and agree modulo the maximal ideal. Telescoping their polynomial differences multiplies $z - z'$ by a square matrix reducing to F_z; that matrix is invertible in the local quotient. Thus the substituted z's agree there as well. The formulas agree on denominators by their unit residues. Taking limits identifies the completed local ring with P. The ring A' is noetherian by the polynomial-ring argument of Lemma 291.1, since a discrete valuation ring has only principal ideals. It injects into its completion by the intersection-of-powers argument in Lemma 294.1. The power series ring over the domain D is a domain by its first nonzero homogeneous term, and p is nonzero there. This proves the assertion at y.

For a prime of a special affine chart select a closed point of its closure after extension of the fiber to $\overline{\mathbf{F}}_p$. Such a point exists by the affine point argument in Lemma 280.1, since extension of scalars on the nonzero quotient by the prime is faithful. The finitely many coordinates of the closed point generate a finite field over $\mathbf{F}_p$, so it contracts to a closed point of finite residue field containing the original prime. Localization of that closed local domain proves the assertion at the original prime. On the generic fiber use an algebraic closure of $\mathbb{Q}_p$ and a geometric closed point of the closure in an affine chart by the same argument. The closed point there has a power series local completion over the algebraically closed field by Lemma 294.1's local Jacobian computation. Descent to the contracted local ring is injective by the flat local argument above, and localization gives a domain at the original prime. On this fiber p is invertible. □

Proposition 314.2. *For Y of Lemma 314.1, restriction to $\mathbb{Q}_p$ gives a bijection between open and closed subschemes of Y and of its generic fiber. The inverse takes reduced closure. A projective Y therefore gives a smooth projective $\mathbb{Z}_p$-model of every such generic subscheme.*

Proof. Here an open and closed subscheme has the structure induced on the open subset. The noetherian scheme Y has finitely many irreducible components, by the argument in Proposition 291.2. Two different components through one point would give distinct minimal primes in its local ring as in Lemma 294.1, contradicting Lemma 314.1. Thus the components are disjoint and each is both open and closed. All their local rings are domains, so they are reduced. The element p is nonzero in every local ring. On every nonempty affine of a component it is therefore a nonzero element of a domain, whose generic point belongs to the open subset $p \neq 0$. The generic fiber of that component is consequently an irreducible dense open subset. These opens are precisely the irreducible components of the generic fiber, since they form a finite disjoint closed covering of it. An open and closed subset is a union of whole components in either scheme, by irreducibility. The bijection and the description by reduced closure follow. Such a union inherits its Jacobian charts and its closed projective embedding. It is flat over $\mathbb{Z}_p$: its chart rings have no p-torsion by the lemma, and a torsion-free module over this discrete valuation ring is flat by the directed union of free finitely generated submodules in Proposition 271.2. □

315. Recovering the finer level from marked columns

Theorem 315.1. *Fix the coarse data $C_b = K(b_\sigma)$ in Proposition 312.1. Over E, write $S_b = \mathcal{X}(C_b, 0)$ for the coarse base and $G_b = \mathcal{X}(C_b, 1)$ for its one-point group scheme over S_b. Let $0 \neq \alpha \in b_\sigma O_E$, put $C_\alpha = K(\alpha)$, and put $S_\alpha = \mathcal{X}(C_\alpha, 0)$. In $L = \widehat{O}_E^n$ let $e_1, \ldots, e_n$ be the standard columns and write $t_j = \alpha^{-1} e_j \bmod L$. The map sending a finer framed torus to its coarse point and the ordered points $P_x(t_1), \ldots, P_x(t_n)$ is an isomorphism from S_α onto an open and closed subscheme of the n-fold product over S_b of $G_b[\alpha] = \ker[\alpha]$. For each $\nu \geq 0$, $\mathcal{X}(C_\alpha, \nu)$ identifies over E with the pullback of the ν-fold coarse group power to this subscheme.*

Proof. The full α-torsion label group is $A_\alpha = \alpha^{-1}L/L$. Indeed in the coordinate formula for P_x of Lemma 269.1, the action of α multiplies the finite adelic lift by α; the adelic quotient there labels all torus torsion. Alternatively its α-torsion is directly $\alpha^{-1}\Lambda_x/\Lambda_x$, with the identification given by the finite lattice residues as in Lemma 283.1. This is finite, since a nonzero rational integer multiple of $1/\alpha$ belongs to O_E. The group C_b acts on these labels by its matrices. Inside C_b the simultaneous stabilizer of the listed n columns is exactly C_α: fixing them means $(k-I)e_j \in \alpha L$ for every j, which is the congruence in Definition 239.1. It fixes every member of A_α, and C_α is the kernel of the action there.

For every tuple $\mathbf{s} = (s_1, \dots, s_n) \in A_\alpha^n$ there is a map $j_{\mathbf{s}}$ from S_α to the indicated product. In a ball chart use the coarse frame and the points $P_x(s_i)$ as its marked slots. These are holomorphic in the base since the real torsion periods are fixed linear combinations with rational coefficients of its holomorphic period columns in Proposition 252.2. Right multiplication of a frame by $k \in C_\alpha$ does not change these marks; rational left changes identify the points by the same linear lattice isomorphism as the underlying group. Thus the maps are well defined on the quotient. They take values in the closed kernel product scheme, by the torsion equalities. They are algebraic over $\mathbb{C}$ by Theorem 267.2, first as morphisms into the product of the coarse group over its base. The conjugation of all the marks for a fixed frame in Theorem 288.2 is the identical list of labels at its reconstructed frame. Consequently Lemma 289.1 descends $j_{\mathbf{s}}$ to E, compatibly with the forgetful projection.

Fix one complex coarse point represented by a frame x and its complex structure. Its lifts to S_α are represented by xk, $k \in C_b/C_\alpha$, with that complex structure. The only additional possible identifications would be rational stabilizers of the coarse point, trivial by Proposition 239.2. Thus the tuple values of $j_{\mathbf{s}}$ on this fiber are exactly the C_b-orbit of $\mathbf{s}$, in the label coordinates of x. For $\mathbf{t} = (t_1, \dots, t_n)$ each value occurs once by the stabilizer calculation above. The finitely many orbits on A_α^n exhaust the geometric complex fiber, with disjoint sets of tuple values.

Each of these orbit images is closed: its map is from a projective variety, so project its closed graph and apply the fiberwise homogeneous minor argument in Lemma 280.1. This describes the closed image over E as well and commutes with field extension, since emptiness of a projective fiber is witnessed by spanning all monomials in one homogeneous degree, with the same matrix rank after field extension. The union over the other orbits therefore gives a closed complement to the $\mathbf{t}$-image. Indeed any nonempty intersection or uncovered locally closed subset after extension to $\mathbb{C}$ would contain a complex point by the affine point argument of the same lemma. Extension of fields is faithful on nonzero affine coordinate rings, so the same partition holds over E. Give the image its open subscheme structure. The ambient characteristic-zero kernel product is reduced: the differential of $[\alpha]$ on the fibers is invertible by the norm identity of Lemma 313.1. The kernel charts in Proposition 313.2 over the smooth generic base give smooth charts for the product; their local rings over $\mathbb{C}$ are domains by the chart argument in Lemma 294.1. Reducedness over E follows by faithful scalar extension.

In the local ball charts of Theorem 252.3 both the fine cover and these tuple values consist of disjoint sheets over the base. For the tuple values this follows also by applying the local inverse argument of Theorem 257.2 to the zero-dimensional relative charts and continuing each labelled section; the complete list above accounts for the fiber. On each fine sheet $j_{\mathbf{t}}$ is the identity in its base coordinate. The bijection onto the indicated union thus has holomorphic inverse. The union is a closed projective smooth variety, so the inverse is algebraic by Theorem 267.2. Uniqueness makes it commute with all coefficient automorphisms fixing E, and Lemma 289.1 descends it.

For the marked powers, retain the same ν torus points when forgetting from C_α to C_b. In a ball chart the base sheets change and the identification on each torus power is the identity. Thus the resulting map to the stated fiber product is a biholomorphism, including $\nu = 0$. It is algebraic with algebraic inverse by the same theorem on holomorphic maps,

using the smooth product charts of Theorem 268.3. Forgetting the level and slots commutes with coefficient automorphisms by Theorem 289.2; uniqueness of the inverse again gives descent. □

316. Descent from the completed local integers

Lemma 316.1. *Let $B \to D$ be flat and let $A = B[X_1, \ldots, X_s]/I$ with I finitely generated. Let y be a prime of A admitting a lift y' in $A' = D \otimes_B A$. If the latter scheme at y' has a smooth Jacobian chart of relative dimension d over D, then near y the given presentation itself has $s - d$ defining equations with an invertible $(s - d)$-rowed minor of relative partial derivatives. These equations may be chosen from any finite set of generators of I.*

Proof. Put $P' = D[X_1, \ldots, X_s]$ and $I' = IP'$. Localize P' at the inverse image of y'; write M for the resulting local ring, $J = I'M$, $A'_0 = M/J$. Consider $C = M/J^2$, a square-zero extension of A'_0 with kernel $V = J/J^2$. There is a D-algebra section $A'_0 \to C$. Indeed use the prescribed smooth chart at y'; locally it presents A'_0 by free and dependent variables and equations with invertible dependent-variable Jacobian. Lift all their values from A'_0 to C. The errors in their equations belong to V, and changing the dependent variables by elements of V changes the errors by exactly that Jacobian, since $V^2 = 0$. Its determinant is a unit in C, as it is one in A'_0 and the kernel is nilpotent. Solve for corrections killing the errors. All elements inverted in the chart and then in its local ring map to units for the same reason, giving the required section.

Use the section to lift the images of the original variables X_i, and denote the difference between X_i in C and this lift by $v_i \in V$. The gradient gives an A'_0-linear map

$$J/J^2 \longrightarrow (A'_0)^s, \qquad f \longmapsto (\partial_i f)_i.$$

Here derivatives are reduced modulo J; the product rule defines them on the localized ring and kills J^2 after reduction. This map has left inverse sending the i-th basis vector to v_i. In fact evaluate f at the lifted coordinates from the section; the value is zero since $f \in J$. Evaluating instead at the original X_i adds $\sum_i (\partial_i f) v_i$ by the first-order polynomial identity. It also holds for the inverted denominators by $(c + v)^{-1} = c^{-1} - c^{-2}v$ when $v^2 = 0$. This gives the claimed left inverse.

After taking the residue field at y', the dimension of V tensored with that field is therefore the rank of the gradient rows of generators of I. Indeed those generators still generate J, and the split injection remains injective. This rank is $s - d$. The kernel of the matrix on column vectors is the relative tangent space: an infinitesimal lift of the residue point to its field with a square-zero element is determined by increments of the s variables satisfying precisely the linearized equations. In the smooth chart the d free increments are arbitrary and the other increments are uniquely determined. Both calculations describe the same lifts with coefficients from D fixed.

Flatness identifies $I \otimes_B D$ inside P' with I' and likewise identifies the quotient by the square with $(I/I^2) \otimes_B D$, by tensoring the inclusions and quotient sequences. Consequently the dimension of $(I/I^2) \otimes_A \kappa(y)$ and the rank of the gradient matrix over $\kappa(y)$ are both $s - d$, since extension to $\kappa(y')$ preserves rank. Choose $s - d$ generators giving independent rows there and a corresponding unit minor. They span $(I/I^2) \otimes \kappa(y)$: the gradient map on that space has rank equal to its dimension. For the prime $\mathfrak{q}$ over y in $B[X]$, this space is $I_{\mathfrak{q}}/\mathfrak{q}I_{\mathfrak{q}}$ because $I \subseteq \mathfrak{q}$. The finite-module argument in Lemma 271.1 applied to the quotient of $I_{\mathfrak{q}}$ by the selected equations shows they generate there. Clear denominators in the expressions for all original generators and invert the chosen minor. On this neighborhood the asserted chart results. □

Corollary 316.2. *Let v be a degree-one prime with $E_v = \mathbb{Q}_p$, let $R = (O_E)_v$, and let Y/R be of finite type with a base change to $\mathbb{Z}_p$ admitting smooth Jacobian charts. Such charts already exist over R, with the same relative dimensions. Its local rings are domains, Y is*

flat over R, and every open and closed subscheme of Y_E extends uniquely to an open and closed subscheme of Y, by reduced closure.

Proof. By Proposition 82.3 and the equality of completions, R is a discrete valuation ring with uniformizer p, residue $\mathbf{F}_p$ and completion $\mathbb{Z}_p$. The latter is flat over R by the torsion-free argument of Proposition 271.2. Every point of Y lifts after this extension. Indeed the tensor product of its residue field with $\mathbb{Z}_p$ over R is that same field on the special fiber and on the generic fiber it is the tensor product of two extension fields over E, which is nonzero by taking bases. A maximal ideal of this nonzero ring gives the lift. On an affine chart, finite type is finite presentation since R and its polynomial rings are noetherian by Lemma 291.1. Apply Lemma 316.1. At each lift the induced map on local rings is flat and local, hence injective by the argument of Lemma 314.1. The ring upstairs is a domain in which $p \neq 0$ by that lemma; so is the ring downstairs. This proves flatness over R by the same torsion-free criterion. The component argument of Proposition 314.2 uses just these properties and noetherianity. Applying it over R proves the remaining assertions. □

317. A smooth model at the free-weight place

Theorem 317.1. *For the approximants and ordered choices in Proposition 312.1 and Lemma 313.1, let $v = w_1$, let $R = (O_E)_v$, and take any of the finer principal levels $\alpha = \mathfrak{n}$ specified there. There is a smooth projective R-model of $S_\alpha = \mathcal{X}(K(\alpha), 0)_E$. Over it there is a smooth projective commutative group with its O_E-action, whose ν-fold power is a smooth projective R-model of $\mathcal{X}(K(\alpha), \nu)_E$ for every $\nu \geq 1$. The projection to the base, the group operations, the actions of O_E and the forgetful maps of slots have their specified generic values.*

Proof. Use the coarse models $G_b \to S_b \to \operatorname{Spec} R$ from Proposition 312.1, base changed from its localization of O_E. They have their group law and scalar maps over R. The two morphisms have smooth relative Jacobian charts after extension to $\mathbb{Z}_\ell$, by that proposition, and over R by Lemma 316.1. For the relative assertion apply the latter result on affine charts of S_b with the flat extension obtained by tensoring the chart ring over R with $\mathbb{Z}_\ell$. The existence of a lift of each point of G_b is the residue-field argument of Corollary 316.2. Charts of finite type are of finite presentation here by the noetherian polynomial argument of Lemma 291.1.

Form the projective R-scheme $Y = (G_b[\alpha])^n_{/S_b}$, where the notation denotes the n-fold fiber product. The closed kernel and its product commute with base change, since the kernel is an inverse image of a section and the product is given by the equalities of the base coordinates. The prime data of the theorem make α invertible on the geometric relative tangent spaces over $\mathbb{Z}_\ell$ by Lemma 313.1. Thus $Y_{\mathbb{Z}_\ell}$ has smooth charts by Proposition 313.2. Apply Corollary 316.2 to Y. By Theorem 315.1 its generic fiber contains S_α as the open and closed part given by the ordered standard columns. Its reduced closure U is open and closed in Y. Consequently U/R has the required smooth charts and is flat. It is projective since it is closed in the projective Y. This gives the fine base at the actual place v.

Over U take $G_b \times_{S_b} U$. Pullback of the projective embedding and of the relative Jacobian charts gives a smooth projective group over U; its operation and each scalar map are base changes of those on G_b. For a finite power over U the relative equations in the point variables have block diagonal Jacobian minors. Combining with the smooth chart of U/R gives a unit minor over R with the expected number of free variables, as in Proposition 312.1. The product has a projective embedding over R: embed it as a closed subscheme of the product of the projective models of U and the coarse group powers using the base diagonal equations. It is flat over R by Corollary 316.2 applied to these charts after completion. The generic identifications with the stated powers and their maps are the last assertion of Theorem 315.1, which retains the actual marked points in each slot. □

Corollary 317.2. *For each fine base in Theorem 317.1 one may take its models, its one-point group and all their finite powers over a single localization $O_E[1/c]$ with $0 \neq c \in O_E \setminus w_1$. There the base and the group are projective and have smooth relative Jacobian charts. The fibers over w_1 and the generic fiber agree with those of the theorem. For any one ν, at all but finitely many rational primes the marked power identifies with the earlier model at its same level used in Theorem 301.2. The finite exceptions in this last identification can depend on the approximant and on ν.*

Proof. Take the projective equations of the fine base and the projective coarse group with its projection, and the graphs of the finitely many maps defining the group operations and an integral basis of its scalars. All are of finite presentation over the noetherian R. Include the relative smooth Jacobian charts, their inverted minors, the group and scalar identities, and the identifications with the fiber products on each of their charts from Theorem 317.1. Finite covers suffice since the schemes and charts are quasi-compact, by noetherianity. Each coefficient and identity in these presentations uses finitely many denominators of O_E outside w_1. For the covering assertions use the expressions of one as a sum of the covering functions with chart coefficients, and for the algebra identities use finite expressions in the defining ideal, as in Proposition 270.2. Multiplying these finitely many denominators together gives $c \notin w_1$, after including the denominators of the coarse data as well. Keeping these equations and maps over $O_E[1/c]$ gives the asserted models. The powers and their slot operations are then formed by fiber products, so no further denominator depends on their number. The generic and w_1-local identifications follow by the original chart equalities after localization at w_1.

For fixed ν the two projective models over their common generic field are isomorphic by Theorem 315.1. Include both directions of this generic isomorphism, their composites and chart covers in the denominator-clearing argument of Theorem 290.2, over the intersection of the two initial open bases. This requires finitely many nonzero denominators and yields an isomorphism on the resulting open. Only rational primes dividing their norms or those of the original exclusions can be lost. The prescribed-place assertion comes directly from the first construction. □

318. Cohomology of differential forms

Definition 318.1. For a commutative k-algebra A over a field k, write $\Omega^1_{A/k}$ for the A-module on symbols da, $a \in A$, with additivity, $dc = 0$ for $c \in k$ and $d(ab) = a\,db + b\,da$. Put $\Omega^h_{A/k} = \bigwedge^h_A \Omega^1_{A/k}$ for $h \geq 0$, with $\Omega^0_{A/k} = A$; the exterior power is formed by the alternating multilinear relations. For a smooth separated variety X/k of finite type, take a finite ordered affine open cover $\mathcal{U} = (U_i)$. On an affine intersection $U_J = \bigcap_{j \in J} U_j$ use the module $\Omega^h(U_J) = \Omega^h_{A_J/k}$ with $A_J = \Gamma(U_J, \mathcal{O}_X)$, zero on an empty intersection. In degree $q \geq 0$ form the direct sum

$$C^q(\mathcal{U}, \Omega^h) = \bigoplus_{i_0 < \cdots < i_q} \Omega^h(U_{\{i_0,\ldots,i_q\}}).$$

Its differential δ takes alternating restrictions as in Definition 292.1, now without the augmented term. Write $H^q(X, \Omega^h_{X/k})$ for its cohomology, under the canonical cover identifications below. The same cochains will be used for systems $\mathcal{F}$ of modules on affines glued by restriction, for which restriction to each principal open is localization. Such a system is called quasi-coherent; finite modules on the affine charts over a noetherian scheme give coherent systems.

Lemma 318.2. *The modules of forms in Definition 318.1 agree on localization. On a smooth relative dimension-d Jacobian chart Ω^1 over the base has the d free coordinate differentials as a basis. For a quasi-coherent module on an affine scheme U and a finite affine open cover $\mathcal{V}$ of U, augmenting its cover cochains by $\mathcal{F}(U)$ in degree -1 gives an exact complex.*

Proof. A homomorphism $A \to B$ sends da to $d\bar{a}$ in the notation of the definition, by its relations. Inverting any element f forces $d(f^{-1}) = -f^{-2}df$. This rule extends derivations uniquely: the formula $d(a/f^j) = f^{-j}da - jaf^{-j-1}df$ respects equality of fractions by differentiating a cleared equality after inverting f. The universal presentation therefore identifies the localized differentials with those of A_f. Localization commutes with the alternating multilinear presentation of every exterior power. On a polynomial quotient with generators x_i and defining equations F_j, the dx_i generate with relations $\sum_i (\partial_i F_j) dx_i = 0$; the polynomial product rule gives the expression for every differential, and assigns a derivation for every list of values satisfying these relations. The same description holds relative to a base ring. An invertible dependent-variable minor solves the relations for the corresponding dx_i in the free coordinate differentials, without further relations. This proves the basis assertion.

A finite intersection of affine opens on a separated scheme is affine: for two opens it is the inverse image of the closed diagonal inside their affine product, hence a closed affine subscheme there. Induct for more opens. In the augmented cover complex over $U = \operatorname{Spec} A$, fix a prime $\mathfrak{p}$ of A. There is a principal neighborhood $D(f)$ of it contained in one member V_i, since principal opens form an affine basis. Localization at f restricts every cochain term to the corresponding intersection with $D(f)$: the intersection before restriction is affine, and f gives a principal open on it. On this new cover $V_i \cap D(f) = D(f)$. Regard cochain entries as alternating in their indices, zero on repetition. A contraction is obtained by evaluating an entry with i inserted first, including in degree zero the value at i for the augmentation. Restrictions agree on its domains since that member is all of $D(f)$. Deleting i returns the original entry and the other deletions cancel in pairs in $\delta H + H\delta$. The localized complex is therefore exact, also after further localization at $\mathfrak{p}$. Localization commutes with kernels and images by clearing denominators as in Lemma 292.2; every cohomology module thus vanishes at all primes. If an element of a module is nonzero, its proper annihilator belongs to a maximal ideal at which the element stays nonzero. This proves the stated exactness. In particular on affine principal covers the degree-zero gluing identifies the common module of forms, giving the asserted chart interpretation. □

Proposition 318.3. *The groups in Definition 318.1 are independent of the finite affine cover. Algebraic maps between these varieties induce functorial pullbacks on the groups of forms. For a field extension k'/k one has canonical isomorphisms*

$$k' \otimes_k H^q(X, \Omega^h_{X/k}) \simeq H^q(X_{k'}, \Omega^h_{X_{k'}/k'}),$$

compatible with those pullbacks.

Proof. For two covers $\mathcal{U}, \mathcal{V}$, take in bidegree $a, b \geq 0$ the cochains on intersections using $a+1$ ordered indices from $\mathcal{U}$ and $b+1$ from $\mathcal{V}$. The total differential is $\delta_U + (-1)^a \delta_V$, whose square is zero by commutation of restrictions. Augment vertically by the $\mathcal{U}$ cochains at $b = -1$. Each column is exact by Lemma 318.2, applied on the affine intersection from $\mathcal{U}$. The augmented total complex is exact. Given a cocycle, its part of smallest a is vertically closed, so subtract a total boundary using a vertical primitive with the prescribed sign. At the bottom of an exact column a closed entry is already zero. Continuing in increasing a terminates because the cover is finite. Thus restriction from the $\mathcal{U}$ complex to the unaugmented total complex at $b = 0$ induces an isomorphism on cohomology. Explicitly a cocycle in the total complex has an augmented primitive; its $b = -1$ part is a cocycle giving that class by restriction up to the degree sign, and exactness applied in the reverse direction gives injectivity. Interchanging the covers gives the same assertion for $\mathcal{V}$ at $a = 0$; the sign convention is interchanged by multiplying bidegree (a, b) by $(-1)^{ab}$.

When $\mathcal{V}$ refines $\mathcal{U}$, choose $i(j)$ with $V_j \subseteq U_{i(j)}$. From the total complex to the $\mathcal{V}$ complex, evaluate an (a, b)-array on the two lists

$$i(j_0), \ldots, i(j_a) \quad \text{and} \quad j_a, \ldots, j_{a+b},$$

and restrict to their common $\mathcal{V}$-intersection. For a total cochain sum over $a + b$ equal to its degree. Alternating extensions of the entries give a cochain map. In its differential, deleting an index before the shared index is a horizontal deletion and deleting one after it is a vertical deletion with the factor $(-1)^a$. The two terms at the junction in the sum of these differentials have the same restrictions and opposite signs. This verifies the identity including degrees zero and one. On cochains from $\mathcal{V}$ it is the identity; on those from $\mathcal{U}$ it is exactly restriction by $i(j)$. Hence every such refinement induces the same canonical isomorphism on cohomology. A third cover refines any two by all their intersections, and refinement maps compose by restriction, giving the asserted compatible identifications.

For a morphism refine the source cover by affine opens mapping into members of the chosen target cover. A finite such refinement exists since affine opens form a basis and the source is noetherian and quasi-compact, by the polynomial noetherian argument of Lemma 291.1. Pullback of functions and differentials then gives the cochain map with those index choices. Two choices for the map of indices yield homotopic restrictions: on a degree q array use the sum over $0 \le l < q$ with sign $(-1)^l$ of the entries at

$$i(j_0), \dots, i(j_l), i'(j_l), \dots, i'(j_{q-1}).$$

Pull back to the indicated common source intersection. In $\delta H + H\delta$, all deletions off the doubled index cancel, and those at that index telescope to the difference of the two pullbacks. Refining on the source and target therefore gives the same map; for composites choose compatible refinements and use composition on differentials. Finally the base-changed affine cover and all its intersections are their scalar extensions. The differential presentation and exterior powers give the base-changed modules of relative forms. Tensoring over a field preserves kernels and images by taking vector space bases, so the isomorphism follows term by term on the cover complex and on its cohomology. □

319. Finite Hodge groups and descent along a level cover

Proposition 319.1. *Let X be a closed subscheme of $\mathbf{P}^N_k$ over a field k, and let $\mathcal{F}$ be a coherent module in Definition 318.1. The groups $H^q(X, \mathcal{F})$ of that definition are finite dimensional over k for all $q \ge 0$. They vanish for $q > N$.*

Proof. Use the $N + 1$ standard affine charts. The proof of the cover comparison in Proposition 318.3 applies to any of the indicated modules by Lemma 318.2. View the module on $\mathbf{P}^N$ by taking on any affine the sections on its closed intersection with X. On each standard affine this is a finite module over its polynomial coordinate ring via the quotient defining X, and principal restrictions are still localizations. The two standard cover complexes are identical. Their terms above N are zero.

For a coherent module $\mathcal{M}$ on $\mathbf{P}^N$ choose finitely many generators in each standard chart. Each generator s on the i-th chart extends, after twisting, to a global section: on the j-th chart multiply its restriction to the overlap by a power of X_i/X_j and clear the localization denominator. Increase to a common power on all charts and use the frames of $\mathcal{O}(t)$ there. On each overlap the resulting difference becomes zero after inverting X_i, and increasing the twist by another common power of X_i kills all those differences. This is precisely the localization argument of Lemma 296.1, which uses only finite generators and restriction to principal opens in this step. The resulting finitely many sections, with their respective twists, give a surjection $\mathcal{E} \to \mathcal{M}$ from a finite sum of lines $\mathcal{O}(-t)$. The kernel $\mathcal{K}$ is coherent: on every standard chart it is a submodule of a finite module over a noetherian ring by Lemma 291.1, and these kernels localize. Generation on each chart is immediate from its own selected generators, whose powers of X_i are units in that chart frame. Thus the three cover complexes are termwise short exact, including on the intersections by localization.

For any integer b, compute the cover complex of $\mathcal{O}(b)$ in Laurent monomials of total degree b, by Definition 292.1. When the set of negative exponents is proper and nonempty, the contraction of Lemma 292.2 leaves no cohomology. When that set is empty, the same

contraction on the augmented complex leaves in the unaugmented complex just one copy of k in degree zero for that monomial. When every exponent is negative only the top intersection occurs, giving one copy in degree N. There are finitely many all-nonnegative and all-negative exponent lists of a fixed sum. This proves finiteness for each line and hence for $\mathcal{E}$. From the termwise short exact sequence the groups

$$H^q(\mathbf{P}^N, \mathcal{E}) \longrightarrow H^q(\mathbf{P}^N, \mathcal{M}) \longrightarrow H^{q+1}(\mathbf{P}^N, \mathcal{K})$$

are exact at the middle: lift a cocycle to an $\mathcal{E}$-array and take its differential in $\mathcal{K}$, then use the cocycle lifting argument of Lemma 292.2. Starting above N and descending, suppose degree $q+1$ is finite for every coherent module. Both outer spaces are then finite dimensional, proving the same at degree q for every $\mathcal{M}$. Apply the conclusion to the module supported on X. □

Proposition 319.2. *Let k be a field of characteristic zero and let $\pi : W \to X$ be a finite locally free cover of smooth projective k-varieties with relative étale Jacobian charts. Suppose a finite group D acts over X and that $(w, d) \mapsto (w, wd)$ identifies $W \times D$ with $W \times_X W$. For all q, h there is a trace map*

$$\operatorname{Tr}_\pi : H^q(W, \Omega^h_{W/k}) \longrightarrow H^q(X, \Omega^h_{X/k})$$

with $\operatorname{Tr}_\pi \pi^* = |D| I$ and $\pi^* \operatorname{Tr}_\pi = \sum_{d \in D} d^*$. In particular the pullback identifies the space on X with the D-invariant subspace on W. This applies to every generic cover π_m of Theorem 302.1 and after extension of its ground field.

Proof. For an affine $U = \operatorname{Spec} A$ of X, write $\pi^{-1}U = \operatorname{Spec} B$. Then $\Omega^1_{B/k} = B \otimes_A \Omega^1_{A/k}$. To check the natural map locally, a k-derivation of A into a module over a relative étale chart extends uniquely to its coordinate ring: the derivatives of its extra variables solve the linear equations obtained by differentiating the defining relations, whose square Jacobian is invertible. Existing derivatives of coefficients give the inhomogeneous terms. Denominators extend by the quotient rule of Lemma 318.2. This verifies the equality by the universal differential presentations, and on taking exterior powers it holds for Ω^h as well. A map of modules is an isomorphism when its kernel and cokernel vanish at every prime, by the annihilator argument in the same lemma.

Take the trace of multiplication by b on the locally free A-module B. These matrix traces are independent of changes of basis, by $\operatorname{tr}(PQ) = \operatorname{tr}(QP)$ obtained by interchanging the two finite sums, and glue to an A-linear map $B \to A$. Tensoring it with the identity on the forms on U gives the trace on forms. These maps commute with restriction and scalar extension because the local matrices do. On a finite affine cover of X and its inverse-image affine cover of W they therefore induce the asserted cohomology map, independent of the cover by Proposition 318.3.

Every geometric fiber has $|D|$ points, each of length one by the relative étale charts and the local tangent argument of Proposition 313.2; the cover is nonempty over each point, and the given product isomorphism gives a free transitive action there. The rank of B is thus $|D|$; multiplication by a scalar from A has trace that scalar times $|D|$. For the other identity, change base to B on the first copy in the fiber product. The given isomorphism identifies the algebra $B \otimes_A B$ with $\prod_{d \in D} B$, the second B acting in its d-th component by $d^* b$. The trace after this change is the sum of those components, by its diagonal multiplication matrix. Applying the same equalities to forms and then to cover complexes gives the two identities. Since $|D|$ is invertible in k, the first one gives injectivity and the second writes every invariant class as the pullback of its normalized trace. The converse inclusion follows from $\pi d = \pi$. The covers cited in the statement have precisely these properties by Theorem 302.1. □

320. Hecke operators on the groups of differential forms

Theorem 320.1. *Fix a neat admissible level $C = K(\mathfrak{n})$, a marked power $X = \mathcal{X}(C,\nu)_{\mathbb{C}}$, and integers $h, q \geq 0$. Put $V^{h,q} = H^q(X, \Omega^h_{X/\mathbb{C}})$. For each $H \in \mathcal{H}^+$ of Definition 245.1 there is a unique operator T_H on this finite-dimensional space satisfying, on each sufficiently fine cover,*

$$\pi_m^* T_H = \sum_{gC} H(gC)\, u_{g,\mathbf{1}}^*.$$

Only the support of H occurs. One has $T_{HK} = T_H T_K$, linearity in H, and the identity coset operator is the identity. For each $\mathbf{a} \in O_E^{\nu}$, let $d_{\mathbf{a}}$ be scalar multiplication slot by slot on X. Then

$$\sum_{gC} H(gC)\, u_{g,\mathbf{a}}^* = \pi_m^* T_H d_{\mathbf{a}}^*.$$

The constructions with coefficients in any subfield of $\mathbb{C}$ containing E extend by scalars to $\mathbb{C}$.

Proof. Finiteness follows from Proposition 319.1 and the smooth-chart description of forms in Lemma 318.2. The sum in the first display is invariant under the deck transformations of the refined cover. In fact, right translation by $k \in C$ on its source changes the target represented by xg into the target represented by xkg. The torus marks agree under this equality by Proposition 254.1. Left translation by k permutes the right cosets in the support and preserves their coefficients by Definition 245.1. Thus Proposition 319.2 gives a unique descended operator, obtained by applying $\mathrm{Tr}_{\pi_m}/[C : C_m]$ to the sum. Pulling back to a common deeper level identifies both sums since their maps there have the same frame and slot formulas. The pullback from X to that common level is injective by the same trace proposition, proving independence. The formula gives linearity, and for the identity coset the map is π_m.

For multiplication choose a level m fine enough for the support of K. Choose $l \geq m$ so fine that $g^{-1} C_l g \subseteq C_m$ for the finitely many chosen representatives for H, and fine enough for all products of these representatives with those for K. This is possible by the congruence neighborhood argument of Proposition 254.1. The same frame formula gives a map v_g from the l-level to the m-level with all slot scalars one: the inclusion just imposed gives well-definedness, and the period charts give holomorphy. The map is algebraic over E by the proof of Theorem 289.2, using this finer congruence in the target. On the l-level the unscaled g-map to X is $\pi_m v_g$, and composing v_g with the unscaled s-map from the m-level to X gives the gs-map there. These identities follow by multiplying the frame and retaining the same torus vectors modulo the successive enlarged lattices. By pullback functoriality in Proposition 318.3,

$$\pi_l^*(T_H T_K) = \sum_{gC} H(gC) v_g^* \pi_m^* T_K = \sum_{gC,sC} H(gC) K(sC)\, u_{gs,\mathbf{1}}^*.$$

Grouping equal final cosets gives precisely the product of Definition 245.1. Injection of π_l^* proves the claim. The identity for scalar slots follows from $u_{g,\mathbf{a}} = d_{\mathbf{a}}\, u_{g,\mathbf{1}}$ in the period formula. Descent over E of all maps is in Theorem 289.2, and the field-extension statements for pullback and trace follow from Propositions 318.3 and 319.2. □

Corollary 320.2. *For an approximating form and its θ in Theorem 251.3, fix one tested index $h = h_j$. Let F be a finite polynomial in words in the safe generators with $F(\theta) = 1$, and write $F(\tau)$ for its value in $\mathcal{H}^+$ as in Definition 251.2. If $F((T_\tau)_\tau)$ acts as zero on $V^{h,q}$ for $0 \leq q \leq \dim X$, then for every left multiple $H = BF(\tau)$, every $\mathbf{a}$ and every such q the map*

$$\sum_{gC} H(gC)\, u_{g,\mathbf{a}}^* : V^{h,q} \longrightarrow H^q(\mathcal{X}(C_m,\nu)_{\mathbb{C}}, \Omega^h)$$

is zero at every sufficiently fine common level. This includes the regular multiple and the scalar slots of Corollary 255.2.

Proof. Theorem 320.1 turns word evaluation into composition in its given order and sends the constant identity to the identity. Thus $T_H = T_B F((T_\tau)_\tau) = 0$ on each stated space. Apply its last displayed pullback formula, including when a slot scalar is zero. The cited corollary takes H of this form and $\mathbf{a} = 2\boldsymbol{\alpha}$. □

321. Exterior products and the coefficient of a scaled graph

Proposition 321.1. *Let X, Y be smooth separated varieties of finite type over a field k. For $a, r \geq 0$ there is a natural isomorphism*

$$\bigoplus_{\substack{i+j=a \\ b+c=r}} H^b(X, \Omega^i_{X/k}) \otimes_k H^c(Y, \Omega^j_{Y/k}) \xrightarrow{\simeq} H^r(X \times_k Y, \Omega^a_{(X\times Y)/k}).$$

On the summand i, j the forms on the first factor precede those on the second. Denote the image of an elementary tensor by $v \boxtimes w$.

Proof. On affine factors of rings A, B, a k-derivation on $A \otimes_k B$ is determined by two derivations on A, B, by the product rule applied to $(a \otimes 1)(1 \otimes b)$. Conversely any such pair with values in an $A \otimes B$-module extends by that rule; its formula is k-bilinear on a, b. The corresponding presentation of $\Omega^1_{(A\otimes B)/k}$ is the direct sum of the two extended differential modules. Its a-th exterior power is the direct sum of the exterior products of degrees $i + j = a$. On smooth coordinates the ordered wedges of the two differential bases in Lemma 318.2 give a basis for this equality, and the maps glue since they are pullback followed by wedge. On an affine rectangle the sections of the summand are $\Omega^i(A) \otimes_k \Omega^j(B)$, by the tensor presentation. Write $\mathcal{M}$ for this summand and fix the pair i, j.

Choose finite affine covers (U'_i), (V'_j) of the two factors. Form the double complex of $\mathcal{M}$ on products of their respective intersections, with differential $\delta_U + (-1)^b \delta_V$ in bidegree (b, c). Its total complex computes the cohomology of $\mathcal{M}$ on the product. To see this, use a third index of cochains from the ordinary rectangle cover $\mathcal{W} = (U'_i \times V'_j)_{i,j}$. In tridegree (b, c, e) take sections on the intersection of the double rectangle with each $(e + 1)$-fold intersection from $\mathcal{W}$. Use total signs $1, (-1)^b, (-1)^{b+c}$. Restriction at $e = 0$ embeds the double complex in this triple calculation at the level of cocycle classes. In fact it induces an isomorphism: augment by the double complex at $e = -1$. For fixed first and second index lists the remaining augmented column is exact by Lemma 318.2 on the affine rectangle. The total cycle-elimination argument of Proposition 318.3, in increasing $b + c$, proves the assertion.

In the other direction, fix one intersection from $\mathcal{W}$. It is contained in the product of the first and second members of any one rectangle in its list. Therefore the augmented complex in the second cover index contracts by inserting that second member, for every fixed first index list, even after intersecting with all the first-index opens. The same holds for the first index after dropping the second, by inserting the first member of the selected rectangle. These are the contractions in Lemma 318.2; their augmented terms are sections on the intersection using no opens from the dropped index. Applying the same finite cycle-elimination argument successively to the two augmented indices proves that restriction from the ordinary $\mathcal{W}$ complex gives an isomorphism onto the cohomology of the triple complex as well. Equivalently one can first eliminate at each fixed third degree, then in increasing third degree, since taking its differential only increases that index. This proves the comparison with the ordinary cover in Definition 318.1.

The double complex is the tensor product over k of the two factor complexes for degrees i, j. For a complex of vector spaces choose in each degree a complement of the boundaries in the cycles, and a complement of the cycles in the whole space. Its differential identifies the latter complement with the boundaries in the next degree. The complex is thus a direct sum of spaces with zero differential representing its cohomology and of pairs joined by an isomorphism. Tensoring such a pair with any other complex gives a contractible complex: use the inverse on the paired component as the homotopy there, and the opposite signs

in consecutive degrees cancel the terms from the other differential. Applying this to both factors shows that tensoring representatives of cocycles induces the displayed isomorphism. The map on classes is independent of the complements. All comparisons above are given by restrictions and tensoring representative arrays. For a pair of algebraic maps into X, Y, refine the two source covers to map into the two target covers as in Proposition 318.3; their product rectangles refine the corresponding product cover. Restriction and pullback commute in all three indices, so both comparison isomorphisms commute with these maps on cohomology. This proves naturality and independence of the chosen covers. □

Corollary 321.2. *Let X, W be smooth projective over k, put $d = \dim X$, and let $u_i, c : W \to X$ be morphisms for i in a finite set, with coefficients $\beta_i \in k$. For $0 \le h \le d$, suppose $\sum_i \beta_i u_i^*$ is zero on $H^q(X, \Omega^h_{X/k})$ for every $0 \le q \le d$. On degree-d forms of $X \times X$ let S_T multiply the summand of first form degree j by T^j, where T is an indeterminate. For every $\delta \in H^d(X \times X, \Omega^d_{(X\times X)/k})$ one then has*

$$[T^h] \sum_i \beta_i (u_i, c)^* (S_T \delta) = 0 \quad \text{in } H^d(W, \Omega^d_{W/k}).$$

Proof. By Proposition 321.1 the coefficient at T^h is a sum of external products $v \boxtimes w$ before pullback, with $v \in H^q(X, \Omega^h)$ and $w \in H^{d-q}(X, \Omega^{d-h})$, $0 \le q \le d$. Write $\Delta_W : W \to W \times W$ for the diagonal. Since $(u_i, c) = (u_i \times c)\Delta_W$, the weighted sum on each product is

$$\Delta_W^* \left(\Big(\sum_i \beta_i u_i^* v \Big) \boxtimes c^* w \right)$$

by the naturality just proved and Proposition 318.3. It vanishes by the hypothesis. Summing gives the result. □

322. Cohomology with support and the gluing of local classes

Definition 322.1. Let Z be a closed subset of a separated finite-type k-scheme X, and let $\mathcal{F}$ be a coherent module. Choose finite ordered affine covers (U_i) of X and (V_j) of $X \setminus Z$; an empty open may have the empty cover. In bidegree $(a, 0)$, $a \ge 0$, take the ordinary U-cochains of degree a. In (a, b), $b \ge 1$, take the sum of the modules on $U_I \cap V_J$ for $|I| = a + 1$, $|J| = b$, with ordered nonempty lists in both indices. The horizontal differential is the U-cover differential. In the vertical direction use restriction from $b = 0$ to $b = 1$, then the V-cover differential. The total differential is $\delta_U + (-1)^a \delta_V$, with the augmentation included in δ_V. Its square is zero by the restriction identities. Denote its cohomology in degree r by $H^r_Z(X, \mathcal{F})$. Projection to $b = 0$ gives a map to ordinary cohomology, called forgetting the support.

Proposition 322.2. *Definition 322.1 is independent of both covers. Its groups for differential forms admit pullback by $f : X' \to X$ with $f^{-1}Z \subseteq Z'$, taking values in the groups with support Z'; these pullbacks commute with forgetting support. In particular support can be enlarged. The construction is also functorial for module homomorphisms on a fixed scheme.*

Let $d \ge 1$. Assume that on every nonempty finite intersection U_I one has $H^b_{Z\cap U_I}(U_I, \mathcal{F}) = 0$ for $0 \le b < d$. Then restriction identifies $H^d_Z(X, \mathcal{F})$ with the lists of classes in $H^d_{Z\cap U_i}(U_i, \mathcal{F})$ whose two restrictions on every $U_i \cap U_{i'}$ agree.

Proof. For fixed U_I, its vertical complex is the complex for support on this affine using the single chart and the cover by $U_I \cap V_j$ of its complement. Changing the V-cover by refinement induces an isomorphism on the cohomology of this column. Indeed its degrees $b \ge 1$ comprise the ordinary cover complex shifted by one, compared in Proposition 318.3; the degree-zero term and its restriction are left compatible with that refinement. In degree zero the kernel consists of the sections vanishing on the complement. In degree one take the ordinary degree-zero cohomology modulo the restricted sections, while in higher degrees the

groups are the ordinary ones with the indicated shift. This proves the column claim. For a refinement of the U-cover, fix instead the vertical degree. At $b = 0$ this is the ordinary cover comparison on X, and at $b > 0$ it is the sum of such comparisons on the affines V_J, covered by their intersections with the U_i.

A map of bounded double complexes inducing isomorphisms on cohomology of all columns induces one on total cohomology. For clarity, attach to each column map $L \to M$ its complex in degree j with terms $M^j \oplus L^{j+1}$ and differential $(m, l) \mapsto (dm + f(l), -dl)$. This complex is exact precisely when f induces an isomorphism: a closed pair first gives a closed l mapping to a boundary, and after removing a boundary for l the remaining closed m tests surjectivity. Conversely applying exactness to $(m, 0)$ and to pairs lifting a kernel class gives the two implications. With the total signs these column complexes assemble to the same construction on totals, up to the sign change $(-1)^a$ on the shifted component in column a. Exact columns give exact total complex by eliminating a cocycle in increasing horizontal degree as in Proposition 318.3. The row version follows by interchanging the indices. This proves invariance under each refinement. Choices of assignments for a refinement or a pair pullback give the homotopy of the same proposition in a changed index; in the vertical index multiply by $(-1)^a$ and use zero homotopy on the augmented entry. Restrictions in the unchanged index commute with it, so the other terms cancel under the total signs. Intersecting covers and composing the chosen restrictions gives independence and functoriality. The hypothesis on $f^{-1}Z$ permits independent affine refinements of the two source covers over the two target covers. Projection to $b = 0$ is compatible with all these maps. Module homomorphisms act on all terms by restriction.

For the gluing assertion represent the given classes by vertical d-cocycles at $a = 0$. Their horizontal differential is a vertical boundary in column one by agreement on double overlaps. Choose vertical primitives there with sign giving cancellation in the total differential. After choosing columns through $a - 1$, the next uncancelled term in column $a \geq 2$ is vertically closed since the total differential squares to zero. Its vertical degree is $d + 1 - a < d$, so it too has a primitive when that degree is positive; at degree zero a closed entry is already zero under the hypothesis. Continuing produces a total cocycle. Conversely any total cocycle supplies such an agreeing list. If its initial classes vanish, remove vertical boundaries in column zero by subtracting total boundaries. At every subsequent nonzero column the remaining first entry is vertically closed in degree $d - a < d$ and can likewise be removed, with the degree-zero case necessarily zero. The finite number of columns gives injectivity. A restriction to U_i computes the initial class by the projection to that chart, using the comparison with its single-open cover. □

323. Local symbols for a transverse system of equations

Lemma 323.1. *Let $U = \operatorname{Spec} A$ be smooth over an algebraically closed field k, let Z be the common zero locus of $f_1, \dots, f_d \in A$ with $d \geq 1$, and suppose their differentials at every closed point of Z are linearly independent. Let M be a finite locally free A-module, also denoting its corresponding sheaf on U by M. Then $H^b_Z(U, M) = 0$ for $b < d$. The top group in degree d is the quotient of $M_{f_1 \cdots f_d}$ by the sum of the images of the localizations omitting one factor, taking the unlocalized M when $d = 1$. In this quotient write*

$$\left[\frac{\omega}{f_1, \dots, f_d}\right] \quad \text{for the class of} \quad \frac{\omega}{f_1 \cdots f_d}, \qquad \omega \in M.$$

The class depends only on $\omega \bmod (f_1, \dots, f_d)M$. These groups commute with restriction to principal opens.

Proof. Use the single chart U and the ordered principal opens $D(f_j)$ on its complement in Definition 322.1. The term with subset J is $M_{\prod_{j \in J} f_j}$ in degree $|J|$, also for the empty subset in degree zero. Hence the stated top quotient follows from its last differential. Multiplying a

numerator by f_j makes the fraction come from the term omitting j. The complex localizes termwise, proving the restriction claim.

For $N \geq 1$ use the finite complex with one copy of M in every subset position and differential from J to $J \cup \{j\}$ given by multiplication by f_j^N with the same incidence sign. Its square is zero by commutation of these multiplications. In passing from N to $N+1$ multiply the copy at J by $\prod_{j \in J} f_j$. These maps commute with the differentials, and their direct limit is the above localized complex by sending an entry at J to its fraction with denominator $(\prod_{j \in J} f_j)^N$. For a direct sequence cohomology commutes with this limit: a differential which is zero in the limit is already zero at a later stage, and an equality to a boundary can likewise be tested at a common later stage for its two representatives.

We check the finite complexes at maximal ideals. Away from Z some f_j is a unit, and insertion of that index followed by division by f_j^N contracts the complex as in Lemma 318.2. At a closed point of Z, complete the independent differentials of the f_j to a basis of cotangent coordinates using linear combinations of the smooth chart parameters. The completed local ring is a power series ring in those parameters by the local computation of Lemma 294.1. It is also a power series ring in the f_j and the chosen remaining coordinates, centered at the point. Indeed this substitution has invertible linear part, so it has an inverse modulo each successive power of the parameter ideal by solving for the homogeneous correction with that same linear part. The inverses are compatible and give the power series inverse. The completed quotient by any initial list $f_1^N, \ldots, f_i^N$ in this power series ring has unique expressions with the first i exponents less than N, by separating monomials. Thus the next f_{i+1}^N acts injectively there.

In the local ring the quotient by that initial ideal injects into the corresponding quotient of the power series ring. If a local element lies in the ideal upstairs, approximate its finitely many coefficient multipliers modulo every power of the maximal ideal by local elements, using the truncated chart isomorphisms. Its class in the noetherian local quotient then lies in every maximal ideal power and is zero by the intersection argument of Lemma 294.1. Thus each next parameter power acts injectively on the successive local quotients and on the corresponding quotients of the free local module M.

Starting with degree zero for no parameters, adding one parameter gives the two copies of the preceding complex on subsets omitting and containing its new index, with multiplication by f_i^N between them up to the indicated degree sign. The cocycle and boundary calculation for the two-term cone in Proposition 322.2 shows that if the preceding cohomology is concentrated in its top degree with module $M/(f_1^N, \ldots, f_{i-1}^N)M$, the new groups in the last two degrees are its kernel and cokernel under f_i^N; all lower ones vanish. The kernel is zero by the preceding injectivity, and the cokernel is the next quotient. Induction and passage to the direct limit prove the assertion at the closed point. All maximal ideals here are closed k-points by the affine point argument in Lemma 280.1; vanishing at all of them implies vanishing over A by the annihilator argument of Lemma 318.2. □

Lemma 323.2. *In the setting of Lemma 323.1, if $g = Cf$ for two ordered columns of d functions with C invertible over A, then*

$$\left[\frac{\omega}{f_1, \ldots, f_d}\right] = \left[\frac{\det(C)\,\omega}{g_1, \ldots, g_d}\right].$$

In particular the local class with numerator $df_1 \wedge \cdots \wedge df_d$ is independent of the chosen generating parameters under such a change.

Proof. The localizations of the comparison maps commute with the common cover refinements in Proposition 322.2. It is enough to prove the identity near every maximal ideal, by the restriction and localization calculations in Lemma 323.1. Over a local ring an invertible matrix is a product of row swaps, unit row scalings and additions of a multiple of one row to another. A unit in the first column exists by invertibility on the residue field; after moving it to the first row, normalize it, clear the column, and continue on the remaining

invertible block, then clear the entries above the pivots. These finitely many operations already hold on a neighborhood after inverting their denominators. A swap changes the sign of the ordered top cover entry; a unit scaling changes its denominator by that unit. These give the formula in those cases.

For a row addition let $g_i = f_i + a f_j$ with $j \neq i$ and all other functions fixed. Refine both covers by replacing their i-th open by $D(f_i g_i)$, keeping all the other members. These opens still cover the complement of Z: where none of the other functions is nonzero, $g_i = f_i$ in the residue field. On the common top intersection the difference of the two fractions is

$$\frac{a\,\omega}{f_i g_i \prod_{k \neq i,j} f_k}.$$

It extends over the intersection omitting j of that common cover, hence is a boundary, by using its appropriate signed entry in the preceding degree. This proves equality for an addition. Multiplication of determinants gives the formula for the original change. Finally $dg_i = \sum_j C_{ij} df_j \bmod (f_1, \dots, f_d)\Omega^1$. Wedging the d equalities thus gives numerator $\det(C)\, df_1 \wedge \cdots \wedge df_d$ modulo that ideal on the differential module, which suffices by the last assertion about numerators in Lemma 323.1. □

324. The differential classes of the diagonal and of points

Proposition 324.1. *Let Y be a smooth separated variety over an algebraically closed field k, and let D be a closed subscheme. Suppose that near each point of D its ideal is generated by $d \geq 1$ functions whose differentials are independent there. There is a unique class $\delta_D \in H^d_{|D|}(Y, \Omega^d_{Y/k})$ whose restriction to each such affine chart, with its ordered equations $f_1, \dots, f_d$, is*

$$\left[\frac{df_1 \wedge \cdots \wedge df_d}{f_1, \dots, f_d}\right].$$

On an open disjoint from D its restriction is zero.

Proof. Take a finite affine cover by the indicated opens and by opens in the complement of D. After restricting to any affine intersection meeting D, the chosen equations from its first member of the cover still give the ideal and their differentials remain independent along D. By Lemma 323.1 the supported groups below d vanish there. On an intersection disjoint from D the vertical augmented complex of Definition 322.1 is an ordinary augmented cover of the entire affine, exact by Lemma 318.2. Thus it also has zero group in degree d.

On a double intersection meeting D, let f, g be the two generating lists. At any closed point of D write $g = Cf$ in the local ring, since the f's generate the ideal. On the cotangent space there $dg = C\, df$ since all f_j vanish. Both lists of covectors are independent, so the square matrix C has invertible determinant there. Localize on a principal neighborhood to obtain the equations and unit determinant on that neighborhood. Lemma 323.2 gives identical classes there. At the other closed points some generator is a unit locally, giving zero supported cohomology. The two restrictions agree on the full affine intersection, by the localization and annihilator arguments of Lemma 323.1. Proposition 322.2 now gives the class and its uniqueness. Adding any other such chart to the cover proves the claim there too. □

Corollary 324.2. *Let X/k be smooth projective of pure dimension $d \geq 1$ over an algebraically closed field. Its diagonal in $X \times X$ has the class of Proposition 324.1. Write δ_X for its image in $H^d(X \times X, \Omega^d_{(X\times X)/k})$. Near any diagonal point, if $t_1, \dots, t_d$ are free smooth-chart parameters on X, the defining functions in its symbol can be taken to be*

$$f_j = t_j \circ \mathrm{pr}_1 - t_j \circ \mathrm{pr}_2.$$

For a closed point z of a smooth projective d-dimensional W/k of pure dimension, let $\mathrm{cl}_W(z) \in H^d(W, \Omega^d_{W/k})$ be the class obtained by forgetting support for the reduced point

subscheme. If Z is a finite set of such points, the unique class supported on Z with local point symbols multiplied by scalars c_z has ordinary image $\sum_{z\in Z} c_z \operatorname{cl}_W(z)$.

Proof. Take a smooth chart with coordinates t free and s dependent and equations $P(t,s)=0$ with unit P_s minor (the square matrix in the dependent variables). For a pair of points in this chart the ideal of the diagonal is generated by $t-t'$, $s-s'$, since equating all coordinates and their inverted functions identifies the two evaluations. Write the difference $P(t,s)-P(t',s')$ as matrices times $t-t'$ and $s-s'$ by changing the variables one at a time. The second matrix reduces on the diagonal to P_s. On its determinant open the equations therefore express $s-s'$ in the ideal of $t-t'$. The differentials $dt_j - dt'_j$ are independent on the diagonal, by restricting to tangent vectors in the first factor. Hence the proposition applies, with the asserted equations.

For a point the same calculation sets the second chart coordinates to the constants at z, so the centered free parameters generate its ideal on a neighborhood. On the complement the singleton class has empty support. Enlarge that support to Z by Proposition 322.2. Its local restriction at any other member of Z is zero, as seen on a neighborhood disjoint from z. Choose affine neighborhoods of the finitely many points each excluding all the others, with their centered parameters, and complete the cover by affines in the complement. The vanishing below d in Lemma 323.1 and Proposition 322.2 give uniqueness for the specified list on this cover. The sum of the enlarged singleton classes with coefficients c_z has that list. Forgetting support gives the stated linear combination, also for the empty set where the augmented cover complexes are exact. □

325. Pulling the diagonal class back to a correspondence

Theorem 325.1. *Let X, W be smooth projective of pure dimension $d \geq 1$ over an algebraically closed field k of characteristic zero. Let $u, c : W \to X$ have finite set Z of coincident closed points z, meaning $u(z)=c(z)$, and suppose $du_z - dc_z$ is invertible there. With S_T, δ_X from Corollaries 321.2 and 324.2, one has in $H^d(W, \Omega^d_{W/k})[T]$*

$$(u,c)^* S_T \delta_X = \sum_{z\in Z} \frac{\det(T du_z - dc_z)}{\det(du_z - dc_z)} \operatorname{cl}_W(z).$$

The ratio is formed using any bases of T_zW and $T_{u(z)}X$; the two changes of basis cancel in it. If dc_z is invertible and $A_z = du_z (dc_z)^{-1}$, this ratio is $\det(I - TA_z)/\det(I - A_z)$.

Proof. The first-factor degree decomposition in Proposition 321.1 is a decomposition of the differential module itself. The multiplication by T^j in degree j is therefore a polynomial of module homomorphisms commuting with restrictions and denominators. Apply it before forgetting the support of the diagonal class in Corollary 324.2. The pair map (u,c) pulls this supported class back to support Z by Proposition 322.2. Indeed the inverse image of the diagonal is closed, and its set outside the finitely many indicated points is empty since every nonempty locally closed set has a closed point by Lemma 280.1's affine calculation.

At z use free target parameters $t_1, \ldots, t_d$ near its common image. Put $g_j = t_j \circ u - t_j \circ c$. The pullback of the support symbol in those coordinates is represented near z by

$$\left[\frac{\bigwedge_{j=1}^d (T\, d(t_j \circ u) - d(t_j \circ c))}{g_1, \ldots, g_d} \right].$$

In fact, scaling the first-factor form degree multiplies each occurrence of a differential from that factor by T under wedge expansion. The denominators are functions, hence are only pulled back. The cover on the complement by the nonvanishing of the original coordinate differences pulls back to the corresponding principal opens for g_j, giving this precise description in Definition 322.1.

The differentials of the g_j at z form a basis because $du_z - dc_z$ is invertible. Thus the g_j generate the maximal ideal of the local ring: their images span it modulo its square,

and one uses the finite-module determinant argument of Lemma 271.1. They generate the reduced point ideal on a neighborhood, by clearing denominators in expressions for finite local chart generators as in Lemma 316.1. One may shrink the affine neighborhood to exclude other coincidences. Relative to the basis $\bigwedge_j dg_j$, the numerator at that point is the scalar $\det(T du_z - dc_z)/\det(du_z - dc_z)$: write both wedges in any ordered local cotangent basis, using the target rows dt_j. By Lemma 323.1 the value of the numerator modulo the point ideal suffices coefficient by coefficient. By Lemma 323.2 the resulting unscaled point symbol is the intrinsic one of Corollary 324.2. That corollary now identifies the ordinary image of the global supported class with the stated sum. When dc_z is invertible, factor it on the right in both determinants. The two common factors and the two signs $(-1)^d$ cancel, giving the final expression. □

Corollary 325.2. *For a finite list u_i with a common $c : W \to X$ satisfying Theorem 325.1, let Z_i be their respective coincidence sets. If $\sum_i \beta_i u_i^*$ vanishes on $H^q(X, \Omega^h_{X/k})$ for $0 \le q \le d$ with $0 \le h \le d$, then*

$$\sum_i \beta_i \sum_{z \in Z_i} [T^h] \frac{\det(T du_{i,z} - dc_z)}{\det(du_{i,z} - dc_z)} \operatorname{cl}_W(z) = 0 \quad \text{in } H^d(W, \Omega^d_{W/k}).$$

In particular this applies to the regular correspondences and torus slots of Corollary 255.2 under the vanishing assumption of Corollary 320.2.

Proof. Apply Corollary 321.2 to δ_X and use the theorem on each term. For the specified Hecke covers $c = \pi_m$ is étale by Theorem 268.3 and the two dimensions agree. Their coincidences are finite by Theorem 254.3, and $\det(I - A_z) \neq 0$ by Theorem 255.1. These verify the required transversality for each $u_{g,2\boldsymbol{\alpha}}$. The stated pullback vanishing is precisely the conclusion of Corollary 320.2. □

326. Smooth forms representing the classes with support

Definition 326.1. Let M be the complex manifold of a smooth projective complex variety or any open subset of it. For an open $U \subseteq M$ let $\mathcal{A}^{a,j}(U)$ be the smooth complex differential forms of holomorphic degree a and antiholomorphic degree j. In local holomorphic coordinates their basis factors are $dz_i, d\bar{z}_i$ in the specified numbers; put these spaces equal to zero for negative degrees. Use $\bar{\partial}$ of Definition 285.1 with its signed product rule and $\bar{\partial}^2 = 0$. For closed $Z \subseteq M$ define the relative smooth complex in degree r by

$$\mathcal{A}^{a,r}(M) \oplus \mathcal{A}^{a,r-1}(M \setminus Z), \qquad (A, B) \longmapsto (\bar{\partial} A,\ A|_{M \setminus Z} - \bar{\partial} B).$$

Projection to the first component forgets the support. For a larger closed Z' restrict the second component to $M \setminus Z'$.

Lemma 326.2. *For a smooth projective complex W, $M = W(\mathbb{C})$, there is a natural linear map from $H^r(W, \Omega^a)$ to the cohomology of $(\mathcal{A}^{a,\bullet}(M), \bar{\partial})$. From $H^r_Z(W, \Omega^a)$ for an algebraic closed support there is likewise a map to the relative smooth cohomology of Definition 326.1. These maps commute with restriction, enlargement and forgetting of support. On analytic open neighborhoods a restricted Čech representative may be computed on any finite analytic refinement by the same construction.*

Proof. Every finite open cover on any such complex open manifold has a smooth partition of unity. Here are details for the covers used below. For an open subset of the compact projective manifold use the sets at distance at least $1/j$ from the complement, in a metric induced by a projective embedding as in Definition 263.1. These compact sets exhaust the open and each lies in the interior of the next. For the whole compact manifold one compact set suffices. Cover the successive compact differences $K_j \setminus \operatorname{int} K_{j-1}$ by finitely many smaller coordinate balls, each with its closed larger ball inside a cover member and in $\operatorname{int} K_{j+1} \setminus K_{j-2}$, taking the first missing lower sets empty. Center a smooth nonnegative

bump in each larger ball, positive on the smaller one. In scaled real coordinates one can use $\exp(-1/(1-|x|^2))$ for $|x|<1$ and zero outside; all derivatives extend by zero because an exponential of $-1/t$ tends to zero faster than every power of $t>0$. The collection is locally finite by the two compact cutoffs. Its sum is positive. Divide by that sum and collect the functions assigned to each cover member. Their supports are closed and contained in the assigned members by local finiteness.

On any such cover the augmented Čech complex for smooth forms of fixed bidegree is exact. If the partition is ρ_j, the contraction of an alternating array has entry $\sum_j \rho_j c_{j,I}$ on the intersection indexed by I, with summands extended by zero there outside their j-th opens. The support condition makes these extensions smooth. In $\delta H + H\delta$ the deletions other than the inserted index cancel and the remaining sum is multiplication by $\sum_j \rho_j = 1$, including the augmented term.

In the ordinary cover cochains add a second index j for smooth forms of type (a,j) and use $\bar\partial$ with the total sign. Holomorphic a-forms are $\bar\partial$-closed, so inserting an algebraic array at $j=0$ gives a cochain map, by the analytic coordinate pullback formulas for df. The augmentation from the single global smooth complex into this total complex gives an isomorphism on cohomology, by the exact augmented rows and finite filtration argument of Proposition 322.2. Invert this isomorphism on cohomology to obtain the ordinary map.

For support use a third index j in Definition 322.1, with smooth forms of type (a,j) on each intersection and $\bar\partial$ carrying the sign $(-1)^{i+b}$ for horizontal cover degree i and vertical augmented degree b. The same insertion of holomorphic forms is a cochain map. In the U cover index the augmentation on M at $b=0$ and on V_J at $b>0$ gives isomorphisms on total cohomology by the partition contraction. There remains the complex consisting of global smooth forms at $b=0$ and smooth cover cochains of the complement at $b\geq 1$. Augment the latter by global forms on the complement, placed at $b=1$. This again induces an isomorphism: on the subcomplex with $b\geq 1$ it is the ordinary smooth cover comparison just proved, and on the quotient at $b=0$ it is the identity. The conclusion on the full complex follows by lifting cocycles in the two terms, or by the exact-cone argument of Proposition 322.2 filtered by these two parts. At $b=0,1$ the resulting differential is exactly that of Definition 326.1. All comparison arrows being inverted are restrictions of global forms onto the cover, and commute with refinements, restriction to an open and restrictions on the complement. The partitions prove these arrows are isomorphisms on cohomology and do not enter their definition. This also proves compatibility with forgetting support, since projection to $b=0$ commutes with the comparisons. The same smooth comparisons apply to analytic covers and their refinements, proving the local assertion. □

Lemma 326.3. *Let Z be compact in M, and U an open neighborhood of Z. Restriction from M to U induces an isomorphism on the relative smooth cohomology of Definition 326.1. For a relative cocycle on U represented by $(A,0)$ with A supported in Z, its inverse is represented by the extension of $(A,0)$ by zero.*

Proof. Choose a smooth χ supported in U, one on a neighborhood of Z, by the partition construction in Lemma 326.2 for $U, M\setminus Z$, first replacing U by a relatively compact neighborhood inside it if needed. On arbitrary pairs on U send (A,B) to the extension by zero of

$$(\chi A + \bar\partial\chi\wedge B,\ \chi B).$$

The term involving $\bar\partial\chi$ vanishes near Z, so it is defined smoothly over U. The formula is a cochain map by the signed product rule and $\bar\partial^2=0$. On either side the identity minus the composite with restriction is the differential times the homotopy plus that homotopy times the differential, where the homotopy sends (A,B) to $((1-\chi)B,0)$. Indeed $(1-\chi)B$ extends by zero near Z, and the two first-component derivatives of B cancel, leaving $(1-\chi)A-\bar\partial\chi\wedge B$; the second component is $(1-\chi)B$. These formulas prove both assertions. □

327. A degree functional for classes of points

Lemma 327.1. *Let W be smooth and projective over $\mathbb{C}$, of pure dimension $d \geq 1$, and put $M = W(\mathbb{C})$. Integration in the complex orientation gives a linear functional on $H^d(W, \Omega^d_{W/\mathbb{C}})$ by the map of Lemma 326.2. If a supported class has smooth representative $(A, 0)$ on an analytic neighborhood of its compact support, its image under this functional is the integral of A on the neighborhood.*

Proof. Use the coordinate order $(\Re z_1, \Im z_1, \ldots, \Re z_d, \Im z_d)$. A holomorphic coordinate change with complex derivative D has real determinant $\det D \overline{\det D} > 0$: over $\mathbb{C}$ the complexified real derivative in the coordinates $dz_j, d\bar{z}_j$ has the two conjugate blocks. The set M is compact, being closed in complex projective space, which is the continuous image of a compact unit sphere. Use a finite coordinate cover with a smooth partition of unity as in Lemma 326.2. Multiply a top form by the partition functions, integrate in the resulting oriented coordinates and add. For two such choices multiply the two partitions. On their intersections the local integrals agree by the change of variables proved in Lemma 127.1, and summing first over either partition gives independence.

If B has bidegree $(d, d-1)$, then $\mathrm{d}B = \bar{\partial} B$, since the component of holomorphic degree $d+1$ vanishes. Writing $B = \sum_j \rho_j B$ as above reduces the integral of $\mathrm{d}B$ to the differentials of compactly supported forms in coordinate domains. Extend the coefficients by zero in the real coordinates. Each integral of a first partial derivative vanishes by one-variable integration and Lemma 123.1. Thus integration descends to the smooth cohomology, giving the asserted functional by composition. For a relative cocycle $(A, 0)$ the restriction of A to the complement of the support is zero. Lemma 326.3 identifies its global image with the extension by zero, whose integral is the same by coordinate independence. □

Proposition 327.2. *For W, d of Lemma 327.1 and any $z \in W(\mathbb{C})$, the functional there takes the value*

$$c_d = (-1)^{d(d+1)/2} (2\pi i)^d$$

on the ordinary class $\mathrm{cl}_W(z)$ of Corollary 324.2. Consequently there is a linear functional $\deg_W : H^d(W, \Omega^d_{W/\mathbb{C}}) \to \mathbb{C}$ with $\deg_W(\mathrm{cl}_W(z)) = 1$ on every component.

Proof. Choose the algebraic coordinates for the point symbol from Corollary 324.2, translated to vanish at z. They give local holomorphic coordinates by the inverse construction in Lemma 127.1: the inverse has complex-linear derivative everywhere. Use a coordinate polydisc $P = \prod_{i=1}^d \{|z_i| < R_i\}$ inside this neighborhood. Write $\eta = dz_1 \wedge \cdots \wedge dz_d$. On the ordered cover by $z_i \neq 0$ of the punctured polydisc the local representative is $\eta / \prod_i z_i$ in augmented degree d. This calculation on an analytic restriction represents the smooth image by Lemma 326.2. Enlarge the support to the compact polydisc $K = \prod_i \{|z_i| \leq r_i\}$, with $0 < r_i < R_i$. The new complement is covered in order by the opens $D_i = \{|z_i| > r_i\}$ in P. Restriction to their intersections represents the enlarged class.

On the i-th disc choose a smooth radial function f_i equal to zero near zero and equal to one for $|z_i| \geq r_i$. Such a cutoff is obtained by integrating a nonnegative smooth bump of positive integral supported inside $(0, r_i^2)$, normalizing its integral to one, and using $|z_i|^2$ as argument. The function $e_i = f_i / z_i$, extended by zero near the center, is smooth. Consider smooth $(0, j_i)$-forms on this disc in degree $b_i = 0$ and on its annulus in degree $b_i = 1$. Use the differential given by restriction in b_i and $(-1)^{b_i} \bar{\partial}$ in j_i. The degree-one cochains given respectively by $1/z_i$ on the annulus and by $-\bar{\partial} e_i$ on the disc are cocycles. Their difference is the total differential of e_i on the disc, since $e_i = 1/z_i$ on the annulus.

In the tensor product of the complexes over $\mathbb{C}$, the differential on factor i carries sign $(-1)^{\sum_{k<i} (b_k + j_k)}$. There is a cochain map to the augmented cover complex of smooth forms on P and the D_i, as follows. The set $J = \{i : b_i = 1\}$ labels the intersection. Pull back the antiholomorphic factors in their given order to this intersection, wedge them, then wedge with η on the right. Multiply by $(-1)^{\sum_{k<l} j_k b_l}$. For a restriction in factor i the change in this

last exponent is $\sum_{k<i} j_k$, so the resulting sign is the alternating deletion sign $(-1)^{\sum_{k<i} b_k}$ of the cover. For a derivative in factor i the change is $\sum_{l>i} b_l$. Combining the tensor sign and the factor sign $(-1)^{b_i}$ with this gives $(-1)^{|J|+\sum_{k<i} j_k}$, exactly the total $\bar\partial$ and wedge signs. This proves the claim.

Tensor products of the cohomologous cocycles give cohomologous results. Replacing one factor by its difference adds the differential of the same tensor with its primitive in that factor, up to the tensor sign. At both ends the extra sign of the map is one. Thus the original class on the enlarged support has the smooth representative $(A, 0)$, where

$$A = (-1)^d(\bar\partial e_1 \wedge \cdots \wedge \bar\partial e_d) \wedge \eta.$$

Indeed A vanishes on every D_i and the augmented cover map from relative pairs in Lemma 326.2 sends this pair to its degree-zero cover component. Enlarging support preserves the ordinary class by the same lemma. By Lemma 327.1 it remains to integrate A on P.

On the i-th disc the two-form $\bar\partial e_i \wedge dz_i = \mathrm{d}(e_i\, dz_i)$ has compact support. In polar coordinates write $e_i\, dz_i = a\, dt + b\, d\theta$ with radius t. Its differential has coefficient $\partial_t b - \partial_\theta a$ in $dt \wedge d\theta$. The second term integrates to zero by periodicity and the first gives the outer circle integral, since e_i vanishes near zero. On any circle of radius between r_i and R_i, the one-form is $dz_i/z_i = i\, d\theta$. This gives $2\pi i$ for the disc integral. Moving each dz_i next to $\bar\partial e_i$ costs the sign $(-1)^{d(d-1)/2}$. Product integration by Lemma 123.1 now gives c_d in the complex orientation. Divide the functional by this nonzero constant. □

328. Detection of the prescribed form degrees

Theorem 328.1. *Let X, W be smooth projective complex varieties of the same pure dimension $d \geq 1$. Let $c, u_i : W \to X$ be morphisms for $1 \leq i \leq m$, let $\beta_i \in \mathbb{C}$, and fix $0 \leq h \leq d$. Suppose that $Z_i = \{z \in W(\mathbb{C}) : u_i(z) = c(z)\}$ is finite for each i, and that $du_{i,z} - dc_z$ is invertible for every $z \in Z_i$. If*

$$\sum_{i=1}^{m} \beta_i u_i^* : H^q(X, \Omega^h_{X/\mathbb{C}}) \longrightarrow H^q(W, \Omega^h_{W/\mathbb{C}})$$

is zero for each $0 \leq q \leq d$, then

$$\sum_{i=1}^{m} \beta_i \sum_{z \in Z_i} [T^h] \frac{\det(T du_{i,z} - dc_z)}{\det(du_{i,z} - dc_z)} = 0.$$

An empty coincidence set contributes zero. If dc_z is invertible the fraction is $\det(I - TA_{i,z})/\det(I - A_{i,z})$ for $A_{i,z} = du_{i,z}(dc_z)^{-1}$.

Proof. Corollary 325.2 gives a zero relation with the same coefficients on the classes of the points in $H^d(W, \Omega^d_{W/\mathbb{C}})$. Apply the single linear functional $\deg_W$ of Proposition 327.2 to this finite relation. Every point on every component has value one. The determinant expression for invertible dc_z follows by right composition with $(dc_z)^{-1}$ in numerator and denominator, as in Theorem 325.1. □

Theorem 328.2. *Use the approximating form, the central correction $\theta_{w,r} = c_w^{-r} t_{w,r}$ for $0 \leq r \leq n$, and the integers $h_j = \mu + k_{v_0,j}$ of Theorem 251.3. Take $X = \mathcal{X}(C, \nu)_{\mathbb{C}}$ at its neat level and put $d = \dim X$. Let $T_{w,r} = T_{\tau_{w,r}}$ be the operators of Theorem 320.1, with $T_{w,r} = 0$ and $\theta_{w,r} = 0$ for $r > n$. For each $1 \leq j \leq n$ one has $0 \leq h_j \leq d$. For every finite complex linear combination F of ordered words in the safe generators, constants included,*

$$F(\theta) \neq 0 \implies F((T_{w,r})_{w,r}) \neq 0 \quad \text{on} \quad \bigoplus_{q=0}^{d} H^q(X, \Omega^{h_j}_{X/\mathbb{C}}).$$

Proof. Nonnegativity was proved in Theorem 251.3. For $F = 1$, Corollary 255.2 gives a nonzero fixed-point sum at each h_j. Its numerators use the h_j-th exterior power of an endomorphism on a d-dimensional cotangent space. That power is zero for $h_j > d$, proving the upper bound. The dimension is positive since the fiber contains $\nu \geq 1$ copies of the positive-dimensional torus in Theorem 252.3.

Divide the given F by $F(\theta)$, so that $F(\theta) = 1$. Suppose the operator is zero on the displayed direct sum for some j. By Corollary 255.2, choose a regular left multiple $H = BF(\tau)$ and a tuple $\boldsymbol{\alpha}$ for which its sum at that h_j is nonzero. Choose a common level sufficiently fine for Corollary 320.2 and the correspondence calculation. Further refinement multiplies the local sum by the degree of the new cover and the normalizing index by the same degree, by the coincidence count in Theorem 254.3. The asserted vanishing of $F(T)$ implies the vanishing of the linear combination of the pullbacks $u^{*}_{g,2\boldsymbol{\alpha}}$, with coefficients $H(gC)$, from every $H^{q}(X, \Omega^{h_j})$, $0 \leq q \leq d$, by Corollary 320.2.

Use these maps in Theorem 328.1 with $c = \pi_m$. The dimension, finiteness and transversality hypotheses were checked in Corollary 325.2. The resulting zero sum is precisely the numerator of the normalized sum for $H, \boldsymbol{\alpha}, h_j$: the coefficient of $\det(I - TA_z)$ is $(-1)^{h_j} \operatorname{tr}(\bigwedge^{h_j} A_z^{\vee})$ by the exterior-power calculation in Corollary 255.2. The normalizing index $[C : C_m]$ is a positive integer. This contradicts the choice of the sum. □

329. Simultaneous quotients at a detected degree

Lemma 329.1. *The elements $\tau_{w,r}$ of Definition 251.2 commute pairwise in $\mathcal{H}^{+}$. Their operators on the coherent groups of Theorem 320.1 commute over the field E of the period model.*

Proof. By Theorem 241.2 each sum is a central translate of the image, under b_w, of the right cosets of an integral spherical double coset for $K_w = \mathrm{GL}_n(O_w)$. The central factors commute with all representatives. If $\{w, \bar{w}\}$ and $\{v, \bar{v}\}$ are disjoint, the two noncentral factors have disjoint support in the product of the local groups and commute term by term. For a single pair, use $a \mapsto b_w(a)$ from Proposition 240.4. At a safe pair its inverse image of C is K_w: membership forces integrality and integral invertibility at w, and $b_w(K_w) \subseteq C$ by Lemma 245.2. The description of u_w and its conjugate inverse transpose at the mate shows that $b_{\bar{w}}(a) = b_w(\bar{a}^{-\mathsf{T}})$. This group isomorphism carries $K_{\bar{w}}$ onto K_w. Hence both unshifted sums on this pair are images of local spherical sums for the same K_w.

Multiplication of such sums in the full group of similitudes agrees with multiplication of those local sums followed by b_w on representatives. Indeed, sums over pairs of right cosets group according to $b_w(a)b_w(b)C = b_w(ab)C$, and equality of the final cosets is precisely equality modulo K_w at the chosen member. The proof of Lemma 111.1 gives commutativity of this local convolution by transpose on each double coset. For the mass-one subgroup its convolution coefficients are exactly these coset counts, since integration over a right-factor coset counts it once. Restoring the central translates gives the asserted equality of products in $\mathcal{H}^{+}$, including the zero generators. Their cohomological actions and their descent to E follow from Theorem 320.1, since the coset functions have integral coefficients. □

Theorem 329.2. *Keep the data of Theorem 328.2. View $E \subset \mathbb{C}$ by the embedding defining X, and denote its projective model over E by X_E. The field $L = E((\theta_{w,r})_{w,r})$ generated by all the tested eigenvalues is finite over E. For every $h = h_j$ set*

$$D_h = \bigoplus_{q=0}^{d} H^{q}(X_E, \Omega^{h}_{X_E/E}).$$

There is a nonzero quotient Q_h of D_h with an L-vector space structure for which $T_{w,r}$ acts by $\theta_{w,r}$. In particular, $[L : E] \leq \dim_E D_h$ for every such h. For any field embedding $\iota : L \to k$, the simultaneous quotient of $D_h \otimes_{E,\iota} k$ with eigenvalues $\iota(\theta_{w,r})$ is nonzero. This quotient has a nonzero summand in at least one cohomological degree $0 \leq q \leq d$.

Proof. The space D_h is finite dimensional by Proposition 319.1. Proposition 318.3 identifies its complex scalar extension with the sum in Theorem 328.2, and Theorem 320.1 identifies the operators there. By Lemma 329.1 their image generates a commutative unital E-algebra $A_h \subseteq \mathrm{End}_E(D_h)$. If a polynomial over E in finitely many generators vanishes on D_h, it still vanishes over $\mathbb{C}$, so the detected degree theorem makes its value at θ zero. Evaluation therefore gives a unital E-algebra map $A_h \to \mathbb{C}$. Its image is $E[(\theta_{w,r})_{w,r}]$, independent of h and finite dimensional over E. Multiplication by any nonzero member of this image is injective, hence surjective as an endomorphism of a finite-dimensional space. The image is a field, equal to L.

Let I_h be the kernel of this evaluation and put $Q_h = D_h/I_hD_h$. Its action by $A_h/I_h = L$ has the stated scalar values. Suppose $I_hD_h = D_h$. A basis $v_1, \ldots, v_a$ of D_h then satisfies $v_i = \sum_j b_{ij}v_j$ for $b_{ij} \in I_h$, by expressing the elements of I_hD_h in that basis. The space is nonzero by the detected degree theorem applied to $F = 1$. Multiplying the matrix equation by the adjugate of $(\delta_{ij} - b_{ij})$ shows that its determinant kills every v_i. It is therefore zero in the concrete endomorphism algebra A_h, but its image in L is one. This proves $Q_h \neq 0$ and the degree bound follows from $\dim_E Q_h = [L : E] \dim_L Q_h$.

After extension to k, the map $D_h \otimes_E k \to Q_h \otimes_{L,\iota} k$, $v \otimes t \mapsto [v] \otimes t$, is onto since the classes $[v]$ span Q_h. Its target is nonzero by extending an L-basis. Thus the quotient by the sum of all $(T_{w,r} - \iota(\theta_{w,r}))(D_h \otimes_E k)$ is nonzero as well. The operators preserve each cohomological degree by Theorem 320.1. The sum of their images after subtracting the scalars therefore splits by those degrees, proving the last assertion. □

330. Étale cochains and geometric stalks

Definition 330.1. In each residue characteristic, including zero, fix an algebraically closed field K large enough to contain an isomorphic copy of every residue field of the schemes and charts in question. One can take the algebraic closure of a purely transcendental field on a set larger than the cardinalities of the base fields and rings in use. An embedding of a transcendence basis extends to an embedding of the algebraic extension by adjoining roots, as in Lemma 16.2. Use all absolute morphisms $\mathrm{Spec}\, K \to X$ in that characteristic. Write $P(X)$ for the union of these sets of points of X.

Use the small étale site of X. Its objects are étale schemes $U \to X$, its arrows are morphisms over X, and covering families are jointly surjective étale maps. The calculations may use separated objects of finite presentation and affine open refinements. Here étale charts are locally given over a base ring by the same number of equations and polynomial variables, with the square Jacobian determinant inverted, as in Proposition 313.2. Products over the base and compositions have such charts by scalar extension and by the block triangular Jacobian. The diagonal of an étale chart is open near each of its points. Indeed, subtract its equations in two sets of variables. The matrix of the differences equals the Jacobian on the diagonal, so where its determinant is invertible all coordinate differences vanish. This applies after the principal localizations of a chart. It shows also that a map between étale objects is étale, by factoring it as its graph and a projection.

An abelian sheaf F assigns abelian groups to these objects, contravariantly, with unique gluing of matching sections on covers. For $x \in P(X)$ let F_x be the direct limit of $F(U)$ over the pointed neighborhoods (U, u), $u \in P(U)$ over x, with restriction to smaller neighborhoods. For an abelian group Λ write $\underline{\Lambda}_X$ for the sheaf of locally constant sections with values in Λ, obtained by gluing constant sections. All sheaves below are abelian.

Lemma 330.2. *The pointed neighborhoods in Definition 330.1 are filtered in the direction of restriction. A sheaf map is injective, surjective as a sheaf map, or an isomorphism precisely when all its stalk maps have that property. Kernels and cokernels of sheaf maps have stalks*

equal to the respective kernels and cokernels on stalks. For every F the rule

$$G_X(F)(U) = \prod_{u \in P(U)} F_{x(u)}, \qquad x(u) \in P(X) \text{ the image of } u,$$

with restriction by reindexing, defines a sheaf, and the map $F \to G_X(F)$ taking germs is injective. The functor G_X takes a short exact sequence to a sequence exact on sections on every U. The functor $Q_X F = \operatorname{coker}(F \to G_X(F))$ on sheaves is exact.

Proof. A fiber product with its given pair of lifts refines two neighborhoods. The equalizer of two arrows agreeing at a lift contains a neighborhood of it by openness of the diagonal in Definition 330.1. An étale cover of U is jointly onto $P(U)$: over a chosen point of U some fiber is nonempty, and after extension to the algebraically closed field it has a rational point by the affine calculation of Lemma 280.1. Conversely each ordinary point admits a lift from the chosen field in its characteristic, by its size. A family of neighborhoods of all the chosen lifts therefore covers U. Equality of germs gives local equality on such a cover and thus equality of sections. Lifting a germ of a section gives a local lift after another restriction, so surjectivity on stalks is equivalent to local surjectivity.

For clarity, kernels are computed on sections. For a cokernel take the quotient presheaf and sheafify. For any abelian presheaf, form a new presheaf from matching families on covers, identifying two families when their restrictions agree on a common refinement. This is separated because local agreements can be refined on each member and the resulting covers composed. After repeating the construction it is a sheaf. Indeed matching families of sections of the separated presheaf glue by choosing representatives on covers of each member. Their restrictions agree on overlaps by separatedness and the local agreements, so their union is a matching family on the composite cover. Compatible maps to any sheaf extend uniquely by gluing at both steps. Neither step changes the stalks: a matching family has a representative on a member through each lift, and equality is detected after refinement near that lift. Taking stalks commutes with kernels and ordinary quotients on groups since a relation among finitely many germs holds on one common pointed neighborhood. This proves the exactness tests, including the identification of the stalk of $\underline{\Lambda}_X$ with Λ.

A matching family of elements of $G_X(F)$ gives at each $u \in P(U)$ a value independent of a chosen lift to a cover member. Any two lifts pair to a point of their fiber product, where the prescribed values agree. Conversely these values give the unique glued element. The preceding germ test proves injectivity of $F \to G_X(F)$. Products of exact sequences of abelian groups are exact, by choosing lifts in each factor, proving the assertion about G_X.

Take stalks of a short exact sequence of sheaves and of its image under G_X. Both sequences are exact and the vertical germ maps into G_X are injective. Their quotients form a short exact sequence. To check the middle, lift a representative whose image belongs to the last germ subgroup, subtract a lift in the middle germ subgroup, then lift the difference in the first product group. For injectivity the same diagram identifies the intersection of the first product group with the middle germ subgroup as the first germ subgroup. Surjectivity follows from surjectivity of the last product map. The stalk test gives exactness of Q_X. □

Definition 330.3. For $b \geq 0$ write

$$C^b_{\mathrm{et}}(X, F) = G_X Q_X^b F(X), \qquad d^b : \ G_X Q_X^b F \longrightarrow Q_X^{b+1} F \longrightarrow G_X Q_X^{b+1} F.$$

The first arrow is the quotient and the second is the germ inclusion. The complex of sheaves begins with $F \to G_X F$. It is exact there and in every subsequent degree, by the quotient definition and injectivity in Lemma 330.2. Define $H^b_{\mathrm{et}}(X, F)$ by the cohomology on global sections of this resolution, called the Godement resolution. For constant coefficients abbreviate $H^b_{\mathrm{et}}(X, \underline{\Lambda}_X)$ to $H^b_{\mathrm{et}}(X, \Lambda)$.

Proposition 330.4. *For a map $f : Y \to X$ and an abelian sheaf B on Y put $(f_* B)(U) = B(Y \times_X U)$. A sheaf map $F \to f_* B$ induces a cochain map $C^\bullet_{\mathrm{et}}(X, F) \to C^\bullet_{\mathrm{et}}(Y, B)$. These*

maps preserve identity and composition. In particular each f induces functorial pullback on the cochains with constant coefficients Λ.

Proof. Base change takes covers and their overlaps to covers and overlaps, so f_*B is a sheaf. A representative on a pointed neighborhood of $x = f(y)$ maps to a section on its base change with the indicated lift of y. Passing to germs gives $F_x \to B_y$. At $v \in P(Y \times_X U)$ use the factor of $G_X F(U)$ at its projection to U and apply this stalk map at its projection to Y. This defines $G_X F \to f_* G_Y B$, and on sections from F it commutes with the germ inclusions. Composing with the target quotient therefore factors through the sheaf quotient $Q_X F \to f_* Q_Y B$. Iterating the two constructions gives the maps at every degree and the quotient and germ squares give compatibility with d^b. For a composite the chosen projections compose, the stalk maps compose by restriction of representatives, and the maps on quotients compose by the uniqueness of factorization. This gives the stated identities on global cochains. For constants use the map carrying a locally constant value to that same value on the inverse image of its defining neighborhood. □

331. Restriction to a union of closed subschemes

Lemma 331.1. *Let $j : D \hookrightarrow W$ be a closed immersion and B an abelian sheaf on the small étale site of D. At a point $x \in P(W)$ the stalk of j_*B is zero if x lies outside D, and is B_x at the unique lift to D otherwise. In particular j_* is exact and there are natural identifications*

$$G_W j_* B = j_* G_D B, \qquad Q_W j_* B = j_* Q_D B.$$

Proof. Outside D a pointed neighborhood can be restricted to its complement. A sheaf of groups has just the zero section on the empty object by gluing on its empty cover. Inside D an arrow from the spectrum of a field whose point lies in D kills its defining ideal, giving the unique lift. Every pointed étale neighborhood on D can be refined, near this lift, by the restriction of one on W. Choose an affine open of W around the image, of ring A, with D cut out by I. Choose in the given neighborhood a Jacobian chart over A/I at the lift. A further principal restriction of the base may be lifted by lifting its defining element from A/I to A. Lift the coefficients of the chart equations and its inverted functions to A, and invert the lifted Jacobian determinant. Reduction modulo I then gives exactly the chosen restricted chart, including its lift from the field. The resulting scheme is étale over W by Definition 330.1.

Thus every germ on D is a germ of j_*B. For injectivity, start with a section on the restriction to D of a given pointed U over W. If its germ on D vanishes, it vanishes on some smaller étale neighborhood over $U \times_W D$. Apply the same chart lifting with U as base and $U \times_W D$ as closed subscheme. This gives a pointed restriction over U on whose closed part the section is zero, which proves injectivity. Two germs can be compared in one neighborhood by the product in Lemma 330.2.

The stalk description and that lemma give exactness. In $G_W(j_*B)(U)$ only the points of $P(U)$ factoring through D can have nonzero factors. They identify with $P(U \times_W D)$, with the stalks just computed. This identifies the product on every U with $(j_*G_D B)(U)$ and respects the inclusion of germs. Taking sheaf quotients, using exactness of j_*, gives the formula for Q. □

Proposition 331.2. *Let $j_a : D_a \hookrightarrow W$ be a finite set of closed immersions whose underlying subsets cover W. Let Λ be an abelian group. For each $b \geq 0$ the combined restriction*

$$C^b_{\mathrm{et}}(W, \underline{\Lambda}_W) \longrightarrow \prod_a C^b_{\mathrm{et}}(D_a, \underline{\Lambda}_{D_a})$$

is injective, with the restrictions of Proposition 330.4. In particular suppose $f_i : W \to Z$ is a finite list of morphisms, $m_i \in \mathbb{Z}$, and for each a

$$\sum_i m_i [f_i j_a] = 0 \quad \text{in } \mathbb{Z}[\mathrm{Hom}(D_a, Z)].$$

Here brackets denote the basis of the free abelian group. Then $\sum_i m_i f_i^* = 0$ as cochain maps on constant coefficients and on every $H^b_{\mathrm{et}}(Z, \Lambda)$ with target $H^b_{\mathrm{et}}(W, \Lambda)$.

Proof. The restriction of the constant sheaf on W to the closed pieces gives the sheaf map

$$\underline{\Lambda}_W \longrightarrow \prod_a (j_a)_* \underline{\Lambda}_{D_a}.$$

At a chosen geometric point its stalk has at least one component equal to the identity on Λ, by Lemma 331.1. Finite products commute with stalks by common pointed refinement. Thus the map is injective. Apply the exact functor Q^b_W of Lemma 330.2. This functor preserves finite direct sums since taking germs, their products, and the sheaf quotients are additive. By Lemma 331.1, its target identifies with $\prod_a (j_a)_* Q^b_{D_a} \underline{\Lambda}_{D_a}$. Applying G_W and taking sections on W preserves injectivity by its point product formula. The same identifications give exactly the displayed target. At a closed piece the map is the restriction defined in Proposition 330.4, because the stalk maps and their iterated quotients were induced by that restriction.

Compose the cochain sum with each j_a^*. By strict functoriality the resulting sum is $\sum_i m_i (f_i j_a)^*$. Grouping identical morphisms gives the zero map by the hypothesis in the free group. The injectivity just proved forces the original cochain sum to be zero in each degree. Passing to cohomology proves the final assertion. □

Theorem 331.3. *At every degree-one prime and level covered by Proposition 303.1, use its $k, Z, \mathcal{W}_k, \Phi$ and maps u_A. For every abelian group Λ and every $b \geq 0$,*

$$\sum_{A \subseteq \mathbf{F}_p^n} (-1)^{\dim A} p^{\binom{\dim A}{2}} u_A^* (\Phi^*)^{n - \dim A} = 0 \quad \text{as maps} \quad H^b_{\mathrm{et}}(Z, \Lambda) \longrightarrow H^b_{\mathrm{et}}(\mathcal{W}_k, \Lambda).$$

The sum is over linear subspaces, including zero and the whole space.

Proof. Use the reduced irreducible components of $\mathcal{W}_k$ as the closed pieces. They are finite in number and cover its underlying space by Corollary 303.3. On each such D, that corollary gives cancellation in the free group on morphisms from D to $Z \times_k Z$. Composing its basis morphisms with the second projection sends this zero sum to the required one on the maps $\Phi^{n - \dim A} u_A|_D$. Proposition 331.2 and the order of pullbacks in Proposition 330.4 give the result on $\mathcal{W}_k$. □

332. Representatives on finite systems of étale neighborhoods

Lemma 332.1. *Let W be a scheme of finite type over a field. From a jointly surjective family of étale maps to W one may select finitely many affine chart neighborhoods in the members which still cover W. More generally a cover of the underlying space by images of finite-type affine charts over affine opens of W has a finite subcover.*

Proof. Use noetherian induction on closed subsets of W. Such induction is valid by finite generation of polynomial ideals and the descending chain condition on closed subsets as in Proposition 291.2. It suffices to treat an irreducible reduced closed subset D, since there are finitely many irreducible components. Choose a chart whose image contains the generic point of D, and work over an affine open in D, of ring A, containing that point. After base change the chart has a nonempty affine part with a nonempty fiber over $K = \mathrm{Frac}(A)$. Its nonzero coordinate algebra remains nonzero after tensoring with $\bar{K}$, as is seen on a vector space basis. It has a $\bar{K}$-point by the affine point argument of Lemma 280.1. The finitely many coordinates of this point belong to a finite extension of K. The images of the finitely many affine generators, including any inverted coordinates, are algebraic over K. Clear the denominators in their monic equations by localizing A at one nonzero element. They generate a finite algebra B over this localized ring inside the field extension, because bounded powers of integral generators span their products. The map from the chart algebra to B gives a map back to the chart on spectra.

At every prime of the localized A, the localization of B is nonzero since it contains the local base. Its residue algebra is nonzero since it is finite: if its maximal base ideal generated the module, a matrix of coefficients in that ideal would express a finite generating list in terms of itself. Its identity minus that matrix has unit determinant and would kill the list. A maximal ideal in the nonzero residue algebra gives a point above the chosen prime. The selected chart therefore covers a nonempty open in D. The complement in D is a proper closed subset and the induction finishes. Affine étale charts over affine base opens give the indicated finite-type family in the first assertion by Definition 330.1. □

Definition 332.2. An augmented system over a separated finite-type X over a field has $U_{-1} = X$ and schemes U_i étale over X for $i \geq 0$, with face maps ∂_j for $0 \leq j \leq i$ from U_i to U_{i-1}. They satisfy $\partial_j \partial_l = \partial_{l-1} \partial_j$ for $j < l$ whenever the composites are defined. Put $M_0 = X$. For $i \geq 1$, the matching object M_i is the subscheme of U_{i-1}^{i+1} over X whose tuple $(a_0, \ldots, a_i)$ satisfies $\partial_j a_l = \partial_{l-1} a_j$ for $j < l$. We use systems with $U_i \to M_i$ jointly covering in each degree, allowing U_i to be a disjoint union of finitely many étale charts. A finite prescribed part can be continued by $U_i = M_i$ in higher degrees. If the earlier objects are separated of finite type and étale over X, the same holds for M_i. Indeed each equality in its definition is a pullback of the diagonal of an earlier étale object over X. That diagonal is closed by separatedness and open by Definition 330.1. A cover of M_i by affine étale charts therefore continues the system with the same properties.

Proposition 332.3. *For X as in Definition 332.2, an abelian group Λ, $n \geq 0$, and $y \in H^n_{\mathrm{et}}(X, \Lambda)$, one may choose such a system through degree $n+1$ and a section $\beta \in \underline{\Lambda}_X(U_n)$, constant on each of finitely many disjoint chart members, with $\sum_j (-1)^j \partial_j^* \beta = 0$ on U_{n+1}. Its class maps naturally to y. Pullback of the charts and section represents pullback of y, and a map between augmented systems over X preserves the class so represented.*

Proof. Put $\mathcal{G}^b = G_X Q_X^b \underline{\Lambda}_X$ from Definition 330.3. For a chosen system use the arrays $C^{a,b} = \mathcal{G}^b(U_a)$, $a, b \geq 0$. The total differential is $\delta + (-1)^a d$, where $\delta = \sum_j (-1)^j \partial_j^*$ increases a; the face identities give $\delta^2 = 0$. Restriction into $a = 0$ induces an isomorphism from the cohomology on X to that of this total complex. To prove it, augment the array at $a = -1$ by $\mathcal{G}^b(X)$. Each horizontal row is exact. Indeed, by the point product formula in Lemma 330.2 it splits into products indexed by $x \in P(X)$. For each x, the terms are functions on the sets of lifts of x to U_a, valued in the single group $(Q_X^b \underline{\Lambda}_X)_x$. Every matching tuple of lifts has a filler by the covering property. Choose a lift at degree zero. For a simplex t of degree a choose $H(t)$ of degree $a+1$ with zeroth face t and j-th face $H(\partial_{j-1} t)$ for $j > 0$. Start at degree -1 with the chosen lift. Induction using the face identities makes these a matching tuple. On the free groups on these sets the alternating boundary satisfies $\partial H(t) = t - H(\partial t)$. Evaluation gives a contraction of the augmented row of functions. The augmented total is consequently exact by the row filtration argument of Proposition 322.2, which uses only finitely many bidegrees in any fixed total degree. This proves the comparison. The section of $\underline{\Lambda}_X$ in degree n is sent to $C^{n,0}$ by germs. A face cocycle there defines a class by inverting this comparison.

To obtain a representative, start with a global closed $c \in \mathcal{G}^n(X)$ representing y. For $n > 0$ it lifts locally under $d : \mathcal{G}^{n-1} \to \mathcal{G}^n$ by exactness of the sheaf resolution. Choose a cover $U_0 \to X$ by finitely many such charts using Lemma 332.1. Subtract from the cocycle the total coboundary of its chosen primitive there. Only the term in bidegree $(1, n-1)$ remains, and it is the signed face sum of the primitives, already defined on M_1. In general, in bidegree $(a, n-a)$ the remaining section on M_a is vertically closed. For $a = 0$ this is the original condition and thereafter it follows from commutation with face restriction and cancellation of the two alternating faces on matching tuples. When $a < n$, lift locally under $(-1)^a d$, again choosing a finite cover of M_a, and subtract the resulting total coboundary. Its face sum gives the sole remaining section, now defined on M_{a+1}. These subtractions take place in the eventual total complex since they use only already chosen faces. When $a = n$,

the vertical kernel is $\underline{\Lambda}_X$. Choose $U_n \to M_n$ to be a finite chart cover on each member of which that section has a constant value. This also applies at $n = 0$ directly to c in the vertical kernel. The final section has alternating face sum zero on M_{n+1} by the matching identities (at $n = 0$ it comes from X). Take $U_{n+1} = M_{n+1}$. The subtractions prove that β represents y.

Under changing a system by a map of augmented systems over X, restriction on each array commutes with the augmentation. Under $f : Y \to X$ use the cochain maps of Proposition 330.4 on the arrays, defined there on every U_a by passage to $Y \times_X U_a$. They respect the augmentations, faces, and inclusions of locally constant sections. The isomorphism on cohomology just proved therefore gives both naturality assertions. □

333. The action of geometric Frobenius on the special fiber

Lemma 333.1. *Let X be a scheme in characteristic $p > 0$, with p prime, and denote the absolute Frobenius by F_X, the endomorphism given on functions by $a \mapsto a^p$. For every étale $U \to X$ the canonical map*

$$U \longrightarrow X \times_{F_X, X} U$$

over X, whose second projection is F_U, is an isomorphism.

Proof. Use affine charts $A \to B$ of Definition 330.1. Write the chart with variables $t_1, \ldots, t_m$, equations $f_1, \ldots, f_m$, and inverted function g including the Jacobian determinant in the localization. For $B' = A \otimes_{A, F_A} B$ we use the convention that scalars of B move to A by F_A. It has variables t'_i, equations $f_j^{(p)}(t')$ and inverted $g^{(p)}(t')$, where the superscript raises only the coefficients to p-th powers. The ring map in question is $B' \to B$, identity on the left A, $t'_i \mapsto t_i^p$.

Form the auxiliary algebra

$$S = B'[v_1, \ldots, v_m]/(v_i^p - t'_i : 1 \le i \le m).$$

Successive division by the monic equations gives a free B'-basis v^a with $0 \le a_i < p$. The ideal $J = (f_j(v) : 1 \le j \le m)$ satisfies $z^p = 0$ for all $z \in J$, since $f_j(v)^p = f_j^{(p)}(t') = 0$. It is nilpotent, as every product of more than $m(p-1)$ of its generators repeats one at least p times. Here $g(v)^p = g^{(p)}(t')$ is a unit. Eliminating t' identifies S/J with B, by $v_i \mapsto t_i$.

The identity of $B = S/J$ lifts to an A-map $s : B \to S$. Start with any lifts of the t_i. At each successive quotient J^a/J^{a+1}, $a \ge 1$, the errors in the equations can be corrected by multiplying their negatives by the inverse Jacobian in that square-zero layer. The Jacobian and the inverted coordinates are units since they reduce to units modulo the nilpotent ideal. The iteration terminates when the ideal power is zero. This is the coordinate correction used in Lemma 316.1.

In S each derivation $\partial/\partial v_i$ fixing B' is well defined, since the derivatives of the power relations vanish. Applied to the equations for $s(t_j)$, it gives the invertible Jacobian times their derivatives equal to zero. The coefficients from A are fixed. Hence all $s(t_j)$ are in the common kernel of these derivations. On the free basis v^a that kernel consists of B', since $1, \ldots, p-1$ are units and distinct terms of positive v_i-degree have distinct derivatives in coordinate i. The inverses of $s(g)$ are also killed by the derivations, so s lands in B'. In S, the difference $s(t_i) - v_i$ lies in J, hence $s(t_i)^p = v_i^p = t'_i$. The map $s : B \to B'$ is therefore inverse on both sides to the proposed map. The affine identifications agree on overlaps since they invert the canonical map. Indeed the preimage of each chart under F_U is the same open, as a^p lies in a prime exactly when a does. Thus the local inverses cover both sides. □

Proposition 333.2. *Let X be separated of finite type over a field of characteristic $p > 0$. The action F_X^* is the identity on $H^b_{\mathrm{et}}(X, \Lambda)$ for every b and every abelian group Λ. If $X = X_0 \times_{\mathbf{F}_p} \overline{\mathbf{F}}_p$, write Φ for the base change of F_{X_0}, and s_φ for the endomorphism given on the scalar field by $\varphi(a) = a^p$ and on X_0 by the identity. Then $\Phi^* = (s_{\varphi^{-1}})^*$.*

Proof. Represent a class by $U_\bullet, \beta$ as in Proposition 332.3. Pullback by F_X uses the base changes of that system. Lemma 333.1 identifies the original system with this pullback via a map over X commuting with all faces. The composite on each U_i with the projection back to U_i is F_{U_i}, which preserves its underlying points and each of its disjoint chart members. The constants of β on those members are unchanged. The naturality in the representing proposition now proves that F_X^* fixes the class.

On each extended affine chart $A_0 \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$, the two commuting ring maps giving Φ and s_φ apply the p-th power on the first and second factors respectively. Their composite is absolute Frobenius. Consequently $\Phi^*(s_\varphi)^* = 1$ by the first assertion and Proposition 330.4. The inverse to $(s_\varphi)^*$ is $(s_{\varphi^{-1}})^*$, proving the formula. □

Corollary 333.3. *In Theorem 331.3 take $k = \overline{\mathbf{F}}_p$ and let $r_{\mathrm{sp}}(\gamma) = (s_{\varphi^{-1}})^*$ act on the special target at a geometric Frobenius datum (w, h, γ) of Definition 221.1. Then for all b and Λ*

$$\sum_{A \subseteq \mathbf{F}_p^n} (-1)^{\dim A} p^{\binom{\dim A}{2}}\, u_A^*\, r_{\mathrm{sp}}(\gamma)^{\,n-\dim A} = 0 \quad \text{on } H^b_{\mathrm{et}}(Z, \Lambda),$$

with values on the special cover $\mathcal{W}_k$.

Proof. The residue field action of γ^{-1} is φ by Definition 221.1. The coarse special target is defined over $\mathbf{F}_p$ and its Φ is the base change in Proposition 303.1. Substitute Proposition 333.2 in Theorem 331.3. □

334. Continuity and integral coefficients

Proposition 334.1. *Let E be a number field, X a separated scheme of finite type over E, and $\bar{X} = X_{\bar{E}}$ for the algebraic closure in Definition 221.1. For $g \in G_E$ write $s_g : \bar{X} \to \bar{X}$ for the map on affine charts induced by $a \otimes t \mapsto a \otimes g(t)$. For each abelian group Λ and $b \geq 0$,*

$$r_b(g) = s_g^* : H^b_{\mathrm{et}}(\bar{X}, \Lambda) \longrightarrow H^b_{\mathrm{et}}(\bar{X}, \Lambda)$$

defines an action on the discrete group in which each element has open stabilizer. Morphisms over E and homomorphisms of constant coefficient groups induce equivariant maps.

Proof. For composition $gh = g \circ h$ one has $s_{gh} = s_h \circ s_g$ on schemes. Pullback reverses this order by Proposition 330.4, giving the representation property and inverses. Equivariance follows from the commuting squares for a morphism over E and from the functorial stalk and quotient arrows for a coefficient map.

Choose a representative system and its constants for one class as in Proposition 332.3. Only degrees through $b+1$ are needed. The charts and all their face maps and identities are described by finitely many coefficients in $\bar{E}$. To see this on each member, use finite affine presentations for the schemes and refine to affine opens on which a face map lands in a target chart. All overlaps have finite affine covers since the schemes are separated of finite type over a field. The transition maps and the mutual inverse and composite identities can be written with finitely many polynomials, inverted denominators and ideal-membership identities, as in the chart proof of Proposition 270.2. This applies also to the matching objects since they are formed by finitely many products and equality conditions in Definition 332.2. Adjoin all these coefficients to E and take a finite normal extension containing them, using the power-basis argument of Lemma 16.2. Each automorphism fixing it preserves the chart members, all their maps, and the individual constants assigned to them. Equivalently its pullback system is identified with the original by the isomorphisms on these presentations, with the same cocycle. Naturality in the representing proposition fixes the original class. The subgroup so obtained is open by Definition 221.1. □

Definition 334.2. For a scheme Y and a prime ℓ, put

$$L^b_\ell(Y) = \varprojlim_{a \geq 1} H^b_{\mathrm{et}}(Y, \mathbb{Z}/\ell^a\mathbb{Z}), \qquad V^b_\ell(Y) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} L^b_\ell(Y).$$

The transition maps are coefficient reduction. Use the inverse limit topology of the discrete groups on the left. Compatible residues give its $\mathbb{Z}_\ell$-module structure by scalar multiplication in each coordinate.

Proposition 334.3. *If $m = \dim_{\mathbf{F}_\ell} H^b_{\mathrm{et}}(Y, \mathbf{F}_\ell)$ is finite, then $L^b_\ell(Y)$ is generated by at most m elements over $\mathbb{Z}_\ell$. Its kernel on passage to $H^b_{\mathrm{et}}(Y, \mathbb{Z}/\ell^a\mathbb{Z})$ is $\ell^a L^b_\ell(Y)$ for every $a \geq 1$. In particular $\dim_{\mathbb{Q}_\ell} V^b_\ell(Y) \leq m$. For $Y = \bar{X}$ of Proposition 334.1 the action on $L^b_\ell(Y)$ is continuous with its inverse limit topology, and under this finiteness hypothesis the action on $V^b_\ell(Y)$ is continuous in matrix entries.*

Proof. Write $H_a = H^b_{\mathrm{et}}(Y, \mathbb{Z}/\ell^a\mathbb{Z})$ and $L = L^b_\ell(Y)$. For positive a, t, the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z}/\ell^t\mathbb{Z} \xrightarrow{\ell^a} \mathbb{Z}/\ell^{a+t}\mathbb{Z} \longrightarrow \mathbb{Z}/\ell^a\mathbb{Z} \longrightarrow 0$$

is exact on sheaf stalks, hence on sheaves. Each functor $G_Y Q^j_Y$ takes it to a sequence exact on sections, by Lemma 330.2. Consequently $H_t \to H_{a+t} \to H_a$ is exact in the middle. Indeed, for a cocycle mapping to a coboundary, lift the primitive to the middle complex and subtract its differential. The remaining representative lifts to the first complex and is closed there by injectivity. In degree zero no subtraction is needed. Taking $a = 1$ and inducting gives $|H_t| \leq |H_1|^t < \infty$.

Suppose $x = (x_j) \in L$ has $x_a = 0$. For each t the set of $y_t \in H_t$ mapping to x_{a+t} under the displayed inclusion is nonempty by exactness and is finite. Reduction sends each such set into the preceding one. There is a compatible choice at every stage: among finitely many elements at the first stage some admits descendants arbitrarily far up, and among its finitely many immediate descendants one has the same property. Inducting gives $y \in L$. Reducing the equality at $a + t$ further to t gives $\ell^a y_t = x_t$. Thus $x = \ell^a y$; the converse follows on reducing to a.

It follows that $L/\ell L$ embeds in H_1. Choose $r \leq m$ lifts of a basis of its image. Expand any element x modulo ℓ in these lifts, using digits between 0 and $\ell - 1$. The residual difference is ℓ times an element of L. Repeat on that element. The successive digits for each basis coefficient give a member of $\mathbb{Z}_\ell$, by the convergent digit expansions in Lemma 16.2. Their linear combination equals x in each coordinate and hence in L. This proves finite generation and the rational dimension bound.

For $Y = \bar{X}$, continuity of the action on each discrete factor follows from Proposition 334.1. At (g, x), holding the coordinate x_a fixed and restricting g to its coset of the stabilizer of x_a fixes the result at that coordinate. This gives joint continuity on the inverse limit. In $V^b_\ell(Y)$ choose a vector space basis from the images of a finite generating set of L. In these coordinates the image of L lies in $\ell^{-c}\mathbb{Z}^r_\ell$ for some $c \geq 0$, using the actual basis size in place of r if necessary. If g fixes the reductions of the chosen lifts of the basis modulo level a, its matrix differs from the identity by entries in $\ell^{a-c}\mathbb{Z}_\ell$, by the kernel assertion. The stabilizers form an open finite intersection. Letting a increase proves continuity at the identity and hence everywhere by translation. □

335. The Hecke action on geometric étale cohomology

Proposition 335.1. *Let $\pi : Y \to X$ be finite étale and surjective with an action over X of a finite group D such that $\coprod_{g \in D} Y \to Y \times_X Y$, $y \mapsto (y, gy)$ on the g-th copy, is an isomorphism. For every abelian group Λ the constant-coefficient complexes of Definition 330.3 have a trace from Y to X such that*

$$\mathrm{Tr}_\pi \pi^* = |D| I, \qquad \pi^* \mathrm{Tr}_\pi = \sum_{g \in D} g^*.$$

It commutes with homomorphisms of coefficients. Consequently the pullback on V^b_ℓ of Definition 334.2 is injective and identifies the space on X with the D-invariant space on Y.

Proof. Over the étale cover Y of X, π is the disjoint union of $|D|$ copies of the identity by the product isomorphism. For a sheaf B on Y one therefore has

$$(\pi_* B)_x = \prod_{v|x} B_v \qquad (x \in P(X)).$$

Indeed choose a lift of x to Y. The pointed neighborhoods lying over that lift are cofinal by taking fiber products. On such neighborhoods the direct image is a finite product over the sheets and the stalk is the product of their stalks by common refinement. A pointed neighborhood on one sheet is also étale over X, and two morphisms of pointed neighborhoods agreeing at the lift agree there after restriction by the diagonal calculation in Definition 330.1. This identifies each stalk with the stated B_v. In particular π_* is exact and $G_X \pi_* B = \pi_* G_Y B$ by taking the product over the lifts on each object. Taking sheaf quotients gives $Q_X \pi_* B = \pi_* Q_Y B$ by exactness and Lemma 330.2.

On a splitting neighborhood sum the entries of $\pi_* \underline{\Lambda}_Y$ over its sheets. Changes of splitting permute the sheets, so the sums glue to a sheaf map $\pi_* \underline{\Lambda}_Y \to \underline{\Lambda}_X$. Apply the iterated germ and quotient functors and use the identifications just proved to get a map from the cochains on Y to those on X, commuting with the differentials by their construction. Restriction of any germ resolution to an étale object is its resolution there: stalks are the same by the pointed-neighborhood argument above, the G formula on each further étale object is then identical, and taking Q commutes with restriction by the stalk test. Hence on a split neighborhood this cochain map is ordinary summation and pullback is the diagonal on the sheets. Summation after the diagonal multiplies by $|D|$, and the diagonal after summation adds all sheet permutations from this regular group action. These are identities on the complexes of sheaves, since they hold on a cover, so on global cochains as well. The same sum commutes with all changes of coefficients.

Apply cohomology, take compatible inverse limits, and extend scalars to $\mathbb{Q}_\ell$. Division by $|D|$ is allowed even when ℓ divides $|D|$. The first identity gives injectivity and the second expresses any invariant as the pullback of its normalized trace. Every pullback is invariant since $\pi g = \pi$. □

Theorem 335.2. *For the marked power X_E and its neat level in Theorem 329.2 put $V^b = V_\ell^b((X_E)_{\bar{E}})$, $b \geq 0$. Define the rational positive Hecke algebra by the coset sums and multiplication of Definition 245.1 with rational coefficients. It acts on V^b by operators T_H^{et} commuting with G_E. On each sufficiently fine common generic level,*

$$\pi_m^* T_H^{\mathrm{et}} = \sum_{gC} H(gC)\, u_{g,\mathbf{1}}^*.$$

The action is unital and multiplicative and extends by scalars to $\mathbb{Q}_\ell$. The safe operators $T_{w,r}^{\mathrm{et}} = T_{\tau_{w,r}}^{\mathrm{et}}$ commute.

Proof. The maps and deck actions are defined over E by Theorem 302.1; the product isomorphism for their generic torsors is the one verified in Proposition 319.2. The finite coset sum of pullbacks is invariant on its target under C/C_m, because a deck translate permutes gC by left multiplication and preserves H, by the frame identity in Theorem 320.1. Proposition 335.1 thus gives a unique descended operator. Injectivity on common refinements makes it independent of the cover. The map for the unit coset is π_m, so the unit acts by identity. Every pullback is equivariant for the G_E-actions of Proposition 334.1 on each coefficient group and thus on the rational limit. Equivariance of the descended operator follows by injectivity of π_m^*.

For two coset sums choose successively deep levels on which the first set of representatives maps to a cover carrying all the second maps. Their composite maps have frame labels gst when the starting label is g and the successive right representatives are s, t, by the algebraic map identity proved in Theorem 320.1. Thus after pullback the composition of the two operators is the double sum of the maps for stC, each with its counted product coefficient.

Functoriality is Proposition 330.4 on finite coefficients, followed by passage to V^b. Cancel the injective pullback to obtain multiplication. Linearity follows from the defining sum and its unique descent. The relations of Lemma 329.1 are identities of the rational coset counts, hence hold on V^b. □

336. The congruence on functions on the components

Lemma 336.1. *Let X be smooth, separated and of finite type over an algebraically closed field k. Its connected components are finitely many open and closed irreducible subschemes. For any abelian group Λ, $H^0_{\mathrm{et}}(X,\Lambda)$ is the group of Λ-valued functions on these components. If X is the geometric generic fiber of a fixed projective model over a localization of the integers of a number field E, then the partition into components extends over a finite normal extension B/E and a further localization $O_B[1/N]$. It may be chosen with the action of $\mathrm{Gal}(B/E)$ permuting the extended subsets in the same way as the original components.*

Proof. The local calculation in Lemma 294.1 gives a local domain at each closed point and shows that the finitely many irreducible components are disjoint and reduced. Its proof up through disjointness uses only the smooth affine charts. Each component is therefore open and connected. By Definition 330.3 the degree-zero kernel is the group of global sections of the constant sheaf. On each component such a section is constant. Indeed take an étale cover of this component by charts with constant values. Each nonempty chart maps dominantly on at least one irreducible piece of its source to the base component. To verify this, take a k-point in the chart using Lemma 280.1. Its completed local ring is isomorphic to that at its image: the extra variables in the étale presentation are solved, uniquely to each order, in the same free parameters by the Jacobian computation in Lemma 294.1. Both local rings inject into their completions by that lemma's local argument. On the irreducible chart piece through the point the map from the local domain on the base is thus injective, so its generic point maps to the generic point of the base component. Any two nonempty chart members now have nonempty fiber product over the generic point, since the tensor product of two nonzero algebras over its field is nonzero by a vector space basis. The matching condition forces their constants to agree. Conversely an assignment on the open components is a section by gluing. This proves the description.

For the extension, use a finite affine cover of the fixed model. On each extended generic affine chart the component subsets are defined by idempotents, with product zero for distinct subsets and sum one. Indeed on disjoint complementary opens the pair of regular functions one and zero glues to the required idempotent. Express these finitely many idempotents and their overlap identities in the coordinate presentations of the model. Adjoin their coefficients to E and take a finite normal extension B containing them. The finitely many identities for their products, sums, overlaps, and conjugates by $\mathrm{Gal}(B/E)$ persist after inverting one positive integer N: clear the denominators of coordinates and of the certificates of equality in the localized quotient algebras, as in Proposition 270.2. Alternatively, each difference already over the number ring and zero after extension to its fraction field is killed by one nonzero integer after clearing the norm of the denominator. The idempotents partition the model after this single localization. The conjugate identities give the asserted action. □

Theorem 336.2. *Fix a marked coarse target X_E in Theorem 335.2. On the finite-dimensional representation $V^0 = V^0_\ell((X_E)_{\bar E})$, for every geometric Frobenius datum (w,h,γ) at a safe degree-one prime of norm p outside a fixed finite set one has*

$$\sum_{r=0}^{n} (-1)^r p^{\binom{r}{2}}\, T^{\mathrm{et}}_{w,r}\, r_0(\gamma)^{\,n-r} = 0.$$

Here r_0 denotes the action of Proposition 334.1 in degree zero. The exclusions may depend on X_E and are independent of the fine covers chosen to represent these operators.

Proof. Take the coarse model and the fixed exclusions of Theorem 301.2. Add the primes for the component partition of Lemma 336.1 of that model. For a remaining w let $A = (O_{\bar{E}})_{\ker h}$. This is a valuation ring of $\bar{E}$ stabilized by γ, by Lemma 310.1. Its residue field identifies via h with $\overline{\mathbf{F}}_p$. Indeed its image is algebraic over $\mathbf{F}_p$; it is algebraically closed because a monic polynomial over the residue field lifts to a monic polynomial over A, and all roots in $\bar{E}$ are integral in A. A root of negative valuation would make its highest power the unique term of least valuation. Reduction thus splits the original polynomial. On this residue field γ acts by inverse p-power Frobenius. The coarse model over A has the open and closed component labels extended by the lemma, since it is the base change of the same model from the localized number ring.

Take any $a \geq 1$, $\Lambda = \mathbb{Z}/\ell^a\mathbb{Z}$, and any degree-zero class on the geometric generic fiber. By the lemma it is a function of those labels. Its values extend to a section on the model over A by assigning the same values on its open and closed pieces. Choose a projective fine cover $\mathcal{W}$ and its integral diagram for this w by Theorem 302.1. On $\mathcal{W}_A$ pull back the extended function after the appropriate powers of the scalar automorphism s_γ of the coarse model over the underlying integer ring. Sum these functions with the coefficients of Corollary 333.3. The result is given on a finite partition into open and closed subsets, obtained by intersecting the partitions for the summands. Its restriction to the special fiber is zero by that corollary, including the indicated residue action of γ.

Every $\bar{E}$-point of the generic fine cover extends to an A-point. In its projective equations scale homogeneous coordinates by one of least valuation so that they are integral and one is a unit. All defining homogeneous relations still vanish in A. The spectrum of A is connected since it is a domain. An A-point therefore lies in just one piece of the finite partition. Its generic and special values agree, so its generic value is zero. All degree-zero classes on the smooth generic cover are detected on such points by the lemma. Consequently the congruence after fine-level pullback is zero at each finite coefficient level. In the integral inverse limit and after extension to $\mathbb{Q}_\ell$ it is still zero. By Lemma 269.1 summing the unscaled legs in dimension r gives the defining pullback of $T^{\mathrm{et}}_{w,r}$. Cancel the injective π^*_m by Proposition 335.1 to get the formula. Finally V^0 is a finite sum of copies of $\mathbb{Q}_\ell$ indexed by the geometric components, by the lemma at all coefficient levels. □

337. Restriction of a relation from a valuation model

Proposition 337.1. *Fix $b \geq 0$, a safe degree-one prime w of norm $p \neq \ell$ in Theorem 331.3, and a geometric datum (w, h, γ). Use $A = (O_{\bar{E}})_{\ker h}$ and the coarse model $\mathcal{Z}_A$ and projective fine model $\mathcal{W}_A$ of Theorem 302.1. For either model Y, let j_Y and i_Y be the inclusions of its geometric generic and special fibers. Write $V(Y) = V^b_\ell(Y)$. Suppose the following two relations hold for this diagram:*

$$j^*_{\mathcal{Z}} V(\mathcal{Z}_A) = V((\mathcal{Z}_A)_{\bar{E}}),$$

$$\ker\left(i^*_{\mathcal{W}} : V(\mathcal{W}_A) \to V((\mathcal{W}_A)_{\overline{\mathbf{F}}_p})\right) \subseteq \ker j^*_{\mathcal{W}}.$$

Then on $V((\mathcal{Z}_A)_{\bar{E}})$,

$$\sum_{r=0}^{n} (-1)^r p^{\binom{r}{2}} T^{\mathrm{et}}_{w,r}\, r_b(\gamma)^{n-r} = 0.$$

The same conclusion after pullback to the fine cover, with coefficients $\Lambda = \mathbb{Z}/\ell^a\mathbb{Z}$ for a single $a \geq 1$, holds if the two relations are satisfied with $H^b_{\mathrm{et}}(-, \Lambda)$ in place of V.

Proof. The proof of Theorem 336.2 identifies the residue field of A and the action of γ on it. Since all models and maps descend to the original localization of O_E, γ acts on the scalar ring of each model over A; write s_γ for the resulting scheme map on $\mathcal{Z}_A$. Restriction to either fiber commutes with its pullback, using the induced action on that fiber, by Proposition 330.4. Each integral u_A of the Hecke diagram likewise commutes with restriction. To avoid conflict with the valuation ring, index these maps here by subspaces $D \subseteq \mathbf{F}_p^n$ and write them as u_D.

Take a generic class $x = j_{\mathcal{Z}}^{*} y$, which is possible by the first relation. On the integral fine cover form the class

$$z = \sum_{D \subseteq \mathbf{F}_p^n} (-1)^{\dim D} p^{\binom{\dim D}{2}}\, u_D^{*}(s_\gamma^{*})^{n-\dim D} y.$$

Its restriction to the special fiber is zero by Corollary 333.3 on each finite coefficient group, also after inverse limits and rational extension by Definition 334.2. The second relation therefore gives $j_{\mathcal{W}}^{*} z = 0$. This is exactly the vanishing on the generic fine cover. For rational local coefficients, group the subspaces by dimension using Lemma 269.1. Theorem 335.2 identifies the result with the pullback of the claimed polynomial on x, and Proposition 335.1 makes that pullback injective. At a single finite coefficient level the argument up to $j_{\mathcal{W}}^{*} z = 0$ already proves its assertion. □

Corollary 337.2. *For fixed w, b in Proposition 337.1, suppose its two relations hold at an unbounded set of finite coefficient levels a, using one fine level carrying the specified correspondence. Then the rational generic operator polynomial in that proposition vanishes on the full $V_\ell^b((\mathcal{Z}_A)_{\bar{E}})$.*

Proof. Take an element of the integral inverse limit. At every specified a, the fine-level sum on its a-th coordinate vanishes by the finite-coefficient assertion. At any smaller level the same sum vanishes by coefficient reduction, since all its maps are pullbacks and its coefficients are integers. Unboundedness covers every coordinate. Thus the fine-level sum vanishes on the integral limit and, by linearity, on its rational scalar extension. Group terms and cancel π_m^{*} as in Proposition 337.1. □

338. Finite precision in the rank criterion

Theorem 338.1. *Take E, M, n, r, Q, κ with the diagonal span, local diagonal family, and n prescribed free weights of Theorem 232.3, over $k_0 = K_\ell \subset k = C_\ell$ as in Definition 233.1. Write $\|B\| = \max_{ij} |B_{ij}|_\ell$ in the given basis. Assign a monic polynomial $P_v \in K_\ell[X]$ of degree n at each degree-one prime outside a finite set. Suppose that for every real $\eta > 0$,*

$$\|P_v(r(\gamma))\| \le \eta \quad \text{for most Frobenius data } (v, h, \gamma)$$

in the sense of Definition 221.3. Then $M = n$ and r has polynomials P_v in the approximation sense of that definition.

Proof. Conjugating by the fixed Q multiplies the matrix norm bound by at most $\|Q\|\|Q^{-1}\|$, by the ultrametric inequality on a matrix product. Hence the hypothesis at all precisions applies to $s = QrQ^{-1}$. Fix a power minor Δ of order $n+1$ from Definition 231.2. Adding to its last column the corresponding linear combination of the preceding columns leaves its determinant unchanged. Monicity permits replacing that last column of $s(\gamma)^n$ by the same entries of $P_v(s(\gamma))$.

Suppose $\Delta(s(g)) \neq 0$ at some $g \in G_E$. On a nonempty open neighborhood of g, its absolute value stays at least $c > 0$, and the entries of the first n columns of powers are bounded by $B \ge 1$, by continuity. Take $\eta < c/B^n$ in the conjugated hypothesis and choose its open subset and finite exclusions there. Lemma 221.2 supplies a geometric Frobenius datum meeting them. Expanding the altered determinant, every product has absolute value at most $B^n \eta < c$. This contradiction proves that all these minors vanish everywhere on $s(G_E)$, with the conclusion vacuous if there are no such row selections.

The rank argument in Theorem 232.3, starting with that simultaneous vanishing, uses only the local diagonal family, freeness, and the diagonal span. Indeed its polynomial substitution, followed by Theorem 231.3, makes each weight-block compression scalar. The coordinate projections in the span then force one dimension per weight. It gives $M = n$. The same argument at order n in the proof of Theorem 232.3 shows that every nonempty open

set contains a point at which $I, s(g), \dots, s(g)^{n-1}$ are independent. These two conclusions precede the use of a polynomial in the final coefficient comparison of that proof.

For arbitrary nonempty open $V \subset G_E$ choose such a point in V and the n entries witnessing independence. On some nonempty open $U \subseteq V$ the resulting square submatrix of the power columns is invertible and every entry of its inverse has absolute value bounded by a constant $C \geq 1$, by continuity of its adjugate divided by its determinant. Write $D_\gamma(X) = \det(XI - s(\gamma))$. It annihilates $s(\gamma)$ by the adjugate argument in Lemma 231.1. The coefficient vector of $P_v - D_\gamma$, of length n in ascending powers, is sent by that submatrix to the selected entries of $P_v(s(\gamma))$. Given a desired error $\delta > 0$, use the conjugated hypothesis at δ/C inside U. Multiplying by the bounded inverse makes every coefficient error at most δ for these data. The leading coefficients already agree. Determinants are invariant under conjugation, giving exactly the prescribed quantifier for r. □

Corollary 338.2. *For an approximating form in Corollary 238.4 put $c_v = \iota(\omega(z_v))$ and $Q_v(X) = c_v^{-n} P_v(c_v X)$. Suppose a continuous finite-dimensional K_ℓ-representation V has a local frame containing its n prescribed weights free through $n(n+1)/2$. It suffices for attachment of this approximant that for every $\eta > 0$, $\|Q_v(r_V(\gamma))\| \leq \eta$ for most Frobenius data. If the condition holds for all forms in the list of Theorem 230.3, that theorem yields the representation for the initial eigensystem.*

Proof. Pass to the direct sum V_0 of distinct simple constituents as in Lemma 232.2. In a fixed basis adapted to the invariant filtration, the polynomial of each diagonal block is the diagonal block of the polynomial of the full matrix. Its norm is at most that of the full matrix in this basis. Changing to this basis costs at most a fixed product of two matrix norms, which is absorbed by decreasing the chosen positive tolerance. The local-frame argument of Theorem 238.3 retains all the required weights on V_0, and Theorem 234.4 gives its diagonal family. The cited constituent lemma gives the diagonal span. Theorem 338.1 therefore gives rank n and Frobenius polynomials Q_v in the approximation sense.

Multiply the action on V_0 by the continuous character a_ω of Theorem 311.2. Its Frobenius value is c_v outside a fixed finite set. For any chosen coefficient $j \leq n$, the error for the new characteristic polynomial is the error for Q_v multiplied by c_v^{n-j}, by scaling the determinant. Inside any starting open set shrink to a nonempty open on which $|a_\omega|$ is bounded, by continuity. Apply the approximation within it after dividing the desired tolerance by a bound at least one for $|a_\omega|^{n-j}$. This proves the required coefficient bound for P_v. Applying the attachment theorem to the list proves the last assertion. □

339. A norm identity for the quadratic character

Definition 339.1. Let K be the imaginary quadratic field of Definition 304.1 and let p split into $\mathfrak{u}, \bar{\mathfrak{u}}$ in K. Use the isomorphisms of both completions with $\mathbb{Q}_p$, induced by the respective extensions of the identity on $\mathbb{Q}$. For a fractional ideal $\mathfrak{a}$ put

$$N\mathfrak{a} = \prod_{\mathfrak{v}} |O_K/\mathfrak{v}|^{v_{\mathfrak{v}}(\mathfrak{a})} \in \mathbb{Q}_{>0}, \qquad \nu_p(x) = \frac{x_{\mathfrak{u}} x_{\bar{\mathfrak{u}}}}{N\mathfrak{a}_x} \in \mathbb{Q}_p^\times$$

for a finite idele x and its fractional ideal $\mathfrak{a}_x$ from that definition. Denote by $\chi_{p,K} : G_K \to \mathbb{Z}_p^\times$ the action on all p-power roots of unity in $\bar{K}$, taking compatible exponents as in Definition 233.1.

Theorem 339.2. *The character ν_p descends continuously to $\mathcal{C} = \mathbb{A}_{K,f}^\times / K^\times$. For the homomorphism r_K of Proposition 306.2 one has*

$$\nu_p(r_K(g)) = \chi_{p,K}(g) \qquad (g \in G_K).$$

Proof. The local exponents of $\mathfrak{a}_x$ are those of x, by the ideal computation in Proposition 304.3. Thus the norm factor is multiplicative and is locally constant on the finite ideles, being unchanged on a coset of $\widehat{O}^\times$. For $a \in K^\times$ the principal-ideal formula of Proposition 82.3

gives $N(aO) = a\bar{a}$, since the two embeddings are complex conjugates and $a\bar{a} > 0$. The product of the two p-adic embeddings is this same rational number. Hence principal ideles have value one, and continuity of the numerator gives continuity on the quotient by its quotient topology.

At a degree-one prime $\mathfrak{v}$ of rational norm P outside the fixed exceptions in Theorem 310.2, the ramified primes of K, and p, that theorem gives $r_K(\gamma) = [t_{\mathfrak{v}} b]$ with b a unit idele supported above P. Such degree-one primes split in K by the residue degree calculation of Lemma 219.3. The numerator of ν_p is one and the ideal is $\mathfrak{v}$. Its value is therefore P^{-1}.

The value of $\chi_{p,K}$ on the same geometric Frobenius is $P^{-1} \in \mathbb{Z}_p^{\times}$. Indeed the reduction of p^a-th roots of unity in residue characteristic P is injective for all a. For a root reducing to one, the geometric sum $1 + \zeta + \cdots + \zeta^{p^a - 1}$ reduces to the nonzero p^a; multiplying it by $\zeta - 1$ gives zero and forces $\zeta = 1$. Taking the ratio of two roots proves injectivity. By Definition 221.1 the residue of $\gamma^{-1}\zeta$ is that of ζ^P. Hence γ^{-1} has exponent P at every level and γ has exponent P^{-1}. The character $\chi_{p,K}$ is continuous because each finite root group is contained in a finite Galois extension. So both characters in the claimed identity are continuous into a Hausdorff group. Their equality set is closed and meets every nonempty open by Lemma 221.2 and the values just computed. It is the whole group. □

340. Inertia of the central twist

Lemma 340.1. *Let $p \nmid N_K$ split in K, choose the prime $\mathfrak{u}$ and the residue map h as in Theorem 310.2. For $s \in I_h$, write $r_K(s) = [b]$ with b a unit idele supported above p, uniquely as in that theorem. In the level-six normalized coordinates of Y_1, the action on every torsion point is*

$$sP_1(t) = P_1(bt).$$

For every $a \geq 1$, specialization at h is injective on the subgroup labelled by the $\bar{\mathfrak{u}}$-component of $\mathcal{T}_K[p^a]$. In particular $b_{\bar{\mathfrak{u}}} = 1$.

Proof. The unit b is one at the primes of six and $\mathfrak{a}_b = O$. Its period group is thus the same Y_1, and $P_b(t) = P_1(bt)$ by Proposition 304.3. In particular its normalizing labels of orders two and three coincide with the original ones. The normalized equations are identical by Lemma 305.1. For any given torsion point choose a level divisible by six and its order. The action on the finite rigid tuple in Proposition 306.2 identifies the conjugate point in these coordinates with $P_b(t)$. The origin is also preserved. This proves the first formula on the same cubic.

Choose $\alpha \in O$ congruent to zero modulo $\bar{\mathfrak{u}}^a$ and to one modulo $\mathfrak{u}$, using the coprime ideal decomposition in Proposition 82.3. All labels in the stated component are killed by α, since p is a uniformizer at both primes. On the model of the cubic over the valuation ring in Lemma 310.1, $[\alpha]$ extends by Lemma 307.1. Its derivative on the generic tangent line at the origin is the scalar α, by the complex uniformization in Proposition 304.3. The identity conormal line is free with coordinate $t_0 = X/Y$ by the proof of Lemma 307.2. The same scalar equality holds on this line over the valuation ring since it embeds in its fraction field. Consequently, near the closed identity, $[\alpha]^* t_0 = \alpha t_0 + t_0^2 H$ with H regular. If a point in the kernel reduces to the identity, its parameter lies in the maximal ideal and $\alpha + t_0 H$ evaluates to a unit. The identity $[\alpha]^* t_0 = 0$ at the point then forces $t_0 = 0$. The local cubic equation identifies the point with the origin, as in Lemma 307.2. Taking differences gives the required injectivity.

Every torsion point has algebraic homogeneous coordinates by Theorem 305.2. Coordinates may be scaled to be integral at h, one a unit. The inertia action fixes the residue of each such coordinate by Lemma 310.1. For labels in the indicated component, $P_1(bt)$ is in the same subgroup and has identical reduction to $P_1(t)$. Injectivity and the bijective labelling imply $bt = t$. These labels run through $p^{-a} O_{K_{\bar{\mathfrak{u}}}} / O_{K_{\bar{\mathfrak{u}}}}$ in that one component. Thus $b_{\bar{\mathfrak{u}}}$ is one modulo p^a for every a, giving equality. □

Theorem 340.2. *For $s \in I_h$ as in Lemma 340.1 let $e_{\mathfrak{u}}(z)$ denote the idele equal to $z \in \mathbb{Z}_p^{\times} = O_{K_{\mathfrak{u}}}^{\times}$ at $\mathfrak{u}$ and one elsewhere. Then*

$$r_K(s) = [e_{\mathfrak{u}}(\chi_{p,K}(s))].$$

For a character a_ω of Theorem 311.2 with coefficient prime $\ell \neq p$, one has for every $s \in I_h \cap G_E$

$$a_\omega(s) = \iota\big(\omega(e_{\mathfrak{u}}(\chi_{p,K}(s)))\big).$$

If no prime of E above $\bar{\mathfrak{u}}$ divides the defining level $\mathfrak{n}$ of the central test vector, this value is one for all such s.

Proof. The preceding lemma identifies the representative b with one outside $\mathfrak{u}$. Its ideal is the unit ideal. Definition 339.1 therefore gives $\nu_p([b]) = b_{\mathfrak{u}}$, and Theorem 339.2 identifies this coordinate with $\chi_{p,K}(s)$. In the formula for ω_ℓ in Proposition 311.1 the two ℓ-coordinates of b are one. The composition defining a_ω gives the second formula, without any exclusion for the conductor of ω at p.

For the last assertion, the scalar similitude $s(e_{\mathfrak{u}}(z)) = \overline{e_{\mathfrak{u}}(z)}I$ of Proposition 241.1 has unit entries on its diagonal and equals the identity except at the places over $\bar{\mathfrak{u}}$. Its multiplier is a rational profinite unit, since the two conjugate factors have product the norm at p and are one away from p. Under the level assumption this matrix is congruent to the identity modulo $\mathfrak{n}$ at every dividing prime and belongs to $K(\mathfrak{n})$ by Definition 239.1. It fixes the nonzero level vector with central eigenvalue $\omega(e_{\mathfrak{u}}(z))$, so that eigenvalue is one. □

341. A tame representative for the marked scalar

Proposition 341.1. *In Theorem 113.2 put $p = g_0$, $m = p^{r!} - 1$, and use its ξ, T. Its inertia element s may be chosen to have order prime to p on every finite Galois extension of E in $\bar{E}$, while preserving its asserted ratios for all divisors of m. For this choice put*

$$\alpha = T^{-1}(\exp(2\pi i/(p-1))) \in \mathbf{F}_p^{\times}.$$

For $p > 2$, the p-adic cyclotomic value of s is the unique $b \in \mathbb{Z}_p^{\times}$ with $b^{p-1} = 1$ and $\bar{b} = \alpha$.

Proof. Take an element s_0 and ξ supplied by Theorem 113.2. On a finite normal subextension where s_0 has order $N = p^a d$, $p \nmid d$, use the power s_0^k with $k \equiv 0 \bmod p^a$ and $k \equiv 1 \bmod d$. Existence and uniqueness modulo N follow from Lemma A.1. On any smaller normal subextension the order divides N, so these choices are compatible. They define one field automorphism of the union $\bar{E}$, since restrictions and their inverses agree and field operations occur in finite subextensions. Its order on the selected extension divides d. Preservation of the valuation and identity on residues hold element by element, since they hold for every power of s_0.

The m-th roots of unity are fixed by s_0: reduction on them is injective by the geometric-sum argument of Theorem 339.2, and inertia fixes their reductions. Thus with $\zeta = s_0(\xi)/\xi$ one has $s_0^k(\xi)/\xi = \zeta^k$. The root ζ has order m by the prescribed value under T. On a normal extension containing ξ, ζ that order divides N by the power formula, hence divides d. The chosen exponent preserves the ratio. Taking powers of ξ gives all the stated ratios also for s.

Let ϵ be a primitive p-th root of unity in $\bar{E}$ and normalize the valuation by $v(p) = 1$. It reduces to one since $X^p - 1 = (X-1)^p$ in the residue field. For $1 \leq j < p$ the quotient $(1-\epsilon^j)/(1-\epsilon)$ is a unit reducing to j. Evaluating $\prod_{j=1}^{p-1}(X - \epsilon^j) = (X^p - 1)/(X-1)$ at one gives $v(\epsilon - 1) = 1/(p-1)$. The element $t = \xi^{m/(p-1)}$ has the same valuation. Inertia fixes the residue of the unit $(\epsilon - 1)/t$. Consequently

$$\overline{s(t)/t} = \overline{(s\epsilon - 1)/(\epsilon - 1)} = \overline{\chi_{p,K}(s)},$$

where $K \subseteq E$ and we restrict the global cyclotomic character of Definition 339.1. The second equality is the preceding geometric sum for the exponent of $s\epsilon$ modulo p. The first

value is α by the ratio for $p-1$; it belongs to $\mathbf{F}_p^{\times}$ since its order divides $p-1$ and that group contains all the roots of $X^{p-1}-1$.

On the roots of order p^j, the order of the action of s is prime to p by its finite quotient property. The kernel of reduction of $(\mathbb{Z}/p^j\mathbb{Z})^{\times}$ onto $\mathbf{F}_p^{\times}$ has order p^{j-1}, by counting lifts of one. Thus the prime-to-p order in every such quotient divides $p-1$. It follows on taking compatible residues that $b=\chi_{p,K}(s)$ satisfies $b^{p-1}=1$. If two such values have the same reduction, their quotient is a $(p-1)$-st root reducing to one, which is one by the geometric-sum calculation. This proves uniqueness. □

Corollary 341.2. *Fix K and the marked $p=g_0$ of Proposition 341.1. At sufficiently large scales $p\nmid N_K$. For the chosen $\mathfrak{u}=w|_K$, any approximating central character with coefficient prime $\ell\neq p$ and the above s satisfies*

$$a_\omega(s)=\iota\big(\omega(e_{\mathfrak{u}}(b))\big),\qquad b^{p-1}=1,\quad \bar{b}=\alpha.$$

In particular the order of $a_\omega(s)$ divides $p-1$.

Proof. The integer N_K depends only on the fixed field and its finite level-six list by Theorem 310.2. The lower bound for g_0 in Theorem 11.4 tends to infinity. The selected place is split as in Theorem 113.2. Apply Theorem 340.2 to the s of the proposition, and then raise its value to the $p-1$-st power. □

342. Coset multiplicities in the finite part of a period

Definition 342.1. Let H be a Hausdorff topological group, S a closed subgroup, and $J\subseteq A$ compact open subgroups of H. Write $L=S\cap A$ and $H/J=\{xJ : x\in H\}$ with its left H-action. For a finite L-stable subset $T\subseteq H/J$, define

$$m_T(t)=\#\{rL\in S/L : r^{-1}t\in T\}\qquad (t\in H/J).$$

The membership test is independent of the representative because T is L-stable. We call T a cut for $W\subseteq H$ if $W/J:=\{wJ : w\in W\}$ lies in ST.

Lemma 342.2. *In Definition 342.1, m_T is constant on each S-orbit and zero outside ST. Fix $t=xJ$ and put $S_t=S\cap xJx^{-1}$. If $T\cap St=Lt$, then*

$$m_T(t)=[S_t : S_t\cap L]<\infty.$$

In particular m_T is finite everywhere for any finite T.

Proof. Translation of rL on the left by $s\in S$ identifies the counting sets for t and st. If $T\cap St=Lt$, then $r^{-1}t=lt$ for some $l\in L$ exactly when $r\in S_tL$. The cosets rL represented in this product are parametrized by the left cosets of $S_t\cap L$ in S_t. The latter is compact since S is closed, and the former subgroup is open in it. Its disjoint open cosets have a finite subcover. For arbitrary finite T, split $T\cap St$ into its finitely many L-orbits. The same computation, with a representative of each such orbit in place of t and then left translation, gives a finite count for each piece. □

Theorem 342.3. *Let J,A,S,H,L be as in Definition 342.1. Let $D\subseteq H$ be compact, let B be a subgroup containing A and D, and suppose $W\subseteq SD$. There is a cut $T\subseteq B/J$ for W, independent of the integer Q below, with the following property. If $Q\geq 1$ is an integer and for each $x\in D$ there is an integer $a_x\geq 0$ such that*

$$(xux^{-1})^{Q^{a_x}}\in A\qquad (u\in J),$$

then there is an integer $c\geq 0$ such that $m_T(t)\mid Q^c$ on ST. In particular $m_T(t)=1$ there when $Q=1$.

Proof. The image of D in the discrete space H/J is finite. From that image choose $t_i=x_iJ$, $x_i\in D$, one in each S-orbit met. Set $T=\bigcup_i Lt_i$. Each of these orbits is finite by compactness of L and discreteness of H/J. Since $L\subseteq A\subseteq B$, the cut lies in B/J. It

covers W/J because $W \subseteq SD$. Lemma 342.2 computes the multiplicity on St_i as the index of $M_i = S_{t_i} \cap L$ in S_{t_i}.

Let P_i be the image of S_{t_i} in the permutation group of the finite set S_{t_i}/M_i. Every $g \in S_{t_i}$ is of the form $x_i u x_i^{-1}$, $u \in J$; its $Q^{a_{x_i}}$-th power belongs to M_i. Applying this to $h^{-1}gh$ for every $h \in S_{t_i}$ shows that the same power of g acts trivially on every coset. Every prime $r \mid |P_i|$ divides Q. Indeed, cyclic rotation acts on the $|P_i|^{r-1}$ tuples of length r in P_i having product one. Orbits have size 1 or r, and fixed tuples are the constant tuples $(v, \dots, v)$ with $v^r = 1$. The all-identity tuple therefore has another fixed tuple beside it. Its entry has order r, and its $Q^{a_{x_i}}$-th power is one. Transitivity gives $[S_{t_i} : M_i] \mid |P_i|$ by counting the fibers of evaluation at M_i. Factorization in Lemma A.1 and finiteness of the list of i give a single c. The empty cut permits $c = 0$. □

343. Integration over a finite cut

Lemma 343.1. *Let G_1 be a second countable locally compact Hausdorff group and S_1 a closed subgroup equipped with left Haar measure σ_1. Here a left Haar measure is a left translation invariant Borel measure finite on compact sets and positive on every nonempty open set. There is a positive Borel function α on G_1 such that*

$$\int_{S_1} \alpha(u^{-1}x)\,\mathrm{d}\sigma_1(u) = 1 \qquad (x \in G_1).$$

Proof. The interiors of a sequence of compact sets C_i, $i \geq 1$, cover G_1. Indeed choose relatively compact open neighborhoods covering the group and extract a countable subcover by assigning a cover member to each basis element contained in one. For given x, if $u_0^{-1}x \in C_i$ then all u satisfying $u^{-1}x \in C_i$ belong to $u_0(S_1 \cap C_i C_i^{-1})$. Its measure b_i is finite and independent of x, u_0. Put $h = \sum_{i \geq 1} 2^{-i}\mathbf{1}_{C_i}/(1 + b_i)$ and $c(x) = \int_{S_1} h(u^{-1}x)\,\mathrm{d}\sigma_1(u)$. Then $c(x) \leq 1$, and $c(x) > 0$ because x belongs to the interior of some C_i. Product integration as in Lemma 123.1 gives measurability of c. Changing u by left translation gives $c(sx) = c(x)$ for $s \in S_1$. Hence $\alpha = h/c$ works. □

Theorem 343.2. *In addition to the groups in Definition 342.1 and Lemma 343.1, assume H is locally compact and second countable. Fix a compact open subgroup K of H and a level A. Use left Haar measures on G_1, H, S_1, S, normalized at the finite places by $\mu_H(K) = \sigma_S(L) = 1$. Put $G = G_1 \times H$, $U = S_1 \times S$, with their product measures. Let $\Gamma \subseteq U$ be a countable subgroup acting on G, U by left translation with measurable fundamental domains D_G, D_U, respectively. Fix a measurable $a : G_1 \to \mathbb{C}$. There exists $\kappa \in \mathbb{C}$ depending only on a, the fixed groups, measures and cutoff such that the following holds.*

Suppose $J \subseteq A \cap K$ is a compact open subgroup, $T \subseteq H/J$ is finite and L-stable, and $f : G \to \mathbb{C}$ is measurable, left Γ-invariant and integrable over D_G. Suppose $f(x,h) \neq 0$ implies $hJ \in ST$. For a function $b : T \to \mathbb{C}$, suppose for almost every (x,h) with $hJ \in T$,

$$\int_{D_U} f(u(x,h))\,\mathrm{d}u = [K : J]\,a(x)b(hJ).$$

Then

$$\int_{D_G} f(y)\,\mathrm{d}y = \kappa \sum_{t \in T} \frac{b(t)}{m_T(t)}.$$

Proof. Write $\psi(t) = 1/m_T(t)$ on T, zero elsewhere. The multiplicities on T are positive and finite by Lemma 342.2. For $hJ \in ST$, the $s \in S$ such that $s^{-1}hJ \in T$ form $m_T(hJ)$ disjoint left cosets of L, each of measure one. On them $\psi(s^{-1}hJ) = 1/m_T(hJ)$. Thus the integral of this function over S equals $\mathbf{1}_{ST}(hJ)$ for every h. Put $A_0(x,h) = \alpha(x)\psi(hJ)$. The preceding computation and Lemma 343.1 give $\int_U A_0(u^{-1}y)\,\mathrm{d}u = \mathbf{1}_{ST}(hJ)$ at $y = (x,h)$.

For any nonnegative measurable left Γ-invariant F, Lemma 123.1 gives

$$\int_G A_0(g) \int_{D_U} F(ug)\,\mathrm{d}u\;\mathrm{d}g = \int_{D_G} F(y) \int_U A_0(u^{-1}y)\,\mathrm{d}u\;\mathrm{d}y.$$

Indeed change g to $u^{-1}g$ in the outer group integral, split $G = \coprod_{\gamma\in\Gamma} \gamma D_G$, and use $F(\gamma y) = F(y)$. The remaining integral over D_U of $\sum_\gamma A_0(u^{-1}\gamma y)$ is the integral over $U = \coprod_\gamma \gamma^{-1} D_U$, by left translation. This also proves independence of the domains by the decomposition in Lemma 21.4. Taking $F = |f|$ gives finiteness on the left and the right. The same identity follows for f by decomposing its real and imaginary parts. The right side now equals $\int_{D_G} f$ by the support hypothesis.

Every coset hJ has measure $[K : J]^{-1}$, by partitioning K into translates of J. Insert the given period formula on the left. If $b(t) \neq 0$ for some t, the integral with $|f|$ bounds $\int_G A_0(g)| \int_{D_U} f(ug)\,\mathrm{d}u|\,\mathrm{d}g$. The contribution from that coset shows $\int_{G_1} \alpha|a| < \infty$. We can take $\kappa = \int_{G_1} \alpha a$ when this integral is absolutely convergent, and $\kappa = 0$ otherwise. In the latter case all $b(t)$ vanish and both sides are zero. This single choice of κ gives the formula for all J, T, f, b. □

Corollary 343.3. *Under the hypotheses of Theorem 342.3, use its cut in Theorem 343.2 whenever the support and period hypotheses there hold. All the finite coefficients $1/m_T(t)$ belong to $\mathbb{Z}[1/Q] = \{z/Q^e : z \in \mathbb{Z},\ e \geq 0\}$. Their sum is at most $|T|$, and for $Q = 1$ each is exactly one.*

Proof. Theorem 342.3 gives $1/m_T(t) = (Q^c/m_T(t))/Q^c$ with integral numerator. Every positive integer $m_T(t)$ has reciprocal at most one. Theorem 343.2 uses precisely these reciprocals. □

344. Local depth in the denominator of a finite period

Lemma 344.1. *Let F be a discretely valued field with valuation ring O, uniformizer π, and residue field of order q. For $h \geq 1$ write $K_h = \{g \in \mathrm{GL}_n(O) : g \equiv I \bmod \pi^h\}$. Then K_1/K_h has order $q^{n^2(h-1)}$. If $x \in \mathrm{GL}_n(F)$ and $d, d' \geq 0$ satisfy $\pi^d x, \pi^{d'} x^{-1} \in M_n(O)$, then $xK_h x^{-1} \subseteq \mathrm{GL}_n(O)$ for $h \geq \max(1, d + d')$.*

Proof. For $j \geq 1$, reduction of $(g - I)/\pi^j$ modulo π is an additive homomorphism from K_j with kernel K_{j+1}. The quadratic term in multiplication vanishes since $2j \geq j + 1$. It is onto $M_n(O/\pi)$ because $I + \pi^j M$ has unit determinant for every integral lift M. Multiplying the successive indices proves the order formula. For $g \in K_h$ write

$$xgx^{-1} = I + \pi^{h-d-d'}(\pi^d x)((g - I)/\pi^h)(\pi^{d'} x^{-1}).$$

This is integral and its determinant is a unit, so the adjugate formula gives an integral inverse. □

Proposition 344.2. *Let E be a CM field, with conjugation $w \mapsto \bar{w}$ on its finite places. In standard coordinates let H be the finite-adelic unitary group for the identity Gram matrix, with $K = \{x \in H : x_w \in \mathrm{GL}_n(O_w)\ \forall w\}$. Thus $x_{\bar{w}}$ is the inverse conjugate transpose of x_w. Let P be a finite set of split places with $P \cap \bar{P} = \varnothing$, $B = \{x \in H : x_w \in \mathrm{GL}_n(O_w)\ (w \notin P \cup \bar{P})\}$, and $J \subseteq K$ compact open, reducing to the identity at every $w \in P$. Let $S \subseteq H$ be closed, $D \subseteq B$ compact and $W \subseteq SD$. There is an $(S \cap K)$-stable cut $T \subseteq B/J$ for W whose multiplicities divide*

$$\prod_{w\in P} q_w^{n^2(h_w-1)} \qquad \text{on } ST,$$

where $q_w = |O_w/\pi_w|$. Here take any integers $d_w, d'_w \geq 0$ bounding the denominators of all x_w, x_w^{-1}, respectively, for $x \in D$, and $h_w = \max(1, d_w + d'_w)$.

Proof. For each w, the entries of x_w and of its inverse range over compact sets. The increasing open fractional ideals $\pi_w^{-r} O_w$, $r \geq 0$, cover them. This gives the bounds. The local matrix rings are inverse limits of their finite reductions, with the determinant-unit condition closed and open. The product of the local integral groups is compact by successive subsequence extraction in its countably many finite reductions. The unitary equations are closed. Openness follows from the restricted product topology of Definition 21.3.

Choose representatives $x_i J$ and the union T of their $(S \cap K)$-orbits as in Theorem 342.3, with $A = K$. Put $S_i = S \cap x_i J x_i^{-1}$. The map

$$s \longmapsto \left((x_i^{-1} s x_i)_w \bmod K_{h_w} \right)_{w \in P}$$

is a homomorphism from S_i into $\prod_{w \in P} K_1 / K_{h_w}$ in the respective local matrix groups. An element of its kernel is integral at each $w \in P$ by Lemma 344.1, and then at each conjugate place by the unitary equation. At all other places x_i and $x_i^{-1} s x_i \in J$ are integral with integral inverses. Thus the kernel lies in $S_i \cap K$. Counting cosets in this chain shows that $[S_i : S_i \cap K]$ divides the order of the finite target. This is exactly the claimed bound by Lemmas 342.2 and 344.1. □

Corollary 344.3. *For $P = \varnothing$ in Proposition 344.2, the cut lies in K/J, every multiplicity on it is one, and $|T| \le [K : J]$. At any conductor, if $J \subseteq A \subseteq K$ are compact open, $W \subseteq SA$, and the cut is formed using A itself for D and $S \cap A$ for L, the same conclusions hold with A/J and $[A : J]$. In both cases the finite coefficients of Theorem 343.2 are integers.*

Proof. For the first assertion the product bound is one and $B = K$. In the second case $x_i \in A$ and $J \subseteq A$ imply $S_i \subseteq A$, so Lemma 342.2 gives multiplicity one. In each case sum over a subset of the indicated finite coset space and apply Theorem 343.2. □

345. Extension of primitive hermitian frames

Lemma 345.1. *Let k be a finite field of order q. Let R be its quadratic field extension with conjugation $z \mapsto z^q$, or $k \times k$ with the two coordinates exchanged. Write $A^* = \bar{A}^{\,t}$. Let $V = R^n$ with a nonsingular hermitian matrix $J = J^*$. If $P, Q \in M_{n,r}(R)$ have left inverses and $P^* J P = Q^* J Q$, some $g \in \mathrm{GL}_n(R)$ satisfies $g^* J g = J$, $gP = Q$.*

Proof. The trace $z \mapsto z + \bar{z}$ is onto k. In the field case its kernel has at most q elements by its polynomial and the image lies in k. In the product case use $(z, 0)$. Consequently each hermitian matrix D can be written $M + M^*$: choose lifts of the diagonal entries under the trace, copy the upper off-diagonal entries and set the lower ones to zero.

All nonsingular hermitian forms on free R-modules of the same rank are isometric. In the field case, a nonzero nonsingular space contains a vector of nonzero norm. Otherwise expansion of the norm of $v + \lambda w$ gives $\mathrm{Tr}(\lambda v^* J w) = 0$ for all λ, v, w, contradicting trace surjectivity and nonsingularity. The norm on $R^\times$ maps onto $k^\times$, since its kernel has size at most $q + 1$ by the equation $z^{q+1} = 1$. Scale the vector to have norm one and split off its orthogonal complement. Induction gives an orthonormal basis. In the product case the form is a perfect pairing of the two k-coordinate spaces, together with its transpose. A basis on the first and its dual on the second give the identity matrix.

The proposed isometry on the column spans is well defined because both matrices have left inverses. In the first span split off a complement U of the radical D_0 of the restricted pairing. The form on U is nonsingular because the quotient by the radical has that property. In the product case choose a complement on each coordinate. The radical dimensions there agree by equality of the ranks of a pairing matrix and its transpose. Make the corresponding choices in the second span by the prescribed isometry. In the ambient orthogonal complement of U, let $v_1, \dots, v_e$ be a basis of D_0. The linear equations $v_i^* J w_j = \delta_{ij}$ have solutions there since the form there is nonsingular and the v_i are independent in each field coordinate. Write v, w for these column matrices. The replacement $w \mapsto w - vM$, with $M + M^* = w^* J w$, makes the Gram matrix of w zero while preserving $v^* J w = I$. Thus v, w form a nonsingular summand of Gram matrix $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Repeat for the image of the radical, using the ordered image basis. Identify the two summands, use the given isometry on U, and identify the remaining nonsingular orthogonal complements by the equal-rank assertion above. The direct sum extends the prescribed map on every column. □

Theorem 345.2. *Let O be a complete discrete valuation ring with finite residue field and uniformizer π. Let B be the ring of integers in an unramified quadratic extension with residue conjugation of order two, or $O \times O$ with exchange of factors. Suppose $J \in \mathrm{GL}_n(B)$ is hermitian, $P, Q \in M_{n,r}(B)$ have left inverses over B, and $P^*JP = Q^*JQ$. There is $g \in \mathrm{GL}_n(B)$ with $g^*Jg = J$, $gP = Q$.*

Proof. Complete P to a basis C by extending its independent columns after reduction in each residue field and lifting the remaining columns. The determinant is then a unit. Lemma 345.1 gives a residue isometry $\bar{g}$ taking the reduced P to the reduced Q. Take a matrix C_1 with first r columns exactly Q and other columns lifting those of $\bar{g}\bar{C}$. Thus $C_1^*JC_1 \equiv C^*JC \bmod \pi$. Suppose C_j is constructed with this congruence modulo π^j, $j \geq 1$, and with the prescribed first columns. Reduce $(C^*JC - C_j^*JC_j)/\pi^j$ modulo π to obtain a hermitian D with zero leading r-square block. The uniformizer belongs to O and hence is fixed by conjugation. Choose $M + M^* = D$ with its first r columns zero. Take the upper right block of D unchanged and apply the trace construction of Lemma 345.1 to the lower right square block. Solve $(\bar{C}_j^*\bar{J})\bar{Z} = M$ in the residue algebra, then lift Z with its first r columns zero. Expansion gives $C_{j+1} = C_j + \pi^j Z$ with $C_{j+1}^*JC_{j+1} \equiv C^*JC \bmod \pi^{j+1}$, since $2j \geq j + 1$. The limit C_∞ exists by completeness, has the prescribed columns, and is invertible by reduction. Now $g = C_\infty C^{-1}$ has both required properties. □

Corollary 345.3. *For the arithmetic field E of Lemma 57.1, put $\widehat{O}_E = \prod_w O_w$. If two n-by-r matrices over $\widehat{O}_E$ have left inverses and the same hermitian Gram matrix for the identity form, there exists a single element of K in Proposition 344.2 taking the first ordered frame to the second.*

Proof. Group the local factors by the places of the real subfield. Lemma 57.1 gives exactly the two integral rings of Theorem 345.2, including above two. Apply that theorem at every factor with $J = I$. Their product belongs to K because each local isometry and its inverse are integral. □

346. Bessel orbits and the quotient lines of a flag

Definition 346.1. Over the field E of Lemma 57.1, let V have a nonsingular hermitian form of matrix J_0 and a basis consisting of a basis of X and a string $e_0, \ldots, e_{s-1}$, $s \geq 0$. Assume X is perpendicular to the string and its string entries $h(e_i, e_j)$ vanish unless $i + j = s - 1$. Fix an invertible finite adelic frame C with $C^*J_0C = I$. Use the standard finite-adelic H, K of Proposition 344.2. Write $r = \lceil s/2 \rceil$, $F_{-1} = 0$, and $F_i = \mathrm{span}_E(e_0, \ldots, e_i)$, $0 \leq i < r$. The finite Bessel subgroup S consists of $u \in H$ such that $CuC^{-1}e_i - e_i \in F_{i-1} \otimes_E \mathbb{A}_{E,f}$ for each $0 \leq i < r$. Here $\mathbb{A}_{E,f}$ is the restricted product of the finite completions of E with respect to their valuation rings. These triangular conditions define a closed subgroup, since they give closed linear equations, persist under products and hold for inverses by triangular substitution.

At a finite place w put $\Lambda_x(w) = C_w x_w O_w^n$ for $x \in H$, $n = \dim_E V$. Write $I_i(x; w)$ for the image of $\Lambda_x(w) \cap F_i(E_w)$ in $F_i(E_w)/F_{i-1}(E_w) = E_w e_i$, identified with a fractional ideal in E_w by its coefficient of e_i. Denote its exponent by $b_i(x; w)$, so that $I_i(x; w) = \pi_w^{b_i(x;w)} O_w$.

Lemma 346.2. *The ideals in Definition 346.1 are nonzero. For any generators $\beta_i(w)$ of them one can choose $v_i(x; w) \in \Lambda_x(w) \cap F_i(E_w)$ with coefficient $\beta_i(w)$ on e_i. These columns extend to a basis of $\Lambda_x(w)$. Their hermitian Gram matrix, assembled over the conjugate places, is zero except possibly for the last diagonal entry when s is odd. That entry is $\bar{\beta}_{r-1}\beta_{r-1}h(e_{r-1}, e_{r-1})$.*

Proof. The lattice contains $\pi_w^a O_w^n$ and is contained in $\pi_w^{-a} O_w^n$ in the displayed rational basis for some a. Thus the image is a nonzero bounded fractional ideal. Its least valuation gives a generator in a discrete valuation ring. Lifts v_i exist by definition. Successively

subtract multiples of these lifts, starting at the largest index, to see that they are a basis of $\Lambda_x(w) \cap F_{r-1}(E_w)$, with the empty list used for $r = 0$. Their reductions in $\Lambda_x(w)/\pi_w\Lambda_x(w)$ are independent. Indeed if a combination belongs to $\pi_w\Lambda_x(w)$, dividing by π_w places it in the same flag intersection. Uniqueness of its basis coordinates forces all original coefficients to be divisible by π_w. Extend the reduced list to a basis and lift its additional columns. The resulting determinant in lattice coordinates is a unit. Finally every index in the early columns is at most $r-1$. Two such indices can sum to $s-1$ only when s is odd and both equal $r-1$. The claimed Gram formula follows by expansion. □

Theorem 346.3. *In Definition 346.1, two elements x, x' lie in the same double coset in $S\backslash H/K$ if and only if $I_i(x;w) = I_i(x';w)$ for all $i < r, w$.*

Proof. Right translation by K preserves the lattice. Left translation by $u \in S$ sends it to its translate by CuC^{-1}, which preserves F_i and acts as one on F_i/F_{i-1}. This proves necessity.

For sufficiency choose the same generators $\beta_i(w)$ for both lattices, taking one at almost every place since the frames there are invertible integral matrices in the displayed basis. Lemma 346.2 gives the early columns v, v' and equality of their Gram matrices. In the respective standard coordinates $P = (Cx)^{-1}v$, $Q = (Cx')^{-1}v'$, the columns have left inverses over every O_w, because each frame extends to a basis of its lattice. Their local left inverses assemble over $\widehat{O}_E$. Both frames use the identity form in these coordinates. Corollary 345.3 gives $k \in K$ taking P to Q. Hence $C(x'kx^{-1})C^{-1}$ takes every v_i to v_i'. It preserves the flag by their triangular bases, and on F_i/F_{i-1} it sends $\beta_i e_i$ to the same element. As each $\beta_i(w) \neq 0$, all these quotient actions are identity. Thus $x'kx^{-1} \in S$, as required. □

347. Compact representatives at the twist places

Theorem 347.1. *Use the string flag of Definition 346.1. Let P be a finite set of split places disjoint from $\bar{P}$, and $k \geq 0$ an integer. Suppose $W \subseteq H$ satisfies, for every $x \in W$ and $0 \leq i < r$,*

$$|b_i(x;w) - b_i(1;w)| \leq k\,\mathbf{1}_{P\cup\bar{P}}(w) \qquad \text{at every finite place } w.$$

Then there is a compact set $D \subseteq B$ as in Proposition 344.2 such that $W \subseteq SD$. It is the union of at most $(2k+1)^{r|P\cup\bar{P}|}$ translates of K, each represented by a group element which is identity outside $P \cup \bar{P}$. In particular, for every compact open $J \subseteq K$ reducing to one at P, the same D yields a cut with coefficients in $\mathbb{Z}[1/Q]$, where $Q = \prod_{w\in P} q_w$.

Proof. The possible tuples of differences at the indicated places belong to a product of $r|P \cup \bar{P}|$ intervals of integers of length $2k+1$. Choose a representative $x \in W$ of each occurring tuple. Keep its components at $P \cup \bar{P}$, replacing all others by identity, and denote the result by x°. The replacement respects the unitary equation because it is made on entire conjugation orbits. The definition of b_i uses only $C_w x_w$. Thus the new representative has the old tuple at the indicated places and the identity tuple elsewhere. For any $y \in W$ of that type all its line ideals agree with those of x°. Theorem 346.3 places it in $Sx^\circ K$. Take the union of the $x^\circ K$ for D. Proposition 344.2 and Corollary 343.3, or the displayed integer product in that proposition directly, give the asserted localization. □

Proposition 347.2. *Keep E and the marked split primes g_0, g_1 of Proposition 15.2. Write $f = [E:\mathbb{Q}]/2$. For a unitary group in standard finite coordinates of rank n let $A_g \subseteq K$ be the kernel of reduction at the f marked places over each g_d, one in every conjugate pair. Then*

$$[K : A_g] = \prod_{d=0}^{1}\prod_{u=0}^{n-1}(g_d^n - g_d^u)^f \;\leq (g_0 g_1)^{fn^2}.$$

For $P = \varnothing$ in Theorem 347.1 and $J = A_g$, the sum of the absolute finite coefficients of Theorem 343.2 is at most this index. For a string of length zero the same assertions hold with $r = 0$.

Proof. The residue fields and the number of marked places are those of Proposition 15.2. At a marked split place choose any invertible residue matrix and lift its entries to the integers. The lift has unit determinant. Use its inverse conjugate transpose at the conjugate place and identity outside the chosen pairs to see that simultaneous reduction is onto the full product. Its kernel is A_g. In an n-dimensional space over a field of g_d elements, the next column after u independent columns can be any of the $g_d^n - g_d^u$ vectors outside their span. This gives the order and its upper bound. For empty P all representatives in the proof of Theorem 347.1 are identity. Apply Corollary 344.3. If $s = 0$ there are no flag conditions, $S = H$, and the same proof applies. □

348. The arithmetic frames for the row lifts

Lemma 348.1. *Use $g = g_0 g_1, \pi_g, \pi_N, \varpi, \mathcal{A}$ from Theorem 11.4, Propositions 12.2 and 18.4, and Definition 44.2. Write $\omega = (1 + \sqrt{-d})/2$, $D_c = P'_{N_1}(\zeta_{N_1})$ and $\jmath = -1/\sqrt{-d}$. Let $\alpha_w = D_c^{-1}$ where ω reduces to zero, and let $\alpha_w = 1$ elsewhere. Let δ_g have component $\bar{\pi}_g$ at the marked places dividing $\bar{\pi}_g$, and component one elsewhere. The finite ideles*

$$a_f = \alpha/\bar{\delta}_g, \qquad u_j = (a_f/\bar{a}_f)^{\lfloor (m-j+1)/2 \rfloor} \quad (0 \le j \le m)$$

satisfy $u_j \bar{u}_j = 1$. At each w, $a_f \bar{a}_f$ has the same valuation as $c = -(\jmath g \varpi)^{-1}$.

Proof. By Proposition 44.1, D_c is a unit away from N_1 and has the same valuation on conjugate branches. At a prime dividing N_1, exactly one branch reduces ω to zero by Proposition 44.3. Hence $\alpha\bar{\alpha}$ generates $D_c^{-1}\widehat{O}_E$, in agreement with Lemma 47.1. Lemma 13.2 gives the same fractional ideal for $\delta_g\bar{\delta}_g$ and g. Proposition 18.4 gives $\jmath\varpi = -\zeta_{N_1}^{h'_0} D_c$, a unit times D_c. Dividing the local ideals proves the valuation identity. Conjugating $a_f/\bar{a}_f$ inverts it, which proves the norm claim. □

Definition 348.2. Let $r = k + m - 1$ be the row rank of Proposition 15.2, and let t be the block size of the arithmetic data. For $0 \le j < m$ put $s = m - 1 - j$, $d_j = k + j(t+1)$. Use a definite space $X_j = E^{d_j}$ with identity Gram matrix and a string of length s, perpendicular to X_j. Its antidiagonal entries are one, except that a middle entry of an odd string is $-N$. Denote the resulting rational space by Y_j, with dimension $d_j + s = r + jt$. Set $Y_m = E^{r+mt}$ with the positive identity form. In Y_j let b be the last basis vector of X_j, and use $e_0, \dots, e_{s-1}$ for the string. Define the following ordered integral finite-adelic lattice Λ_j by its basis, with scales from Lemma 348.1. For even s, take the standard basis of X_j and string columns $u_j a_f^{-1} e_i$ for $2i < s$, $u_j \bar{a}_f e_i$ for $2i \ge s$. For odd s, start with the standard basis of X_j and the columns $u_j e_i$, then replace b by

$$v = b/\bar{\pi}_N + u_j e_{(s-1)/2}/N.$$

For $j = m$ use the standard integral lattice.

Theorem 348.3. *For $j < m$ the number of negative real directions of Y_j at each infinite place is $\lceil s/2 \rceil$. Its lattice has unimodular hermitian Gram matrix G_j. In the basis matrix B_j of Definition 348.2 there is a change $T_j \in \mathrm{GL}_{r+jt}(\widehat{O}_E)$ such that $C_j = B_j T_j$ is a finite frame for the identity form. One may prescribe $(T_j)_w = I$ at every marked place $w \mid \bar{\pi}_g$. For $0 \le i < \lceil s/2 \rceil$ the reference ideals $I_i(1; w)$ of Definition 346.1 are generated by $(u_j)_w$ if s is odd, and by $(u_j/a_f)_w$ if s is even. At $j = m$ the negative count is zero and the identity frame works.*

Proof. In each distinct paired string plane (e_i, e_{s-1-i}), the sum and difference divided by $\sqrt{2}$ have respective norms one and minus one. For odd s the middle vector divided by $\sqrt{N}$ has norm minus one. These vectors together with the standard X_j-basis are orthogonal, giving the claimed signature. In finite coordinates the scaled distinct string pairs still have pairing one by $u_j \bar{u}_j = 1$. In the odd case the Gram block on $v, u_j e_{(s-1)/2}$ equals $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$

by $\pi_N\bar{\pi}_N = N$. The remaining definite basis vectors are orthonormal. This gives an integral Gram matrix with unit determinant, including for the empty string.

At a split pair choose the correction on one member equal to identity, taking the marked member if present. On its conjugate choose the correction with conjugate transpose $G_{j,w}^{-1}$. Both resulting Gram matrices are identity by hermitian symmetry. At a nonsplit place Lemma 57.1 gives an unramified quadratic integer extension. Choose a residue basis with Gram matrix identity by the classification in Lemma 345.1. Lift this basis with Gram congruent to identity modulo the uniformizer. Apply the correction in Theorem 345.2 with zero prescribed columns, using I as target Gram matrix: at step h write the reduced error as $M + M^*$, solve for the correction and pass to the limit. All the resulting corrections and their inverses are integral, giving the asserted adelic frame.

The corrected basis generates the original lattice. In the odd case, a vector in an early flag span must have zero coefficient of v, since its b-coordinate vanishes. Its independent early columns are therefore precisely the $u_j e_i$. The even case follows from the diagonal columns directly. Projection to consecutive lines gives the ideals as stated. □

349. The column level inside the finite Bessel subgroup

Proposition 349.1. *For $0 \le j < m$ let $K_{c,j} \subset U(X_j)(\mathbb{A}_f)$ consist of integral unitary matrices reducing to identity at the marked places over g and satisfying $b^*(h-I) \in \pi_N \widehat{O}_E^{1\times d_j}$, where b is the last coordinate vector in Definition 348.2. Under extension by the identity on the string, $K_{c,j}$ lies in the Bessel subgroup, preserves Λ_j, and acts as identity on its reduction at every marked place over g. These conditions hold in the frames C_j of Theorem 348.3.*

Proof. The conditions define a subgroup. The row condition states that $b^*h = b^*$ in the reduction, hence also $b^*h^{-1} = b^*$ there. Taking adjoints gives $(h-I)b \in \bar{\pi}_N \widehat{O}_E^{d_j}$. The ideals of $\pi_N, \bar{\pi}_N$ have disjoint support by Lemma 13.2 and $\pi_N\bar{\pi}_N = N$. The last entry of $(h-I)b$ therefore belongs to $N\widehat{O}_E$, since it belongs to both ideals. Applying the same facts to h^{-1} will give the reverse lattice inclusion.

All string vectors are fixed, giving Bessel membership by Definition 346.1. For even string length the standard definite lattice and the scaled string lattice are direct summands, so preservation follows. For odd length the pure definite vectors in Λ_j have integral coordinates, with the last coordinate in $\pi_N\widehat{O}_E$. In fact a multiple zv of the glued column has last definite coordinate $z/\bar{\pi}_N$. Its string component can be cancelled by an integral multiple of $u_j e_{(s-1)/2}$ precisely when $z \in N\widehat{O}_E$. For an ordinary definite column e other than b, $(h-I)e$ meets these pure definite conditions by the row congruence. On v the difference is $(h-I)b/\bar{\pi}_N$, integral by the adjoint condition and with last entry divisible by π_N by the preceding product divisibility. This proves both inclusions.

At a marked place over g, $\pi_N, \bar{\pi}_N$ are units by Proposition 13.5. In the scaled lattice basis, a pure definite difference with coordinates (d', d_b) has, in the odd case, coefficients $d', \bar{\pi}_N d_b, -d_b/\pi_N$ in the ordinary definite, glued, and middle string columns. Every such difference coming from $h - I$ is zero on reduction, also after division by $\bar{\pi}_N$ on v. The even case is immediate on its direct summands. Changes of integral basis preserve the identity reduction. □

Lemma 349.2. *Let Λ^{old} be the row lattice of Definition 15.1, using the string order in Definition 348.2. Let A^{old} be its marked principal level and A^{new} the corresponding level on Λ_0, viewed in the same rational coordinates. Write*

$$b_0 = \mathrm{diag}(u_0^{-1} I_{X_0}, I_{\mathrm{string}}).$$

Then $b_0 \in S_f$, where S_f denotes the Bessel subgroup in rational coordinates, and $A^{\mathrm{new}} = b_0 A^{\mathrm{old}} b_0^{-1}$. If $P(y,h)$ is right A^{old}-invariant and supported over $S_f A^{\mathrm{old}}$ in its finite variable, $P'(y,h) = P(y, hb_0)$ is right A^{new}-invariant and supported over $S_f A^{\mathrm{new}}$. Right translation

intertwines left multiplication by every rational element, so it also commutes with absolutely convergent left periodizations.

Proof. The string length at zero is odd since $m = 2\nu + 2$. The finite isometry acting by u_0 on the string and by one on X_0 takes the old basis, including its glued column, to precisely the basis of Λ_0. It factors as the central scalar $u_0 I$ times b_0, which fixes every string vector. The level is characterized by lattice preservation and trivial action on the specified reduced lattices, so transport of basis gives the conjugation formula. The two assertions on functions now follow from $b_0 \in S_f$ and $A^{\mathrm{old}} b_0^{-1} = b_0^{-1} A^{\mathrm{new}}$. Left and right translations commute termwise. □

350. The initial Bessel function of the row columns

Theorem 350.1. *Use the row group, rational points Γ, old principal level $A^{\mathrm{old}} = K_Y$, and class representatives h_c of Definition 21.3. Let $S(F)$ be the rational Bessel subgroup in Corollary 204.5. Choose for each column class a complex linear functional λ_c on the value space of F_∞ in Definition 204.1. Define*

$$b(y,h) = \sum_c \mathbf{1}_{h_c A^{\mathrm{old}}}(h)\, \beta_c(y) \lambda_c(F_\infty(y)),$$

$$P(y,h) = \sum_{\delta \in S(F)} b(\delta_\infty y, \delta_f h).$$

The latter series converges absolutely, is continuous, is left $S(F)$-invariant and right A^{old}-invariant, and vanishes unless $h \in S_f A^{\mathrm{old}}$. Its periodization over $S(F)\backslash\Gamma$ equals the automorphic column with these coordinates in Theorem 204.4. For all d and y,

$$P(y, h_d) = \lambda_d(F_\infty(y)).$$

Moreover $P(n_\infty y, n_f h) = \chi_\infty(n_\infty)\chi_f(n_f) P(y,h)$ for the initial unipotent group and the characters of Proposition 21.2.

Proof. The finite class set is finite by Proposition 13.5. Absolute convergence follows by restricting the absolutely convergent sum in Theorem 204.4. Partition its terms by the left cosets of $S(F)$ to obtain the asserted periodization. Reindexing gives rational left invariance, and the right level follows from the indicators. Each h_c belongs to S_f by its definition as a middle isometry, proving the support condition. When $h = h_d$ the proof of Corollary 204.5 identifies the contributing rational elements as precisely Γ_d, with class $c = d$. On this group the character is one by Proposition 21.2, and F_∞ transforms by this character by Theorem 122.2. The cutoffs sum to one by Proposition 202.3. This gives the displayed evaluation.

Every $h \in S_f A^{\mathrm{old}}$ has the form $\delta_f h_d a$, with $\delta \in S(F)$ and $a \in A^{\mathrm{old}}$. Indeed, first factor the Bessel component as $u_f t_f$ for a unipotent and a definite middle isometry. To see this, the preserved isotropic flag gives an induced middle isometry on $P_0^{\perp}/P_0$, with P_0 the span of the outer isotropic vectors. It fixes the class of z, so acts as a definite isometry on its orthogonal complement. After removing this isometry the action on the successive flag quotients is identity, also on the dual lines by unitarity. This is the unipotent group of Definition 20.3. Decompose the definite part as $t_f = \eta_f h_d k_c$ by Proposition 13.5. The embedded k_c belongs to A^{old} by Proposition 66.1. Approximate u_f by v_f for $v \in U(F)$ using Lemma 21.1, close enough that $u_f \eta_f h_d \in v_f \eta_f h_d A^{\mathrm{old}}$. This gives the required $\delta = v\eta$.

On this coset rational left invariance gives

$$P(y,h) = \lambda_d(F_\infty(\delta_\infty^{-1} y)).$$

Thus P is continuous there, and it vanishes on the remaining open level cosets. The character χ_∞ is unchanged under conjugation by $S(F)$. Unipotent conjugation preserves every group character. Conjugation by a middle isometry preserves the simple-root entries defining the

character because it fixes each of the string vectors and their coordinate functionals. The transformation law for F_∞ therefore proves the assertion for n_∞ on each coset. For $v \in U(F)$, rational invariance gives $P(y, v_f h) = P(v_\infty^{-1} y, h) = \chi_f(v_f) P(y, h)$, using Proposition 21.2. Density in Lemma 21.1 and continuity give the finite assertion everywhere. The real and finite actions commute. □

Corollary 350.2. *For the stage-zero function $P'(y,h) = P(y, hb_0)$ in Lemma 349.2, one may take $D = A^{\mathrm{new}}$ in the finite period cut. This works simultaneously for all ramified columns, all twist levels $J \subseteq A^{\mathrm{new}}$, and every product $P'z$ with a function z. The cut lies in A^{new}/J, and $m_T = 1$ on its saturation by the Bessel group.*

Proof. Theorem 350.1 gives the old support for all coordinate functionals. Transport it with Lemma 349.2. Multiplying by z preserves support containment. Apply Corollary 344.3 with $A = A^{\mathrm{new}}$. □

351. Duality on opposite string lines

Definition 351.1. For Y_j of Definition 348.2, $j < m$, order the blocks as the first $\lceil s/2 \rceil$ string lines, the definite block X_j, and the remaining string lines in increasing order of their string index. Denote the span of all blocks preceding e_i by V_i^-, and let $V_i^+ = V_i^- + Ee_i$. For the local lattice $L_w = (C_j)_w x_w O_w^{r+jt}$ at $x \in H$, define $I_i(x; w)$ by the image of $L_w \cap V_i^+(E_w)$ in V_i^+/V_i^-, and $b_i(x; w)$ by its valuation as in Definition 346.1. For $i < \lceil s/2 \rceil$ these are the ideals and exponents of that definition. For every i write $i^* = s - 1 - i$.

Proposition 351.2. *In Definition 351.1, for every x, w and $i \neq i^*$ one has*

$$b_i(x; w) + b_{i^*}(x; \bar{w}) = 0.$$

For $i = i^*$ one has $b_i(x; w) + b_i(x; \bar{w}) \geq -v_w(N)$. A negative sum is therefore possible only for odd s and a conjugate pair over N, in which case it equals -1.

Proof. The local lattices are in perfect hermitian duality between $w, \bar{w}$, using conjugation to identify coefficient fields. Indeed their coordinate matrices are those of the frame in Theorem 348.3 multiplied by a standard unitary x, so their pairing in integral coordinates is the identity matrix. For $i \neq i^*$, direct inspection of the antidiagonal form gives

$$(V_i^-)^\perp = V_{i^*}^+, \qquad (V_i^+)^\perp = V_{i^*}^-.$$

The central definite block belongs to a prefix exactly when its paired prefix excludes it. A possible odd middle vector is handled by the same observation for these noncentral i. On the two line quotients the induced pairing of the indicated basis vectors is one. Pairing any two integral lifts therefore gives $b_i(x; w) + b_{i^*}(x; \bar{w}) \geq 0$.

Choose a lattice lift v of a generator of $I_i(x; w)$. Choose a basis of $L_w \cap V_i^-(E_w)$ by successive projection onto local coordinate lines, as in Lemma 23.1. Together with v it is a basis of $L_w \cap V_i^+(E_w)$. This list extends to a basis of L_w by the reduction argument of Lemma 346.2, since the intersection with a vector subspace is saturated. Assign value one to v and zero to the other members of the extended basis. Perfect duality gives a vector in the opposite integral lattice representing this functional, with the conjugate-linear convention on the appropriate side. It annihilates V_i^-, so lies in $V_{i^*}^+$. The pairing of its quotient coordinate with the generator is one. This gives the opposite inequality.

For the middle line choose the early lifts of Lemma 346.2 in both components. Their pairing has valuation $v_w(N) + b_i(x; w) + b_i(x; \bar{w})$ and is integral. By Lemma 13.2, N is of valuation one at its split places, and of valuation zero elsewhere. The exponents are integers. □

Corollary 351.3. *For noncentral pairs the deviations $\Delta_i(x; w) = b_i(x; w) - b_i(1; w)$ satisfy $\Delta_i(x; w) = -\Delta_{i^*}(x; \bar{w})$. For a middle line $\Delta_i(x; w) + \Delta_i(x; \bar{w}) \geq 0$ unless the moved lattice has the exceptional value -1 from Proposition 351.2.*

Proof. Subtract the two identities in Proposition 351.2. For a middle line the reference generator is u_j by Theorem 348.3, of norm one by Lemma 348.1. Its pair of valuations sums to zero. □

352. A local test at the middle line

Proposition 352.1. *Let $j < m$ have odd string length s, and let $i_0 = (s-1)/2$. Suppose $b_{i_0}(x;w) + b_{i_0}(x;\bar{w}) = -1$ at a pair above N. Let $K^{(w)} \subseteq H$ be the subgroup supported on that pair and integral there in the standard finite coordinates of Y_j. Let $K_X^{(w)}$ be the analogous subgroup on X_j, embedded by acting trivially on the string in rational coordinates and then conjugating into the standard coordinates. There exists a single $r_0 \in S$ for which*

$$(r_0 x)^{-1} K_X^{(w)} (r_0 x) \subseteq K^{(w)}.$$

Proof. Change the lattice only on the indicated pair. In rational coordinates give X_j its standard lattice there. For every early i choose generators β_i of the two component ideals of x, and use $\beta_i e_i$ as a basis vector in the new lattice. For $i < i_0$ give its partner the basis vector $\bar{\beta}_i^{-1} e_{i^*}$. The distinct pairs have Gram block $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The middle vector has Gram entry $-N\beta_{i_0}\bar{\beta}_{i_0}$, a unit by the assumed sum and Lemma 13.2. This is a unimodular lattice on the split pair. Choosing a basis in one component and the dual basis in the other identifies its pairing with the identity, as in Theorem 348.3. Together with the original coordinates outside the pair this gives an element $x' \in H$ whose moved lattice is the new one.

Its early flag ideals are visibly those of x at the pair and are unchanged elsewhere. Theorem 346.3 gives $x' = r_0 x k_0$ for $r_0 \in S,\ k_0 \in K$. The new lattice is fixed by the entire $K_X^{(w)}$: its definite summand is standard and its string vectors are fixed. Consequently $x'^{-1} K_X^{(w)} x' \subseteq K^{(w)}$. Conjugation by k_0 preserves this local direct factor of K, giving the assertion. □

Lemma 352.2. *For the odd string of Proposition 352.1, let U be the matrices whose difference from identity strictly raises the grades $s-1-2i$ on the string, with all grades zero on X_j. Let $H_X \subset S$ be the embedded finite definite group, and $\chi_U : U \to \mathbb{C}^\times$ a character invariant under H_X. Fix $z \in K_X^{(w)}$ central in H_X. Let $\phi \mapsto f_\phi$ be a linear family of functions on H and R a linear operator on their parameter space. Assume*

$$f_\phi(uh) = \chi_U(u) f_\phi(h), \quad f_\phi(hk) = f_\phi(h), \quad f_{R\phi}(h) = f_\phi(zh) \qquad (u \in U,\ k \in K^{(w)},\ h \in H).$$

At every exceptional x in that proposition one has $f_\phi(x) = 0$ for $\phi \in \mathrm{im}(1-R)$.

Proof. Here $S = UH_X$. The positive string grades span the isotropic flag, and its perpendicular modulo its span has the orthogonal middle decomposition $X_j \oplus E e_{i_0}$. An element of S acts as one on the line generated by e_{i_0} in this quotient. Removing the induced action on X_j leaves identity on all graded pieces, including the negative ones by duality. It preserves the filtration by its flag conditions and unitarity. This proves the decomposition, also when the positive flag is empty. The character $\widetilde{\chi}(uh_X) = \chi_U(u)$ is a character of S because the decomposition is unique and H_X normalizes U, χ_U. For r_0 of Proposition 352.1 the element $v = r_0^{-1} z r_0 z^{-1}$ belongs to U by centrality in the quotient. Since it is a commutator, $\chi_U(v) = 1$. There is $k \in K^{(w)}$ with $vzx = xk$. Therefore $f_{R\psi}(x) = f_\psi(zx) = f_\psi(vzx) = f_\psi(xk) = f_\psi(x)$. Subtract and use linearity. □

Corollary 352.3. *For the ramified columns of Theorem 13.6, at every marked pair over N their inclusion in $\mathrm{im}(1-R)$ in Lemma 352.2 holds when R is translation by the corresponding residue generator. Thus the two equivariances and the right level in that lemma suffice to rule out a middle-line defect for such columns.*

Proof. The action of each finite residue unit group on the ramified columns has no trivial character by Theorem 13.6. On a character summand its generator acts by a scalar different

from one, and $1 - R$ is invertible there. The finite character decomposition follows by Lemma 7.1. Apply Lemma 352.2. □

353. A lattice inequality for a map between consecutive gradings

Definition 353.1. Consider Y_j, Y_{j+1} of Definition 348.2 with $s = m - 1 - j \geq 2$, so that the next string has length $s' = s - 1$. Use grades $g_i = s - 1 - 2i$ on the source string and $g'_i = s - 2 - 2i$ on the target string, with grade zero on both definite blocks. Define $T : Y_j \to Y_{j+1}$ by zero on X_j and on e_{s-1}, and by $Te_i = t_i e'_i$ for $0 \leq i < s'$. Put $h_i = h(e_{i^*}, e_i)$, equal to $-N$ when $g_i = 0$ and to one otherwise. The scalars $t_i = h_i \tau_i$ are given by

$$\tau_i = \begin{cases} c, & s \text{ even and } g_i > 0, \text{ or } s \text{ odd and } g_i \leq 0, \\ 1, & \text{otherwise,} \end{cases} \qquad c = -(\jmath g \varpi)^{-1}.$$

Every nonzero entry of T lowers the grade by one. Write $\mathcal{N}$ for the space of maps whose entries of negative grade difference vanish. For a finite twist set P as in Theorem 347.1, let δ_P be a finite idele with uniformizer components at P and one elsewhere. Put $d_P = \delta_g \delta_P$ and $a_P = \bar{d}_P/\alpha$, using Lemma 348.1. Thus $v_w(a_P) = -v_w(a_f) + \mathbf{1}_{\bar{P}}(w)$.

Proposition 353.2. *Take x, x' at the two consecutive stages, with moved local lattices L_w, L'_w in the notation of Definition 351.1. Suppose a finite adelic map $Z \in T + \mathcal{N}(\mathbb{A}_{E,f})$ satisfies $a_P Z L_w \subseteq L'_w$ at every place, with the local component of the scalar and map understood. Then, for every $0 \leq i < s'$,*

$$b'_i(x'; w) \;\leq\; v_w(a_P) + v_w(\tau_i) - b_{s-1-i}(x; \bar{w}).$$

Here primes on b indicate the target space.

Proof. At $\bar{w}$ take an adapted lattice vector $v \in V^+_{i^*}$ whose e_{i^*}-coefficient β generates $I_{i^*}(x; \bar{w})$. Use the extension of a lattice basis in Proposition 351.2 to choose $y \in L_w$ with $h(v, y) = 1$, annihilating $V^-_{i^*}$. From the antidiagonal form, all components of y have grade at least g_i. Indeed all string components of lower grade pair with earlier string vectors than e_{i^*}, and if $g_i > 0$ then X_j also occurs before e_{i^*}. The e_i-coefficient of y is $1/(h_i \bar{\beta})$ because the earlier terms of v pair to zero with y. When $g_i = 0$ there may also be components of y in X_j.

The map $Z - T$ has image of grade at least g_i on y. In grade $g_i - 1$ of Ty the sole component is $\tau_i e'_i/\bar{\beta}$, since T kills X_j. Hence Zy lies in the target prefix $(V'_i)^+$, with this coefficient on e'_i. In particular if $g_i - 1 = 0$ its grade-zero component has zero projection onto the target definite block, which follows the middle string line in the block order. Since $a_P Z y \in L'_w$, its projected coefficient belongs to $I'_i(x'; w)$, proving the upper bound by taking valuations. □

Lemma 353.3. *For the reference lattices the following equality holds for all the indices of Proposition 353.2:*

$$b'_i(1; w) = v_w(\tau_i) - v_w(a_f) - b_{s-1-i}(1; \bar{w}).$$

Consequently for the deviations of Corollary 351.3 that proposition gives

$$\Delta'_i(x'; w) \leq -\Delta_{s-1-i}(x; \bar{w}) + \mathbf{1}_{\bar{P}}(w).$$

Proof. The defining basis in Definition 348.2 is triangular for the full block order, with invertible diagonal blocks over the local fields. Thus each string diagonal generates its projected ideal: working backward, a vector in a prefix must have zero coefficients on all later basis columns. Denote these diagonal generators by γ_i and by γ'_i on the next string. If $s = 2h + 1$, then $u_{j+1} = u_j$ and each $\gamma_i = u_j$. For $i < h$ the next generator is u_j/a_f and $\tau_i = 1$. For $i \geq h$ the next generator is $u_j \bar{a}_f$ and $\tau_i = c$. If $s = 2h$, then $u_{j+1} = u_j \bar{a}_f/a_f$, and this generates every target string ideal. For $i < h$ the reversed source generator is $u_j \bar{a}_f$ and $\tau_i = c$. For $i \geq h$ the reversed generator is u_j/a_f and $\tau_i = 1$. In all four cases the ideal generated by γ'_i equals the one generated by $\tau_i/(a_f \bar{\gamma}_{s-1-i})$, using $u_j \bar{u}_j = 1$ and equality of

the valuations of c and $a_f\bar{a}_f$ from Lemma 348.1. This gives the first identity. Subtract it from Proposition 353.2 and insert $v_w(a_P)$ from Definition 353.1. □

354. One-sided movement of the string ideals

Theorem 354.1. *Under Proposition 353.2, assume the middle-line exception of Proposition 351.2 does not occur at either of the two points. Then at every finite place w and every $0 \le i < s-1$ one has*

$$\Delta_{i+1}(x;w) - \mathbf{1}_P(w) \;\le\; \Delta'_i(x';w) \;\le\; \Delta_i(x;w) + \mathbf{1}_{\bar{P}}(w).$$

Consider stages $0,\dots,j$ with $0 \le j < m$, starting at x_0 with all deviations zero. If the map and middle-line conditions hold at every transition with nonempty target string, then

$$-j\,\mathbf{1}_P(w) \le \Delta_i(x_j;w) \le j\,\mathbf{1}_{\bar{P}}(w) \qquad (0 \le i < m-1-j).$$

Proof. Write $i^* = s-1-i$ in the source and $i^\dagger = s-2-i$ in the target. By Corollary 351.3, $\Delta_i(x;w) + \Delta_{i^*}(x;\bar{w}) \ge 0$. Combining this with Lemma 353.3 gives the stated upper bound. Apply that lemma at the target index $i^\dagger$ and place $\bar{w}$, and negate the inequality. The lower bound follows from $\Delta'_i(x';w) \ge -\Delta'_{i^\dagger}(x';\bar{w})$ by the same corollary and $s-1-i^\dagger = i+1$. The second assertion follows by induction on the number of steps. At empty target string it is vacuous. □

Corollary 354.2. *For fixed j, P let W be any set of stage-j finite points reached by chains satisfying Theorem 354.1. Its compact representatives in Theorem 347.1 can be chosen in at most*

$$(j+1)^{\lceil (m-1-j)/2 \rceil\,|P\cup\bar{P}|}$$

translates of the standard integral subgroup. The cut for the common stage level, the kernel of reduction at the marked places and P, has every coefficient in $\mathbb{Z}[1/Q]$ with $Q = \prod_{w\in P} q_w$. When $P = \varnothing$ the sum of its absolute coefficients is at most $(g_0 g_1)^{f(r+jt)^2}$.

Proof. At each place in the indicated union exactly one indicator in Theorem 354.1 can be one. There are at most $j+1$ possible integer values there and a single value elsewhere. Apply the representative construction of Theorem 347.1 with this count. Proposition 344.2 controls the coefficients. For empty P use Proposition 347.2 and the stage rank in Definition 348.2. □

355. The annihilator of the positive-grade lattice

Definition 355.1. For consecutive $Y = Y_j, Y' = Y_{j+1}$ set $V = \mathrm{Hom}_E(Y, Y')$. For maps u, v let

$$H(u,v) = \mathrm{tr}(J^{-1}u^*J'v), \qquad [u,v]_p = \mathrm{Tr}_{E\otimes\mathbb{Q}_p/\mathbb{Q}_p}\big(\jmath H(u,v)\big),$$

where J, J' are their rational Gram matrices. This is a hermitian form and the second pairing is alternating and $\mathbb{Q}_p$-bilinear. Indeed cyclic permutation under matrix trace gives $H(v,u) = \overline{H(u,v)}$, and $\bar{\jmath} = -\jmath$. Write $\mathcal{P}$ for the space of strictly positive-grade maps, and $\mathcal{N}$ for the nonnegative-grade maps as in Definition 353.1. For a finite adelic additive subgroup D put

$$D^\circ = \{v \in V(\mathbb{A}_{E,f}) : \psi_f(\jmath H(u,v)) = 1 \text{ for all } u \in D\}.$$

The additive character ψ_f is defined in Proposition 18.5.

Lemma 355.2. *Let L, L' be the source and target lattices moved by x, x', and let $\mathcal{M} = \mathrm{Hom}_{\widehat{O}_E}(L, L')$ in the rational finite coordinates, meaning the adelic linear maps taking L into L'. Then $\alpha\mathcal{M}$ is self-dual for Definition 355.1, and $(d_P\alpha\mathcal{M})^\circ = (\alpha/\bar{d}_P)\mathcal{M}$. Over $\mathbb{Q}_p$ the exact orthogonal complement of $\mathcal{P}(E\otimes\mathbb{Q}_p)$ for $[\,,\,]_p$ is $\mathcal{N}(E\otimes\mathbb{Q}_p)$.*

Proof. In the moved isometric coordinates of Theorem 348.3, $\mathcal{M}$ consists of integral matrices and $H(u,v)$ is the sum of corresponding products $\bar{u}_{ij}v_{ij}$. By Lemma 348.1 and Proposition 18.4, $\jmath\alpha\bar{\alpha}\widehat{O}_E = \varpi^{-1}\widehat{O}_E$, the trace dual of $\widehat{O}_E$. Proposition 18.5 applied entrywise gives exact self-duality, and moving the factor d_P across the pairing gives the second assertion. In particular the local rational pairing is nonsingular: if a vector paired to zero against all vectors, all its rational scalar multiples would lie in a fixed dual lattice, forcing it to vanish.

In the rational graded coordinates J, J' pair opposite grades, including perfect pairings on grade zero. The same holds for their inverses. Expansion of the trace gives zero between map grades d, e unless $d + e = 0$. The pairing between each grade and its opposite is perfect by the nonsingularity just proved and the direct-sum decomposition. Thus annihilation of every strictly positive grade is precisely vanishing of all negative-grade components. □

Proposition 355.3. *With $D = d_P\alpha\mathcal{M}$ and $P_f = \mathcal{P}(\mathbb{A}_{E,f})$, one has*

$$(D \cap P_f)^\circ = (\alpha/\bar{d}_P)\mathcal{M} + \mathcal{N}(\mathbb{A}_{E,f}).$$

In particular, if $\psi_f(\jmath H(u,T)) = 1$ for every $u \in D \cap P_f$, there exists a Z satisfying the full map hypothesis of Proposition 353.2.

Proof. At each rational prime p let D_p be the local lattice, $P_p = \mathcal{P}(E \otimes \mathbb{Q}_p)$. A character test confined to p says exactly $[u,v]_p \in \mathbb{Z}_p$ for all $u \in D_p \cap P_p$, by Proposition 18.5. The latter intersection is a full lattice of P_p, since multiplying any finite basis by a large power of p puts it in D_p. It is saturated in D_p: if $py \in P_p$ then $y \in P_p$. Choose its $\mathbb{Z}_p$-basis by successive coordinate projections as in Lemma 23.1. Extend this to a basis of D_p by lifting a complementary residue basis as in Lemma 346.2. Extend the $\mathbb{Z}_p$-valued functional $u \mapsto [u,v]_p$ on the intersection by giving it value zero on the extra basis vectors. Its rational extension is represented by a vector z_p via the nonsingular local pairing of Lemma 355.2. Then z_p is in the dual lattice and $v_p - z_p$ annihilates P_p exactly because it vanishes on its spanning lattice. Here v_p is the local component of v. At almost all p, v_p itself is in the dual lattice, since the coordinate frames and all idele scalars are integral units or invertible integral matrices there. We may use $z_p = v_p$ there, so the components assemble into finite adeles. Lemma 355.2 gives the stated decomposition. Both summands on the right pass the character test on the left, proving the converse. For $v = T$ take $Z = (z_p)_p$. Then $Z - T \in \mathcal{N}(\mathbb{A}_{E,f})$ and $(\bar{d}_P/\alpha)Z \in \mathcal{M}$, as required. □

356. A nonzero constant term and its finite frequency

Definition 356.1. Let $V, H, \jmath$ be as in Definition 355.1. Write $\mathbb{A}_E$ for the full adeles. Use the additive character

$$\Psi(v) = \exp(-2\pi i \operatorname{Tr}_{E_\infty/\mathbb{R}}(v_\infty))\psi_f(v_f), \qquad E_\infty = E \otimes_{\mathbb{Q}} \mathbb{R}.$$

It is one on E by Proposition 18.5. With $\omega = (1+\sqrt{-d})/2$ put $B(x,y) = \Psi(\jmath\bar{\omega}H(y,x))$ on $V(\mathbb{A}_E)$. Use B_f for its restriction to finite vectors, identified with full vectors having zero infinite component. An adelic theta function here means a continuous F on $V(\mathbb{A}_E)$ obeying $F(v+r) = B(r,v)^{-1}F(v)$ for $r \in V(E)$. Define $(D_zF)(v) = B(v,z)F(v+z)$. For the positive-grade subspace $\mathcal{P}$ and rational T of Definition 353.1, its constant term at T is

$$I_T(F) = \int_{\mathcal{P}(E)\backslash\mathcal{P}(\mathbb{A}_E)} (D_TF)(v)\,\mathrm{d}v$$

with quotient probability measure.

Lemma 356.2. *The integral in Definition 356.1 exists. Each D_z preserves the transformation condition there and*

$$D_xD_y = B(x,y)D_{x+y}, \qquad D_xD_y = \Psi(\jmath H(y,x))\,D_yD_x.$$

For $u \in \mathcal{P}(\mathbb{A}_E)$, $I_0(D_uF) = I_0(F)$.

Proof. The function B is multiplicative under addition in either variable. Factoring $B(v+r,z) = B(v,z)B(r,z)$ cancels the added factor in $B(r,v+z)^{-1}$, so D_zF obeys the same condition. Expansion gives the product formula. To compute the commutator use $H(x,y) = \overline{H(y,x)}$, invariance of trace under conjugation, $\bar{\jmath} = -\jmath$ and $\omega + \bar{\omega} = 1$. Hence $B(x,y)/B(y,x) = \Psi(\jmath H(y,x))$. All the formulas use the integral trace-one element ω of Lemma 348.1.

The hermitian pairing vanishes between two positive-grade maps by Lemma 355.2, or by the same grade expansion over E. Thus on this subspace $B = 1$, and the restriction of D_TF is rationally periodic. In any rational basis of its underlying $\mathbb{Q}$-space, one coordinate of the additive adelic quotient has representatives $[0,1) \times \widehat{\mathbb{Z}}$. To find one, clear the finitely many finite denominators by a rational translation by the Chinese remainder construction of Lemma 21.1, then translate by an integer to choose the real representative. A difference of two representatives which is rational is an integer at the finite places, and equals zero by the real interval. The quotient is compact as the image of $[0,1] \times \widehat{\mathbb{Z}}$ in each coordinate. On this domain take product of length measure and the measures giving every residue coset modulo n in $\widehat{\mathbb{Z}}$ mass $1/n$. These finite measures are compatible under reduction. Equivalently use independent uniform digits in the p-adic expansions at all primes. Addition permutes the residues at each finite level and hence preserves this measure on its generating cylinder sets, by the measure extension argument of Lemma 123.1. The product measure on the fundamental domain gives an invariant probability on the quotient: translation moves the domain to another such domain and its pieces are translated back by rational elements as in Lemma 21.4. The same construction works in finitely many coordinates, including the zero-dimensional case. Continuity and compactness give integrability. Translation on this quotient now proves the last identity. □

Theorem 356.3. *With the lattices L, L' and $\mathcal{M}$ of Lemma 355.2, suppose $D_uF = F$ for $u \in d_P\alpha\mathcal{M}$ placed at the finite adeles. If $I_T(F) \neq 0$, the local lattice inequality of Proposition 353.2 holds at every place.*

Proof. For $u \in d_P\alpha\mathcal{M} \cap \mathcal{P}(\mathbb{A}_{E,f})$, Lemma 356.2 gives

$$I_T(F) = I_0(D_uD_TF) = \psi_f(\jmath H(T,u))I_0(D_TD_uF) = \psi_f(\jmath H(T,u))I_T(F).$$

Nonvanishing forces the character to be one. Conjugating the hermitian pairing inverts this character, so the same holds with $H(u,T)$. Proposition 355.3 now gives a map Z with $Z - T$ of nonnegative grade and $(\bar{d}_P/\alpha)ZL \subseteq L'$. Apply Proposition 353.2. □

357. The inclusion datum at a split level

Definition 357.1. For consecutive stage ranks n, n', use the reference finite isometric frames of Theorem 348.3. In the resulting matrix coordinates let $L_0 = \alpha M_{n',n}(\widehat{O}_E)$, with its trace pairing from Definition 355.1. On compactly supported locally constant functions of the full finite matrix variable use $D_zf(v) = B_f(v,z)f(v+z)$, as in Definition 356.1. Put $v_0 = \mathbf{1}_{L_0}$, and let $\mathcal{H}_f$ be the complex span of all D_zv_0.

Let M be the union of the marked places over g and the twist set P, with $P \cap \bar{P} = \varnothing$ and $\prod_{w\in P} q_w$ prime to gN. Thus $M \cap \bar{M} = \varnothing$. Write L_+, L_- for L_0 supported on $M, \bar{M}$, respectively, and set

$$C_+ = L_+/d_PL_+, \qquad C_- = \bar{d}_P^{-1}L_-/L_-.$$

For $a \in C_+$, $z \in C_-$, put $\chi_z(a) = \psi_f(\jmath H(z,a))$ using any lifts. At each marked or twist place identify the two stage index sets with an initial block of n and its enlargement by t indices, as in their ranks in Definition 348.2. Denote by $a_{\mathrm{inc}} \in C_+$ the residue matrix of the standard inclusion $\binom{I_n}{0}$ in the coordinates divided by α.

Proposition 357.2. *The groups C_+, C_- are in perfect duality by χ, and have order $D = ((g_0 g_1)^f Q)^{nn'}$, $Q = \prod_{w \in P} q_w$. The vector*

$$f_a = D^{-1} \sum_{z \in C_-} \chi_z(-a)\, D_z v_0 \qquad (a \in C_+)$$

is well defined. On the span of the $D_z v_0$, $z \in C_-$, the map $D_z v_0 \mapsto (\chi_z(b))_{b \in C_+}$ is a linear isomorphism onto functions on C_+. It sends f_a to the indicator of a. Every coefficient in the formula is a root of unity divided by D. Its coefficient sum in absolute value is one after any automorphism of $\mathbb{C}$.

Proof. The lattice L_0 is self-dual by Lemma 355.2. The trace character on $\jmath H(L_0, L_0)$ is one even after multiplying either argument by an integral element. In particular $B_f = 1$ on $L_0 \times L_0$, since ω is integral. This gives $D_l v_0 = v_0$ for $l \in L_0$. Hermitian pairings between two vectors supported on $\bar{M}$ vanish: conjugating the first argument moves its support into M. It follows from the product law of Lemma 356.2 that $D_z v_0$ is unchanged by adding L_- to z. Distinct classes in C_- give disjoint support cosets of L_0, so these vectors are linearly independent.

The pairing defining χ is trivial on adding L_- to z or $d_P L_+$ to a, by the same duality. If χ_z is identically one, the character tested at z is trivial on all of L_0, since only its M-components pair with z. Hence $z \in L_-$. Locally d_P is a uniformizer times a unit at each member of M, and invertible elsewhere, by Lemma 13.2 and Definition 353.1. Both quotients are products of nn'-dimensional residue vector spaces of the same orders. Proposition 347.2 counts the marked places, giving the formula for D. The resulting D distinct characters are a basis of functions on C_+: two distinct ones are orthogonal because translation by an element where their ratio differs from one multiplies their inner product by that ratio. Thus the character matrix divided by $\sqrt{D}$ is unitary. Its inverse gives the displayed formula and its image is the indicated point mass. Character values here are roots of unity since C_+ is finite. Every automorphism preserves their orders and fixes the integer D, proving the size assertion. □

Corollary 357.3. *The actual residue indicator $f_{a_{\mathrm{inc}}}$ is fixed by D_u for all $u \in d_P L_0$. In its expansion in Proposition 357.2, multiplication by $(g_0 g_1)^{fnn'}$ makes every coefficient integral at all places away from Q. For integral stage isometries k, k', the change of variables $v \mapsto k'^{-1} v k$ on functions sends f_a to $f_{k' a k^{-1}}$. In particular the inclusion indicator is fixed when the two reductions commute with the inclusion on M.*

Proof. Since $u \in L_0$, it fixes v_0. The commutator with each D_z is one by the duality calculation of Lemma 355.2, since $z \in \bar{d}_P^{-1} L_0$. This proves the period assertion termwise. After the indicated multiplication the denominator is $Q^{nn'}$, and roots of unity and their inverses satisfy monic equations over $\mathbb{Z}$. Integral isometries preserve L_0, H and B_f. They therefore preserve its indicator and carry each translated vector $D_z v_0$ to $D_{k' z k^{-1}} v_0$. They likewise preserve the pairing of the two residue modules. Reindex the finite sum of Proposition 357.2 to get the last assertion. □

358. Transport between the finite oscillator lattices

Theorem 358.1. *Use $\mathcal{H}_f, L_0, D_z, v_0$ of Definition 357.1, with the inner product of squared norm $\int |f|^2$ for additive Haar measure giving L_0 mass one. For any finite adelic isometry G of H in Definition 355.1, write $L_1 = G L_0$ and $h = [L_1 : L_0 \cap L_1]$. Then*

$$v_1 = h^{-1/2} \sum_{r \in L_1/(L_0 \cap L_1)} D_r v_0$$

is a unit vector. There is a unique unitary linear bijection W_G of $\mathcal{H}_f$ sending v_0 to v_1 and satisfying $W_G D_z = D_{Gz} W_G$ for all z. Every unitary bijection with this intertwining property is a scalar of modulus one times W_G. If K_0 is an isometry preserving L_0,

$$W_{K_0} f(v) = f(K_0^{-1} v), \qquad W_{GK_0} = W_G W_{K_0}, \qquad W_{K_0 G} = W_{K_0} W_G.$$

Proof. Each lattice is compact open. Its intersection with the other has finite index by an open coset cover. Both are self-dual and B_f is one on pairs in each, by Lemma 355.2, Proposition 357.2 and invariance under G. Hence $D_r v_0$ in the displayed sum is independent of the chosen lift of its coset. Translation preserves additive measure and all factors of B_f have modulus one, so the D_r are unitary. Distinct summands have disjoint supports and norm one. Left translation by any $D_l,\ l \in L_1$, permutes them with factor one. Thus v_1 has norm one and is fixed by L_1.

For $a = 0, 1$, the translates $D_y v_a$ at distinct classes modulo L_a are orthogonal. Indeed the commuting unitary operators $D_l,\ l \in L_a$, act on the translate by the character $l \mapsto \psi_f(\jmath H(y, l))$ by Lemma 356.2. Distinct classes give distinct characters by self-duality. Applying an operator whose eigenvalues differ forces the inner product to vanish. Within one class there is only the relation $D_{y+l} v_a = B_f(y, l)^{-1} D_y v_a$.

The span of the $D_y v_1$ contains v_0. Average $D_l v_1$ over $l \in L_0$. In the displayed expansion the average of the term $D_r v_0$ vanishes for $r \notin L_0$, by translation of a nontrivial character. The identity coset contributes $h^{-1/2} v_0$. This is a finite average: the finitely many characters appearing have open kernels by the local constancy of ψ_f in Proposition 18.5, and the common kernel has finite index. The span is invariant under all D_y by the product rule. Hence the translates of v_1 span $\mathcal{H}_f$, just as those of v_0 do by definition. The map $D_y v_0 \mapsto D_{Gy} v_1$ now respects all relations and sends an orthonormal basis to one, giving the required bijection. An operator commuting with all translations sends v_0 into the L_0-fixed line by the character and spanning calculation for $a = 0$. It must therefore be scalar on the entire span. This gives uniqueness with the indicated normalization.

An integral isometry K_0 fixes v_0, and change of variables by K_0^{-1} intertwines the translations, giving the first formula. The lattices of GK_0 and G coincide. For $K_0 G$ the displayed average is the image under W_{K_0} of that for G, by reindexing cosets. Uniqueness gives the two product formulas. □

Corollary 358.2. *Take G acting on maps by $v \mapsto x' v x^{-1}$ in the reference isometric coordinates of two consecutive stages. In rational coordinates this action sends the reference module of integral maps to $\mathcal{M} = \mathrm{Hom}_{\widehat{O}_E}(L, L')$. The vector $W_G f_{a_{\mathrm{inc}}}$ is fixed by every $D_u,\ u \in d_P \alpha \mathcal{M}$. Multiplication by any complex scalar preserves that conclusion. If a linear map of this finite space into the adelic theta functions of Definition 356.1 intertwines the finite translations, every nonzero I_T of the image of this vector gives the map of Proposition 353.2.*

Proof. Conjugation on rectangular matrices preserves the hermitian trace pairing because both group elements are isometries. For $u \in d_P \alpha \mathcal{M}$, $G^{-1} u$ belongs to the corresponding reference period lattice. Apply Corollary 357.3 and the intertwining identity of Theorem 358.1. The period test survives the intertwining map, so Theorem 356.3 applies. □

359. Adelic realization of the finite period test

Theorem 359.1. *For the rational space V in Definition 355.1, write $V_\infty = V(E \otimes_{\mathbb{Q}} \mathbb{R})$. For a Schwartz function ϕ on this real vector space and $f \in \mathcal{H}_f$ in the reference rational coordinates, set*

$$\Theta_{\phi,f}(y) = \sum_{\gamma \in V(E)} B(\gamma, y) \phi(\gamma_\infty + y_\infty) f(\gamma_f + y_f).$$

A Schwartz function has continuous derivatives of all orders, each bounded after multiplication by any polynomial in real coordinates. This series converges absolutely, locally uniformly in y, and defines an adelic theta function of Definition 356.1. For each fixed ϕ, the map $f \mapsto \Theta_{\phi,f}$ intertwines all finite D_u. If $\phi(y) = \exp(-\|y\|^2)$ in any real Euclidean coordinates, it is injective.

Proof. Every f in Definition 357.1 is bounded, locally constant and compactly supported. On a compact set of y, the rational points contributing to the series have finite components

in a fixed compact set. Hence they lie in $D^{-1}O_E^{nn'}$ in a rational matrix basis, for some integer $D \geq 1$. Indeed the finite adele space is the increasing union of integer-denominator lattices, for example with factorial denominators, and integral at all places means integral in O_E by Proposition 44.3. In real coordinates this is a lattice by Lemma 82.2. A ball of radius $R \geq 1$ contains at most $C(1+R)^{\dim_{\mathbb{R}} V_\infty}$ of its points, by applying the inverse of a real basis matrix and counting integer coordinates. On compact sets of y_∞ the Schwartz bound gives an arbitrarily large inverse power of $1 + \|\gamma_\infty\|$. Summing over dyadic shells proves absolute locally uniform convergence. Each summand is continuous, so the sum is continuous.

For $r, \gamma \in V(E)$, $B(\gamma, r) = 1$ by Proposition 18.5 and rationality of the trace pairing. Reindex γ by addition of r to obtain the factor $B(r, y)^{-1}$ under $y \mapsto y + r$. For a finite u, the identity $B(y, u)B(\gamma, y + u) = B(\gamma, y)B(\gamma + y, u)$ gives $D_u \Theta_{\phi,f} = \Theta_{\phi, D_u f}$ term by term.

The kernel of any such intertwining map is invariant under all D_u. Any nonzero invariant subspace of $\mathcal{H}_f$ is the whole space. In fact expand a nonzero vector in finitely many distinct $D_z v_0$. Averaging over L_0 against the inverse of one occurring eigencharacter isolates that basis vector by the character calculation of Theorem 358.1. The average is a finite sum there. Translating then gives all the basis vectors. For the specified Gaussian the value $\Theta_{\phi,v_0}(0)$ is positive, since all terms are nonnegative and the term at zero equals one. Therefore the kernel is zero. □

Corollary 359.2. *For x, x' at successive stages with nonempty target string let G be their action on the finite Hom space in Corollary 358.2. For every Schwartz ϕ take $F = \Theta_{\phi, W_G f_{a_{\mathrm{inc}}}}$, or any scalar multiple. If $I_T(F) \neq 0$, the deviations satisfy the upper bound of Lemma 353.3. If the middle-line condition holds at x, x', the two-sided bounds of Theorem 354.1 hold.*

Proof. Theorem 359.1 supplies the finite intertwiner in Corollary 358.2. Use the resulting lattice map in Proposition 353.2 and Lemma 353.3. The lower bound is then exactly the deduction of Theorem 354.1. □

360. The split local action on the inclusion datum

Theorem 360.1. *Fix a finite set M of finite places of E disjoint from $\bar{M}$. In the coordinates of Definition 357.1, let X, Y denote the full spaces of matrices supported on $M, \bar{M}$, respectively. Let $\Lambda = X \cap L_0$, $\Lambda^\vee = Y \cap L_0$. Give X its additive measure with Λ of mass one. The factor of $\mathcal{H}_f$ at $M \cup \bar{M}$ identifies isometrically with the locally constant compactly supported functions on X, with vacuum $\mathbf{1}_\Lambda$. For $x \in X$ and $y \in Y$ the identification takes D_x, D_y to*

$$(L_x b)(t) = b(t + x), \qquad (M_y b)(t) = \psi_f(\jmath H(y, t))b(t).$$

If G is an E-linear isometry in the finite coordinates, acting trivially off these pairs, and $A = G|_X$, then W_G of Theorem 358.1 acts on this factor by

$$(U_A b)(t) = J_A^{-1/2} b(A^{-1}t), \qquad J_A = \prod_{w \in M} |\det A_w|_w.$$

Here $|u|_w = q_w^{-v_w(u)}$. In particular $W_G W_{G'} = W_{GG'}$ for isometries supported on these pairs.

Proof. Write $\chi_y(t) = \psi_f(\jmath H(y, t))$. The vanishing of H on $X \times X$ and $Y \times Y$, and the self-duality in Lemma 355.2, give $\Lambda^\vee = \{y : \chi_y(\Lambda) = 1\}$, with the reverse annihilator equal to Λ. The bicharacter B_f on separate pairs of places is one. Accordingly a translated vacuum splits as the product of its translated vacua on $X \oplus Y$ and the complementary places. Products of basis vectors in the two spaces have disjoint coset supports and norm one, giving the asserted factor with the product inner product.

For $x + y$ put $\widetilde{D}_{x+y} = B_f(x, y)^{-1} L_x M_y$. One has $L_x M_y = \chi_y(x) M_y L_x$, and $\chi_y(x) = B_f(x, y)/B_f(y, x)$ by Lemma 356.2. To multiply two such operators reorder the middle $M_y L_{x'}$, acquiring the factor $\chi_y(x')^{-1}$. After factoring out $\widetilde{D}_{x+x'+y+y'}$, the total factor is $B_f(x, y')B_f(y, x') = B_f(x + y, x' + y')$. The vacuum is fixed by each $\widetilde{D}_l$, $l \in \Lambda \oplus \Lambda^\vee$, since

B_f is one on $L_0 \times L_0$. The translates with distinct X/Λ coordinates have disjoint support. On one support coset distinct $Y/\Lambda^\vee$ coordinates have orthogonal characters, by integrating their ratio and translating in Λ. Thus sending $D_z v_0$ on this factor to $\widetilde{D}_z \mathbf{1}_\Lambda$ preserves the norms and the relations within each lattice coset, as listed in Theorem 358.1.

A locally constant compactly supported function meets finitely many cosets of Λ. Compactness gives, on each of them, invariance under $\prod_{w \in M} \pi_w^{e_w} \Lambda_w$ for some $e_w \geq 0$. Indeed finitely many open constancy balls cover the coset, and a common smaller subgroup is contained in the translation subgroup of each ball. The characters from the opposite inverse powers of the uniformizers form a basis of functions on this finite quotient. The annihilator argument and equal-order count of Proposition 357.2 apply with e_w successive residue quotients at w. This proves surjectivity.

An invertible matrix over E_w changes additive measure by $|\det A_w|_w$. Gaussian elimination uses coordinate permutations, row additions and scalings. Permutations preserve the product measure, additions preserve it by translation in the changed coordinate, and multiplication by a uniformizer gives the sublattice of index q_w. A unit permutes every finite residue quotient. These checks on generators give the formula. Consequently U_A is unitary. Direct substitution gives $U_A L_x U_A^{-1} = L_{Ax}$ and $U_A M_y U_A^{-1} = M_{Gy}$, using invariance of H. The inner product of $U_A \mathbf{1}_\Lambda$ with $\mathbf{1}_\Lambda$ is strictly positive, since $A\Lambda \cap \Lambda$ is open. The normalized vacuum for W_G also has positive inner product with the reference vacuum: only its identity coset contributes to the average in Theorem 358.1. The two intertwiners differ by a scalar of absolute value one by that theorem. Positivity forces it to be one. The product law follows from $J_{AA'} = J_A J_{A'}$. □

Corollary 360.2. *For M as in Definition 357.1, the vector f_a on these pairs is $\mathbf{1}_{a+d_P\Lambda}$ in the preceding realization. If the two stage elements act at the w-component by invertible matrices g_w, g'_w of orders n, n', set $A_w t = g'_w t g_w^{-1}$. Their moved indicator on X is*

$$J_A^{-1/2} \mathbf{1}_{A(a+d_P\Lambda)}, \qquad J_A = \prod_{w \in M} |\det g'_w|_w^n |\det g_w|_w^{-n'}.$$

Proof. On Λ the Fourier sum of Proposition 357.2 equals one on the indicated residue coset and zero elsewhere by character orthogonality. Every term vanishes outside Λ. Apply Theorem 360.1. Left multiplication acts on the n columns and right multiplication on the n' rows, giving the determinant factors. □

361. Coefficients of a moved residue coset

Proposition 361.1. *Use M, Λ, A, J_A, a of Corollary 360.2. Choose a representative of a in Λ. Write*

$$H_1 = A(d_P\Lambda), \qquad H_2 = H_1 \cap \Lambda, \qquad u = [H_1 : H_2], \qquad v = [\Lambda : H_2].$$

For any subgroup $H \subset X$ write $H^\vee = \{y \in Y : \chi_y(H) = 1\}$, using the character of Theorem 360.1. The moved inclusion vector for a has an expansion with uv distinct translated vacuum cosets. Every coefficient is a root of unity times $J_A^{-1/2}/v$. Its sum of absolute coefficient values under each field automorphism of $\mathbb{C}$ is $uJ_A^{-1/2}$. Moreover

$$\frac{u}{v} = \frac{J_A}{[\Lambda : d_P\Lambda]}.$$

Proof. The intersection of two compact open lattices has finite index in each by compactness. The cosets of Λ occupied by $Aa + H_1$ have representatives $b \in Aa + H_1$, one for each of the u cosets modulo H_2 there. Inside $b + \Lambda$ the desired indicator is

$$\mathbf{1}_{b+H_2}(t) = \frac{1}{v} \sum_{y \in H_2^\vee/\Lambda^\vee} \chi_y(-b)\, (M_y L_{-b} \mathbf{1}_\Lambda)(t).$$

Indeed H_2 contains a product of uniformizer powers $\Lambda' \subset \Lambda$. The proof of Theorem 360.1 supplies a character basis on Λ/Λ'. Averaging each character under the translations by H_2/Λ' kills it unless it is trivial on that subgroup. The surviving characters are therefore a basis of the functions on Λ/H_2 and number v. Averaging them gives the indicator of its identity element: the columns of their square character matrix are orthogonal by the finite Fourier calculation in Proposition 357.2. Multiplication by $\chi_y(-b)$ centers this indicator at b.

Each $M_y L_{-b}$ is D_{y-b} times a value of B_f by Lemma 356.2. Such values and the values $\chi_y(-b)$ are roots of unity by the finite product of rational fractional-part characters in Proposition 18.5. The first coordinate distinguishes the u supports and the second distinguishes the v modulations on each, giving the claimed distinct cosets. Multiply by the amplitude $J_A^{-1/2}$ from Corollary 360.2. Its square is a positive rational number, so any automorphism sends it to a root with the same absolute value. Coefficient magnitudes now give the stated sum. Finally H_2 has measure $1/v$, and H_1 has measure both u/v and $J_A/[\Lambda : d_P\Lambda]$, which proves the identity. □

Corollary 361.2. *Let $Q = \prod_{w \in P} q_w$ and suppose the finite stage elements are integral away from $P \cup \bar{P}$, in their reference isometric coordinates. For any a, multiplying the coefficients of the moved f_a by $(g_0 g_1)^{fnn'}$ makes them integral at every finite place with residue characteristic prime to Q. The translating vectors can be supported on $P \cup \bar{P}$ and the conjugates of the marked places. At those conjugates they lie in $\bar{d}_P^{-1} L_0$. For $Q = 1$ the sum of the absolute coefficient values under every complex field automorphism is one.*

Proof. On each integral complementary pair or nonsplit place the lattice is fixed, so the action preserves the vacuum by Theorem 358.1. That theorem also allows the integral part and the part supported on $P \cup \bar{P}$ to be composed in either order when their factors are disjoint. At a marked w, $A_w \Lambda_w = \Lambda_w$ and $H_{2,w} = d_P \Lambda_w$. In the preceding expansion the translation of the support there belongs to Λ_w and can be dropped at the cost of a root of unity, by the lattice-coset relation in Theorem 358.1. Only the modulation from the opposite $\bar{d}_P^{-1} \Lambda_w^\vee$ remains. The marked factors of v multiply to $(g_0 g_1)^{fnn'}$ by Proposition 357.2. Every other factor of v, and the numerator and denominator of J_A in lowest terms, has prime divisors only among those of Q: the indices over w are orders of quotients of local lattices killed by a uniformizer power. Their sizes are powers of the residue characteristic by successive residue quotients. Thus $J_A^{-1/2}$ has valuation zero away from Q, by squaring. Roots of unity likewise have valuation zero by raising to their orders. This proves the denominator assertion using Proposition 361.1. If $Q = 1$, all changes are integral, so $u = 1$, $J_A = 1$ in that proposition. □

362. Reduction of the finite oscillator to degree zero

Lemma 362.1. *In the finite adelic space of Definition 355.1 let U be the strictly positive-grade space, W the degree-zero space, and $N_+ = U \oplus W$ the nonnegative-grade space. Let $\pi : N_+ \to W$ be the projection. If L is a moved reference lattice $\alpha\mathcal{M}$ as in Lemma 355.2, then $\Lambda = \pi(L \cap N_+)$ is a compact open self-dual lattice in W for the restricted trace character. The bicharacter B_f is one on $\Lambda \times \Lambda$.*

Proof. The same grade pairing as in Lemma 355.2 gives $U^\perp = N_+$ and $N_+^\perp = U$. Here the orthogonals are taken for the local rational trace pairings and assembled over the finite places. In particular, the hermitian form on N_+ is the pullback of the nonsingular form on W. The projection is open in the direct-sum topology, and $L \cap N_+$ is compact open in N_+. Its image is therefore compact open. The calculation of Proposition 355.3 also gives

$$(L \cap U)^\circ = L + N_+, \qquad (L \cap N_+)^\circ = L + U.$$

Indeed at each prime the intersection with either linear subspace is saturated in L_p. Extend the tested integral functional by zero on an integral basis complement and use self-duality to

represent the extension in L_p, taking the tested vector itself at almost all primes as in that proposition. For $w \in W$ annihilating Λ, the second equality gives $w = l + u,\ l \in L,\ u \in U$; then $l \in N_+$ and $w = \pi l$. Conversely L annihilates itself and the pairing descends on N_+. This proves self-duality. The same descent applies to B_f, which is one on $L \times L$ by Proposition 357.2 and transport of the reference lattice. □

Theorem 362.2. *For $L, U, W, N_+, \pi, \Lambda$ as in Lemma 362.1, form $\mathcal{H}(V, L)$ as the span of the $D_y \mathbf{1}_L$ using B_f. Similarly form $\mathcal{H}(W, \Lambda)$ on W with the restricted bicharacter and denote its translations by D^0_w. There is a unique surjection $R : \mathcal{H}(V, L) \to \mathcal{H}(W, \Lambda)$ with $R\mathbf{1}_L = \mathbf{1}_\Lambda,\ RD_w = D^0_w R$ for $w \in W$, and $RD_u = R$ for $u \in U$. Its kernel is*

$$\operatorname{span}_{\mathbb{C}}\{D_u f - f : u \in U,\ f \in \mathcal{H}(V, L)\}.$$

If $y \notin L + N_+$, then $RD_y \mathbf{1}_L = 0$. Otherwise, for any $y = l + n$ with $l \in L,\ n \in N_+$,

$$RD_y \mathbf{1}_L = B_f(n, l)^{-1} D^0_{\pi n} \mathbf{1}_\Lambda.$$

Proof. Write $e_y = D_y \mathbf{1}_L$. As in Theorem 358.1, the lines $\mathbb{C}e_y$ have disjoint supports at distinct classes in V_f/L, where V_f denotes the ambient finite adelic space. For $l \in L$, $e_{y+l} = B_f(y, l)^{-1} e_y$. First take the quotient by the asserted kernel span and denote classes by brackets. The operators D_u for $u \in U$ represent addition exactly, since B_f is one on $U \times U$. They permute the basis lines along their translation orbits. For $u \in L \cap U$ the action on e_y is multiplication by $\psi_f(\jmath H(y, u))$ by Lemma 356.2. Thus the class vanishes unless $y \in L + N_+$, by Lemma 362.1. On a surviving orbit take $y = l + n$ and write $n = w + u$ with $w = \pi n$. The lattice-coset relation and $B_f(u, w) = 1$ give $[e_y] = B_f(n, l)^{-1}[e_w]$.

Every such orbit contributes exactly one dimension. On the orbit of $w \in W$, the vectors $D_u e_w$ in the same basis line coincide, since their indices differ by an element of $U \cap L$, which fixes e_w. Assigning value one to each of these vectors gives a well-defined invariant functional on the orbit span, and hence $[e_w] \neq 0$. Distinct orbits give independent quotient spaces because every generating difference belongs to the sum of the separate orbit relations, on expanding f in basis lines.

The orbits through $w, w' \in W$ coincide exactly when $w' - w \in (L + U) \cap W = \Lambda$. All $D_w,\ w \in W$, commute with D_u and thus descend. If $\lambda \in \Lambda$, lift it to $l \in L \cap N_+$. Writing $l = \lambda + u$ gives $[e_\lambda] = [e_0]$. Thus the relations on the surviving vectors are exactly

$$[e_{w+\lambda}] = B_f(w, \lambda)^{-1}[e_w] \qquad (w \in W,\ \lambda \in \Lambda).$$

These are the relations among $D^0_w \mathbf{1}_\Lambda$: the vacuum is fixed by Lemma 362.1, and distinct support cosets are disjoint. Sending the bracketed vectors to these translates identifies both spaces, respecting the product law on W. This constructs the claimed surjection with its exact kernel. Every map satisfying the three conditions factors through the quotient and has the prescribed value on each of its spanning translates, giving uniqueness. □

363. An integral lift of the residual space

Proposition 363.1. *Use the notation of Theorem 362.2. Let W_s be another finite adelic hermitian space with the same scalar $\jmath$, bicharacter defined as in Definition 356.1, and compact open self-dual lattice L_s on whose square the bicharacter is one. Suppose $h : W_s \to W$ is an isometry onto W, and $b : W_s \to U$ is linear over the finite adeles of E. Put $B = h + b : W_s \to V_f$ and assume $B(L_s) \subset L$. Let $B^\dagger : V_f \to W_s$ denote its hermitian adjoint, so $H_s(t, B^\dagger y) = H(Bt, y)$. Then*

$$hL_s = \Lambda, \qquad B^\dagger L \subset L_s, \qquad B^\dagger n = h^{-1}\pi n \quad (n \in N_+).$$

If $Z : \mathcal{H}(V, L) \to \mathcal{H}(W_s, L_s)$ is linear and $ZD_u = Z,\ ZD_{hw} = D^s_w Z$ for $u \in U,\ w \in W_s$, there exists a common scalar C such that for every $y \in V_f$

$$ZD_y \mathbf{1}_L = C\zeta_y D^s_{B^\dagger y} \mathbf{1}_{L_s}.$$

One can take $\zeta_y = 0$ for $y \notin L + N_+$ and ζ_y a root of unity for $y \in L + N_+$.

Proof. The hermitian adjoint exists since the form on W_s is nonsingular; in coordinate bases it is a product of the inverse Gram matrix, the conjugate transpose of B, and the ambient Gram matrix. Projecting BL_s gives $hL_s \subset \Lambda$. Both lattices are self-dual by Lemma 362.1 and invariance of the pairing under h. Taking annihilators reverses the inclusion and proves equality. For $l \in L$, the functional given by $B^\dagger l$ has trivial trace character on L_s, since $BL_s \subset L$. This proves the second inclusion. For $n \in N_+$, the pairing with b vanishes, and $H(ht, n) = H_s(t, h^{-1}\pi n)$, proving the last equality.

For $t \in L_s$, $D_{Bt}\mathbf{1}_L = \mathbf{1}_L$; since $Bt = ht + bt$ with mutually orthogonal components, applying Z gives $D_t^s Z\mathbf{1}_L = Z\mathbf{1}_L$. The fixed space is the vacuum line by the character argument in Theorem 358.1: the eigencharacters on the distinct translated basis lines are distinct by self-duality. Write $Z\mathbf{1}_L = C\mathbf{1}_{L_s}$. By its U-invariance Z kills the kernel in Theorem 362.2. For $y = l + n$ as there it sends $D_y\mathbf{1}_L$ to $CB_f(n, l)^{-1}D^s_{h^{-1}\pi n}\mathbf{1}_{L_s}$. Adding $B^\dagger l \in L_s$ to the index of this last translation changes the vector by a value of the finite bicharacter. All such factors are roots of unity by Proposition 18.5, and $B^\dagger y = B^\dagger l + h^{-1}\pi n$. This proves the formula, including when $C = 0$. □

Corollary 363.2. *Under Proposition 363.1, let $\tau \in V_f$ satisfy $B^\dagger\tau = 0$. There are coefficients ϵ_y, each zero or a root of unity, such that*

$$ZD_\tau D_y\mathbf{1}_L = C\epsilon_y D^s_{B^\dagger y}\mathbf{1}_{L_s} \qquad (y \in V_f).$$

If $y \mapsto c_y$ has finite support, this substitution in the sum over $c_y D_y\mathbf{1}_L$, with the common C factored outside, preserves every common algebraic denominator for the c_y and every upper bound for their total absolute value under all field automorphisms of $\mathbb{C}$.

Proof. Multiply the product formula in Lemma 356.2 by the formula of Proposition 363.1 at $\tau + y$. The new character factor is $B_f(\tau, y)$; $B^\dagger(\tau + y) = B^\dagger y$. Roots of unity are integral units in each finite valuation ring and have absolute value one after every field automorphism. Adding any coefficients at coincident images preserves the denominator in each valuation ring, and the triangle inequality preserves the bound. □

364. Combining the two finite oscillator spaces

Theorem 364.1. *For consecutive stage spaces Y, Y', Y'' of Definition 348.2 put*

$$V_1 = \mathrm{Hom}_E(Y, Y'), \quad V_2 = \mathrm{Hom}_E(Y', Y''), \quad V_* = \mathrm{Hom}_E(Y', Y^- \oplus Y'').$$

The superscript minus negates the hermitian form. Give these spaces their trace hermitian forms of Definition 355.1 and use the associated finite bicharacters B_1, B_2, B_*. Write $L_Y, L_{Y'}, L_{Y''}$ for the three stage lattices in the isometric frames of Theorem 348.3, and put $L_i = \alpha\,\mathrm{Hom}(L_{\mathrm{source}}, L_{\mathrm{target}})$ in V_i, $i = 1, 2$, with integral finite-adelic Hom. Define

$$A(a, b) = (a^\dagger, b), \qquad L_* = A(L_1 \oplus L_2) = \bar\alpha\,\mathrm{Hom}(L_{Y'}, L_Y) \;\oplus\; \alpha\,\mathrm{Hom}(L_{Y'}, L_{Y''}).$$

Here $a^\dagger$ is the adjoint with respect to the original forms on Y, Y'. There is an isometric complex linear identification of the algebraic tensor product

$$\mathcal{H}(V_1, L_1) \otimes_{\mathbb{C}} \mathcal{H}(V_2, L_2) \;\longrightarrow\; \mathcal{H}(V_*, L_*).$$

It sends the tensor of the vacua to $\mathbf{1}_{L_*}$ and intertwines $D^1_a \otimes D^2_b$ with $\lambda(a)D^*_{A(a,b)}$, where

$$\lambda(a) = \psi_f\big(-\bar\omega H_1(a, a)\big).$$

The factor $\lambda(a)$ is a root of unity. The identification commutes with the normalized lattice intertwiners for every triple of finite stage isometries acting on the three Hom spaces.

Proof. In isometric coordinates conjugate transpose carries all integral rectangular matrices bijectively onto the integral transposed rectangular matrices. Moving the scalar α through it

gives $\bar{\alpha}$. Thus A is an additive bijection with the asserted lattice image. For $z = (a, b),\ z' = (a', b')$, matrix trace and cyclic permutation give

$$H_*(Az', Az) = -\overline{H_1(a', a)} + H_2(b', b).$$

In fact the trace of $a'a^\dagger$ equals the conjugate of the trace of $a'^\dagger a$. The minus sign is exactly the one on the first output space. Conjugation preserves the trace in ψ_f and negates $\jmath$. Hence the alternating trace pairing is preserved. In particular L_* is self-dual by Lemma 355.2. For the bicharacters, with $s = H_1(a', a)$, the quotient is

$$\frac{B_*(Az, Az')}{B_1(a, a')B_2(b, b')} = \psi_f\big(\jmath(\omega - \bar{\omega})s\big) = \psi_f(-s) = \frac{\lambda(a + a')}{\lambda(a)\lambda(a')}.$$

The middle equality uses $\jmath(\omega - \bar{\omega}) = -1$ from Lemma 348.1. For the last, the two mixed terms are $s, \bar{s}$; under the trace their coefficients $\bar{\omega}, \omega$ add to one. All finite character values are roots of unity by Proposition 18.5. For $a, a' \in L_1$ the pairing belongs to $\alpha\bar{\alpha}\widehat{O}_E$. This is contained in the trace dual since $\jmath\alpha\bar{\alpha}\widehat{O}_E$ is the trace dual and $\jmath^{-1}$ is integral, by Lemma 355.2 and Proposition 18.4. It follows that λ is one on L_1 and the preceding quotient is one on pairs of lattice vectors. Thus B_* is one on $L_* \times L_*$.

The operators $\lambda(a)D^*_{Az}$ now obey the same product law as the tensor operators, and they fix the new vacuum for $z \in L_1 \oplus L_2$. Defining the map on the translated vacuum tensors by this rule respects the lattice-coset relations of Theorem 358.1. It takes the orthonormal basis indexed by pairs of lattice cosets to an orthonormal basis since A is bijective. This proves the identification. For a triple of isometries the first transformed adjoint is $ga^\dagger g'^{-1}$ if the original map becomes $g'ag^{-1}$. Also $\lambda(g'ag^{-1}) = \lambda(a)$. Consequently the two transported operators intertwine the same translations and differ by a scalar of absolute value one, by the vacuum-line argument of Theorem 358.1. Each has positive overlap of the moved vacuum with the original vacuum by the lattice average of that theorem, taking the product of the two overlaps for the tensor operator. Hence the scalar is one. □

365. The rational residual space for a pair of lifts

Proposition 365.1. *Let j be even with $0 \le j \le m - 2$ and set $Y = Y_j,\ Y' = Y_{j+1},\ Y'' = Y_{j+2}$, $s = m - 1 - j$. In $V_* = \mathrm{Hom}(Y', Y^- \oplus Y'')$ of Theorem 364.1, grade a matrix entry by the target grade minus the source grade as in Definition 353.1. Put $X' = X_{j+1}$. Let Q_0 be the grade-zero space of $Y^- \oplus Y''$. The degree-zero subspace of V_* is $\mathrm{Hom}(X', Q_0)$, embedded by zero on the string of Y'.*

If $s \ge 3$, write z, z'' for the central string vectors of Y, Y''. There is an orthogonal decomposition

$$Q_0 = E\eta \oplus E\xi \ \perp\ (X_j^- \oplus X_{j+2}), \qquad \xi = z + z'', \qquad \eta = \frac{z - z''}{2N},$$

where η, ξ have zero norms and pairing one. If $s = 1$, write z for the unique string vector of Y. Then

$$Q_0 = X_j^- \ \perp\ (Y_m \oplus E(z/\pi_N)),$$

and the last displayed vector has norm one. The action of $U(Y_m)$ on the enlarged positive space fixes that vector. The determinant of the enlarged action equals the original determinant. Here $U(S)$ is the group preserving the hermitian form on the indicated space S.

Proof. The odd strings in the outside spaces have even grades, while the even string of Y' has only odd grades. The definite summands have grade zero. Consequently equality of source and target grades forces both to be zero. This also applies when a string is empty, by Definition 348.2. For $s \ge 3$, negating the first outside form gives norms $N, -N$ on z, z''; the vectors are perpendicular. The displayed linear combinations give the indicated pairing and span those two lines since $N \neq 0$. Their orthogonal complement is the sum of the definite blocks. For $s = 1$ the other outside space is the final positive space Y_m; the norm of z/π_N in

Y^- is $N/(\pi_N\bar\pi_N)=1$ by Proposition 12.2. The extension of the last action is block diagonal with one added identity entry. □

Proposition 365.2. *For the stages of Proposition 365.1 let $T: Y\to Y'$ and $T': Y'\to Y''$ be the maps of Definition 353.1 when their target string is nonempty. Set the map to zero if either string is empty. The combined map $\tau=(T^\dagger,T')\in V_*(E)$ annihilates X'. If $s=2d+1\ge 3$, its projection to the grade-zero target Q_0 vanishes on every basis vector of Y' except e'_{d-1}, where its value is $c\xi$. If $s=1$, $\tau=0$.*

Proof. The forward maps kill the corresponding definite summands and have image in the string. The adjoint $T^\dagger$ therefore kills X'. Each nonzero entry of T,T' lowers grade by one. The adjoint $T^\dagger$ also lowers grade by one: pairing a vector of grade h with Ta can be nonzero only if a has grade $1-h$, and this tests precisely the component of $T^\dagger$ in grade $h-1$. For $s=2d+1$, the only basis vector of grade one in Y' is e'_{d-1}. The coefficient of T' there is c, so $T'e'_{d-1}=cz''$. In Y the coefficient at $z=e_d$ is $-Nc$, so that

$$H_Y(z,T^\dagger e'_{d-1})=H_{Y'}(Tz,e'_{d-1})=-N\bar c.$$

Here $H_Y, H_{Y'}$ denote the stage forms, conjugate-linear in the first argument. The adjoint has no component in X_j, since T kills X_j. Its value is thus $\bar c z$. The factors $\jmath,\varpi$ in $c=-(\jmath g\varpi)^{-1}$ both change sign under conjugation by Proposition 18.4 and Lemma 348.1. This gives $\bar c=c$, proving the formula. In the final pair both maps have empty source or target strings. □

Corollary 365.3. *For $s\ge 3$ enlarge the strictly positive-degree subspace $U_>$ of V_* by $\mathrm{Hom}(X',E\xi)$, and for $s=1$ use $U_>$ alone. Call the resulting rational space U_*. It is totally isotropic. Its hermitian orthogonal is $U_*\oplus W_*$, with*

$$W_*=\begin{cases}\mathrm{Hom}(X',X_j^-\oplus X_{j+2}), & s\ge 3,\\ \mathrm{Hom}(X',X_j^-\oplus(Y_m\oplus E(z/\pi_N))), & s=1.\end{cases}$$

The finite oscillator reduction of Theorem 362.2 applies with U_*,W_*,L_* of the present pair.

Proof. In the matrix trace pairing the grades occur only in opposite pairs, by the argument of Lemma 355.2. On degree zero the hyperbolic pair in Proposition 365.1 gives a perfect pairing between the maps into $E\eta$ and $E\xi$, by taking matrix units on the orthonormal basis of X'. The asserted orthogonal and its direct decomposition follow. The proofs of Lemma 362.1 and Theorem 362.2 use only $U^\perp=U\oplus W$, the vanishing of the hermitian pairing of U with that sum, self-duality of the lattice, and triviality of its bicharacter on the lattice square. Theorem 364.1 supplies these lattice properties for L_*. Thus their proofs apply to the indicated subspaces, including the case of an empty string. □

366. The integral image in the residual space

Definition 366.1. Retain j,X',s,W_* of Corollary 365.3, write $\pi=\pi_N$ and let $O=\widehat{O}_E$. In the residual target write e for the last basis vector of X_j^-. For $s\ge 3$ let e' be the last basis vector of X_{j+2}. For $s=1$ take $e'=z/\pi$ in the enlarged positive space $Y_m\oplus E(z/\pi)$. Normalize the coordinates of a residual map W_* by writing its first block as $\bar\alpha x_-$ and its second as αx_+, using the last indicated vector as a basis element in the final pair. For one column in these coordinates define $\mathcal{B}_\pi$ by integrality of every entry outside e,e', and by

$$\bar\pi x_e,\ \bar\pi x_{e'}\in O,\qquad x_e-x_{e'}\in\pi O.$$

The matrix lattice with this condition on every column is denoted by $\mathcal{B}_\pi^{X'}$. Put $\delta=\delta_g/\bar\delta_g$ with δ_g from Lemma 348.1. Let D_δ multiply the e'-entry by δ in the last pair and act as the identity in every middle pair.

Theorem 366.2. *For the reference lattices of Theorem 364.1 and the projection $p: U_*^\perp=U_*\oplus W_*\to W_*$, the image $p(L_*\cap U_*^\perp)$ in the normalized coordinates of Definition 366.1 equals $D_\delta\mathcal{B}_\pi^{X'}$. There is an adelic linear map $B: W_*\to U_*^\perp$ with $pB=1$ taking this image*

lattice back into L_*. It is zero on the string of Y' as a family of maps from Y', that is, $B(w)|_{\mathrm{string}(Y')} = 0$ for all w. Its adjoint for the Hom pairings satisfies $B^\dagger\tau = 0$ for the frequency in Proposition 365.2.

Proof. Each of the three original stage lattices is a direct sum of its intersections with the individual grades. For even string length all columns in Definition 348.2 are homogeneous. For odd string length the changed column uses two vectors of grade zero and every other column is homogeneous. For Y' the degree-zero block X' has the ordinary integral basis. The Hom lattice therefore decomposes into its separate source and target grade blocks, by testing a map on each piece of the source lattice. By the description of L_* in Theorem 364.1, after the first positive-degree reduction, its degree-zero image consists exactly of independent columns on that basis of X' in $\bar\alpha L_Y^0 \oplus \alpha L_{Y''}^0$. Here the superscript zero denotes intersection with the respective grade-zero space. All other grade components of a lift of these columns can be set to zero.

For $s \geq 3$, a lift in $U_*^\perp$ of one normalized column has outside components $\bar\alpha x_- + tz,\ \alpha x_+ + tz''$ for a common t, since the additional isotropic direction is $\xi = z + z''$ by Proposition 365.1. Set $A = \bar\alpha u_j,\ A' = \alpha u_{j+2}$. The column of the original lattice with coefficient $1/\bar\pi$ on the distinguished definite coordinate has coefficient u_j/N on z. Reading its coordinates and the independent column $u_j z$ in Definition 348.2 gives the equivalent conditions

$$\bar\pi x_e,\ \bar\pi x_{e'} \in O, \qquad t/A - x_e/\pi \in O, \qquad t/A' - x_{e'}/\pi \in O,$$

with ordinary integrality on the other normalized entries. The same basis formula at $j+2$ gives the second test on t. Since $u_j/u_{j+2} = a_f/\bar a_f$,

$$A/A' = (\bar\alpha/\alpha)(a_f/\bar a_f) = \delta$$

by Lemma 348.1. Above N this ratio equals one, as the marked primes are prime to N by Lemma 13.2. The two tests there are soluble exactly when $x_e - x_{e'} \in \pi O_w$, and one may take $t = Ax_e/\pi$. Away from N the scalars $\pi, \bar\pi$ are units. Both distinguished entries are then integral and $t = 0$ passes both tests. These local choices are linear in the column and give the desired adelic lift. They prove the claimed lattice equality for the middle pairs.

For $s = 1$ there is no second isotropic direction. In the original outside first space the z-coordinate coming from the positive residual coordinate is $\alpha x_{e'}/\pi$. The original lattice conditions are thus

$$\bar\pi x_e \in O, \qquad \delta^{-1} x_{e'} - x_e \in \pi O, \qquad \delta = \bar\alpha u_j/\alpha.$$

Indeed $j = m - 2$ gives $u_j = a_f/\bar a_f$ in Lemma 348.1. The conditions already imply $\bar\pi\delta^{-1} x_{e'} \in O$, since $\pi\bar\pi = N$ is integral. The other coordinates are ordinary integral in the chosen scales. This is precisely $D_\delta \mathcal{B}_\pi$. Take the lift into the grade-zero block itself.

In both cases let the lifted columns vanish on every other basis vector of Y'. Proposition 365.2 says that τ vanishes on X'. The trace pairing of a map supported on X' with τ is zero: the source Gram matrix and its inverse are block diagonal for X' and its orthogonal string, so all diagonal contributions vanish. The definition of the adjoint and nonsingularity on W_* give $B^\dagger\tau = 0$. □

367. The residual vacuum as a sum of translates

Theorem 367.1. *Use normalized coordinates on W_* as in Definition 366.1. Put L_{st} equal to the integral matrix lattice in these coordinates, $L_{\mathrm{res}} = D_\delta \mathcal{B}_\pi^{X'}$, $d' = \dim_E X'$, and let $v_{\mathrm{st}} = \mathbf{1}_{L_{\mathrm{st}}}$ in $\mathcal{H}(W_*, L_{\mathrm{st}})$. In formulas for the oscillator, use the pullback of the physical hermitian form under coordinate normalization. Define finite sets of representatives as follows.*

(1) *For each column and each $w \mid \bar\pi_N$, choose one representative in $\bar\pi_{N,w}^{-1} O_w$ of each class modulo O_w, with zero representing zero. Insert the chosen element into entries e, e'*

of that column at w and take zero elsewhere. The simultaneous choices form a set $\mathcal{R}$.

(2) *For $s \geq 3$ let $\mathcal{D} = \{0\}$. For $s = 1$ and every column at each $w \mid \pi_g$, choose one representative of each class of $\delta_w O_w / O_w$ in $\delta_w O_w$, with zero representing zero. Insert it only in entry e'. Take zero elsewhere, and call the resulting set $\mathcal{D}$.*

Then $h_N = |\mathcal{R}| = N^{fd'}$ and $h_g = |\mathcal{D}|$ is one for $s \geq 3$ and $(g_0 g_1)^{fd'}$ for $s = 1$. The unique unit vector fixed by L_{res} and with positive real overlap with v_{st} is

$$\begin{aligned} v_{\mathrm{res}} &= (h_N h_g)^{-1/2} \sum_{d \in \mathcal{D}} \sum_{r \in \mathcal{R}} D_{r+d} v_{\mathrm{st}} \\ &= \left(\frac{h_g}{h_N} \right)^{1/2} \frac{1}{h_g} \sum_{d \in \mathcal{D}} D_d \left(\sum_{r \in \mathcal{R}} D_r v_{\mathrm{st}} \right). \end{aligned}$$

Proof. In these coordinates the physical form is $\alpha\bar{\alpha}$ times the trace matrix pairing with signs minus on the first block and plus on the second. Therefore L_{st} is self-dual and B_f is one on its square, by the trace-dual calculation of Lemma 355.2. The corresponding properties for L_{res} follow from Corollary 365.3 and Theorem 366.2.

At $w \mid \pi_N$, the neighboring lattice has integral entries whose two distinguished residues agree, as $\bar{\pi}_{N,w}$ is a unit. At $w \mid \bar{\pi}_N$ it is the sum of the ordinary lattice and the diagonal (t, t) in the two entries with $t \in \bar{\pi}_{N,w}^{-1} O_w$. Indeed $\pi_{N,w}$ is a unit, so the two entries have integral difference. Off N the two conditions in Definition 366.1 give the ordinary lattice. For the last pair the additional factor δ on e' has valuation one at $w \mid \bar{\pi}_g$, minus one at $w \mid \pi_g$, and zero at all other places, by Lemma 348.1. Hence the enlargements of the standard lattice modulo its intersection with L_{res} are exactly the independent diagonal classes of $\mathcal{R}$ and the one-coordinate classes of $\mathcal{D}$. Every $r + d$ belongs to L_{res}. The split degree-one residue rings and their number f over each selected side in Lemma 13.2 give the cardinalities.

For any two self-dual lattices L_0, L_1 with trivial bicharacters on their squares, the normalized sum of translates of the first vacuum over $L_1/(L_1 \cap L_0)$ is a unit vector fixed by L_1. The proof in Theorem 358.1 used just these lattice properties: disjoint supports give the norm, translation permutes the summands, averaging back over L_0 recovers a nonzero multiple of the reference vacuum, and the translates of the sum are an orthonormal basis up to the lattice-coset relations. In that basis self-duality leaves one fixed line. The identity coset yields positive overlap, so the stated vector is the unique normalized choice. The supports of d and r are on disjoint conjugate pairs, whence $D_d D_r = D_{r+d}$ by Lemma 356.2. This proves both expressions. □

Corollary 367.2. *There is a surjective reduction from $\mathcal{H}(V_*, L_*)$ onto $\mathcal{H}(W_*, L_{\mathrm{st}})$ invariant under U_* and equivariant under the physical residual translations, which takes the reference vacuum to v_{res} of Theorem 367.1. Any linear map with these invariances into this same residual space differs from this reduction by a scalar. On the input $D_\tau D_y \mathbf{1}_{L_*}$ its normalized value has the form*

$$\epsilon_y D_{B^\dagger y} v_{\mathrm{res}}, \qquad \epsilon_y = 0 \text{ or a root of unity},$$

where the adjoint translation is expressed in the normalized residual coordinates. The integral lift B is the one in Theorem 366.2.

Proof. Identify the oscillator on L_{res} with that on L_{st} by sending $D_w \mathbf{1}_{L_{\mathrm{res}}}$ to $D_w v_{\mathrm{res}}$. The basis and lattice-coset relations in the proof of Theorem 367.1 give an isometric bijection intertwining translations. Compose with the exact quotient in Corollary 365.3. The vacuum of any map with the stated invariances is fixed by L_{res}, by lifting it inside L_* as in Proposition 363.1. That fixed line and the spanning translates of the quotient prove scalar uniqueness. For the value at the shifted frequency apply Corollary 363.2 and the identity $B^\dagger \tau = 0$ in Theorem 366.2. □

368. Marked denominators under the adjoint

Definition 368.1. For an arrow with its two reference stage lattices call a finite map y an allowed shift for P if it can be written $y = \alpha y_0$, with y_0 zero outside $P \cup \bar{P}$ and the places dividing π_g, and $\bar{\delta}_g y_0$ an integral Hom at every $w \mid \pi_g$. At the twist places impose no size restriction. For definite arrows use the standard integral lattices in the coordinates of Proposition 365.1. The normalization α is that of Lemma 348.1.

Proposition 368.2. *Let y_1, y_2 be allowed shifts on the two original arrows. Apply Theorem 364.1, the normalized shifted reduction of Corollary 367.2, and the inverse of the adjoint identification on the two definite arrows. Up to the common factor $(h_g/h_N)^{1/2}$, the image of their translated vacuum tensor is an average over $d \in \mathcal{D}$, with coefficients zero or roots of unity divided by h_g, of vectors of the form*

$$(D^1_{t_1} \otimes D^2_{t_2}) \sum_{r \in \mathcal{R}} D^1_{\alpha r_-^\dagger} v_1 \otimes D^2_{\alpha r_+} v_2.$$

Here v_i are the two standard definite oscillator vacua, r_-, r_+ are the normalized first and second block matrices of r, and $r_-^\dagger$ is their conjugate transpose on the first block with the original positive forms. Both t_i are allowed shifts of Definition 368.1.

Proof. Write $y = A(y_1, y_2) = (y_1^\dagger, y_2)$. By Theorem 364.1 the first identification introduces only a root of unity. The shifted reduction gives $\epsilon_y D_{B^\dagger y} v_{\mathrm{res}}$ by Corollary 367.2. All components of y above N vanish, since gQ is prime to N. The integral lift in Theorem 366.2 works place by place, so $B^\dagger y$ vanishes above N as well (the adjoint can interchange conjugate places, which are both above the same rational prime). Off N the lift has zero correction, so its adjoint restricts to X' on the source and projects orthogonally onto the residual target. This follows also by pairing rectangular matrix units in the grade-zero blocks. It creates no component away from the conjugate pairs already present.

At the marked pairs all the original lattices split as the ordinary definite block plus the indicated string columns, including a separate $u_j z$ in odd length. Indeed π_N and its conjugate are units there, so the two glued generators in Definition 348.2 generate both b and $u_j z$ individually. In the normalized coordinates of Definition 366.1, the residual first block of $B^\dagger y$ thus has entries in $\delta_g^{-1} O$ supported on $w \mid \bar{\pi}_g$ at the marked pairs: it comes from the conjugate transpose of an integral map divided by $\bar{\delta}_g$. The ordinary second block has entries in $\bar{\delta}_g^{-1} O$ supported on $w \mid \pi_g$ there, directly from y_2.

For $s = 1$, the extra second-block coordinate comes from the z-coefficient of $y_1^\dagger$. At $w \mid \bar{\pi}_g$ this coefficient belongs to $\bar{\alpha} u_j \delta_g^{-1} O_w$. Expressing it in the positive basis vector z/π_N and dividing by α puts the extra coordinate in

$$\pi_N (\bar{\alpha} u_j / \alpha) \delta_g^{-1} O_w = \pi_N \bar{\delta}_g^{-1} O_w = O_w.$$

At $w \mid \pi_g$ this contribution vanishes. The equality of scales is the final-pair computation of Theorem 366.2. The integral extra entries may be removed modulo L_{st} at the cost of a root of unity, by the lattice-coset relation of Theorem 358.1. This cost is independent of r, since every r is supported above N.

Use the sum of Theorem 367.1. Adding $d \in \mathcal{D}$ to $B^\dagger y$ affects only the extra positive coordinate at $w \mid \pi_g$, where its entries are in $\bar{\delta}_g^{-1} O_w$ by the definition of δ. Combining those two translations introduces a root of unity. After the definite inverse adjoint identification the first normalized block is conjugate transposed again and is supported on $w \mid \pi_g$ with denominator $\bar{\delta}_g$ there. Both finite shifts are therefore allowed. Applying that identification to $D_r v_{\mathrm{st}}$ gives exactly the pure tensor in the sum: the first original translate is supported on only one side of the conjugate pairs above N, so its self-pairing and the exponent in λ of Theorem 364.1 are zero. The shifts away from N introduce only their common quadratic phase, independent of r, by the intertwining identity of the same theorem. The factor $1/h_g$ is the one in Theorem 367.1. □

369. A common denominator for the finite tensor

Theorem 369.1. *Put $g = g_0 g_1$, $n_i = r + it$ for $0 \le i \le m$. Choose arbitrary finite isometries at each stage integral off $P \cup \bar{P}$. On each successive arrow take the corresponding moved reference inclusion vector using the normalized intertwiners of Theorem 358.1. For each even $j < m$, combine the two consecutive vectors, apply the normalized residual reduction at τ and return to the definite tensor product as in Proposition 368.2. Let v be the tensor product of these results in order. There is a positive constant C_*, independent of P and the stage isometries, such that v/C_* is a finite linear combination of tensors of the translated sums in Proposition 368.2. Each shift is allowed. The coefficients c can be chosen so that Dc is integral at every finite place away from $Q = \prod_{w \in P} q_w$, where*

$$D = g^{f(\sum_{i=0}^{m-1} n_i n_{i+1} + n_{m-1})} \mid \left(g^{f n_m^2}\right)^m.$$

When $P = \varnothing$ the sum of their absolute values is at most one after every field automorphism of $\mathbb{C}$. If the sums above N are also expanded, the new coefficients satisfy the same denominator bound, and their corresponding untwisted sum is at most

$$N^{f((m/2)k + (t+1)(m/2)^2)}.$$

The resulting individual shifts may also be supported on the places above N.

Proof. For arrow i, Corollary 361.2 gives an expansion in allowed shifts with denominator $g^{f n_i n_{i+1}}$ away from Q. Indeed its bound on the opposite marked components is $\bar{d}_P^{-1} L_0 = \bar{\delta}_g^{-1} L_0$ there, and isometric coordinates identify L_0 with the integral Hom times α by Lemma 355.2. When P is empty the sum of the absolute conjugate coefficients is one. Expand both arrows by bilinearity and use Proposition 368.2. For pair j extract its factor $(h_{g,j}/h_{N,j})^{1/2}$, using the indices of Theorem 367.1 for that pair. The remaining average has coefficient sum at most one, and denominator $h_{g,j}$. This equals one except for the last pair, for which $d' = \dim X_{m-1} = n_{m-1}$ as the string is empty. Consequently $\prod_j h_{g,j} = g^{f n_{m-1}}$, where the product is over the even $j < m$. Take $C_* = \prod_j (h_{g,j}/h_{N,j})^{1/2}$.

The tensor expansion uses products of the individual coefficients. Multiplying the denominators makes each product integral in the stated valuation rings. For each complex field automorphism, the sum of absolute product coefficients factors over the independent choices. Zeros and roots of unity in the residual evaluation can only decrease the resulting upper bound. Grouping equal translates preserves integrality and the upper bound by addition and the triangle inequality. This proves the asserted estimates before the final expansion. Since $t \ge 1$,

$$n_{m-1} n_m + n_{m-1} \le (n_m - 1)(n_m + 1) < n_m^2.$$

Every earlier product $n_i n_{i+1}$ is at most n_m^2. This proves the integer divisibility used in the statement.

For pair j the auxiliary-prime sum has exactly $N^{f d_{j+1}}$ terms of unit coefficient by Theorem 367.1. Multiplying a displacement through any one of these terms gives a root of unity by Lemma 356.2; the shifts of Proposition 368.2 in fact vanish at N. The only newly occupied places are those over N, so the denominator is unchanged. From Definition 348.2, $d_{j+1} = k + (j+1)(t+1)$. The $m/2$ successive odd integers from 1 to $m-1$ sum to $(m/2)^2$, by subtracting consecutive squares. Multiplication of the cardinalities gives the last bound. □

Corollary 369.2. *Replacing each normalized residual map in Theorem 369.1 by any fixed linear residual map with the same invariances changes its output by one common scalar independent of the twist and stage isometries. Suppose further that each moved finite vector is multiplied by an algebraic scalar integral away from Q, with every conjugate of absolute value at most one when $P = \varnothing$. The coefficient and support conclusions, with a common scalar factored outside the expansion, still hold.*

Proof. Corollary 367.2 expresses each fixed residual map as a scalar multiple of its normalized version. Multiply these scalars over the even indices, along with the common constant from Theorem 369.1. This statement allows a zero map, in which case the output tensor is zero. The additional arrow scalars enter multiplicatively by tensor multilinearity. Their product preserves the stated finite integrality and the untwisted absolute-value bound. □

370. Recovering the data at the auxiliary prime

Proposition 370.1. *For each standard definite Hom space use its finite oscillator with reference lattice L_i, and let $K_{N,i}$ be the additive subgroup of vectors in L_i supported at the places above N. Let P_i be averaging D_l over $l \in K_{N,i}$ with mass one. It acts on every vector by a finite average and is the projection onto the subspace fixed by $K_{N,i}$. On the tensor of the two definite oscillators of a pair let $P = P_1 \otimes P_2$. Let S be the sum of operators*

$$S = \sum_{r \in \mathcal{R}} D^1_{\alpha r_-^\dagger} \otimes D^2_{\alpha r_+}$$

with representatives from Theorem 367.1. Then $PSb = b$ for all $b = Pb$. Consequently SP is a projection with image $S(\operatorname{im} P)$. It fixes all the translated sums in Proposition 368.2.

Proof. On $K_{N,i}$ the bicharacter on two lattice vectors is one by Proposition 357.2, so their translations represent addition. On a translated vacuum $D_y v_i$ they act by the character $l \mapsto \psi_f(\jmath H_i(y,l))$ by Lemma 356.2. Each vector is a finite sum of such eigenvectors and these characters have open kernels by Proposition 18.5. Averaging therefore factors through a finite quotient: the common kernel is open of finite index by compactness. A nontrivial character has zero average by translation in that quotient. It follows that P_i retains exactly the basis vectors with $y|_N \in L_i|_N$, where $|_N$ means restriction to all places above N. This last assertion uses local self-duality from Lemma 355.2. The description proves the projection assertion.

For $b = Pb$ the two factors of every surviving basis tensor have integral indices above N. In the sum Sb the zero representative gives b. Each nonzero r has at least one distinguished entry with a nonintegral residue class above N. Thus at least one translated index fails the integrality test after adding r_+ or $r_-^\dagger$ in its coordinates divided by α. Its term is killed by P. This proves $PSb = b$, first on the basis tensors and then on their span. It follows that $(SP)^2 = SP$ and that the image is as stated. The allowed shifts of Definition 368.1 vanish above N, hence commute with S and P, by the product rule for disjoint conjugate pairs. The unshifted vacuum tensor belongs to $\operatorname{im} P$, so the translated sums are fixed. □

Corollary 370.2. *For a tensor product over all even $j < m$ form $\mathcal{P} = \bigotimes_j P_j$ and $\mathcal{S} = \bigotimes_j S_j$ from Proposition 370.1. Every linear combination w of the reduced translated tensors of Theorem 369.1 satisfies $\mathcal{S}\mathcal{P}w = w$. If $w = \sum_t c_t \mathcal{S} b_t$ with b_t allowed translated pure vacuum tensors, then $\mathcal{P}w = \sum_t c_t b_t$. In particular recovery uses the same coefficients, including their finite denominators and their untwisted size bound.*

Proof. The operators at different tensor factors act independently. The allowed translates b_t are all fixed by $\mathcal{P}$. Apply $P_j S_j = 1$ on the image of P_j from Proposition 370.1 in each factor and use tensor multilinearity and then linearity of the sum. □

371. Summing the finite evaluations over the stage cuts

Theorem 371.1. *Use g, Q, n_i, D, C_* of Theorem 369.1. For $0 \le i < m$ let T_i be a finite cut with positive multiplicities $m_i(t)$ whose reciprocals lie in $\mathbb{Z}[1/Q]$, and assume that for $P = \varnothing$ one has $\sum_{t \in T_i} m_i(t)^{-1} \le g^{f n_i^2}$. For example the cuts of Theorem 347.1 with the principal marked level in the untwisted case satisfy these bounds. For any subset $\mathcal{T} \subseteq \prod_i T_i$ choose stage representatives for each tuple integral off $P \cup \bar{P}$, with the last stage representative equal*

to one. Let $v(t)$ be its finite tensor in Theorem 369.1. Then

$$w = \frac{D}{C_*} \sum_{t \in \mathcal{T}} \left(\prod_{i=0}^{m-1} \frac{1}{m_i(t_i)} \right) v(t)$$

is a combination of the allowed translated sums of that theorem, with coefficients integral away from Q. In the untwisted case the sum of their absolute values after each field automorphism is at most

$$D\, g^{f \sum_{i=0}^{m-1} n_i^2} \;\le\; \left(g^{f n_m^2}\right)^{2m}.$$

It satisfies the recovery identity $\mathcal{S}\mathcal{P}w = w$ of Corollary 370.2. If a group permutes $\mathcal{T}$ preserving the product of the multiplicities, and acts linearly on the tensor space sending $v(t)$ to $v(\gamma t)$, then w is fixed by that group.

Proof. The stated cut estimates for the example follow from Proposition 344.2 and Proposition 347.2. At each tuple the factor D clears the required denominators of $v(t)/C_*$ by Theorem 369.1. The rational weights with denominators supported on Q are integral at all remaining finite places. Thus multiplying by them and summing preserves integrality and allowed support. For the untwisted sum, after taking absolute values of the conjugate coefficients, the sum at each tuple is bounded by $D \prod_i m_i(t_i)^{-1}$. These bounds are nonnegative, so their sum over $\mathcal{T}$ is at most the sum over the whole product, which factors into the product of the given one-stage bounds. The last estimate uses $n_i \le n_m$ and the divisibility in Theorem 369.1. The recovery identity follows by linearity from Corollary 370.2. For a group element the permutation simply reindexes the displayed sum, proving invariance. □

Corollary 371.2. *Let $\kappa \in \mathbb{C}$ be independent of P and all tuple choices, and let*

$$z = \kappa \sum_{t \in \mathcal{T}} \left(\prod_{i} m_i(t_i)^{-1} \right) v(t)$$

for a family as in Theorem 371.1. There is a common $c \in \mathbb{C}^\times$ such that cz satisfies all the coefficient, support and recovery conclusions about w in that theorem for the whole family. The same holds after any fixed residual maps and the additional arrow scalars of Corollary 369.2 are used, with the corresponding $v(t)$.

Proof. When $\kappa C_* \neq 0$ take $c = D/(\kappa C_*)$; this gives exactly the vector of Theorem 371.1. When the common factor is zero the family of vectors z vanishes, and $c = 1$ works with empty coefficient lists. For the last assertion multiply κC_* by the fixed residual scalars and apply the coefficient bounds of Corollary 369.2 term by term. □

372. Algebraic characters of the norm-one torus

Definition 372.1. For the CM field E, set

$$T(E) = \{a \in E^\times : a\bar{a} = 1\}, \qquad T_f = \{x \in \mathbb{A}_{E,f}^\times : x\bar{x} = 1\}, \qquad K_T = T_f \cap \widehat{O}_E^\times.$$

Identify T_∞ with $(S^1)^\Phi$ in the embeddings of Proposition 10.3, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For $e = (e_\tau)_{\tau \in \Phi} \in \mathbb{Z}^\Phi$ put $p_e(z) = \prod_\tau z_\tau^{e_\tau}$. Let $\theta : K_T \to S^1$ be a continuous character of finite order, meaning some fixed positive power is identically one. A torus character of these data is a continuous character on $T_\infty \times T_f$ with values in S^1, trivial on the diagonal $T(E)$, whose value on $T_\infty \times K_T$ is $p_e(z)\theta(k)$.

Theorem 372.2. *A torus character of Definition 372.1 exists exactly when*

$$\prod_{\tau \in \Phi} \tau(\zeta)^{e_\tau}\, \theta(\zeta) = 1 \qquad \text{for every root of unity } \zeta \in E.$$

For any such character μ and $x \in T_f$ there are $h \ge 1$ and $a \in T(E)$ such that $x^h/a \in K_T$ and

$$\mu(1, x)^h = \prod_{\tau \in \Phi} \tau(a)^{-e_\tau}.$$

In particular $\mu(1,x)$ is algebraic and every complex conjugate has absolute value one. If x is a unit at every place over a rational prime p, its character value is a unit at every algebraic place over p.

Proof. The ideal of a finite idele x is $I(x) = \prod_w \mathfrak{p}_w^{v_w(x_w)}$ by Proposition 82.3. For $x \in T_f$, the exponents are opposite at conjugate places and zero at every fixed place. Hence $I(x) = J/\bar{J}$ for a fractional ideal J, by choosing one member of each of its finitely many conjugate pairs. The group of all such ideals $J/\bar{J}$, modulo ideals of elements of $T(E)$, is an image of the ordinary ideal class group: the formula $J \mapsto J/\bar{J}$ sends a principal ideal (b) to $(b/\bar{b})$. The ordinary ideal class group is finite by Lemma 82.5. The kernel of I on T_f is K_T, so $T_f/(K_T T(E))$ injects into this finite image.

An element of $T(E)$ integral at every finite place belongs to O_E by Proposition 82.3. In every embedding it has complex absolute value one because conjugation in E induces complex conjugation at each embedding. All its nonnegative powers lie in a bounded part of the lattice O_E of Lemma 82.2. Thus two powers coincide and it is a root of unity. Conversely roots of unity are units of norm one. This proves that the intersection of the diagonal $T(E)$ with $T_\infty \times K_T$ is precisely the group tested in the statement. The test is necessary. If it holds, the character $p_e\theta$ and the trivial character on the diagonal define a character on the subgroup generated by them: two decompositions differ by this intersection and give equal values. That subgroup is open and has finite index by the finite ideal quotient just proved.

For an abelian group and a subgroup of finite index, a character with values in S^1 extends to the whole group. Indeed adjoin one generator of the finite quotient at a time. If its first power in the current subgroup is its n-th power, assign to the generator an n-th root on S^1 of the value there, which exists by writing a complex unit as $e^{i\vartheta}$. On elements ba^j multiply the old value on b by that root to the j-th power. Equality of two expressions reduces to the specified relation at the n-th power, so the rule is a homomorphism. Continuity follows on the open subgroup from the construction on $T_\infty \times K_T$ and on all its cosets by translation. This proves existence.

Finiteness provides $h_0 \geq 1$ with $x^{h_0} = a_0 k$, $a_0 \in T(E)$, $k \in K_T$. Raise this equation to the order of θ to get $x^h = ak'$ with $\theta(k') = 1$. The diagonal law now gives the displayed character power. Its right side is algebraic, and each of its complex conjugates has modulus one because every embedding sends $a\bar{a} = 1$ to the corresponding norm-one equation. An h-th root is again algebraic and has the same modulus property. At primes over p in the last assertion $a = x^h/k'$ is a local unit at every place of E above p. Therefore each conjugate $\tau(a)$ has valuation zero over p, by the correspondence of embeddings and places in Lemma 16.2. Taking valuations in the power formula proves the claim. □

373. The cyclotomic unit and the marked characters

Lemma 373.1. *Write $n = N_1$, $\zeta = \exp(2\pi i/n)$ and $K = \mathbb{Q}(\sqrt{-d})$ as in Definition 10.1. The roots of unity of $E = K(\zeta)$ are exactly the powers of ζ. The element $u = i(\zeta - 1)$, where $i = \zeta^{n/4}$, belongs to $O_E^\times$ and satisfies*

$$u/\bar{u} = \zeta, \qquad N_{E/K} u = 1.$$

Consequently for every root of unity $\xi \in E$ there is a unit u_ξ with $u_\xi/\bar{u}_\xi = \xi$ and $N_{E/K} u_\xi = 1$.

Proof. By Proposition 10.3, $E \subset \mathbb{Q}(\zeta_{dn})$, $d \nmid n$ and $[E : \mathbb{Q}] = 2\varphi(n)$. If a root of order a lies in $\mathbb{Q}(\zeta_{dn})$, so does a primitive root of order $l = \operatorname{lcm}(a, dn)$. Indeed the powers ζ_a, ζ_{dn} of ζ_l have exponents $l/a, l/(dn)$ with greatest common divisor one, and all primitive roots of a given order generate the same cyclic group. Lemma 10.2 gives $\varphi(l) = \varphi(dn)$. Since dn is even, any new prime would be odd and introduce a factor at least two, and raising an existing prime exponent introduces its prime as a factor. This follows by counting units in each prime power quotient and applying the Chinese remainder calculation of Lemma A.1. Thus $a \mid dn$. If also $d \mid a$ and the root belongs to E, then E contains ζ_d and ζ_n, hence ζ_{dn}.

This has degree $(d-1)\varphi(n) > 2\varphi(n)$, since $d \equiv 7 \bmod 8$ by Proposition 10.3. Therefore $a \mid n$.

The relative norm from E to K is the product over the embeddings of Proposition 10.3: over $\mathbb{C}$ evaluation at the distinct primitive n-th roots diagonalizes multiplication in the ζ-power basis, so the product is the multiplication determinant. Pairing inverse roots gives $N_{E/K}\zeta = 1$ as $n > 2$. For the cyclotomic polynomials of Lemma 10.2, factoring $(X^n-1)/(X-1)$ and evaluating at one gives

$$n = \prod_{h \mid n,\ h>1} P_h(1).$$

All factors are positive integers by conjugate pairing, with $P_2(1) = 2$. For $h = p^a,\ a \geq 1$, the quotient $(X^{p^a}-1)/(X^{p^{a-1}}-1)$ gives $P_{p^a}(1) = p$. Their product over all prime-power divisors is already n. As $6 \mid n$ it follows that $P_n(1) = 1$. The even degree $\varphi(n)$ therefore gives $N_{E/K}(\zeta - 1) = 1$. The inverse of $\zeta - 1$ is the product of its other integral relative conjugates. Since i is a root of unity, u is a unit of norm one over K. Direct conjugation gives $\bar{u} = u/\zeta$. Taking powers of u proves the last assertion. □

Proposition 373.2. *Write $A = M_\omega$ as in Theorem 11.4 and $w_0(\tau)$ as in Definition 12.3. At each marked $w \mid \bar{\pi}_g$ choose a residue character ω_w of order A which is an A-th power. Set $\omega_w = 1$ elsewhere. Such choices exist. There are torus characters $\mu^\flat, \chi$ in the sense of Definition 372.1, the first having weight $-w_0$ and trivial character on K_T, the second having weight zero and character $k \mapsto \prod_w \omega_w(k_w \bmod w)$ on K_T. There also exists a character of constant weight one and trivial character on K_T.*

Proof. At a marked $w \mid g_i$ the residue field has g_i elements by Lemma 13.2. Its multiplicative group is cyclic by the proof of that lemma. Theorem 11.4 gives $A^2 \mid g_i - 1$. Send a generator to $\exp(2\pi i/A^2)$ and take the A-th power. This gives the required ω_w. For any root $\xi \in E$, Lemma 373.1 gives $\xi^n = 1$. Since $n \mid A$ by Theorem 11.4, every ω_w is one on the reduction of ξ. The same lemma and $N_{E/K}\zeta = 1$ give $\prod_{\tau \in \Phi} \tau(\xi) = 1$. Finally each $w_0(\tau)$ is h_2 plus an integral multiple of n by Definition 12.3. Thus the product with those exponents at ξ is the h_2-th power of its relative norm and also equals one. Theorem 372.2 now constructs all three characters. □

374. The splitting character on the full idele group

Lemma 374.1. *Let F be the fixed field of conjugation in E of Definition 10.1. Call a prime of F inert if it has just one prime in E. For all $a \in F^\times$,*

$$\prod_{\tau \in \Phi} \operatorname{sgn} \tau(a) = (-1)^{\sum_{\mathfrak{p} \text{ inert}} v_{\mathfrak{p}}(a)}.$$

Proof. One has $E = F(i)$: the real and imaginary parts $(z + \bar{z})/2, (z - \bar{z})/(2i)$ of any $z \in E$ belong to F. All embeddings of F are real and are obtained once by restricting Φ, since each extends in two ways to $F(i)$ as in Theorem 81.1. Also $12 \mid n = N_1$, so Lemma 84.1 gives an unramified quadratic field or a split algebra at every finite place. At an odd prime of residue cardinality q the inert case occurs exactly for $q \equiv 3 \bmod 4$. Every dyadic prime splits here. In fact $E = F(\sqrt{-d})$ by degree two and nonreality. In $\mathbb{Q}_2$ the polynomial $X^2 - X + (d+1)/4$ has roots reducing to zero and one, since $d \equiv 7 \bmod 8$ by Proposition 10.3; its derivative is a unit at both roots modulo two. Successive linear correction as in Lemma 13.2 gives two distinct roots in $\mathbb{Z}_2$, hence in every dyadic completion of F.

At each dyadic prime choose a preimage of a by the norm from the split algebra, for instance $(a, 1)$. By the simultaneous approximation by an element $u \in E^\times$ in the proof of Theorem 85.1 one can arrange $a' = a/(u\bar{u}) \equiv 1 \bmod 4$ in each dyadic local ring. Explicitly, clear the denominators at two, prescribe sufficiently many residue digits at the finitely many primes above two by the Chinese remainder theorem, and divide back by the same power

of two. Then the ratios to the local preimages are one modulo four. The real signs are unchanged and the valuations at inert primes change by even integers. The signed norm $N_{F/\mathbb{Q}}a'$ is one modulo four by the multiplication determinant argument of Theorem 84.2. Its absolute value is the positive norm of the fractional ideal (a'), whose prime factors all have odd residue cardinality. Modulo four each inert prime contributes minus one to that cardinality, and every split one contributes one. This uses the factorization and multiplicativity of Proposition 82.3 also for negative exponents. The signed norm and its negative have opposite residues in $(\mathbb{Z}/4\mathbb{Z})^\times$. The sign of the norm is thus its asserted parity, and the norm is the product of all real conjugates. This proves the formula. □

Theorem 374.2. *There is a continuous character $\upsilon : \mathbb{A}_E^\times \to S^1$ trivial on diagonal $E^\times$ such that*

$$\upsilon(x) = \prod_{\tau\in\Phi} \frac{x_{\infty,\tau}}{|x_{\infty,\tau}|} \quad \text{if } x_f \in \widehat{O}_E^\times, \qquad \upsilon(\bar{x}) = \upsilon(x)^{-1}.$$

At a place fixed by conjugation it takes value -1 on an idele consisting of a uniformizer there and ones elsewhere. On a finite idele x with infinite component one its value is algebraic, all its complex conjugates have absolute value one, and it is an algebraic unit above any rational prime at all of whose places x is a unit.

Proof. Let $\mathcal{I}$ be the group of fractional ideals of E. On its conjugation-fixed subgroup define $\eta(J) = (-1)^{\sum_{w=\bar{w}} v_w(J)}$. For $z \in E^\times$ define

$$b(z) = \frac{\overline{N_{E/K}z}}{|N_{E/K}z|}.$$

If (z) is conjugation-fixed, $z/\bar{z}$ is an integral norm-one unit by Proposition 82.3. It is a root of unity by the argument in Theorem 372.2. Choose u_ξ of Lemma 373.1 for this root. Then $a = z/u_\xi$ lies in $F^\times$, generates the same ideal, and has $N_{E/K}a = N_{E/K}z$. Its relative norm is the product of its real conjugates, by the identification of the embeddings above. At a fixed place the valuation of a in E equals the valuation in F, as the ramification index is one. Consequently $b(z) = \eta((z))$ by Lemma 374.1. For a unit this also proves $b(z) = 1$. Thus b descends to the principal ideals, and agrees with η on the intersection of the two subgroups.

Multiplying these two characters gives a well-defined character on the product of the principal and conjugation-fixed subgroups. Its index in $\mathcal{I}$ is finite by Lemma 82.5. Extend it to a circle character β of $\mathcal{I}$ by the finite generator and root construction of Theorem 372.2. For any J, the ideal $J\bar{J}$ is fixed and has even valuation at every fixed prime, so $\beta(J\bar{J}) = 1$. Define

$$\upsilon(x) = \left(\prod_{\tau\in\Phi} \frac{x_{\infty,\tau}}{|x_{\infty,\tau}|} \right) \beta(I(x_f)).$$

The ideal map I has open kernel $\widehat{O}_E^\times$, so the character is continuous. Its values on the diagonal are one by the definition of b and the relative norm formula in Lemma 373.1. The equation for $J\bar{J}$ gives the conjugation law, and η gives the asserted fixed-prime sign.

If $J = I(x_f)$, choose $h \geq 1$ such that $J^h = (z)$ by ordinary class finiteness. Then

$$\beta(J)^{2h} = \frac{\overline{N_{E/K}z}}{N_{E/K}z}.$$

Both sides show algebraicity of $\beta(J)$. Every complex embedding of the imaginary quadratic field K sends numerator and denominator to numbers of equal modulus. If J has zero valuation above p, then z and its inverse are integral there. The same holds for every conjugate factor in the numerator and denominator of this formula, by Lemma 16.2. Taking valuations and complex absolute values as appropriate gives the last assertions. □

375. The splitting multiplier on the finite representatives

Definition 375.1. In an isometric reference frame divide oscillator coordinates by α, so the reference lattice is O^b, with $O = \widehat{O}_E$ and b the Hom dimension. Let G be a finite adelic isometry acting in these coordinates. At a place w let $a_w(G)$ be the valuation of the determinant of a basis matrix for

$$S_w(G) = \{x \in O_w^b : G_w x \in O_w^b\}.$$

Set $J_G = \prod_w \mathfrak{p}_w^{a_w(G)}$ and $\Delta(G) = \beta(J_G)(NJ_G)^{1/2}$, where NJ_G is the absolute ideal norm and β the finite ideal character of Theorem 374.2. The operator with the splitting multiplier sends v_0 to

$$\frac{\Delta(G)}{|\Delta(G)|^2} \sum_{r \in GL_0/(GL_0 \cap L_0)} D_r v_0$$

and intertwines D_y with D_{Gy}. Here GL_0 denotes the image of the reference lattice L_0 under the given operator G.

Theorem 375.2. *The integers $a_w(G)$ are independent of the bases, nonnegative and zero at almost all places. The operator of Definition 375.1 exists uniquely and equals $\beta(J_G)W_G$. For isometries integral away from $P \cup \bar{P}$ one has*

$$\beta(J_G) = \prod_{w \in P} \beta(\mathfrak{p}_w)^{-v_w(\det G_w)}.$$

On this group the operators form a representation. Their phase $\beta(J_G)$ is an algebraic unit at every finite place away from Q, has all complex conjugates of modulus one, and is one for integral G.

Proof. A local invertible matrix is a product of two integral invertible matrices and a diagonal one with uniformizer powers $\pi_w^{k_i}$. Indeed one can clear denominators first, select an entry of least valuation as pivot, interchange its row and column to the first position and clear its row and column by integral multiples. Iterate on the remaining invertible square block and divide back by the common power. Diagonal units can be absorbed into the integral matrices. In these coordinates $S_w(G)$ is generated by an integral change of basis applied to the diagonal powers $\pi_w^{\max(0,-k_i)}$. Any two of its bases differ by an integral invertible matrix. This proves well-definedness, and gives

$$a_w(G) = \sum_i \max(0, -k_i), \qquad [O_w^b : S_w(G)] = q_w^{a_w(G)}.$$

At almost all places G and its inverse are integral, giving $a_w(G) = 0$. Taking the product of the indices and applying G shows $NJ_G = [GL_0 : GL_0 \cap L_0]$, since scalar division by α does not change the index. The positive normalized average in Theorem 358.1 therefore gives precisely $\beta(J_G)W_G$ with the prescribed value and intertwining law.

In the isometric frames the matrix at the conjugate place is the inverse conjugate transpose. Its diagonal exponents are the negatives of those for G_w, by the same decomposition. As $\beta(\bar{\mathfrak{p}}_w) = \beta(\mathfrak{p}_w)^{-1}$ by Theorem 374.2, the joint contribution of this pair is the value at w raised to the exponent

$$\sum_i \big(\max(0, -k_i) - \max(0, k_i)\big) = -\sum_i k_i = -v_w(\det G_w).$$

If a fixed Gram sign matrix is used, its integral diagonal signs on either side leave the same exponents. The integral complementary places contribute one, giving the displayed character of the determinants.

On $P \cup \bar{P}$ multiplicativity of W_G is Theorem 360.1. On the complementary integral subgroup it is the change of variables of Theorem 358.1. Their actions on the separate factors commute by the tensor factorization in the split theorem. The phase just computed is multiplicative too. Finally J_G has support over Q. Apply the finite character conclusions

of Theorem 374.2 to an idele with ideal J_G, taking components one outside this support. For integral G the ideal is the unit ideal. □

376. The finite action with the prescribed arrow characters

Definition 376.1. Let υ_T be the restriction of Theorem 374.2 to the norm-one torus, and let $\mu^\flat, \chi, \omega_w$ be as in Proposition 373.2. A twist here is a torus character ϑ of weight zero, with finite unit character given by residue characters ϑ_w supported on P, where P is the split set of Definition 357.1. Extend the residue characters by the trivial character at other places. For $1 \le l \le m$, with h of Definition 12.3, set

$$\mu_l = \mu^\flat(\upsilon_T^{-h}\chi\vartheta)^l, \qquad \mu_l^{[n]} = \mu_l\, \upsilon_T^{\lfloor n/2 \rfloor}.$$

On the arrow from rank $n = n_{l-1}$ to $n' = n_l$, for finite stage elements g, g' integral away from $P \cup \bar{P}$, put $G(v) = g'vg^{-1}$. Define

$$\Omega_l(g, g') = \frac{\mu_l^{[n]}(1, \det g')}{\mu_l^{[n']}(1, \det g)}\, \beta(J_G) W_G.$$

All determinants of stage elements are norm-one finite ideles by the hermitian isometry equation. For $b \in \mathbb{Z}$ write $[b]_2 = b - 2\lfloor b/2 \rfloor$.

Theorem 376.2. *The character μ_l has weight $-w_l$ and finite unit characters $(\omega_w\vartheta_w)^l$. The operators Ω_l form a representation of the product of the indicated finite stage groups. The scalar multiplying W_G equals*

$$\frac{\mu_l(1, \det g')}{\mu_l(1, \det g)} \prod_{w \in P} \beta(\mathfrak{p}_w)^{[n']_2 v_w(\det g) - [n]_2 v_w(\det g')}.$$

It is an algebraic unit at all finite places away from Q and has absolute value one under every complex embedding. For integral k, k' whose reductions carry the prescribed inclusion to itself at the marked and twist places, one has

$$\Omega_l(k, k') f_{a_{\mathrm{inc}}} = \left(\prod_{w \mid \bar{\pi}_g \text{ or } w \in P} \frac{(\omega_w\vartheta_w)^l(\det k'_w \bmod w)}{(\omega_w\vartheta_w)^l(\det k_w \bmod w)} \right) f_{a_{\mathrm{inc}}}.$$

Proof. On T_∞, the splitting character restricts to weight one since all coordinates have modulus one. On K_T it is trivial by Theorem 374.2. Multiplication of characters adds their weights and multiplies their finite unit parts. The claimed weights follow from Proposition 373.2 and $w_l = w_0 + lh$ in Definition 12.3. The determinants and G are homomorphisms on the product, where the two actions on Hom commute. Theorem 375.2 therefore gives the representation law.

For any such stage element g, the contributions from the two members of the split pair above $w \in P$ to $\upsilon_T(1, \det g)$ multiply to $\beta(\mathfrak{p}_w)^{2v_w(\det g)}$: the two valuations are opposite, and the two ideal character values are inverse. All other finite determinant valuations are zero. Also $\det G_w = (\det g'_w)^n (\det g_w)^{-n'}$ by the column and row calculation of Corollary 360.2. Combining these two formulas with Theorem 375.2 gives the stated parity exponents. The full ratio of the two μ_l-values satisfies the unit and absolute-value assertions by Theorem 372.2. The local factors at P satisfy them by Theorem 374.2. For an integral pair the splitting phase and the finite υ_T-factors are one. Corollary 357.3 fixes the inclusion vector under W_G for the residues in the statement. Evaluating the remaining unit characters gives the last formula. □

Corollary 376.3. *The coefficient, denominator and recovery conclusions of Theorems 369.1 and 371.1 hold with the finite actions Ω_l of Definition 376.1, for the normalized residual maps and for the fixed scalar changes specified in Corollary 369.2. The finite arrow data depend only on the right level cosets of the stage elements, for the level reducing to identity at all marked and twist places in each stage.*

Proof. Apply the scalar conclusions of Theorem 376.2 in Corollary 369.2 and in the weighted sum therefrom in Theorem 371.1. For two integral right multipliers in the stated levels the residues are identities. Hence the last formula of Theorem 376.2 fixes the original inclusion vector. The representation law then gives the same moved datum on each coset. □

377. Column-level equivariance of the reduced tensor

Proposition 377.1. *For an even $j < m$ and its two successive arrows let $k_i \in K_{c,i}$, $i = j, j+1, j+2 < m$, with $K_{c,i}$ from Proposition 349.1. When $j+2 = m$ put $k_m = 1$. Extend these elements to the original stage spaces by identity on the strings, writing $\widehat{k}_i$ for the extensions. On the definite outside target use k_{j+2} in the middle pairs, and identity on the enlarged target in the last pair. The paired reduction of Proposition 368.2 intertwines the tensor product of the finite actions $\Omega_{j+1}(\widehat{k}_j, \widehat{k}_{j+1})$ and $\Omega_{j+2}(\widehat{k}_{j+1}, \widehat{k}_{j+2})$ with the corresponding actions on the two definite arrows, using their own ranks and the same μ_{j+1}, μ_{j+2}.*

Proof. First omit the scalar characters and consider the lattice actions W alone. The embedded column elements preserve the original lattices by Proposition 349.1, and the definite column elements preserve their standard lattices. Each such action on the corresponding oscillator is consequently the ordinary change of variables by the inverse isometry, by Theorem 358.1. Theorem 364.1 intertwines the two initial changes with the change on V_*. This change fixes τ because the maps T, T' in Proposition 365.2 vanish on the definite inputs and take values in string vectors, which are fixed. It preserves U_* and W_*: it respects the grades and in the middle case fixes both z, z''. Thus it commutes with the projection p onto the residual part on $U_*^{\perp}$. The finite quotient of Theorem 362.2 is equivariant under these changes. Indeed compose its map with the two changes on the respective sides. After transporting the indices back, both maps annihilate positive-translation differences, take the vacuum to the residual vacuum, and act by the same residual translations. Uniqueness in that theorem, applied as in Corollary 365.3, gives equality.

In normalized coordinates the residual action is $t \mapsto (k_j \oplus k_{j+2}) t k_{j+1}^{-1}$, extending the last action as specified. The separate factors $\bar{\alpha}, \alpha$ commute with these block actions. The residual isometry preserves both $L_{\mathrm{res}} = p(L_* \cap U_*^{\perp})$ and L_{st}. It sends the unique normalized L_{res}-fixed vector of Theorem 367.1 to itself: it is unitary, fixes v_{st}, and therefore preserves the positive overlap with v_{st}. This proves equivariance after the identification of Corollary 367.2, and shifting by D_τ preserves it since τ is fixed. The definite inverse adjoint identification intertwines the remaining changes by Theorem 364.1.

For integral elements the splitting multiplier is one by Theorem 375.2, and all powers of v_T are trivial on integral determinants by Theorem 374.2. Each original scalar on arrow l therefore equals the ratio of μ_l on the two determinants as in Theorem 376.2. Extending an element by identity preserves its determinant, also in the final pair by Proposition 365.1. Thus the original and definite products of scalar characters agree. □

Corollary 377.2. *In Theorem 371.1 take the full product of cuts from Definition 342.1, with the embedded column level in $L_i = S_i \cap A_i$ at each stage. This holds for $A_i = K_i$ at positive i, and for the principal marked level in the stage-zero frame. Use Ω_l on the initial data. The resulting definite tensor w is fixed by the product of the $K_{c,i}$, $i < m$, acting simultaneously at the ends of the definite arrows, with last group element one. Its recovered tensor $\mathcal{P}w$ is fixed under the same action.*

Proof. For membership in L_i use Proposition 349.1, including identity reduction at the marked places in the first stage. Each such group permutes its cut by the defining L_i-stability. Its multiplicity stays fixed by Lemma 342.2, since the permutation stays in each S_i-orbit. Acting on the left on the stage representatives acts on the moved initial vectors by the representation law of Theorem 376.2. Changing representatives of the right level cosets

preserves these vectors by Corollary 376.3. Proposition 377.1 therefore gives exactly the tuple permutation law in Theorem 371.1, proving invariance. At N the definite group actions preserve the integral translation subgroup in Proposition 370.1. Conjugating its finite character average by such an action permutes the translations and preserves normalized measure (equivalently, it permutes the classes in each corresponding finite average). The scalar character factors commute with the projections. Thus $\mathcal{P}$ commutes with the full product action and $\mathcal{P}w$ is fixed. □

378. Fourier expansion along a rational isotropic subspace

Theorem 378.1. *Let V be a rational hermitian space over E with nonsingular form H and bicharacter of Definition 356.1. Let $U \subset V$ be a totally isotropic E-subspace and let $U^{\perp}$ be its hermitian orthogonal. Write $[U] = U(E)\backslash U(\mathbb{A}_E)$, with quotient probability measure. Let F be an adelic theta function whose restriction to U is invariant under translation by a compact open subgroup $D \subset U(\mathbb{A}_{E,f})$ and is smooth in its real variables. For $w \in V(E)$ put*

$$A_w = \int_{[U]} (D_w F)(u)\,\mathrm{d}u, \qquad \chi_w(u) = \Psi\big(\jmath H(w,u)\big).$$

Then A_w depends only on $w \bmod U^{\perp}(E)$, and is zero unless $\chi_w|_D = 1$. On all of $[U]$ one has the absolutely and uniformly convergent expansion

$$F(u) = \sum_{w \in V(E)/U^{\perp}(E)} A_w\, \chi_w(u)^{-1}.$$

For a rational E-subspace $U_1 \subseteq U$ and $t \in V(E)$,

$$\int_{[U_1]} (D_t F)(u)\,\mathrm{d}u = \sum_{\substack{w \bmod U^{\perp}(E) \\ w - t \in U_1^{\perp}(E)}} A_w.$$

Proof. On U the rational transformation law and the commutator in Lemma 356.2 give

$$(D_w F)(u) = B(u,w)B(w,u)^{-1} F(u) = \chi_w(u) F(u).$$

The character is one on rational u by Proposition 18.5. Translation by D kills the integral unless $\chi_w|_D = 1$. In a rational basis of the underlying $\mathbb{Q}$-space of U, density of rational vectors in the finite adeles holds by the Chinese remainder proof in Lemma 21.1. Consequently every finite vector is rational modulo D, and $\Lambda = U(E) \cap D$ is dense in D. It is a full lattice in U_∞: in rational coordinates it contains a positive integer multiple of the standard integer lattice and is contained in an integer-denominator lattice, since the compact open D contains and is contained in the corresponding finite lattices. One obtains a smaller and a larger lattice by clearing the finitely many exceptional finite denominators as in Lemma 20.2. Sending u_∞ to the class of $(u_\infty, 0)$ identifies U_∞/Λ with $[U]/D$, by the preceding density and the description of the rational kernel. Integration of a function on this compact real torus agrees with normalized volume on a lattice parallelepiped, by the domain construction in Lemma 356.2, or by translation invariance and subdivision into equal congruent subtiles.

The map $w \mapsto \ell_w$, where $\ell_w(v) = \mathrm{Tr}_{E/\mathbb{Q}}(\jmath H(w,v))$ on $U(E)$, identifies $V(E)/U^{\perp}(E)$ with the full rational dual of that $\mathbb{Q}$-space. Indeed the trace pairing on V is nonsingular: a nonzero hermitian functional takes every value in E, including one with nonzero scaled trace. Restriction of rational linear functionals to a subspace is onto by extending a basis. Its kernel on restricting is the hermitian orthogonal, because testing all E-multiples tests every trace multiple. By Proposition 18.5, χ_w at the finite part of $a \in \Lambda$ equals $\exp(2\pi i \ell_w(a))$. Density in D therefore identifies the surviving indices with the full lattice dual $\Lambda^{\vee}$: every real linear functional integral on a rational lattice is rational on its rational span. At infinity $\chi_w(u) = \exp(-2\pi i \ell_w(u))$. The coefficients A_w are consequently the ordinary normalized Fourier coefficients of a smooth periodic function.

For completeness, integration by parts in a largest frequency coordinate in a lattice basis gives a bound $C_k(1+|w|)^{-k}$ in the dual lattice coordinates, for every k. The boundary terms cancel by periodicity. Counting in coordinate cubes gives a convergent majorant for $k > \dim_{\mathbb{R}} U_\infty$, also covering the constant term separately. The product kernel argument of Lemma 53.2 identifies the uniform sum with the periodic function, since their Fourier coefficients agree. This proves the first formula. For the second multiply by χ_t and integrate over $[U_1]$ term by term. Translation makes the integral of a nontrivial character zero. Triviality on $[U_1]$ is exactly $w - t \in U_1^{\perp}(E)$, as seen by varying the real coordinate along U_1 and taking all trace multiples. This gives the required sum, including $U_1 = 0$. □

Corollary 378.2. *The functions $\Theta_{\phi,f}$ of Theorem 359.1 satisfy the hypotheses of Theorem 378.1 for every rational totally isotropic subspace U.*

Proof. Every finite vector f is a finite sum of translates of a lattice vacuum. It is fixed by a common open subgroup of that lattice by the finite eigencharacter calculation of Theorem 358.1. Intersect with $U(\mathbb{A}_{E,f})$. On U these Heisenberg translations are ordinary translations because $H(U,U) = 0$. Finite equivariance of Theorem 359.1 gives the required invariance. Differentiation of each summand in real coordinates brings derivatives of the Schwartz factor and polynomials in the rational summation coordinates from the character. The lattice counting and Schwartz bound in that theorem still give an absolutely locally uniform sum for every such derivative. This proves smoothness on the real torus. □

379. A common partial sum for two target gradings

Definition 379.1. Choose consecutive spaces Y, Y' in the tower of Definition 348.2. Denote their homomorphism space by $V = \mathrm{Hom}_E(Y, Y')$. Use T from Definition 353.1 for a source string of length at least two, and $T = 0$ for length zero or one. Write g_a, g'_b for the respective grades and a^*, b^* for the indices paired by the Gram matrices. Thus the star reverses each string and fixes each index of the definite block. For $d_b = 0$ at every b, or $d_b = g'_b$ at every b, write U_d for the span of the matrix units E_{ba} with $d_b > g_a$, and N_d for the span of those with $d_b \geq g_a$. Let

$$C = \{(b,a) : g_a > \max(0, g'_b)\}, \qquad U_C = \mathrm{span}_E\{E_{b^*a^*} : (b,a) \in C\}.$$

Say w agrees with T on C if $w_{ba} = T_{ba}$ for all $(b,a) \in C$.

Proposition 379.2. *Suppose F satisfies the hypotheses of Theorem 378.1 on both U_0 and $U_{g'}$. Write $I_w^U(F) = \int_{[U]} (D_w F)(u)\,\mathrm{d}u$. For either choice of d,*

$$\sum_{\substack{w \in V(E)/N_d(E) \\ w|_C = T|_C}} I_w^{U_d}(F) = I_T^{U_C}(F).$$

Both sums converge absolutely. In particular the identity applies to the periodized data of Corollary 378.2.

Proof. In $H(A,B) = \mathrm{tr}(J^{-1}A^*J'B)$, the matrix unit E_{ba} pairs only with $E_{b^*a^*}$, with nonzero coefficient. This follows by multiplying the two antidiagonal or diagonal Gram matrices of Definition 348.2. The signs of both grades reverse under star. Thus U_d is totally isotropic and $U_d^{\perp} = N_d$. The indices common to the two positive spaces are precisely the reversals of C, so $U_C = U_0 \cap U_{g'}$. Orthogonality to U_C means vanishing at every entry in C. It is unchanged on adding any element of N_d. The subspace formula in Theorem 378.1, with $t = T$, now gives the assertion. □

Lemma 379.3. *Let U be a rational totally isotropic subspace of a nonsingular hermitian E-space. For any adelic theta function F, adelic w, and $v \in U(\mathbb{A}_E)$, one has*

$$I_{w+v}^U(F) = B(v,w)^{-1} I_w^U(F).$$

For a rational hermitian isometry G carrying U onto itself, $I_w^U(F \circ G^{-1}) = I_{G^{-1}w}^U(F)$.

Proof. Heisenberg multiplication as in Lemma 356.2 gives $D_v D_w = B(v,w) D_{w+v}$. On U, D_v acts by ordinary translation since the pairing is zero. Invariance of the probability on $[U]$ gives the first formula. For the second, $B(Gx, Gy) = B(x,y)$ gives $D_w(F \circ G^{-1}) = (D_{G^{-1}w} F) \circ G^{-1}$. Rationality makes G an automorphism of the compact quotient. It preserves the invariant probability: for any two invariant probabilities μ, ν on the quotient and continuous h, integrating $h(x+y)$ against their product gives both $\int h \, \mathrm{d}\mu$ and $\int h \, \mathrm{d}\nu$, by Lemma 123.1. Apply this to μ and its transport. Integration proves the claim. □

380. The moment of a surviving frequency

Lemma 380.1. *Use Definition 379.1. For a string of length l define S_l to be zero on the definite part and on its last string vector, if present. For $i+1 < l$ put*

$$S_l e_i = \sigma_l(i) e_{i+1}, \qquad \sigma_l(i) = \begin{cases} -Nc, & l-1-2i = 0, \\ -Nc^2, & l-1-2i = 1, \\ c, & \text{otherwise.} \end{cases}$$

For source and target lengths $s, s' = \max(s-1, 0)$, respectively, $T^\dagger T = S_s$ and $T T^\dagger = S_{s'}$, where the dagger denotes the hermitian adjoint. Every displayed shift coefficient is nonzero.

Proof. For $s \le 1$ all three maps are zero. Otherwise, with the h_i, τ_i, t_i of Definition 353.1, write h'_i for the corresponding target Gram entry. Adjointness at e'_i for $0 \le i < s-1$ gives

$$T^\dagger e'_i = \frac{h'_i}{h_{i+1}} \bar{t}_{s-2-i} e_{i+1} = h'_i \tau_{s-2-i} e_{i+1}.$$

Indeed $h_{s-2-i} = h_{i+1}$ and all the scalars belong to the real subfield. Both adjoints kill the respective definite blocks. The coefficient of $T^\dagger T e_i$ is $h_i h'_i \tau_i \tau_{s-2-i}$. The two source grades in the τ's are g_i and $2 - g_i$. For even s, exactly one is positive except when both are one. In the latter case both factors are c and $h'_i = -N$. For odd s exactly one is nonpositive since the grades are even, and $h_i = -N$ exactly at grade zero. This gives $\sigma_s(i)$. For $i+1 < s'$ the coefficient of $T T^\dagger e'_i$ is $h'_i h_{i+1} \tau_{s-2-i} \tau_{i+1}$, with the two source grades $1 - g'_i, g'_i - 1$. When they differ, exactly one factor is c. The factor h'_i is $-N$ just at $g'_i = 0$. When the two grades are equal they are zero, s is odd, both factors are c, and $h_{i+1} = -N$. This gives $\sigma_{s'}(i)$. The last target string vector maps under $T^\dagger$ to the last source string vector and is therefore killed by $T T^\dagger$. Nonvanishing follows from $cN \neq 0$. □

Definition 380.2. For a source unitary adelic isometry u, say $u - 1$ has degree at least two if every nonzero entry raises the grade by at least two. A rational $w : Y \to Y'$ *survives* for a target grading d of Definition 379.1 if for every such u with $w(u-1) \in U_d(\mathbb{A}_E)$,

$$B(Tu, T) = B(wu, w).$$

Here B is the bicharacter on the homomorphism space in Definition 356.1.

Proposition 380.3. *Suppose w survives for d. Let $x \in Y(E)$ and let k be a source basis index, of grade g_k. Assume that every grade in x is at least $\max(1, 2 + g_k)$, every target grade in wx is strictly greater than $\max(g_k, -1)$, and every sum of a target grade in we_{k^*} and a source grade in x is positive. A zero vector has empty grade support. Then*

$$((T^\dagger T - w^\dagger w) x)_k = 0.$$

Proof. Use $h(v,z) = v^* J z$, linear in z, and put $M = T^\dagger T - w^\dagger w$. For arbitrary $t \in \mathbb{A}_E$ take $y = t e_{k^*}$ and $\kappa = h(y,y)/2$. The hypotheses on grades imply $h(x,x) = h(x,y) = 0$. Write $v^\flat = h(v, \cdot)$ and form

$$R = x y^\flat - y x^\flat - \kappa x x^\flat, \qquad u = 1 + R.$$

Indeed $R + R^\dagger = -2\kappa x x^\flat$ and $R^\dagger R = 2\kappa x x^\flat$, so $u^\dagger u = 1$. The two mixed terms of R raise grade at least two since $g_a - g_k \ge 2$ on the support of x. The last term raises it by at least

two since all grades of x are positive. In particular R is nilpotent and $1+R$ is invertible. The first term in wR sends grade g_k into grades above g_k. For the second term the last support hypothesis gives strictly positive degree. For the third, the output grades are nonnegative and the input grades are negative. Thus $wR \in U_d$.

Write $A = h(y, Mx)$ and $D = h(x, Mx)$. The trace of a rank-one map $v\ell$ is $\ell(v)$ by summation of its diagonal. Hermitian symmetry gives $\operatorname{tr}(MR) = A - \bar{A} - \kappa D$. If $g_k \neq 0$ then $\kappa = 0$. If $g_k = 0$, x has grades at least two, so Tx has strictly positive target string grade, and wx has strictly positive d-grade. Both have norm zero since opposite grades pair. Consequently $D = 0$ in this case. By rationality $\Psi(\jmath\bar{\omega} \operatorname{tr} M) = 1$. The survival identity is therefore $\Psi(\jmath\bar{\omega}(A - \bar{A})) = 1$, by the trace expression for H in Proposition 379.2. Conjugate invariance of Ψ, the relation $\bar{\jmath} = -\jmath$, and $\omega + \bar{\omega} = 1$ turn the left side into $\Psi(\jmath A)$. Here $A = \bar{t}h(e_{k^*}, e_k)(Mx)_k$. The Gram coefficient is nonzero. If $(Mx)_k \neq 0$, varying t at one infinite place gives every real trace value and hence a character value different from one, by Definition 356.1. This contradiction proves the assertion. □

381. The orbit of the positive columns

Lemma 381.1. *Let L be a field of characteristic zero with a nontrivial involution, and let W be a finite dimensional nonsingular hermitian space over L. Every isometry from a subspace $A \subset W$ onto a subspace $A' \subset W$ extends to an isometry of W, including when the restricted forms are degenerate.*

Proof. Write h for the form, linear in its second entry. First suppose the restriction to A is nonsingular. Any nonzero hermitian space with nonzero form contains a vector of nonzero norm. Indeed if every norm vanished, expanding the norm of $v + tw$ would give $\operatorname{Tr}_{L/L^+}(th(v, w)) = 0$ for all t, where L^+ is the fixed field. A nonzero $h(v, w)$ would permit the trace of one, which is two. Splitting off each nonsingular line gives an orthogonal basis of A. Match the basis vectors one at a time in the nonsingular orthogonal complement of the lines already matched.

Here is the required operation in that complement. If v, w have the same nonzero norm, set $z = v - w$ and $a = h(z, v)$. For $a \neq 0$, the transformation

$$L_z(x) = x - z\, h(z, x)/a$$

takes v to w. Expanding $h(L_z x, L_z y)$ leaves the coefficient $-1/a - 1/\bar{a} + h(z, z)/(a\bar{a}) = 0$ of $h(x, z)h(z, y)$, since $h(z, z) = a + \bar{a}$. This is an isometry, invertible by nonsingularity and equal finite dimensions. For $a = 0$, $h(v + w, v) = 2h(v, v) \neq 0$. The same operation now takes v to $-w$. Negating the whole complement then takes $-w$ to w. Extend the operation by the identity on the already matched lines. This proves the nonsingular case.

In general, let R be the radical of A and choose a vector space complement A_0 to R in A. Its restriction is nonsingular, since any vector orthogonal to A_0 in A_0 is orthogonal to all of A. In $A_0^{\perp}$ take a basis $r_1, \ldots, r_b$ of R. The functionals $h(r_i, \cdot)$ are independent by nonsingularity, so choose $t_j \in A_0^{\perp}$ with $h(r_i, t_j) = \delta_{ij}$. Replace t_j by $t_j - \frac{1}{2}\sum_i r_i h(t_i, t_j)$. Their mutual Gram entries become zero and the pairings with the r_i are unchanged, by expansion using $h(R, R) = 0$. These $2b$ vectors span a nonsingular subspace with Gram matrix $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Perform this construction for the images of A_0, r_i in the target. Pairing the two lists extends the prescribed map to an isometry on a nonsingular subspace. The first part of the proof extends this isometry to W. □

Theorem 381.2. *Let $d = 0$ in Definition 379.1 and let w survive in the sense of Definition 380.2. There is a rational unitary isometry δ of Y' with $\delta w e_i = Te_i$ at every positive source index i. In particular δw agrees with T on C.*

Proof. All positive indices belong to the string. Fix positive indices i, j. In Proposition 380.3 take $x = e_i$ and $k = j^*$. Then $g_k = -g_j \leq -1$, $g_i \geq \max(1, 2 - g_j)$, and every target grade

is zero. All three support conditions hold. Pairing the resulting coordinate identity with e_j gives $h'(we_j, we_i) = h'(Te_j, Te_i)$, where h' is the form on Y'.

Let x be a nonzero vector in the positive span with $wx = 0$. Among its nonzero coefficients take the one of least grade, at b. The positive string grades occur with multiplicity one. The next string index $k = b + 1$ exists because b has positive grade. Now $g_k = g_b - 2$, x is supported in grades at least $g_b \geq 1$, and wx has empty support. The last support condition again holds because $d = 0$. The proposition gives $(T^\dagger T x)_k = 0$. Lemma 380.1 identifies this coordinate with $\sigma_s(b) x_b \neq 0$. Thus w is injective on the positive span. So is T, whose positive columns are nonzero multiples of distinct target string vectors by Definition 353.1. For source length zero or one there are no positive columns.

The map $we_i \mapsto Te_i$ consequently defines an isometry between the two image subspaces, by the Gram equalities. Apply Lemma 381.1 over E with conjugation. Every prescribed entry in C occurs in one of these columns. This proves the final claim. □

382. The stabilizer and the negative target rows

Proposition 382.1. *Let $B_Y, B_{Y'}$ denote the groups of unitary isometries preserving the ordered initial string flags and acting trivially on their successive lines, as in Definition 346.1. The same definition applies over E or over its adeles in rational coordinates. Suppose w agrees with T on C in Definition 379.1. Then for $\gamma \in B_Y$ and any target isometry δ,*

$$(\delta w \gamma)|_C = T|_C \quad \Longleftrightarrow \quad \delta \in B_{Y'}.$$

Both Bessel groups preserve the filtrations by decreasing grades. For $\gamma \in B_Y$, $\delta \in B_{Y'}$ one has $\delta T \gamma - T \in N_{g'}$.

Proof. The positive source indices i run from zero to $\lfloor s/2 \rfloor - 1$. There are exactly this many initial target flag lines since $s' = \max(s - 1, 0)$. On each such column the condition on C is precisely

$$we_i = t_i e'_i + \sum_{h<i} b_{hi} e'_h.$$

In fact $g'_i = g_i - 1$, and the only target indices of grade at least $g_i > 0$ are the earlier string vectors. Right multiplication by γ preserves these triangular columns and their nonzero leading coefficients. The first column is fixed modulo its empty prefix precisely when δ fixes e'_0. Inductively, once δ preserves the earlier flag with trivial action on the lines, the condition on the next column says exactly $\delta e'_i - e'_i \in \langle e'_h : h < i \rangle$, by division by $t_i \neq 0$. The reverse implication follows from the same calculation. With no positive columns the target flag is empty and the assertion holds for every target isometry.

For a Bessel isometry the filtration in positive grades is the initial string filtration. The subspace of grades at least a for $a \leq 0$ is the orthogonal complement of the subspace of grades strictly greater than $-a$, by the Gram pairing. Thus the entire grade filtration is preserved. On each negative one-dimensional graded piece the induced action is one because its pairing with the opposite, fixed graded line is perfect. In grade zero the definite block is preserved on the graded quotient: when there is a middle string vector, that vector is fixed on the quotient and spans a nonsingular line, so its orthogonal complement there is preserved. In particular $\gamma - 1$ raises every string vector by at least one grade, and maps the definite block to itself modulo positive grades. The same assertions apply to δ. The map T lowers grades by one, kills the definite block and takes its values in the target string. Accordingly $T(\gamma - 1)$ and $(\delta - 1)T$ have nonnegative degree. Multiplying the former by the filtration-preserving δ gives the final assertion. □

Theorem 382.2. *Let $d = g'$, suppose w agrees with T on C, and suppose w survives for d in Definition 380.2. Then $w - T \in N_{g'}(E)$. Thus $T + N_{g'}(E)$ is the unique surviving frequency class among the classes agreeing on C.*

Proof. It remains to determine w_{ba} when $g'_b < g_a \le 0$. Prove the required equality by increasing induction on the negative target grades g'_b. For this fixed a, every entry of we_a in grades less than g'_b has already been determined. It vanishes, because Te_a is zero or of grade $g_a - 1 \ge g'_b$. The index b is a negative string index. Let i be the source string index bearing the same integer as the positive target index b^*. Then $g_i = 1 - g'_b$. The triangular formula in Proposition 382.1 expresses we_i as $t_i e'_{b^*}$ plus terms of grades strictly greater than $-g'_b$.

Apply Proposition 380.3 to $x = e_i,\ k = a^*$. First $g_i = 1 - g'_b \ge 2 - g_a = 2 + g_k$ and $g_i \ge 1$. Next every grade in we_i is at least $-g'_b > -g_a = g_k \ge 0$. Finally every grade in we_a is at least g'_b, so adding g_i gives at least one. The proposition gives $h'(we_a, we_i) = h'(Te_a, Te_i)$. All components of we_i of grade greater than $-g'_b$ pair to zero with we_a by the previously established vanishing. The sole remaining target vector is $t_i e'_{b^*}$, which reads the b-coordinate of we_a up to conjugation and a nonzero Gram factor. Since $Te_i = t_i e'_{b^*}$, we obtain $w_{ba} = T_{ba}$. This completes the induction, including the vacuous case of no negative target grades.

To check that survival is a property of the class, take any rational $n \in N_{g'}$ and eligible $u = 1 + R$ of Definition 380.2. Eligibility is unchanged for $w + n$ since nR has degree at least two. Both wR and nR are positive and $n \in U_{g'}^{\perp}$ by Proposition 379.2. Thus $H(n, wR) = H(n, nR) = H(wR, nR) = 0$. The identity $H(wu, nu) = H(w, n)$, obtained by cyclicity of the trace, gives $H(w, nR) = 0$ as well. Expanding $H(w + n, (w + n)u)$ now changes $H(w, wu)$ only by rational constants, whose character is one. This proves class invariance. The representative T satisfies the survival identity with equality by its definition and belongs to the triangular set. □

383. A rational unipotent group for each frequency

Theorem 383.1. *Fix either grading d in Definition 379.1 and a rational matrix w. Let J_w be the set of unitary $u = 1 + R$ with R of degree at least two and $wR \in U_d$. Write Δ_w for its rational points and*

$$\Gamma_w = \{\gamma \in B_Y(E) : w(\gamma - 1) \in N_d(E)\}.$$

Then J_w is a group, contained in the source Bessel group. Its adelic points and rational points are obtained by exponentiating the corresponding points of the rational vector space

$$\mathfrak{j}_w = \{A : A^\dagger = -A, \quad \deg A \ge 2, \quad wA \in U_d\}.$$

The notation $\deg A \ge h$ means that all nonzero entries raise the source grade by at least h. The discrete group Δ_w is contained in Γ_w, and Γ_w acts on J_w and Δ_w by conjugation. There is a left and right invariant measure on $J_w(\mathbb{A})$ giving a compact set of representatives up to boundary and giving a measurable fundamental domain of Δ_w volume one. Conjugation by Γ_w preserves this measure.

Proof. All operators of strictly positive degree are nilpotent, with one common bound on their nilpotence as a matrix algebra since the number of grades is finite. The finite power series $\log(1 + R) = R - R^2/2 + \cdots$ and $\exp A = 1 + A + A^2/2! + \cdots$ are inverse there. The scalar formal identities follow by differentiating and comparing constant terms in characteristic zero, and can be reduced modulo the common nilpotence bound. For a unitary $1 + R$, the logarithm is skew adjoint since $\log((1 + R)^{-1}) = -\log(1 + R)$. Conversely the exponential of a skew adjoint matrix is unitary. If wR has positive degree then so does wR^i for $i \ge 1$, and the same applies to A. This proves the two coordinate descriptions. The linear equations defining $\mathfrak{j}_w$ have rational coefficients after writing the entries in a $\mathbb{Q}$-basis of E. Elimination in that basis gives the same parametrization over $\mathbb{Q}$, every completion and the full adeles, since the finitely many coefficients have bounded rational denominators. For R, S as above, $R + S + RS$ has the required degree and $w(R + S + RS)$ is positive. The finite inverse series gives the same conclusion for $(1 + R)^{-1} - 1$. Thus J_w is a group. Raising grade on the initial string flag implies membership in B_Y by Proposition 382.1.

Every member of B_Y preserves grades in the filtered sense of that proposition. Hence N_d and U_d are stable under right multiplication by it and its inverse. It follows that Γ_w is a subgroup. If $\gamma \in \Gamma_w$, write $w\gamma = w + n$, $n \in N_d$. For $1 + R \in J_w$ one has

$$w\gamma R\gamma^{-1} = (wR + nR)\gamma^{-1} \in U_d, \qquad \deg(\gamma R\gamma^{-1}) \geq 2.$$

Apply this also to γ^{-1} to obtain normalization. Rational points are preserved and $\Delta_w \subseteq \Gamma_w$ since $U_d \subseteq N_d$.

Choose a $\mathbb{Q}$-basis of $\mathfrak{j}_w$ in order of increasing degree, by successively choosing representatives of bases of its successive quotients by degree. In these coordinates of the logarithm, multiplication adds the two coordinates of degree h plus a polynomial in the coordinates of strictly lower degree. Indeed every other term of $\log(\exp A \exp B)$ contains at least two matrix factors of degree at least two. Replacing one argument by itself plus a matrix of degree h changes every nonlinear term only in degrees greater than h. Rational linear projection onto the degree quotient gives the assertion. In particular product additive measure in these coordinates is left and right invariant by successive translations of each block, using Lemma 123.1. Inversion is negation on logarithms and preserves that measure.

In each coordinate use the fundamental set $[0, 1) \times \widehat{\mathbb{Z}}$ for $\mathbb{Q}$ in its adeles from Lemma 356.2. Successively left multiply by rational elements supported in the current block of logarithmic coordinates, bringing that block into its fundamental set. Such multiplication leaves all lower blocks fixed. Conversely two points in the product of these sets differing by rational left multiplication agree in the first block, forcing that block of the multiplier to vanish. Repeating proves uniqueness. The product has compact closure and volume one, and discreteness follows also from discreteness of each rational additive diagonal. Conjugation by γ acts linearly on logarithms by $A \mapsto \gamma A\gamma^{-1}$, an invertible rational map of $\mathfrak{j}_w$. In any rational basis its modulus is one on the adeles. Row additions and permutations preserve product additive measure. Multiplication by $a \in \mathbb{Q}^\times$ changes the real measure by $|a|$ and the finite measure by $\prod_p p^{-v_p(a)} = |a|^{-1}$, by successive residue indices and the factorization in Lemma A.1. Elimination into these operations proves the assertion. □

Corollary 383.2. *On $J_w(\mathbb{A})$ the ratio $\eta_w(u) = B(Tu, T)/B(wu, w)$ is a continuous unitary character trivial on Δ_w. It is nontrivial precisely when w fails the test of Definition 380.2.*

Proof. By rationality $B(w, w) = B(T, T) = 1$. Given $u, v \in J_w(\mathbb{A})$, $w(u - 1)$ and $w(v^{-1} - 1)$ both belong to the totally isotropic U_d. Right unitary invariance of H gives

$$H(w, w(u - 1)(v - 1)) = H(w(v^{-1} - 1), w(u - 1)) = 0.$$

Consequently $u \mapsto B(wu, w)$ is multiplicative by expansion of $uv - 1$. For T, the identical calculation uses the positive space for the target grading g', since T lowers its grade by one and $u - 1$ and $v^{-1} - 1$ raise source grades by two. Continuity and modulus one follow from the bicharacter definition. Its value on rational pairs is one by Proposition 18.5. The last assertion is the stated test. □

384. Orbit integration and cancellation

Lemma 384.1. *Let Γ be a countable group acting freely on a measured space Z, preserving measure, with measurable fundamental domain D. Let it act on a countable set I, and suppose the measurable functions k_i on Z satisfy $k_{\gamma i}(\gamma z) = k_i(z)$. Suppose $C \subset I$ and $S \leq \Gamma$ satisfy $\gamma i \in C$ if and only if $\gamma \in S$ for every $i \in C$. Then*

$$\int_{\Gamma\backslash Z} \sum_{i \in \Gamma C} k_i(z)\, \mathrm{d}z = \int_{S\backslash Z} \sum_{i \in C} k_i(z)\, \mathrm{d}z$$

whenever either integral of the corresponding sum of absolute values is finite. Those two nonnegative integrals always agree. For $C = \{w\}$ the subgroup is the stabilizer of w.

Proof. The distinct sets γC are disjoint and indexed by $\gamma \in \Gamma/S$: if two translates meet, their quotient takes one member of C into C and hence lies in S. For coset representatives γ, the sets $\gamma^{-1}D$ together form a fundamental domain for S. By equivariance, integration over D of the sum on γC equals integration over $\gamma^{-1}D$ of the sum on C. The sum on C is S-invariant. Additivity for nonnegative functions and then absolute convergence give the formula, by Lemma A.2. Independence of the domains follows by partitioning into their intersections with group translates as in Lemma 21.4. □

Theorem 384.2. *Let G have a left invariant measure and let the countable subgroup Γ act by left translations with measurable fundamental domain D. Let J be a measurable subgroup normalized by Γ and put $\Delta = \Gamma \cap J$. Assume that J has a left and right invariant measure, preserved by inversion and by conjugation under Γ, and a measurable left fundamental domain D_J for Δ of measure one. All groups and maps are Borel in second countable locally compact groups with measures finite on compact sets. For every measurable left Γ-invariant k integrable on D, let*

$$Qk(x) = \int_{D_J} k(jx)\,\mathrm{d}j.$$

The integral defining Qk exists almost everywhere on D. It gives a Γ-invariant integrable function up to null sets, and

$$\int_D Qk(x)\,\mathrm{d}x = \int_D k(x)\,\mathrm{d}x.$$

If $k(jx) = \eta(j)k(x)$, where η is a nontrivial unitary character of J trivial on Δ, then $\int_D k = 0$.

Proof. Put $a = \mathbf{1}_D$, so $\sum_{\gamma\in\Gamma} a(\gamma y) = 1$. The subgroup Δ is normal in Γ. Changing variables by conjugation shows that $Qk(\gamma x) = Qk(x)$ where the integral is defined: $\gamma^{-1}D_J\gamma$ is another left fundamental domain and $k(jx)$ is left Δ-invariant in j. First take $k \geq 0$, allowing infinite integrals. Left invariance on G and product integration give

$$\int_D Qk(x)\,\mathrm{d}x = \int_G k(y)a'(y)\,\mathrm{d}y, \qquad a'(y) = \int_{D_J} a(j^{-1}y)\,\mathrm{d}j.$$

Here and below nonnegative sums and integrals interchange by Lemma 123.1. Choose representatives s for $\Delta\backslash\Gamma$. Summing $a'(\gamma y)$ first over $\gamma = \delta s,\ \delta \in \Delta$, tiles J by the sets $\delta^{-1}D_J$, giving $\sum_s \int_J a(j^{-1}sy)\,\mathrm{d}j$. Conjugation by s changes this to $\sum_s \int_J a(sv^{-1}y)\,\mathrm{d}v$. Use the right fundamental domain D_J^{-1}, which has measure one by inversion. Write $v = v_0\delta$. The elements $s\delta^{-1}$ run over all of Γ once, because Δ is normal. The last sum equals

$$\int_{D_J^{-1}} \sum_{\gamma\in\Gamma} a(\gamma v_0^{-1}y)\,\mathrm{d}v_0 = 1.$$

Partitioning the G-integral by the translates of D now gives $\int_D k$. Applying this equality first to $|k|$ and then to positive and negative parts of the real and imaginary components proves the integrability and equality claimed for complex k. For the character assertion the integral of η over D_J is unchanged on replacing that domain by $D_J j_0$. Right invariance multiplies its value by $\eta(j_0)$, so choosing $\eta(j_0) \neq 1$ gives zero. Hence $Qk = 0$. □

Corollary 384.3. *Let $B = B_Y(\mathbb{A})$ and $\Gamma = B_Y(E)$ in Proposition 382.1. On $I = V(E)/N_d(E)$ act by $\gamma[w] = [w\gamma^{-1}]$. Assume B is given left Haar measure with a measurable fundamental domain of Γ. Suppose k_i is an equivariant family as in Lemma 384.1 with $\sum_i |k_i|$ integrable on that domain. On an orbit suppose one rational representative w satisfies $k_{[w]}(ux) = \eta_w(u)k_{[w]}(x)$ for $u \in J_w(\mathbb{A})$. If w fails the survival condition, the integral of the sum over its entire orbit is zero.*

Proof. The stabilizer of $[w]$ is Γ_w of Theorem 383.1, since right multiplication by γ preserves N_d. Unfold the single orbit by Lemma 384.1. Its integral is that of $k_{[w]}$ on $\Gamma_w\backslash B$ and

converges absolutely. Here $\Gamma_w \cap J_w(\mathbb{A}) = \Delta_w$: a rational matrix satisfying the defining equations adelically satisfies them over its field of definition. Theorem 383.1 provides the required normalization and measure, and Corollary 383.2 makes η_w nontrivial. Apply Theorem 384.2. □

385. A rational polarization adapted to the constant term

Definition 385.1. Regard the space V of Definition 379.1 as a rational vector space by restriction of scalars. Denote its positive, zero and negative degree subspaces for a fixed d by U, W, U_-. Define rational bilinear forms

$$b(v,z) = \mathrm{Tr}_{E/\mathbb{Q}}(j\bar{\omega}H(z,v)), \qquad a(v,z) = b(v,z) - b(z,v).$$

Let $\mathbf{e}$ be the additive character of $\mathbb{A}_\mathbb{Q}$ with real component $t \mapsto \exp(-2\pi i t)$ and with finite component $\exp(2\pi i r)$ when the finite input is congruent to $r \in \mathbb{Q}$ modulo $\widehat{\mathbb{Z}}$. Such r exists and is unique modulo $\mathbb{Z}$ by Lemma 356.2. The trace definition in Proposition 18.5 gives $B(v,z) = \mathbf{e}(b(v,z))$ adelically.

Lemma 385.2. *There exist rational subspaces $P_W, L_W \subset W$ such that $P = U_- \oplus P_W$, $L = U \oplus L_W$ are complementary maximal isotropic subspaces for a. The form a pairs P, L perfectly. In these coordinates $a(t,U) = 0$ for $t \in P$ if and only if $t \in P_W$.*

Proof. As in Lemma 356.2, $a(v,z) = \mathrm{Tr}_{E/\mathbb{Q}}(jH(z,v))$. It is alternating and nonsingular: for nonzero v the hermitian functional in z takes all values in E, including one with nonzero scaled trace. Proposition 379.2 pairs opposite basis indices with nonzero Gram coefficients. Thus opposite degree pieces are perfectly paired also for a, and W is a nonsingular alternating space. In W take a nonzero p, choose l with $a(p,l) = 1$, and remove the plane they span by orthogonal projection. Indeed the two coefficients in that projection are obtained by pairing with l, p. Repeat on its nonsingular perpendicular until exhausted. Taking the spans of the resulting p's and l's gives P_W, L_W, also when $W = 0$. Perfect pairing between U_- and U gives the last assertion. □

Theorem 385.3. *Use P, L of Lemma 385.2. A test function on $P(\mathbb{A}_\mathbb{Q})$ is a finite sum of products of a real Schwartz function and a locally constant compactly supported finite function. Write $v = x + l$ according to P, L and $z = x' + l'$. Put $b_0(v,z) = a(x,l')$, $q(v) = -(b - b_0)(v,v)/2$, and*

$$(\rho(v)f)(t) = \mathbf{e}(q(v) + a(t,l))f(t+x), \qquad \theta_P(f) = \sum_{t \in P(\mathbb{Q})} f(t).$$

Then $\rho(v)\rho(z) = B(v,z)\rho(v+z)$. The function $F_f(v) = \theta_P(\rho(v)f)$ is an adelic theta function in Definition 356.1 and $D_z F_f = F_{\rho(z)f}$. Its constant term, for every adelic w, is

$$I_w^U(F_f) = \sum_{t \in P_W(\mathbb{Q})} (\rho(w)f)(t).$$

In particular the right side uses only the restriction of $\rho(w)f$ to $P_W(\mathbb{A}_\mathbb{Q})$.

Proof. The alternating parts of b, b_0 coincide since P, L are complementary isotropic spaces. Hence $b - b_0$ is symmetric and $q(v) + q(z) - q(v+z) = (b - b_0)(v,z)$. In a product of two displayed translations the ordinary translation and modulation produce the additional exponent $a(x,l') = b_0(v,z)$. This proves the multiplication law.

Finite support places all rational summation indices in a single lattice in $P(\mathbb{R})$, by clearing denominators. Uniformly over compact adelic translating sets this same conclusion holds by adding the compact set of finite shifts to the finite support. Polynomial decay of arbitrarily high order on the real lattice gives locally uniform absolute convergence, as in Theorem 359.1. Finite characters are locally constant. Real derivatives introduce only polynomial factors or Schwartz derivatives, giving the same convergence and continuity. For $r \in V(\mathbb{Q})$ its two components in P, L are rational. Every character factor of $\rho(r)$ on an argument $t \in P(\mathbb{Q})$ is

therefore one. Translation of $P(\mathbb{Q})$ by x_r preserves the summation set, so $\theta_P(\rho(r)f) = \theta_P(f)$. The multiplication law now gives $F_f(v+r) = B(r,v)^{-1}F_f(v)$ and $D_zF_f = F_{\rho(z)f}$.

On U the quadratic term q is zero since $H(U,U) = 0$ and $U \subset L$. Restriction of $F_{\rho(w)f}$ to U is thus a sum of $(\rho(w)f)(t)\mathbf{e}(a(t,u))$. The compact set of representatives for $[U]$ from Lemma 356.2 permits termwise integration by the uniform bound. A character has integral zero unless it is trivial, by translating the probability integral. For rational t triviality means $a(t,U) = 0$, as is seen by varying the real arguments. These are exactly $t \in P_W(\mathbb{Q})$ by Lemma 385.2. Every remaining character has integral one, proving the formula. □

386. The parabolic operator in rational coordinates

Theorem 386.1. *Let G be an adelic isometry of b in Definition 385.1 preserving L in Lemma 385.2. Write*

$$G(x+l) = Ax + Cx + Dl, \qquad Ax \in P, \quad Cx, Dl \in L.$$

Put $K_G(t,t') = a(t, CA^{-1}t')$ on P. For an adelic invertible A write $|\det A|_{\mathbb{A}}$ for the product of its real and normalized finite absolute determinants. For the test functions of Theorem 385.3 set

$$(\mathcal{W}_Gf)(t) = |\det A|_{\mathbb{A}}^{-1/2}\mathbf{e}(-K_G(t,t)/2)\, f(A^{-1}t).$$

These operators form a unitary representation on the test functions with their additive L^2 norm, and $\mathcal{W}_G\rho(z) = \rho(Gz)\mathcal{W}_G$. If $G-1$ raises the degrees on V by at least one, it preserves L and

$$I_w^U(F_{\mathcal{W}_Gf}) = I_{G^{-1}w}^U(F_f) \qquad (w \in V(\mathbb{A})).$$

For rational G preserving L one also has $F_{\mathcal{W}_Gf}(v) = F_f(G^{-1}v)$.

Proof. Preservation of a gives $a(Ax, Dl) = a(x,l)$ and $a(Ax, Cx') = a(Ax', Cx)$. Thus K_G is symmetric. Matrix multiplication for two such maps G_1, G_2 gives $A_{12} = A_1A_2$, $C_{12} = C_1A_2 + D_1C_2$. Consequently

$$K_{12}(t,t) = K_1(t,t) + K_2(A_1^{-1}t, A_1^{-1}t),$$

proving the representation law. A real linear substitution changes volume by the absolute determinant, and at finite places the measure change follows by row elimination and the residue-index argument in Theorem 360.1. This gives unitarity. The character factor preserves the test space since at infinity it has polynomially bounded derivatives and at the finite places it is locally constant. On the compact finite support only finitely many constancy cosets are required.

With q from Theorem 385.3 one has $q(Gz) - q(z) = a(Ax, Cx)/2$ for $z = x + l$. This follows from invariance of b and $b_0(Gz, Gz) = a(Ax, Cx) + a(x, l)$. The ratio of the quadratic phases at t and $t + Ax$ is $\mathbf{e}(a(t, Cx) + a(Ax, Cx)/2)$. Combining the last two identities with $a(A^{-1}t, l) = a(t, Dl)$ gives the asserted intertwining by evaluating both sides at t.

When $G - 1$ raises degrees, G and its inverse preserve U, take each vector in W to itself modulo U, and hence preserve $L = U + L_W$. The induced operator A on V/L is unipotent, so has determinant one: in a basis ordered by degree it is triangular with diagonal one. For $t \in P_W$ one has $At = t$, $Ct \in U$. In particular $A^{-1}t = t$ and $a(t, Ct) = 0$. It follows that $(\mathcal{W}_Gf)|_{P_W} = f|_{P_W}$. Move $\rho(w)$ through $\mathcal{W}_G$ by the intertwining identity and use the constant term formula of Theorem 385.3. Finally, for rational G, the adelic modulus is one by the rational product calculation of Theorem 383.1, and the quadratic phase is one at rational t. Reindexing $P(\mathbb{Q})$ by A gives $\theta_P\mathcal{W}_G = \theta_P$. Intertwining once more gives the rational assertion. □

Corollary 386.2. *For $v \in J_w(\mathbb{A})$ let G act on V by $z \mapsto zv^{-1}$. For rational w,*

$$B(Tv, T)\, I_w^{U_d}(F_{\mathcal{W}_Gf}) = \eta_w(v) I_w^{U_d}(F_f).$$

If $d = g'$ and v' is a target unitary with $v' - 1$ of target degree at least two, let $G'z = v'z$. Then

$$I_T^{U_{g'}}(F_{\mathcal{W}_{G'}f}) = \Psi\left(\frac{\jmath}{2}\operatorname{tr}(S_{s'}v')\right) I_T^{U_{g'}}(F_f).$$

For source v, the factor $B(Tv, T)$ in the first identity equals $\Psi(\frac{\jmath}{2}\operatorname{tr}(S_s v))$.

Proof. Both maps preserve b by cyclicity of the hermitian trace. Right multiplication by $v^{-1}-1$ raises Hom degree by at least two, since it raises the grade of a source argument before the Hom matrix is applied. The same degree increase holds for left multiplication by $v' - 1$ with $d = g'$. Theorem 386.1 therefore changes the respective frequencies to wv and $v'^{-1}T$. Their differences from the original frequencies belong to the positive space, in the first case by definition of J_w and in the second since T lowers grade by one. Lemma 379.3 now contributes the inverses of $B(wv, w)$ and $B(v'^{-1}T, T)$, respectively, using $B(w, w) = B(T, T) = 1$ by rationality.

For any shift S of Lemma 380.1 and any unitary $u = 1 + R$ of the corresponding space with $\deg R \geq 2$, one has $\operatorname{tr}(S) = 0$ and $\operatorname{tr}(SR^l) = 0$ for $l \geq 2$, by strict increase of grade. The finite inverse series gives $\operatorname{tr}(Su^{-1}) = -\operatorname{tr}(Su)$. Since $S^\dagger = S$, adjointness and cyclicity also identify the left side with $\overline{\operatorname{tr}(Su)}$. For an anti-invariant value z, the traces to $\mathbb{Q}$ of $\jmath\bar{\omega}z$ and $\jmath z/2$ agree, since $\jmath z$ is real and $\omega + \bar{\omega} = 1$. Apply this with $S = T^\dagger T = S_s$ on the source and $S = TT^\dagger = S_{s'}$ on the target. This proves both formulas, with the inverse in the target factor canceling the sign. □

387. Changing the rational polarization

Lemma 387.1. *With the character* $\mathbf{e}$ *of Definition 385.1, give $\mathbb{A}_\mathbb{Q}$ product additive measure with Lebesgue measure at infinity and $\widehat{\mathbb{Z}}$ of finite measure one. For a test function of one adelic variable put $\widehat{f}(t) = \int_{\mathbb{A}_\mathbb{Q}} \mathbf{e}(ty)f(y)\,\mathrm{d}y$. This is a test function and $\sum_{r\in\mathbb{Q}} \widehat{f}(r) = \sum_{r\in\mathbb{Q}} f(r)$, absolutely. Fourier transformation in one coordinate of a several-variable test function preserves the test space and its sum over all rational coordinates.*

Proof. A locally constant compactly supported finite function is a finite linear combination of indicators of $u + d\widehat{\mathbb{Z}}$ with $u \in \mathbb{Q}$, $d \in \mathbb{Q}_{>0}$: cover its support by finitely many constancy cosets, refine to a common open lattice, and take rational representatives as in Lemma 356.2. For $f = \phi \otimes \mathbf{1}_{u+d\widehat{\mathbb{Z}}}$, the finite Fourier transform is $d^{-1}\mathbf{e}_f(tu)\mathbf{1}_{d^{-1}\widehat{\mathbb{Z}}}(t)$. Indeed the finite volume is d^{-1} by the real and finite modulus calculation in Theorem 383.1; translation kills any character nontrivial on $d\widehat{\mathbb{Z}}$, whose dual is $d^{-1}\widehat{\mathbb{Z}}$ by the one-coordinate annihilator calculation of Proposition 18.5. The real transform $\widehat{\phi}(t)$ is Schwartz: derivatives insert powers of the real integration variable and repeated integration by parts gives every required decay bound. Thus the sum of transforms is

$$d^{-1}\sum_{m\in\mathbb{Z}} e^{2\pi i m u/d}\widehat{\phi}(m/d).$$

The smooth periodic sum $\sum_n \phi(x+dn)$ has Fourier coefficients $d^{-1}\widehat{\phi}(m/d)$ by integrating over one real period. Its Fourier series converges to the function by the coefficient estimate and the product kernel proof in Theorem 378.1. Evaluation at u gives $\sum_n \phi(u+dn) = \sum_{r\in\mathbb{Q}} f(r)$. For several coordinates choose product finite constancy lattices and apply the one-coordinate argument with the remaining variables fixed. Partial derivatives in the remaining real variables obey the same decay estimates, uniformly after multiplication by any polynomial in those variables. Thus all sums over rational coordinates converge absolutely, permitting their interchange. □

Theorem 387.2. *Let P, L be complementary rational maximal isotropic spaces for a in Lemma 385.2, with dual rational bases. For $v = x + l$ set $\rho_a(v) = \mathbf{e}(b(v, v)/2)\rho(v)$, and*

$F^a_f(v) = \theta_P(\rho_a(v)f)$. *Every rational linear automorphism G preserving a induces a unique operator $\mathcal{T}_G$ on test functions such that*

$$F^a_{\mathcal{T}_G f}(v) = F^a_f(G^{-1}v).$$

The operators are a unitary representation and intertwine $\rho_a(z)$ with $\rho_a(Gz)$. For G also preserving b the same statements hold for F_f, ρ. In particular if G preserves U_d, $I^{U_d}_w(F_{\mathcal{T}_G f}) = I^{U_d}_{G^{-1}w}(F_f)$. The space of functions F_f obtained in this way is independent of the rational polarization.

Proof. In the dual bases write $a(x,l) = x \cdot l$. The phase in $\rho_a(x+l)f(t)$ is $\mathbf{e}(x \cdot l/2 + t \cdot l)$. For any fixed adelic x, integration of $\mathbf{e}(-x \cdot l/2)F^a_f(x+l)$ over $[L]$ returns $f(x)$. Indeed each rational summation index r contributes its character $\mathbf{e}(r \cdot l)$, leaving only $r = 0$. The absolute convergence is that of Theorem 385.3. This gives injectivity. Squaring absolute values and using the same character orthogonality gives $\int_{[L]} |F^a_f(x+l)|^2 \, \mathrm{d}l = \sum_{r \in P(\mathbb{Q})} |f(x+r)|^2$. Integrate over $[P]$ using the product fundamental domains of Lemma 356.2. Hence the squared norm of f equals the integral of $|F^a_f|^2$ over $V(\mathbb{Q})\backslash V(\mathbb{A}_\mathbb{Q})$. Its integrand is periodic, by the rational translation law and unit modulus.

For rational G preserving L, the formula of Theorem 386.1 applies with the same A, C, D, K_G and rational determinant of adelic norm one. For ρ_a the calculation works for every such symplectic G, since the change in its constant quadratic phase is $a(Ax, Cx)/2$ by $a(Ax, Dl) = a(x,l)$. The phase in $\mathcal{W}_G$ is one at rational arguments and $\theta_P \mathcal{W}_G = \theta_P$. For the coordinate exchange $s_i : (x_i, l_i) \mapsto (l_i, -x_i)$ use partial Fourier transformation in coordinate i. Translation by x_i then gives multiplication by $\mathbf{e}(-t_i x_i)$, while multiplication by $\mathbf{e}(t_i l_i)$ gives translation by l_i. Their product together with the half phase above gives precisely $\rho_a(s_i v)$. Lemma 387.1 makes θ_P invariant under this operator as well. Intertwining and invariance of θ_P give the displayed identity for both types of operation.

These operations generate the rational symplectic group. Indeed put $M = GL$, $\dim(M \cap L) = n - r$, $n = \dim L$. A rational change of basis on L with inverse transpose on P can carry $M \cap L$ to the span of the last $n - r$ basis vectors. Orthogonality to that intersection and the dimension of M imply that its projection to P is the span of the first r basis vectors. Representatives in M of these vectors may be adjusted by $M \cap L$ to have L-components only in the first r positions. Their coefficients there form a symmetric matrix by isotropy. The shear with that matrix, extended by zero on the remaining positions, preserves L and sends $s_1 \cdots s_r L$ to M in the adjusted basis. Returning to the original basis, the remaining factor preserves L. This proves generation. Compose the indicated operators. Injectivity of $f \mapsto F^a_f$ proves independence of the word and the representation law, including inverses. Unitarity follows from the squared norm identity because G preserves quotient probability by the argument of Lemma 379.3.

For G preserving b, the common factor $\mathbf{e}(-b(v,v)/2)$ gives the assertions for F_f, ρ. Use Lemma 379.3 for the constant term. Finally choose a rational symplectic map h carrying dual bases of one polarization to those of a second. Transferring k by $k'(t') = k(h^{-1}t')$ gives $F^{a,\mathrm{second}}_{k'}(v) = F^{a,\mathrm{first}}_k(h^{-1}v)$ by reindexing the rational sum. Taking $k = \mathcal{T}_{h^{-1}} f$ identifies the two functions; the reverse inclusion follows using h^{-1}. Multiplying by the same quadratic factor proves the last claim. □

388. Compact representatives and absolute Fourier bounds

Proposition 388.1. *For every stage of Definition 348.2 the rational Bessel subgroup is discrete in the adelic Bessel group B of Proposition 382.1. It has a measurable left fundamental domain with compact closure and finite positive Haar measure.*

Proof. By the filtration proof in Proposition 382.1, the induced action on grade zero restricts to a unitary operator on X and is the identity on a middle string line if present. On every nonzero graded line the induced action is the identity. The map to $\mathrm{U}(X)$ so obtained has a

section acting trivially on the string. Its kernel J consists precisely of the unitary matrices with $u-1$ of degree at least one. These statements hold rationally and adelically. Thus each element is uniquely nh with $n \in J,\ h \in \mathrm{U}(X)$. For J, use the logarithmic proof of Theorem 383.1 with the filtration starting at degree one and the entire space of skew adjoint strictly grade-raising matrices. The exponential and logarithm are finite, and every nonlinear term containing a degree a argument now has degree at least $a+1$. The identical blockwise additive reduction therefore gives a rational fundamental set with compact closure in $J(\mathbb{A})$.

The real unitary group of the definite X is compact, being a closed bounded set of matrices in the identity Gram coordinates. Theorem 23.2 gives finitely many rational classes modulo the finite compact integral unitary group. This latter group is compact since it is closed in the product of integral matrix spaces, each the inverse limit of its finite residue quotients. Compactness of the countable product also follows by successive diagonal extraction of convergent subsequences. Thus the real group times a finite union of compact finite cosets meets every rational orbit on $\mathrm{U}(X)(\mathbb{A})$. Rank zero gives one point. First left multiply nh by a rational definite unitary bringing h into this compact set, conjugating n in the process. Then left multiply by a rational member of J to bring the new n into its compact set. The product of the two compact sets meets every rational orbit of B.

For clarity, a left Haar measure on B in coordinates hn is the product of those on $\mathrm{U}(X)$ and J: left multiplication changes h by a left translation and changes n by a left translation depending on h. The measure on J was constructed in logarithmic coordinates. At the real unitary factors use the chart density of Lemma 21.4. At the finite group give its integral subgroup the probability assigning each coset of a normal principal congruence subgroup the reciprocal of its index. The nested finite residue quotients give that probability by uniform successive fiber choices as in Lemma 356.2. Sum its left translates over cosets of the integral subgroup to get left invariance. Continuity of the two coordinate factorizations follows from matrix multiplication and the graded block projection. Discreteness and finiteness of the rational matrices meeting a given compact set follow by clearing the finite denominators and bounding the archimedean entries as in Lemma 21.4. Enlarge the compact covering set slightly to a relatively compact open neighborhood. In that neighborhood take a countable cover by open sets disjoint from their distinct rational translates, and keep successive pieces outside the translates of previous ones as in that lemma. The resulting measurable left fundamental domain has compact closure. It has positive measure since countably many translates of it cover a nonempty open set, and finite measure by compact closure. □

Proposition 388.2. *Choose the polarization of Lemma 385.2 for d. For a test function f in this realization,*

$$\sum_{w \in V(E)/N_d(E)} |I_w^{U_d}(F_f)| \le \sum_{t \in P(\mathbb{Q})} |f(t)|.$$

For a rational basis of P, set $n = \dim_{\mathbb{Q}} P$ and use the maximum norm on its real coordinates. Suppose the finite support of f lies in $b_1^{-1}\widehat{\mathbb{Z}}^n,\ b_1$ a positive integer. For any integer $k > n$ the right side is at most

$$C_{n,k}\, b_1^n \sup_{t_\infty, t_f} (1 + \|t_\infty\|)^k |f(t_\infty, t_f)|,$$

where $C_{n,k}$ depends only on the displayed indices. This also holds uniformly over any family with one choice of b_1 and bounded displayed supremum.

Proof. Use the rational representatives $w \in U_-(\mathbb{Q})$ for the quotient, since $N_d = W + U$. In the notation of Theorem 385.3, $w \in P(\mathbb{Q})$ acts by pure translation: its L-component is zero and the constant phase is one by rationality. Consequently $I_w^{U_d}(F_f) = \sum_{\xi \in P_W(\mathbb{Q})} f(w+\xi)$. The rational points of $P = U_- \oplus P_W$ are partitioned in this way, proving the first bound. The rational indices in finite support belong to $b_1^{-1}\mathbb{Z}^n$ by rational prime factorization. In a real box of radius $R \ge 1$ there are at most $(2b_1R+1)^n \le (3b_1R)^n$ such points. Sum first over the unit box and then over shells $2^j \le \|t_\infty\| < 2^{j+1}$, $j \ge 0$, assigning boundary points

once. The geometric series with ratio $2^{n-k} < 1$ proves the claim. For $n = 0$ the point at zero gives the same assertion directly. □

389. Unfolding one period

Theorem 389.1. *Use consecutive spaces Y, Y' and put $B = B_Y(\mathbb{A})$, $\Gamma = B_Y(E)$, $G' = \mathrm{U}(Y')(\mathbb{A})$, $\Gamma' = \mathrm{U}(Y')(E)$, and $\Gamma'_B = B_{Y'}(E)$, with rational points understood as unitary isometries over the real subfield. Use a left fundamental domain D_B with compact closure as in Proposition 388.1, a left Haar measure on G', and a measurable fundamental domain D' for Γ' there. Let $p : B \to \mathbb{C}$ be continuous, left Γ-invariant, with*

$$p(ur) = \Psi\big(\tfrac{1}{2}\operatorname{tr}(S_s u)\big)\, p(r) \quad (\deg(u-1) \geq 2).$$

Let $y : G' \to \mathbb{C}$ be measurable and left Γ'-invariant.

Let $F_{r,h}$, for $r \in B, h \in G'$, be theta functions in the common space of Theorem 387.2, jointly continuous in all variables. Suppose that rational translates satisfy

$$F_{\gamma r, \delta h}(v) = F_{r,h}(\delta^{-1} v \gamma) \quad (\gamma \in \Gamma,\ \delta \in \Gamma').$$

For either grading d, write $f^d_{r,h}$ for its test function in the corresponding polarization with translation space P_d. Under left translation of r by a unitary u with $\deg(u-1) \geq 2$, assume this test function transforms by $\mathcal{W}_{z \mapsto zu^{-1}}$ of Theorem 386.1. For $d = g'$ assume the same for left translation of h by a target unitary v with $\deg(v-1) \geq 2$, using $\mathcal{W}_{z \mapsto vz}$. Suppose $M_d(r,h) = \sum_{\xi \in P_d(\mathbb{Q})} |f^d_{r,h}(\xi)|$ is measurable and bounded on compact sets for both gradings. Assume the single absolute bound

$$\int_{D'} |y(h)| \int_{D_B} |p(r)| M_0(r,h)\, \mathrm{d}r\, \mathrm{d}h < \infty.$$

Define $p'(h) = \int_{D_B} p(r) I_T^{U_{g'}}(F_{r,h})\, \mathrm{d}r$. It is continuous and left Γ'_B-invariant and has the target generic character

$$p'(vh) = \Psi\big(\tfrac{1}{2}\operatorname{tr}(S_{s'} v)\big)\, p'(h) \quad (\deg(v-1) \geq 2).$$

The following integrals converge absolutely and agree:

$$\int_{D_B} p(r) \int_{D'} F_{r,h}(0) y(h)\, \mathrm{d}h\, \mathrm{d}r = \int_{\Gamma'_B \backslash G'} p'(h) y(h)\, \mathrm{d}h.$$

The inner integral on the left is taken for almost every r with respect to $|p(r)|\, \mathrm{d}r$.

Proof. Put $I_d = V(E)/N_d(E)$, $C_d = \{[w] \in I_d : w|_C = T|_C\}$. For rational source and target translations preserving U_d, Lemma 379.3 makes

$$k^d_{[w]}(r,h) = p(r) y(h) I_w^{U_d}(F_{r,h})$$

equivariant under $[w] \mapsto [\delta w \gamma^{-1}]$. This holds for all $\delta \in \Gamma'$ when $d = 0$, and for $\delta \in \Gamma'_B$ when $d = g'$, by Proposition 382.1. The constant terms are continuous in r, h before multiplication by y, by integration on the compact additive quotient and joint continuity. Proposition 388.2 bounds the sum of absolute values of each family by $|p(r)y(h)| M_d(r,h)$. On $J_w(\mathbb{A})$ in Theorem 383.1 the source transformation is multiplication of $k^d_{[w]}$ by η_w, by Corollary 386.2.

At fixed h start with $d = 0$. The sum over I_0 is $p(r)y(h)F_{r,h}(0)$ by Theorem 378.1. Let $D_0 = \Gamma' C_0$, the union of the target translates. The set C_0 is stable under Γ by Proposition 382.1. The source and target actions commute, so $I_0 \setminus D_0$ is also stable under Γ. Every representative there fails the test, for a survivor admits a rational target isometry bringing it into C_0, by Theorem 381.2. On each source orbit in the complement, apply Corollary 384.3. Its hypotheses on absolute integrability hold on D_B by compact closure and the bound just proved. Add over the countably many orbits. Thus the integral over D_B of the sum over I_0 equals the integral of the sum over D_0.

Still for $d = 0$, integrate each k^0_i over the chosen D_B to obtain a function of h. These functions are equivariant under Γ' by the rational covariance with $\gamma = 1$. For $i \in C_0$, $\delta i \in C_0$

holds precisely for $\delta \in \Gamma'_B$ by Proposition 382.1. Apply Lemma 384.1 to C_0 on G'. The assumed integral of $|yp|M_0$ bounds the sum of absolute values on the original domain, also if the absolute value is taken before integrating over D_B. Thus on $\Gamma'_B\backslash G'$ the integral of the sum over C_0 is absolutely convergent and equals the original integral. The same bound permits the initial interchange of D_B, D' by Lemma 123.1.

By Proposition 379.2 the sum over C_0 equals the sum over $C_{g'}$ at each r, h. This use of the two polarizations is justified by Theorem 387.2. In $C_{g'}$, the subset obtained by removing $[T]$ is stable under the rational source group: it preserves $C_{g'}$ and fixes $[T]$ by Proposition 382.1. Every class in that subset fails the test by Theorem 382.2. The cancellation corollary eliminates these source orbits at each fixed h, with absolute convergence on D_B from the bound for $M_{g'}$. The result of the D_B-integration is precisely $p'(h)y(h)$, proving both the identity and the asserted integrability on the larger target quotient.

Finally the integrand defining p' at T is rationally invariant in the source, because its frequency changes only by $N_{g'}(E)$. The same calculation gives rational Γ'_B-invariance on the target. Joint continuity and compact closure of D_B give continuity of p' by bounded convergence on compact parameter neighborhoods. For the target unipotents of degree two or more, pull the constant phase of Corollary 386.2 through the compact integral. This gives the claimed character. □

390. The cocycle on three finite lattices

Lemma 390.1. *Work at one finite place of the real subfield, taking both branches together in the split case. Let L be the reference lattice of Theorem 358.1. For isometries g, h, put $L_1 = gL$, $L_2 = ghL$ and $m_a = [aL : aL \cap L]$. The unitary scalar $\xi(g,h)$ in $W_gW_h = \xi(g,h)W_{gh}$ satisfies*

$$\xi(g,h) = \left(\frac{m_{gh}}{m_g m_h}\right)^{1/2} \sum_{x \in D} \Phi(x), \qquad D = \frac{L_2 \cap (L + L_1)}{L_2 \cap L_1}.$$

For x represented in its numerator choose $r \in L_1$ with $x + r \in L$. Then $\Phi(x) = B_f(x,r)$ is well defined and $\Phi(0) = 1$.

Proof. The scalar exists and has modulus one by the uniqueness and unitarity in Theorem 358.1, applied locally by its same lattice proof. For the unit vacuum v_0, the coefficient of v_0 in $W_{gh}v_0$ is $m_{gh}^{-1/2}$, by its lattice sum. In $W_gW_hv_0$ that coefficient is

$$(m_g m_h)^{-1/2} \sum_{x \in L_2/(L_2 \cap L_1)} \sum_{\substack{r \in L_1/(L_1 \cap L) \\ x+r \in L}} B_f(x,r).$$

Indeed the first translations in W_hv_0 are taken over to the x's by g, and $D_xD_r = B_f(x,r)D_{x+r}$. The vacuum coefficient of $D_{x+r}v_0$ is one for $x + r \in L$ and zero otherwise by the self-dual lattice basis in that theorem. There is precisely one admissible r-class for each $x \in D$. For $j \in L_1 \cap L$, $B_f(x,j) = 1$, since $x + r \in L$, $r \in L_1$ and B_f is one on the square of each of the three lattices. Thus the phase is unchanged by adding j to r. For $k \in L_1 \cap L_2$, replace (x,r) by $(x + k, r - k)$. The phase ratio is $B_f(x,k)^{-1}B_f(k,r-k) = 1$. This proves the formula and independence of all representatives. For $x = 0$ choose $r = 0$. □

Theorem 390.2. *On the full finite adelic unitary group of the reference hermitian space in Definition 375.1, the operators $\mathcal{V}_g = \beta(J_g)W_g$ form a unitary representation. At a nonsplit place with residue cardinality q in the real field and $a_g = v_w(J_g)$ at the unique place w of E, the sum of Lemma 390.1 is*

$$(-q)^{\,a_g + a_h - a_{gh}}.$$

Proof. At a nonsplit place Lemma 374.1 gives an unramified quadratic extension, $q \equiv 3 \bmod 4$, and odd residue characteristic. The diagonal reduction of Theorem 375.2 gives $m_g = q^{2a_g}$, since the index is computed over the quadratic residue field. Put $t = a_g + a_h - a_{gh}$.

It is nonnegative: passing from L_2 to $L_2 \cap L_1$ costs m_h, and intersecting next with L costs at most m_g by inclusion into $L_1/(L_1 \cap L)$. The index $[L_2 : L_2 \cap L]$ is thus at most $m_g m_h$. The overlap lemma implies $|\sum_D \Phi| = q^t$.

The local integer rings act on D. For an integral unit z, multiplication by z replaces r by zr in the phase and scales $H(r,x)$ by $z\bar{z}$. There is an integral unit z of norm -1. The residue norm onto the real residue group is surjective by the cyclicity proof in Lemma 345.1. Choose a lift of a suitable residue element. If its norm already equals -1 modulo π^a, multiply by $1 + b\pi^a$ with b in the real integer ring and π a real uniformizer. Modulo π^{a+1} the norm correction has linear coefficient $2b$. Since two is a unit one can correct each successive residue digit. Completeness gives the desired norm. This permutation sends Φ to its complex conjugate, so the sum is real.

Multiplication by $i \in E = E^+(i)$ has norm one, preserves Φ, and squares to negation. Every nonzero element of D has orbit of length four: two is invertible on this finite module, so only zero equals its negative. Each value of Φ is a root of unity, since the finite additive character on any fixed fractional local ideal factors through a finite quotient by Proposition 18.5. It follows that $(\sum_D \Phi - 1)/4$ is an algebraic integer. Indeed it is a sum of one root from each nonzero orbit. All these roots are powers of a single root ζ of finite order b, and every element of $\mathbb{Z}[\zeta]$ is integral, by its multiplication action on the quotient $\mathbb{Z}[X]/(X^b - 1)$ and the monic characteristic polynomial. The rational values $\sum_D \Phi = \pm q^t$ therefore satisfy $\sum_D \Phi \equiv 1 \bmod 4$: a rational number satisfying a monic integral polynomial is an integer by clearing a fraction in lowest terms. Since $-q \equiv 1 \bmod 4$, there is only one such sign. Thus $\xi(g,h) = (-1)^t$. The ideal phase at this place is $(-1)^{a_g}$ by Theorem 374.2. It cancels the scalar in the composition exactly.

At a split pair the proof of Theorem 375.2 gives multiplicativity for arbitrary local isometries there by the Schrödinger change of variables. Finally at almost every place g, h are integral, act on the local vacuum by the identity, and have unit ideal phase. Every translated adelic vacuum is an elementary tensor with vacuum factors at almost every place. The lattice sum and the diagonal ideal calculation factor into the indicated local operations on these vectors. Multiplicativity at each place consequently proves it on their span, the full finite space of Theorem 358.1. □

391. The finite integral vacuum in a polarization

Proposition 391.1. *At the finite adeles choose a rational symplectic polarization P, L as in Theorem 387.2. Let Λ be the reference lattice in the symplectic space, self-dual for $\mathbf{e}_f(a(\cdot,\cdot))$, with $B_f = 1$ on $\Lambda \times \Lambda$. Denote by ρ the finite factor of Theorem 385.3. The common fixed vectors of $\rho(\Lambda)$ among the locally constant compactly supported functions on P_f form a line. Each unit vector in this line gives a unique unitary Heisenberg intertwiner from the finite lattice space of Theorem 358.1 onto these test functions, sending the reference vacuum to that vector. If P, L are hermitian isotropic and $\Lambda = (\Lambda \cap P_f) \oplus (\Lambda \cap L_f)$, the normalized indicator of $\Lambda \cap P_f$ is such a unit vector.*

Proof. Use the abbreviations $A = \mathrm{pr}_{P_f} \Lambda$, $D = \Lambda \cap L_f$. These are compact open additive lattices. The annihilator of A in L_f is D: a member pairs integrally with A if and only if it annihilates Λ and hence lies in Λ. The reciprocal annihilator is A. For detail, a compact open additive lattice in rational coordinates lies between $b\widehat{\mathbb{Z}}^n$ and $b^{-1}\widehat{\mathbb{Z}}^n$ for an integer $b > 0$. On their finite quotient the character pairing on the dual coordinates is perfect by the one-coordinate character in Proposition 18.5 and a product over the coordinates. Every character of a subgroup of a finite abelian group extends to that group: adjoin one generator at a time, choosing a suitable root of the prescribed character value on its first multiple in the old subgroup. This proves the reciprocal annihilator assertion by separating any nonzero coset by a cyclic character and extending it. It also shows that every continuous character of an additive compact open lattice with open kernel is the restriction of $\mathbf{e}_f(a(v,\cdot))$ for some

finite v: use a small standard lattice in its kernel and a large standard lattice containing it, with the same finite dual calculation.

For $l \in D$, the action is multiplication by $\mathbf{e}_f(q(l) + a(t, l))$. The factor $\mathbf{e}_f(q(l))$ on D is a character, since $B_f = 1$ on Λ^2. It is locally constant by continuity of the finite character. Choose t_0 representing its inverse by the preceding duality. A fixed function must be supported on $t_0 + A$. For every $x + l \in \Lambda$, its value at $t_0 + x$ is forced to be its value at t_0 times $\mathbf{e}_f(-q(x + l) - a(t_0, l))$. This prescription is well defined: two lifts differ by D, whose multiplier at t_0 is one, and ρ is an ordinary representation on Λ. The same representation law gives invariance under every member of Λ on the resulting coset. The resulting function is locally constant, since the phase is continuous with open level sets on compact Λ and its projection to A is open: the projection of any open ball in the open subgroup Λ contains an open ball. With value $\mathrm{vol}(A)^{-1/2}$ at t_0 the function has squared norm one. This proves existence and uniqueness up to a scalar.

Translates indexed by distinct classes modulo Λ are orthogonal eigenvectors for Λ, with distinct characters by self-duality and the commutator in Lemma 356.2. They span the test space. Indeed any test function is fixed by a small open subgroup of Λ, by compact finite support, local constancy and the formula for ρ. More explicitly, choose a common translation lattice for its finitely many support cosets, make q integral on the small subgroup and make all modulations trivial on that compact support. The action on its orbit span then factors through a finite abelian group. Averaging against each of its characters decomposes the function into eigenvectors: summing the averages returns the identity by finite character orthogonality, with a full list of characters obtained by extension from cyclic factors as above. Every occurring character is realized by a translate, and translating back reduces its eigenspace to the one-dimensional fixed space. The translate basis has the same phases under ρ as the lattice basis under D_z, by the same multiplication law. This gives the unitary intertwiner. In the split lattice case $q(x + l) = -b(l, x)$ by hermitian isotropy of P, L. This has trivial finite character for integral x, l in the given summands, so one may take $t_0 = 0$ and the normalized indicator. □

Proposition 391.2. *Assume the final split lattice condition of Proposition 391.1 and that P, L are E-linear. If G preserves each summand, write A for its restriction to P. In this Schrödinger realization*

$$\mathcal{V}_G = v_f(\det_E A)^{-1}\mathcal{W}_G,$$

where $\mathcal{W}_G$ is the finite normalized change of variables of Theorem 386.1 and v_f is the finite factor in Theorem 374.2.

Proof. On $\Lambda_P = \Lambda \cap P_f$ and $\Lambda_L = \Lambda \cap L_f$ the commutator gives perfect additive lattice duality by the preceding proposition. Locally the hermitian pairing identifies the latter lattice with the module dual of the former, scaled by a fixed fractional ideal. Indeed integrality of its scaled trace against every integral multiple of every basis vector says that the corresponding E-valued pairing lies in the scalar trace-dual ideal, by the argument of Proposition 18.5. The action on L is consequently inverse conjugate transpose to A in dual lattice bases. Write $k_{w,i}$ for the diagonal valuations of A_w on Λ_P obtained by integral elimination as in Theorem 375.2. At w the other summand has valuations $-k_{\bar{w},i}$, up to order. Writing $J_- = \prod_{w,i} \mathfrak{p}_w^{\max(0,-k_{w,i})}$ and $J_+ = \prod_{w,i} \mathfrak{p}_w^{\max(0,k_{w,i})}$, we get $J_G = J_-\bar{J}_+$ and the determinant ideal of A is J_+/J_-. Since $\beta(\bar{J}) = \beta(J)^{-1}$, the phase $\beta(J_G)$ is $v_f(\det_E A)^{-1}$. The saturation index ideals are intrinsic to the lattice: integral changes of basis and their inverses preserve their determinant valuations by the same elimination proof. Finally $\mathcal{W}_G$ sends the integral indicator to a positive multiple of the indicator of $A\Lambda_P$. Its overlap with the original vacuum is positive. Uniqueness of the positive-overlap intertwiner in Theorem 358.1 therefore identifies it with W_G. Use Theorem 390.2 to obtain the formula. □

392. Real symplectic transformations of a Gaussian

Lemma 392.1. *In real symplectic coordinates $(x,l) \in \mathbb{R}^n \oplus \mathbb{R}^n$ with $a((x,l),(y,k)) = x^T k - l^T y$, use the real factor*

$$\rho_a(x,l)f(t) = e^{-2\pi i(x^T l/2 + t^T l)} f(t+x).$$

Fourier transformation with kernel $e^{-2\pi i t^T y}$ is a unitary bijection of the Schwartz functions with the L^2 norm. Two unitary bijections implementing the same change of variables on ρ_a which agree on $\phi_0(t) = 2^{n/4}e^{-\pi t^T t}$ agree on all Schwartz functions. A unitary bijection commuting with ρ_a and preserving the Schwartz space is scalar.

Proof. The transformed function is Schwartz by differentiation and integration by parts as in Lemma 387.1. The Gaussian integral calculation in Lemma 4.2 gives $\widehat{e^{-\pi\epsilon t^T t}}(v) = \epsilon^{-n/2}e^{-\pi v^T v/\epsilon}$ by taking products. For Schwartz f, h insert $e^{-\pi\epsilon t^T t}$ in $\int \widehat{f}(t)\overline{\widehat{h}(t)}\,\mathrm{d}t$. Product integration changes the resulting integral to $\int f(x)\overline{h(y)}\epsilon^{-n/2}e^{-\pi\|x-y\|_2^2/\epsilon}\,\mathrm{d}x\,\mathrm{d}y$. It tends to $\int f\bar{h}$: write $y = x + \sqrt{\epsilon}u$ and use the integrable majorant $|f(x)|\sup|h|e^{-\pi\|u\|_2^2}$. The original integral tends to the unweighted one by Schwartz decay. The same insertion in the inverse transform of $\widehat{f}$ gives convolution with the same Gaussian and tends to $f(x)$ for every x. This proves isometry and inversion, also for a single variable with the remaining ones fixed.

For two implementers as stated their quotient A commutes with every $\rho_a(v)$ and fixes ϕ_0. For a Schwartz function h, unitarity implies that $(A-1)h$ is orthogonal to every modulation $e^{-2\pi i t^T l}\phi_0(t)$. The Fourier transform of $((A-1)h)\phi_0$ is zero, hence this product vanishes by inversion. Since $\phi_0 > 0$, equality follows. For the last assertion, differentiating the commutation identities along real translations and modulations shows that the commuting operator commutes on Schwartz functions with ∂_j and t_j: the difference quotients converge in L^2 by the Schwartz estimates and unitarity preserves their limits. The equations $(\partial_j + 2\pi t_j)A\phi_0 = 0$ imply $A\phi_0 = c\phi_0$ by multiplying by $e^{\pi t^T t}$. Its modulus is one since $\|\phi_0\|_2 = 1$. The preceding uniqueness now gives $A = cI$. □

Proposition 392.2. *Let $\mathfrak{H}_n$ consist of complex symmetric n-by-n matrices Z with positive definite imaginary part. Put $\phi_Z(t) = 2^{n/4}e^{\pi i t^T Z t}$, so $\phi_{iI} = \phi_0$ of Lemma 392.1. For each real symplectic $G = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ there is a unitary bijection M_G of the Schwartz functions with $M_G\rho_a(v) = \rho_a(Gv)M_G$. It satisfies*

$$M_G\phi_Z = c_G(Z)\phi_{GZ}, \qquad GZ = (C+DZ)(A+BZ)^{-1}, \qquad c_G(Z)^2\det(A+BZ) = \epsilon_G,$$

where c_G is continuous and ϵ_G is a constant of modulus one. Every other such implementer differs by a unit scalar.

Proof. For vectors (t, Zt) in the complex graph, the alternating form vanishes on that graph, and $a(\bar{v}, v)/(2i) = \bar{t}^T \operatorname{Im}(Z)t > 0$ for nonzero t. A real symplectic transformation preserves both properties. If $(A+BZ)t = 0$, its image has zero first coordinate, contradicting the strict inequality. Hence the new graph is that of GZ, with symmetry and positive imaginary part by the same tests. Projecting the graph after two transformations gives

$$\det(A_{12} + B_{12}Z) = \det(A_1 + B_1(G_2Z))\det(A_2 + B_2Z)$$

for $G_{12} = G_1G_2$.

Use the generators in the rational symplectic proof of Theorem 387.2, now over $\mathbb{R}$. For $B = 0$ the quadratic change of variables in Theorem 386.1, with real measure and the symmetric phase, is unitary, intertwines the translations and gives $c_G = |\det A|^{-1/2}$. For the exchange of x_j, l_j, use the partial Fourier transform of Lemma 392.1. Completing its scalar Gaussian integral gives $c_G(Z) = (-iZ_{jj})^{-1/2}$, with square root positive at $Z_{jj} = i$. To verify the integral, the Fourier transform in Lemma 4.2 gives

$$\int_{\mathbb{R}} e^{-\pi u y^2 + 2\pi i z y}\,\mathrm{d}y = u^{-1/2}e^{-\pi z^2/u} \qquad (\operatorname{Re} u > 0,\ z \in \mathbb{C}).$$

It holds first for positive real u and real z, then for complex z and then for $\operatorname{Re} u > 0$ by the identity principle of Lemma A.3. Gaussian domination on compact parameter sets justifies holomorphy by differentiation under the integral. Here $u = -iZ_{jj}$ and z is the sum of the cross terms minus the Fourier variable. The resulting quadratic exponent is the graph transform: with v_j the old coordinate, the new one is $(Zv)_j$, so substituting $v_j = ((Zv)_j - \sum_{k\neq j} Z_{jk}v_k)/Z_{jj}$ gives its new dual coordinate $-v_j$ and the unchanged dual coordinates $(Zv)_k$, $k \neq j$. The integrated exponent has precisely these linear derivatives after division by $2\pi i$. For this exchange $\det(A + BZ) = Z_{jj}$, so the constant ϵ_G equals i. For $B = 0$ it is $\operatorname{sgn}(\det A)$. Composition using the graph determinant identity proves the assertion for any word. Scalar uniqueness is Lemma 392.1. □

393. The archimedean Weil representation of a unitary group

Theorem 393.1. *At a complex place write the hermitian form in coordinates $z = (z_+, z_-)$ with signs $+1, -1$ of multiplicities p, q, and $n = p + q$. Put $j_0 = \operatorname{Im}(\tau\jmath) > 0$ for the chosen CM embedding, and use real Darboux coordinates*

$$x + il = \sqrt{2j_0}\,(z_+, \bar{z}_-).$$

Then the trace alternating form in Definition 385.1 is $x^T l' - l^T x'$. For the unitary $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ in these signed complex coordinates, there is a unique unitary Schwartz operator $\mathcal{V}_g^\infty$ implementing g on the real Heisenberg translations and satisfying

$$\langle \phi_0, \mathcal{V}_g^\infty \phi_0\rangle = (\det_{\mathbb{C}} \delta)^{-1}.$$

The inner product is conjugate linear in the first entry, and an empty determinant is one. The operators form a representation continuous on Schwartz functions in their Schwartz seminorms. In the symmetric gauge of Lemma 392.1 the coefficient of a translation on the same vacuum is $\exp(-\pi j_0 \|z\|_2^2)$.

Proof. The trace of $\jmath H(z', z)$ at this place is $2j_0 \operatorname{Im} H(z, z')$. Splitting into the two sets of signs gives the stated real alternating form. The element $\jmath = -1/\sqrt{-d}$ in Lemma 348.1 has positive imaginary part at the embeddings with $\sqrt{-d} = i\sqrt{d}$. In the symmetric gauge, direct Gaussian integration with t shifted by $x/2$, as in Lemma 4.2, gives $e^{-\pi(\|x\|_2^2 + \|l\|_2^2)/2}$ for the translation coefficient. Returning to ρ multiplies each translation by $\mathbf{e}_\infty(-b(v, v)/2)$; g preserves b, so both gauges have the same implementing operators.

Write A, B, C, D for the real symplectic blocks of g. For $Z \in \mathfrak{H}_n$ of Proposition 392.2 set $H_0(Z) = \langle \phi_0, \phi_Z\rangle$. Then

$$H_0(Z)^2 = \det\left((I - iZ)/2\right)^{-1}.$$

In fact this is the integral of a Gaussian with complex symmetric matrix $I - iZ$ of positive real part. Make that real part the identity by its positive real square root, and orthogonally diagonalize the remaining real symmetric imaginary part, by the orthogonal eigenbasis argument in Lemma 79.2 (real eigenvectors are obtained by taking real and imaginary parts). The one-dimensional integrals in Proposition 392.2 give the squared determinant formula with the indicated power of two. This calculation also gives nonvanishing and continuity of H_0.

The action on $\zeta = x + il$ is $P\zeta + Q\bar{\zeta}$, where $P = \operatorname{diag}(\alpha, \bar{\delta})$ and $Q = \left(\begin{smallmatrix}0 & \beta\\ \bar{\gamma} & 0\end{smallmatrix}\right)$. Complexify and evaluate at $l = ix$. Adding and subtracting the two resulting complex coordinates gives

$$A + iB = \bar{P} + Q, \qquad C + iD = i(\bar{P} - Q), \qquad (I - i(g\,iI))/2 = \bar{P}(A + iB)^{-1}.$$

Both α, δ are invertible: the unitary equations give $\alpha^*\alpha = I + \gamma^*\gamma$, $\delta^*\delta = I + \beta^*\beta$. The negative block of g^{-1} is δ^*, hence block elimination gives $\det_{\mathbb{C}} g = \det\alpha/\overline{\det\delta}$, of modulus one. Consequently

$$j_g(iI) = \left(\det\delta\, H_0(g\,iI)\right)^{-1} \quad \text{satisfies} \quad j_g(iI)^2 = \frac{(\det_{\mathbb{C}} g)^{-1}}{\det(A + iB)}.$$

Extend this value continuously to $Z \in \mathfrak{H}_n$ by multiplying it by

$$\exp\Big[-\frac{1}{2}\int_0^1 \operatorname{tr}\{(A+BZ_t)^{-1}B(Z-iI)\}\,\mathrm{d}t\Big], \qquad Z_t = iI + t(Z-iI).$$

The graph proof in Proposition 392.2 gives invertibility on this segment. Differentiating the determinant by multilinearity of its columns shows that its logarithmic derivative is the displayed trace. Thus $j_g(Z)^2 \det(A+BZ)$ is constantly $(\det_{\mathbb{C}} g)^{-1}$. The extension is jointly continuous in g, Z. Its composition quotient $j_{g_1}(g_2 Z) j_{g_2}(Z) / j_{g_1 g_2}(Z)$ has square one by the determinant multiplication in that proposition.

The group $\mathrm{U}(p,q)$ is connected. Indeed, put $J = \operatorname{diag}(I_p, -I_q)$, $H = g^* g$, so $JHJ = H^{-1}$. By diagonalizing the positive hermitian matrix H, its real powers satisfy $JH^tJ = H^{-t}$. Thus $gH^{-t/2}$, $0 \le t \le 1$, is a path in the group to an ordinary unitary member. The latter preserves J and commutes with it, so belongs to $\mathrm{U}(p) \times \mathrm{U}(q)$. In each factor an orthonormal eigenbasis from Lemma 79.2 reduces a path to the identity to paths of individual phases on the circle. Since $\mathfrak{H}_n$ is convex, the continuous composition quotient equals its value one at the identity pair and iI.

In Proposition 392.2 the ratio $j_g(Z)/c_g(Z)$ has constant square of modulus one. Continuity on the connected half space makes it a single scalar. Multiply the implementer there by this scalar. It acts on all ϕ_Z by $j_g(Z)\phi_{gZ}$ and has the specified vacuum coefficient by the definition at iI. The composition formula for j and Lemma 392.1 now give the representation law. Scalar uniqueness of implementers and the nonzero coefficient give uniqueness.

Near the identity the real block A is invertible. The symplectic operator factors as a lower shear with coefficient CA^{-1}, the diagonal transformation A, A^{-T}, and an upper shear with coefficient $A^{-1}B$, in this order. Multiplication verifies the factorization from $G^T a G = a$, and both shear coefficients are symmetric. The upper shear is a Fourier conjugate of a lower shear. The corresponding Schwartz operators depend continuously on the matrix parameters: each derivative of a quadratic phase is a polynomial times that phase, and locally the linear substitutions and their inverses have bounded norms. Split into a compact real box and its complement and use an extra Schwartz power there for convergence in each seminorm. The Fourier transform is continuous in these seminorms by the integration by parts in Lemma 387.1. The correcting scalar is continuous by its nonzero Gaussian coefficient and the formula for $j_g(iI)$. The representation law gives continuity everywhere. For $n = 0$ the group is trivial and the assertions use the constant vacuum one. □

394. The action on a maximal isotropic parabolic

Lemma 394.1. *In a real or finite Schrödinger realization, let two continuous unitary representations of a group of isometries implement the same Heisenberg action. At finite places take continuity on each locally constant test vector. Their ratio is a unitary character. If the group preserves a polarization variable L, one of the two representations may be the operators $\mathcal{W}$ of Theorem 386.1. The character is one on any subgroup contracted to the identity by conjugation by members of this group.*

Proof. The quotient of two implementers commutes with all translations. It is scalar by Lemma 392.1 or by Proposition 391.1 and the vacuum-line argument in Theorem 358.1. Modulus one follows by unitarity. The two group laws give multiplicativity, and evaluation of one nonzero matrix coefficient of the scalar operator gives continuity. A character has constant value under conjugation, so contraction forces its value to equal its value at the identity. For the finite representation of Theorem 390.2, continuity holds because the integral subgroup fixes its vacuum. On any particular translated vacuum a sufficiently deep integral congruence subgroup changes the translating vector by a lattice vector with trivial commutator and multiplication phases, by local constancy in Lemma 356.2. This works simultaneously for a finite linear combination. The same continuity for $\mathcal{W}$ follows from its explicit formula on a locally constant function of compact support. □

Theorem 394.2. *Let a rational hermitian space be the sum of complementary maximal hermitian isotropic subspaces P, L, each of E-dimension b, with a reference lattice splitting as in Proposition 391.1. In its Schrödinger model take the product of the real operators in Theorem 393.1 and the finite operators in Theorem 390.2. For every adelic unitary G preserving L, let A be its action on the quotient by L. The product acts by*

$$\upsilon(\det_E A)^{-1}\mathcal{W}_G.$$

In particular it equals $\mathcal{W}_G$ for every rational unitary preserving L.

Proof. Real Schrödinger coordinates can be changed from the signed coordinates to P, L. More explicitly, a real symplectic map carrying one dual basis to another transfers ordinary coordinate functions by its restriction to the position variables, with the positive constant making the measure change unitary. Compose with an implementer of the inverse symplectic map from Proposition 392.2 in the old coordinates, as in the last step of Theorem 387.2. This identifies the two Heisenberg actions on the same physical vectors. Two such identifications differ by a scalar by Lemma 392.1, so transporting the group operators is unambiguous. At finite places use Proposition 391.1.

On the parabolic the ratio to $\mathcal{W}$ is a character locally by Lemma 394.1. It is one on all the shears $x + l \mapsto x + Cx + l$. Indeed conjugation by the unitary acting as t on P, t^{-1} on L, for t in the real base field at this place, multiplies C by t^{-2}. Take $|t| \to \infty$. Every parabolic element is a shear times a map preserving both summands. At finite places Proposition 391.2 therefore gives the required factor.

We record explicitly why the real comparison can be computed using a different complementary isotropic space. A second complement is the graph $x + E_0 x$ in $P + L$ with $c(x, y) = a(x, E_0 y)$ symmetric. In the symmetric gauge both position coordinates represent $t \in V/L$. The coordinate identification $f(t) \mapsto \mathbf{e}_\infty(c(t,t)/2)f(t)$ intertwines the same translations: substituting $l - E_0 x$ for l in the new phase subtracts $c(t, x) + c(x, x)/2$, exactly the change of the quadratic factor under $t \mapsto t + x$. It transports the canonical parabolic operators to one another. In fact the transported and canonical implementers differ by a scalar, and both have value $|\det_{\mathbb{R}} A|^{-1/2} f(0)$ at zero. The common scalar gauge for b has no effect on this argument.

In signed complex coordinates the positive and negative summands have equal dimension. Projection identifies the isotropic L with the graph of a unitary map from the positive to the negative half: projection to the first is injective by negative definiteness on its kernel, and the equality of norms gives unitarity. A unitary choice of bases makes $L = \{(z, z)\}$. Take the complementary $P_1 = \{(z, -z)\}$. In the real Euclidean coordinates of Theorem 393.1, these are orthogonal real Lagrangians, and the compatible complex structure interchanges them. Thus an orthonormal basis of P_1 and its images under that complex structure form a Euclidean symplectic change of Darboux basis. It preserves the vacuum Gaussian up to phase under the preceding identification. Indeed for a Euclidean symplectic map the graph of iI is preserved since it commutes with the compatible complex structure. Proposition 392.2 then takes ϕ_0 to a scalar multiple of itself of norm one. The signed unitary basis changes have the same property.

If the action A on P_1 is an ordinary unitary matrix, the signed complex matrix is $\operatorname{diag}(A, A)$. The change of variables $\mathcal{W}$ fixes the radial Gaussian, whereas the vacuum coefficient of the prescribed operator is $1/\det A$ by Theorem 393.1. Their ratio is $1/\det A$. If A is diagonal positive real, its action on L is A^{-1}, and the negative block in signed coordinates is $(A + A^{-1})/2$, positive definite. The actual vacuum coefficient is positive. The canonical $\mathcal{W}$ also has positive vacuum coefficient, since it acts by a positive rescaling of a real Gaussian. The ratio is therefore one in this case. Every invertible complex matrix is a product of a unitary, a positive diagonal matrix and a unitary: diagonalize A^*A by the proof in Lemma 79.2, then divide the orthogonal columns by their positive lengths. Multiplicativity gives the full real ratio $(\det A/|\det A|)^{-1}$. Together with the finite factors

this is precisely $v(\det_E A)^{-1}$ by Theorem 374.2. For an adelic element, all but finitely many shear or block factors preserve their local integral vacua, so the product formulas on elementary test tensors extend by linearity. The character is one on rational determinants, as asserted. □

395. Rational invariance by doubling

Lemma 395.1. *A unitary bijection of the adelic test space in Theorem 385.3 commuting with all its Heisenberg translations is scalar. In particular unitary identifications of two rational models intertwining the same Heisenberg translations differ by a scalar.*

Proof. The Λ-fixed subspace for a self-dual finite reference lattice as in Proposition 391.1 consists of $h_\infty \otimes v_\Lambda$, where v_Λ spans the finite fixed line. Indeed express each test as a finite sum with linearly independent real Schwartz coefficients. Invariance forces each finite coefficient to be fixed. The operator and its inverse preserve this subspace, and on its real factor commute with every real translation and modulation up to the common scalar gauge. Lemma 392.1 makes their action there scalar. The finite translates span by Proposition 391.1, giving the result on all tests. Compare two identifications by their quotient. □

Theorem 395.2. *Let V be one of the rational hermitian oscillator spaces, with the signed infinite frames and self-dual reference finite lattice used in Definition 375.1. In any rational symplectic polarization let $\Omega_V(g)$ be the product of the real splittings of Theorem 393.1 and the finite splitting of Theorem 390.2, transported to these coordinates. This is a unitary representation on adelic tests, and for every rational unitary isometry γ one has*

$$\Omega_V(\gamma) = \mathcal{T}_\gamma, \qquad \theta_P(\Omega_V(\gamma)f) = \theta_P(f), \qquad F_{\Omega_V(\gamma)f}(v) = F_f(\gamma^{-1}v).$$

Here $\mathcal{T}_\gamma$ denotes the rationally normalized operator of Theorem 387.2.

Proof. Transport the real factors as in Theorem 394.2, and the finite factor by Proposition 391.1. A real change mixing the archimedean components can also be made by a real symplectic implementer on their direct sum, using Proposition 392.2. Products at the real places act on Schwartz functions of all real variables: the changes of variables, quadratic phases and partial Fourier transforms defining each factor act on its coordinates with the other coordinates as parameters, as in Lemma 387.1. Thus the product has the asserted representation and intertwining laws on the full test space. The unitary identification of rational models preserving F_f in Theorem 387.2 intertwines the physical translations by the formula $D_z F_f = F_{\rho(z)f}$ and injectivity there. It is unitary by the squared norm identity in the same theorem. The local identifications through the fixed real and finite models also intertwine those translations. Hence they transport the product operators in the same way by Lemma 395.1.

First consider a hermitian space which is hyperbolic over E and has a splitting of its reference lattice into complementary rational maximal isotropic spaces as in Theorem 394.2. On its rational unitary group $\Omega(\gamma)\mathcal{T}_\gamma^{-1}$ is a scalar by Lemma 395.1, and these scalars form a character since both actions are genuine. On the stabilizer of the isotropic summand L this character is one by Theorems 394.2 and 386.1.

Take rational dual hermitian bases of the complementary isotropic spaces, so their cross Gram matrix is the identity. For any rational maximal isotropic $M = \gamma L$, a rational block diagonal unitary can take $M \cap L$ to the span of the last $b - r$ coordinates of L, where $b = \dim_E L$. Arbitrary changes of basis on L are available by their inverse conjugate transposes on the complement. As in the graph argument of Theorem 387.2, the remaining basis vectors of M can be taken to be the first r complementary basis vectors plus columns in the first r coordinates of L. Isotropy now says that this coefficient matrix is skew hermitian. The corresponding lower shear, extended by zero in the remaining coordinates, stabilizes L. Thus the parabolic and exchanges of individual dual lines generate the rational unitary

group. For each such pair and nonzero $t \in E$ with $\bar{t} = -t$, the product

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$$

exchanges the two lines. The lower shear is in the parabolic. The upper shears are conjugate to lower shears by interchanging the dual basis vectors, a rational isometry. A character trivial on the parabolic is trivial on all its conjugates. Consequently the scalar character is one in this hyperbolic case.

For general V form $V \oplus V^-$, where V^- is the same space with negative form. At finite places the latter can use the same underlying lattice Λ. A frame correcting its sign is available by a local integral scalar of norm -1: use the unit construction of Theorem 390.2 at a nonsplit place and $(1, -1)$ at a split pair. Choose the primitive sixth root $\xi = e^{\pi i/3} \in E$, which exists since $6 \mid N_1$ as in Lemma 374.1. Then $\xi - 1 = \xi^2$ is a unit. The two rational graphs

$$L = \{(x, x)\}, \qquad P = \{(\xi x, x)\}$$

are hermitian isotropic and complementary. Moreover their intersections with $\Lambda \oplus \Lambda$ split this lattice at every finite place. Given any (x, y) in it, the P-parameter is $(x-y)/(\xi-1) \in \Lambda$ and the L-parameter is $y - (x - y)/(\xi - 1) \in \Lambda$. Thus the hyperbolic case applies.

Return in the doubled space to a product of rational polarizations of V, V^-. On pure test tensors, $\Omega_{V \oplus V^-}(g, 1) = \Omega_V(g) \otimes I$. At finite places the two lattice overlaps multiply and the saturation determinant ideals multiply, by the direct sum in Lemma 390.1 and the definition of J_g. At each real place the product Gaussian and the block diagonal negative determinant give the same normalized operator by Theorem 393.1. The theta sum in the product polarization factors by absolute convergence. Choose a test on V^- with nonzero sum, for example a positive real Gaussian times the indicator of the standard integral coordinates. The rational theta identity for $(\gamma, 1)$ on the double then gives $\theta_P \Omega_V(\gamma) = \theta_P$, after cancellation of this nonzero sum. Intertwining yields $F_{\Omega_V(\gamma) f}(v) = F_f(\gamma^{-1} v)$. The uniqueness in Theorem 387.2 completes the proof. □

396. Operator laws and size for the actual arrow

Theorem 396.1. *For successive Y, Y' put $G(g, h)z = hzg^{-1}$ in rational coordinates, and write n, n' for their E-dimensions. With the characters of Theorem 376.2 use the full adelic operator*

$$\Omega_l(g, h) = \frac{\mu_l^{[n]}(\det h)}{\mu_l^{[n']}(\det g)} \Omega_V(G(g, h)).$$

For a fixed test f, $F_{g,h} = F_{\Omega_l(g,h) f}$ satisfies the rational covariance of Theorem 389.1. On both polarizations it also satisfies all the unipotent transformation laws in that theorem. Its test functions in either polarization are Schwartz-continuous at infinity with locally constant finite vectors, apart from the displayed character factor.

Proof. The determinant ratio is one on rational unitary points by Theorem 376.2 and on unipotents since their determinants are one. The action on the Hom space is a hermitian isometry by cyclicity of the trace. Theorem 395.2 therefore gives rational covariance.

For either d, each source or target unipotent used in Theorem 389.1 acts on the Hom space by a map G for which $G - 1$ raises degree by at least one, by the degree calculation of Corollary 386.2. It preserves the real-symplectic polarization variable $L = U + L_W$. For a scalar t in the multiplicative group of the local real base field, act on each Hom degree j by t^j. This preserves the hermitian form, since only opposite degrees pair. It also preserves L and acts as the identity on the degree zero space. Conjugation scales a strictly grade-raising matrix entry by a positive power of t, so $|t| \to 0$ contracts G to the identity. At any rational finite prime one may take t to be powers of that rational prime in all the local factors, and at the combined real places take one positive real t tending to zero. This argument also works

for a rational polarization of the degree zero part mixing those factors, since the dilation is the identity there. Apply Lemma 394.1 to the unitary subgroup preserving L, comparing the actual and canonical operators. It gives equality for the strictly grade-raising isometries and hence the desired transformations. At finite adeles use finite-support truncations of G, equal to it at a growing finite set of rational primes and identity elsewhere. They converge in the restricted product, as G is integral at almost every place. The continuity in that lemma gives the full identity. The real and finite continuity assertions follow from Theorems 393.1 and 395.2. □

Proposition 396.2. *Fix dual rational coordinates on the underlying symplectic space and put $n_0 = \dim_{\mathbb{Q}} P$. For an adelic unitary G in these coordinates let $b(G)$ be the least positive integer clearing all finite denominators in the matrices G_f, G_f^{-1} over $\widehat{\mathbb{Z}}$. Write*

$$H(G) = b(G)(1 + \|G_\infty\| + \|G_\infty^{-1}\|),$$

using Euclidean operator norms. For a fixed test f, there are $C, a > 0$, independent of G, such that

$$\sum_{\xi \in P(\mathbb{Q})} |(\Omega_V(G)f)(\xi)| \le C H(G)^a.$$

The same bound holds with a unit character multiplying the operator. Rational changes of basis or multiplication of G on either side by a member of a fixed compact set change the size by at most a constant factor.

Proof. It suffices to treat an elementary real-times-finite test and sum the resulting bounds. For its finite factor v choose a positive integer b_0 such that $\rho(b_0\widehat{\mathbb{Z}}^{2n_0})v = v$, meaning invariance under every element of that lattice, by Proposition 391.1. Enlarge b_0 to clear the rational denominators in q, so that its finite character is one there. Put $b = b_0 b(G)$. Since $G_f^{-1} b\widehat{\mathbb{Z}}^{2n_0} \subseteq b_0\widehat{\mathbb{Z}}^{2n_0}$, the moved vector v' is invariant under this smaller lattice of translations and modulations. Its finite support lies in $b^{-1}\widehat{\mathbb{Z}}^{n_0}$ by testing pure modulations, and it is constant on the cosets of $b\widehat{\mathbb{Z}}^{n_0}$ by the pure translations. Each such coset has volume b^{-n_0} in the standard measure. Hence $\sup|v'| \le b^{n_0/2}\|v\|_2$, using unitarity.

At infinity put $R = 1 + \|G_\infty^{-1}\|$. Infinitesimal translations in the symmetric gauge act by ∂_i and $-2\pi i t_i$. Intertwining and unitarity give, for a word of length h in these operators applied to the moved real test, an L^2 bound $C_{h,f} R^h$. Indeed commute each generator across the group operator by replacing its direction by G_∞^{-1} times that direction. Expansion gives at most $(2n_0)^h$ words in the original generators with coefficients bounded by R^h. Differentiation is valid in L^2 by the difference-quotient argument in Lemma 392.1. The real model may be taken in the chosen dual basis by the coordinate identification in Theorem 395.2.

For a real-variable Schwartz function v_1, integrate $\partial_1 \cdots \partial_{n_0} |v_1|^2$ from each coordinate of a point to $+\infty$. The absolute value bounds $|v_1(t)|^2$ by the sum over subsets S of $\|\partial_S v_1\|_2 \|\partial_{S^c} v_1\|_2$, by the product rule and Cauchy–Schwarz. Apply this to the moved real function and to each of its products with t_i^k, $k = n_0 + 1$. The word bounds give a uniform weighted supremum at most $C_f R^{k+n_0}$. Multiplying by the finite supremum and applying Proposition 388.2 in its estimate for the rational lattice sum gives the result, also for the zero-dimensional space directly. Finite denominators in products divide the products of the individual clearing integers, while operator norms are submultiplicative. Both quantities are bounded on compact sets, by finite denominators on compact subsets of the finite adeles. Applying this in both directions proves the final claim. □

397. A shortest-vector bound for rational representatives

Theorem 397.1. *Let Y be one of the positive-rank rational hermitian stages, $\Lambda \subset Y_f$ its fixed integral adelic lattice, and ι a fixed real isometry from Y_∞ to the product of signed*

complex standard spaces. Write $G = \mathrm{U}(Y)(\mathbb{A})$, $\Gamma = \mathrm{U}(Y)(E)$. *For* $h \in G$ *let*

$$L(h) = Y(E) \cap h_f\Lambda, \quad \ell_h(v) = \max_\tau \|(\iota h_\infty^{-1} v)_\tau\|_2, \quad \lambda(h) = \min_{0 \neq v \in L(h)} \ell_h(v).$$

In a fixed rational basis of the underlying $\mathbb{Q}$*-space take* $H(h) = b(h)(1 + \|h_\infty\| + \|h_\infty^{-1}\|)$ *as in Proposition 396.2 and* $H_{\mathrm{red}}(h) = \inf_{\gamma\in\Gamma} H(\gamma h)$. *For constants* $c, C > 0$ *depending only on the fixed space, frame, lattice and coordinates,*

$$c \max(1, \lambda(h)^{-1}) \leq H_{\mathrm{red}}(h) \leq C \max(1, \lambda(h)^{-1}).$$

In the upper estimate some $\gamma \in \Gamma$ *attains the displayed bound with* $H(\gamma h)$. *In rank zero the group is a point and both heights may be taken as one.*

Proof. The image of $L(h)$ in the archimedean coordinates defining ℓ_h is a full real lattice of fixed covolume $v_0 > 0$. Indeed Lemma 20.2 gives the global lattice and its covolume from the finite determinant moduli. For a unitary h_f those moduli are one, since the norm of the hermitian determinant to the real subfield is one, also on a split pair. The same determinant calculation applies to h_∞, and the reference frame is fixed. The minimum is positive and attained by discreteness. It is left Γ-invariant by $L(\gamma h) = \gamma L(h)$. In the chosen rational coordinates Λ is contained in $b_0^{-1}\widehat{\mathbb{Z}}^D$ for some integer b_0, $D = \dim_{\mathbb{Q}} Y$. Every nonzero $v \in L(h)$ consequently has a real coordinate of size at least $1/(b_0 b(h))$, by rational prime factorization. Equivalence of the fixed real norms and $\|h_\infty^{-1} v\| \geq \|v\|/\|h_\infty\|$ give the lower estimate after taking the infimum.

First find a compact set meeting every class with $\lambda(h) \geq 1/2$. In ordinary Euclidean norm the moved lattice then has mutual distances at least $1/2$. It has D independent vectors in a ball of fixed radius. For detail, the average number of its points in a translated ball of radius R, as its center runs over a lattice parallelepiped with probability measure, is $\mathrm{vol}(\mathrm{ball}(R))/v_0$, by tiling. Points in a ball contained in an affine subspace of dimension $e < D$ number at most $(4R + 1)^e$, by taking disjoint e-dimensional balls of radius $1/4$ centered at them and comparing volumes in that subspace. A ball of radius R in the ambient space cuts the affine subspace in a ball of radius at most R. For a fixed sufficiently large R the average, of order R^D, exceeds $(4R+1)^{D-1}$. Some translate then contains points affinely spanning the whole space. Their differences supply independent lattice vectors of norms at most $2R$.

The corresponding rational vectors span Y over E: a proper E-subspace would have smaller real dimension after taking all embeddings. Select $n = \dim_E Y$ giving an E-basis $v_1, \ldots, v_n$. Its Gram entries lie in O_E because $h_f\Lambda$ is integral, and all their complex conjugates are bounded by $4R^2$ by Cauchy–Schwarz in the signed coordinates. There are only finitely many such Gram matrices by the lattice argument of Theorem 23.2. For each occurring Gram matrix choose once a rational basis $t_1, \ldots, t_n$ having that Gram matrix. The map sending v_i to t_i is a rational unitary γ. At infinity $(\gamma h)_\infty^{-1} t_i$ have bounded coordinates. At finite places $(\gamma h)_f^{-1} t_i \in \Lambda$. The finite list of fixed bases and their inverses thus gives one uniform real bound and one integer denominator bound for $(\gamma h)^{-1}$. The same is true for γh by the inverse conjugate transpose formula with the fixed rational Gram matrix. These bounds describe a compact set in the unitary group: at infinity use closed bounded matrix sets and at finite places products of fractional integer lattices, with the unitary equations closed. This proves the thick-part assertion.

For arbitrary h, all vectors in $L(h)$ of ℓ_h-length strictly less than one are mutually hermitian-orthogonal. Their pairing is an algebraic integer with every conjugate of modulus strictly less than one, hence zero by its integral absolute norm. At each infinite place the images of these vectors span a totally isotropic complex subspace I. With J the diagonal matrix of signs, I and JI are Euclidean orthogonal. Let D_τ multiply I by two, JI by $1/2$ and their joint Euclidean orthogonal complement by one. It preserves the hermitian form since J exchanges the first two spaces and preserves the last. Both its norm and its inverse norm

are at most two. Right multiply h_∞ by the inverse of this operation, in the coordinates ι, leaving h_f fixed. In the new coordinates every old vector of length less than one has doubled in length, and every other vector has length at least $1/2$. Thus $\lambda_{\mathrm{new}} \geq \min(2\lambda(h), 1/2)$. Iterate until the thick part is reached. For $\lambda(h) < 1/2$ at most $\lceil \log_2(1/(2\lambda(h))) \rceil$ steps are needed; their product and inverse in signed coordinates have norm at most $\lambda(h)^{-1}$. Choose γ for the resulting thick element as above. Undoing the right multipliers retains the same finite denominators and increases the archimedean bound by at most a fixed constant times $\max(1, \lambda(h)^{-1})$. This gives the requested representative. □

398. Absolute convergence for the prescribed Weil arrows

Proposition 398.1. *For the group G, Γ in Theorem 397.1 there is a measurable left fundamental domain D' with $H(h) \leq 2H_{\mathrm{red}}(h)$ on D'. Its subset $H_{\mathrm{red}} \leq R$ has Haar measure at most $C(1+R)^b$ for $R \geq 1$ and fixed C, b. Call a measurable left Γ-invariant function y rapidly decreasing if for every integer $k \geq 0$ there is c_k with*

$$|y(h)| \max(1, \lambda(h)^{-1})^k \leq c_k \qquad (h \in G).$$

For such a function $\int_{D'} |y(h)| H(h)^a \, \mathrm{d}h < \infty$ for every $a \geq 0$.

Proof. The rational matrix subgroup is countable and intersects every compact set in finitely many points by the denominator and archimedean lattice calculation of Lemma 21.4. Its left action is free. The finite adelic group is second countable, using integral lattices with congruence neighborhoods, a countable union of bounds on denominators, and finitely many residues at any given level. As in that lemma choose a measurable fundamental domain D. The function H is Borel: finite denominator divisibility conditions on matrices and inverses are compact-open conditions in each bounded region, and the real norm is continuous. Its countable infimum H_{red} is Borel and positive. On D select the first member γ of an enumeration with $H(\gamma h) \leq 2H_{\mathrm{red}}(h)$. These Borel pieces translated by their respective γ give a measurable fundamental domain D'.

We bound the Haar measure of $H(h) \leq 2R$ in the entire group. At finite places let K be the stabilizer of the standard underlying $\widehat{\mathbb{Z}}$-lattice of rank $D_0 = \dim_{\mathbb{Q}} Y$. It is compact open. When $b(h) = m \leq 2R$, the moved lattice lies between $m\widehat{\mathbb{Z}}^{D_0}$ and $m^{-1}\widehat{\mathbb{Z}}^{D_0}$. The quotient of these bounds is $(\mathbb{Z}/m^2\mathbb{Z})^{D_0}$ by the Chinese remainder formula of Lemma 20.2. Each subgroup of this quotient has at most D_0 generators: project onto the first coordinate, lift one generator of its cyclic image, subtract multiples and induct on the kernel in the remaining coordinates. Thus the subgroup count is at most $m^{2D_0^2}$, by counting padded generator lists. Two elements with the same lattice differ on the right by K. Summing the count for $m \leq 2R$ bounds the finite measure by a polynomial times the mass of K. The finite Haar measure exists by the compact-subgroup construction in Proposition 388.1, equally for the finite unitary group considered here.

In real matrix coordinates the elements in question and their inverses have norms at most $2R$. Cover the ambient box of such matrices by cubes of diameter at most $\epsilon/(2R)$ in operator norm for a fixed small $\epsilon > 0$, using a coordinate grid. At most $C_1(1+R)^{2D_0^2}$ cubes suffice. If x, y occur in the same cube then $\|x^{-1}y - I\| \leq \epsilon$, so the part of the cube in the group is contained in a translate of a fixed compact neighborhood of the identity. For example the closure of the relative ball of sufficiently small radius has bounded inverse and is compact in matrices. Each such translate has bounded real Haar measure by the chart construction of Lemma 21.4. The product estimate gives the sublevel bound on D'. By Theorem 397.1, $|y(h)| \leq C_k H_{\mathrm{red}}(h)^{-k}$. Sum over the dyadic shells $2^j \leq H_{\mathrm{red}} < 2^{j+1}$ using $k > a + b + 1$ and $H \leq 2H_{\mathrm{red}}$ on the chosen domain. This gives the stated integral. □

Theorem 398.2. *Consider successive spaces Y, Y', the genuine twisted arrow Ω_l of Theorem 396.1, and any fixed test f in its oscillator space. Fix a source unitary adelic element x. Let $p(r)$ satisfy the continuity and generic character conditions of Theorem 389.1, and let y*

be a rapidly decreasing automorphic function on the target in the sense of Proposition 398.1. For $F_{r,h} = F_{\Omega_l(rx,h)f}$ the period identity and the target generic transformation law of Theorem 389.1 hold with absolute convergence. In particular this applies to every continuous automorphic y on the definite terminal target.

Proof. Proposition 388.1 gives compact closure of the source domain. Theorem 396.1 gives rational covariance, unipotent covariance and continuity of the oscillator vectors. Their theta functions are jointly continuous by the locally uniform lattice estimates in Theorem 385.3. Their rational lattice sums of absolute values in both polarizations are bounded on compact sets by Proposition 396.2. They are measurable: each evaluation of the continuous real and locally constant finite test family is continuous, and there are countably many rational indices.

In the Hom coordinates the matrix for $z \mapsto hz(rx)^{-1}$ and its inverse have size H at most a fixed multiple of $H(h)$ on D_B, for each of the two chosen polarizations. Indeed right multiplication by $(rx)^{-1}$ runs through a compact set there. Left multiplication by h consists of its action on the individual columns, in a fixed multiplicative field basis. Passing to the chosen rational matrix basis changes the clearing integer and the two norms by bounded factors, as in Proposition 396.2. The same facts hold for h^{-1}. The scalar character does not affect absolute values. Hence $|p|M_0 \le CH(h)^a$ on D_B, because p is bounded on its compact closure. Proposition 398.1 gives the single absolute bound required in Theorem 389.1, which proves the claim.

For the terminal target the Bessel flag is empty and the group is definite. Proposition 388.1 gives compact representatives, so H_{red} and every continuous automorphic y are bounded there. The lower bound in Theorem 397.1 therefore bounds $\max(1, \lambda^{-1})$ on the whole quotient. This verifies rapid decrease for every such y. □

399. The initial character and the successive Bessel periods

Proposition 399.1. *At stage zero use the Poincaré function of Theorem 350.1 with the finite right translation of Lemma 349.2. Call the resulting function P_0. It is continuous, invariant under the left rational Bessel subgroup, and, for every source unitary u whose difference from one raises grades by at least two,*

$$P_0(ux) = \Psi\left(\frac{\jmath}{2}\operatorname{tr}(S_s u)\right) P_0(x).$$

Proof. Continuity and rational invariance are in the stated base theorem and persist under fixed right translation. For the character calculation $s = 2\nu + 1$, and the initial simple string entries in Proposition 21.2 are $u_{i,i+1}$ for $0 \le i < \nu$, with $e_\nu = z$. Write $R = u - 1$. Between source grades differing by exactly two the inverse series for u gives $R^\dagger = -R$ in those entries, since R^2 raises degree by at least four. With $h_i = h(e_{i^*}, e_i)$, the adjoint formula gives, for $k = s - 2 - i$,

$$R_{k,k+1} = -(h_i/h_{i+1})\overline{R_{i,i+1}} \qquad (0 \le i < \nu).$$

The shift S_s of Lemma 380.1 has coefficient c on the edges $i < \nu$, and also on their mirrors except at $k = \nu$, where it is $-Nc$. For $i = \nu - 1$ the Gram ratio is $-1/N$, exactly compensating this middle coefficient. Summing the trace along the two halves therefore gives $\operatorname{tr}(S_s u) = c(\alpha(u) - \overline{\alpha(u)})$, $\alpha(u) = \sum_{i<\nu} u_{i,i+1}$. When $\nu = 0$ both sides vanish. Taking the trace of $\jmath c(\alpha - \bar\alpha)/2$ is the same as taking the trace of $\jmath c\alpha$ because c is real and $\bar\jmath = -\jmath$. Consequently the displayed phase equals $\Psi(-\alpha(u)/(g\varpi))$ by Lemma 348.1. Its finite part is exactly the character in Proposition 21.2. At each infinite embedding its equality to the archimedean character there follows from the same change of frame: the ratio of consecutive string scales gives $\alpha_\tau = -2\pi i\,\tau(-\alpha/(g\varpi))$ on these entries. This identity of linear coordinate functions holds for all archimedean entries. The base theorem supplies precisely this product of phases. □

Theorem 399.2. *Fix stage elements $x_i \in \mathrm{U}(Y_i)(\mathbb{A})$, consecutive arrow tests f_i, and arrow characters Ω_{l_i} from Theorem 396.1. With P_0 as in Proposition 399.1, define for $0 \le i < m$*

$$P_{i+1}(h) = \int_{[B_i]} P_i(rx_i)\, I_{T_i}^{U_{g'}}\Big(F_{\Omega_{l_i}(rx_i,h)f_i}\Big)\,\mathrm{d}r, \qquad [B_i] = B_{Y_i}(E)\backslash B_{Y_i}(\mathbb{A}).$$

Here g' is the grading on the indicated target stage. Each integral converges absolutely. Each P_i is continuous, left rationally Bessel-invariant and has the trace character of its own shift on every unipotent of degree at least two. It depends only on the preceding $x_0, \dots, x_{i-1}$. Against any rapidly decreasing target automorphic function the one-step identity of Theorem 398.2 holds with $p(r) = P_i(rx_i)$.

Suppose the finite factor of f_i is the prescribed inclusion vector of Definition 357.1, with any real Schwartz factor. At any h with $P_{i+1}(h) \neq 0$ there is $r \in B_{Y_i}(\mathbb{A})$ with $P_i(rx_i) \neq 0$ at which the constant term is nonzero. For a nonempty target string the local lattice inequality of Proposition 353.2 holds at the finite points of rx_i, h, hence gives the same string-ideal inequalities at x_i, h.

Proof. Induct from the seed proposition. At a fixed x_i, right translation preserves the rational invariance and left character, so $P_i(rx_i)$ satisfies the hypotheses on p. Theorem 398.2 gives the target character and invariance, by the compact integral defining its new period, and gives the identity with every rapidly decreasing test. Existence and continuity of the new period already follow from the compact-domain proof of Theorem 389.1, independently of that target test. The data in its definition involve just the preceding stage elements, giving the stated dependence by induction.

Nonzero integral implies the integrand is nonzero somewhere. By Theorem 390.2 the finite part is a scalar multiple of the transported inclusion vector, and the finite and real Heisenberg actions on the theta function intertwine by Theorem 385.3. The inclusion invariance, transported as in Corollary 358.2, therefore supplies the period lattice in Theorem 356.3. Apply that theorem and Proposition 353.2. Finally a Bessel element preserves every string prefix in Definition 351.1 and is the identity on the string line of each quotient. For the negative prefixes this follows by unitary duality exactly as in Proposition 382.1, and the middle line is part of the defining flag. It leaves each projected string ideal unchanged, proving the last clause. □

400. The adjoint identification of two adelic arrows

Theorem 400.1. *Use $V_1, V_2, V_*, L_1, L_2, L_*, A(a,b) = (a^\dagger, b)$ of Theorem 364.1. There is a fixed unitary coordinate identification $\mathcal{J}$ on the adelic Schrödinger test spaces, with that theorem's identification on the finite factors, such that on pure tensors*

$$\theta_*\mathcal{J}(f_1 \otimes f_2) = \theta_1(f_1)\theta_2(f_2),$$
$$\mathcal{J}(\rho_1(a) \otimes \rho_2(b)) = \lambda(a)\rho_*(A(a,b))\mathcal{J}, \qquad \lambda(a) = \Psi(-\bar{\omega}H_1(a,a)).$$

For g, h, k on the three consecutive unitary spaces of ranks n, n', n'', put $G_1 a = hag^{-1}$, $G_2 b = kbh^{-1}$, and $G_* z = (g \oplus k)zh^{-1}$ on the combined space. For the untwisted unitary splittings of Theorem 395.2,

$$\mathcal{J}(\Omega_1(G_1) \otimes \Omega_2(G_2)) = \upsilon_T(\det_E G_1)^{-1}\Omega_*(G_*)\mathcal{J}.$$

These identities hold on all pure tests, including after their independent translations by the stage elements.

Proof. The two-arrow computation of Theorem 364.1 holds for the rational trace forms before completion. On the first block adjunction replaces its hermitian value z by $-\bar{z}$. Thus A preserves the alternating form and is a rational symplectic isometry. In the symmetric gauge carry a product polarization to its image under A by changing variables. The rational theta sum is unchanged by this reindexing, and additive measures of covolume one change

by no global factor by the rational determinant calculation in Theorem 383.1. If another rational polarization is used on V_*, apply Theorem 387.2 to change to it. The product rational sum factors on pure tensors by absolute convergence. In the original gauge the change in $b(v,v)/2$ gives the indicated $\lambda(a)$, because $\jmath(\omega-\bar\omega)=-1$ and $H_1(a,a)$ is real. This is also the direct bicharacter calculation of Theorem 364.1, with the full character Ψ.

The coordinate identification just constructed factors into its real and finite operators: each rational change of variables and each rational partial Fourier transform in Theorem 387.2 has this factorization. It agrees with the finite identification fixed in the statement up to unit scalar by Proposition 391.1, since both implement the same translations with the same quadratic gauge. Transfer that scalar to the real factor. This fixes the product identification with its displayed theta normalization and the prescribed finite factor.

For the operator formula, the second block is unchanged. Orthogonal direct sums give products of the genuine actions by the vacuum and determinant argument in Theorem 395.2. In the first block the finite positive-overlap lattice operators intertwine under adjunction by Theorem 364.1. The saturation ideal on the image lattice is $\bar J_{G_1}$, since the map on that block is conjugate linear and carries the lattice onto its image. In the original lattice, unitary duality interchanges the elementary valuations $k_{w,i}$ with the negatives at $\bar w$, exactly as in Theorem 375.2. At a nonsplit place this also follows from the inverse conjugate transpose in a unimodular reference frame. Consequently the determinant ideal of G_1 is $\bar J_{G_1}/J_{G_1}$, since at w the valuation difference is $\sum_i(\max(0,k_{w,i})-\max(0,-k_{w,i}))$. The ratio of the two phases is therefore $\beta(J_{G_1})/\beta(\bar J_{G_1})=\upsilon_f(\det_E G_1)^{-1}$.

At an infinite place adjunction in signed orthonormal frames is complex conjugation, transposition and row and column signs. It exchanges the positive and negative complex axes for the negated target form, and is a Euclidean symplectic isometry in the Darboux coordinates of Theorem 393.1. It therefore preserves the Gaussian line, as in Theorem 394.2. If the positive and negative diagonal blocks of G_1 are α,δ, the negative block of the transformed isometry has determinant $\overline{\det\alpha}$. Indeed adjunction maps the old positive axes antiunitarily to the new negative axes, with the fixed signs canceling in conjugation. Both operators implement the same physical isometry after the coordinate change. Their scalar ratio is the ratio of their Gaussian coefficients, namely $\overline{\det\alpha}/\det\delta=(\det_{\mathbb C}G_1)^{-1}$ by Theorem 393.1. This is the inverse infinite factor of $\upsilon_T(\det_E G_1)$. The finite and infinite conclusions give the claimed formula. □

Corollary 400.2. *For even $j<m$ use the consecutive pair starting at Y_j. Let U_1,U_2 be the positive Hom grades for the string gradings on its two arrows and let $U_>$ be the positive combined Hom grades in Corollary 365.3. For all tests f_1,f_2,*

$$I_T^{U_1}(F_{f_1})I_{T'}^{U_2}(F_{f_2})=I_\tau^{U_>}(F_{\mathcal J(f_1\otimes f_2)}),\qquad \tau=(T^\dagger,T').$$

Proof. Reversing a pair of indices under the adjoint reverses their individual grades and interchanges source and target. Hence the grade of the Hom entry is unchanged, so A sends $U_1\oplus U_2$ onto $U_>$. The quadratic phase in Theorem 400.1 is one on a positive Hom vector since its norm is zero. It is also one on the rational T. Apply that theorem first to the translations by T,T', then to the positive translations in the two quotient integrals. The rational additive change of variables preserves quotient probabilities as in Lemma 379.3. Product integration gives the identity. □

401. The mean on the Bessel quotient

Lemma 401.1. *For a graded stage of Proposition 388.1 let $H=\mathrm U(X)$ act trivially on its string and let J consist of the unitaries n with $n-1$ strictly grade-raising. Write $[S]=S(E)\backslash S(\mathbb A)$ for any of the rational groups $S=H,J,B$, with $B=J\rtimes H$. Give J the logarithmic measure for which $[J]$ has volume one, and use the chosen definite quotient measure on H. In coordinates nh the product is a left Haar measure on B, and for this*

normalization and every continuous left rationally invariant function K on $B(\mathbb{A})$,

$$\int_{[B]} K(r)\,\mathrm{d}r = \int_{[H]}\int_{[J]} K(nh)\,\mathrm{d}n\,\mathrm{d}h.$$

Proof. The logarithmic description and the rational fundamental sets were established in Proposition 388.1 and Theorem 383.1. Conjugation by h is linear on the logarithm. In the skew adjoint strictly positive matrices the entries from X into each positive string line are free and their adjoint entries are then determined. The other free entries have both indices in the string. Indeed X has grade zero, so no entry inside X occurs, and the adjoint exchanges the block from a negative string line into X with the indicated positive one. Thus conjugation acts on each free row out of X by right multiplication by h^{-1} and fixes the string-only coordinates. The determinant over E on each such row is $(\det_E h)^{-1}$. Its norm to the real subfield is one by taking determinants in the unitary equation. In particular the absolute determinant on the underlying rational coordinates at each rational place is one. This last determinant is the product of the conjugates of the E-determinant: elimination into row additions, swaps and diagonal multiplications reduces it to the definition of the field norm as the determinant of scalar multiplication. This argument applies to the product algebras at a split prime as well. Conjugation by h therefore preserves the logarithmic measure by the change of coordinates in Theorem 386.1.

The group law in the given coordinates is $(n,h)(n',h') = (nhn'h^{-1}, hh')$. The invariance just proved gives left invariance of product measure. Choose left fundamental sets D_H, D_J as in Proposition 388.1. Left rational translation first brings the H-coordinate into D_H, then brings the resulting J-coordinate into D_J. If two elements of $D_J D_H$ are rational translates, comparing their H-coordinates first gives the same h and trivial rational H-multiplier. Comparing the J-coordinates then gives the same n, up to the null boundaries of the chosen sets. The integrand is bounded on their compact closures. Product integration by Lemma 123.1 proves the formula. Invariance of the inner integral under rational changes of the representative h follows by the same conjugation, which also preserves $J(E)$. □

Proposition 401.2. *In a pair of consecutive arrows starting at even j, write $s = 2a+1$ for the initial string length. Use J' for the strictly grade-raising unitary group on the middle space Y'. For $a > 0$ put $D = \mathrm{Hom}_E(X', E\xi)$, with ξ from Proposition 365.1; for $a = 0$ put $D = 0$. Define $b : J'(\mathbb{A}) \to D(\mathbb{A}_E)$ by*

$$b(n)x = c\,\big((n-1)x\big)_{e'_{a-1}}\,\xi \quad (x \in X',\ a > 0), \qquad b(n) = 0 \quad (a = 0).$$

Then b is a continuous surjective homomorphism both adelically and on rational points. For continuous ϕ on the additive quotient $[D] = D(\mathbb{A}_E)/D(E)$,

$$\int_{[J']} \phi(b(n))\,\mathrm{d}n = \int_{[D]} \phi(t)\,\mathrm{d}t$$

with the quotient probabilities on both sides.

Proof. A product of two strictly grade-raising differences raises degree by at least two. It has zero component from X' into the grade-one line Ee'_{a-1}. The degree-one row thus adds under multiplication, which proves the homomorphism identity. For any row $\ell : X' \to E$, put $Ax = c^{-1}\ell(x)e'_{a-1}$ on X' and put $A = 0$ on the string. Then $M = A - A^{\dagger}$ raises grade by one, and its adjoint term is zero on X' since X' is perpendicular to this string line. The unitary $n = \exp M$ belongs to J' by the finite exponential calculation of Theorem 383.1. Its first-degree row is that of M, whence $b(n)x = \ell(x)\xi$. This works over E and over its adeles. All formulas are polynomial in the coordinates, hence continuous. The case $a = 0$ is the unique map onto the zero group.

For adelic n_0, right translation by n_0 preserves the quotient probability on $[J']$. Indeed the logarithmic measure is right invariant by the triangular coordinate proof in Theorem 383.1, and the right translate of a left fundamental set is another such set. Decomposing one

set by left rational translates of the other gives equal integrals for left rationally invariant functions. The pushforward under b is therefore a probability on $[D]$ invariant under $b(n_0)$, hence under all translations by surjectivity. Uniqueness by the double-integration argument of Lemma 379.3 proves the identity. □

402. The definite quotients in two consecutive periods

Theorem 402.1. *Use the spaces Y, Y', Y'', tests f_1, f_2, and maps T, T', τ of Corollary 400.2. Write $H = \mathrm{U}(X_j)$, $H' = \mathrm{U}(X')$. Let k be a unitary on the definite block of Y'', extended by the identity on its string, and in the terminal pair let k be any unitary on Y_m. Fix adelic positions x, x', x'' on the three spaces. Use $p = P_j$ in Theorem 399.2, or any continuous p with its rational Bessel invariance and its character there. Denote the two succeeding period operations of that theorem at x, x' by $p \mapsto p_1 \mapsto p_2$. Use the product measures of Lemma 401.1.*

Write Ω_1, Ω_2 for the two twisted arrow operators. Set

$$u = \mathcal{J}\big(\Omega_1(x, x') f_1 \otimes \Omega_2(x', x'') f_2\big), \qquad \mathcal{G}(g, h, k) = \mathcal{J}\big(\Omega_1(g, h) \otimes \Omega_2(h, k)\big)\mathcal{J}^{-1},$$

using the identification of Theorem 400.1. An element of a definite block in this formula acts trivially on the attached string. Choose a rational polarization adapted to U_*, W_* from Corollary 365.3 and write Z for restriction of its position variable to the residual position space. With its rational theta functional θ_{W_*}, one has

$$p_2(kx'') = \int_{[H]} \int_{[H']} p(gx)\, \theta_{W_*}\big(Z \rho_*(\tau) \mathcal{G}(g, h, k) u\big)\, \mathrm{d}h\, \mathrm{d}g.$$

All integrals are absolutely convergent.

Proof. Expanding the two period operations gives an integral over $r \in [B], r' \in [B']$ with integrand $p(rx)$ times the two lowering-map coefficients at $(rx, r'x')$ and $(r'x', kx'')$. The group law absorbs the three fixed positions into the two tests as in the definition of u. Each rational Bessel translation in either integration variable preserves the integrand. Indeed rational covariance in Theorem 396.1 acts on the frequency by that translation, and the rational change at T lies in the nonnegative space, as in the last paragraph of the proof of Theorem 389.1. The same holds at T'. Absolute convergence and interchanges of integration follow from continuity on the compact representative closures of Proposition 388.1 and Lemma 123.1.

Write $r = ng$ as in Lemma 401.1. Every grade in Y is even. Hence $n-1$ raises degree by at least two. Since T lowers by one, $T(n-1)$ is positive, so $n \in J_T$ of Theorem 383.1. The factor $p(ngx)$ acquires $B(Tn, T)$, and the first coefficient acquires its inverse by Corollary 386.2, where $\eta_T = 1$. The second coefficient is independent of n. The n-integral thus has the value at $n = 1$.

Write $r' = n'h$ with $h \in H'(\mathbb{A})$. At $n' = 1$ the product of coefficients is $I_\tau^{U_>}(F_v)$ for $v = \mathcal{G}(g, h, k)u$, by Corollary 400.2. Inserting n' acts on v by the combined untwisted unitary operator for $Gz = z(n')^{-1}$: all determinant factors from Theorems 400.1 and 396.1 are one for this unipotent. The difference $G - 1$ strictly raises Hom degree. For the grading with positive part $U_>$ it follows that $I_\tau^{U_>}(F_{\Omega_*(G)v}) = I_{\tau n'}^{U_>}(F_v)$. Indeed the dilation by t^i on degree i contracts G in the parabolic preserving the polarization variable. Lemma 394.1 and the restricted-product argument of Theorem 396.1 identify its operator with $\mathcal{W}_G$. The stated coefficient formula is Theorem 386.1.

Put $b = b(n')$ as in Proposition 401.2. The difference $\tau(n' - 1) = b + v_+$ has $v_+ \in U_>$. Indeed it has degree at least zero, and by Propositions 365.1 and 365.2 its degree-zero part is precisely the displayed row into $E\xi$. For the final pair the difference vanishes. The adjoint in the negated first block gives

$$\tau_*^\dagger \tau = -TT^\dagger + (T')^\dagger T' = 0$$

by Lemma 380.1 applied to the common middle space. In particular $H_*(\tau, \tau(n'-1)) = 0$, so $B(b+v_+, \tau) = 1$. Moreover $B(v_+, b) = 1$ by orthogonality of positive and zero degrees. The Heisenberg law and invariance of the positive quotient mean now give

$$I^{U_>}_{\tau n'}(F_v) = I^{U_>}_0\big(F_{\rho_*(b)\rho_*(\tau)v}\big).$$

The last expression is continuous and periodic in b modulo $D(E)$. For a rational $d \in D$, its translation factor on $U_> + D$ is one by total isotropy of that sum and rational theta invariance in Theorem 385.3. Proposition 401.2 replaces the n'-integral by the additive mean over $[D]$. Because $U_* = U_> \oplus D$ is totally isotropic, averaging over $[U_>] \times [D]$ gives $I^{U_*}_0(F_{\rho_*(\tau)v})$. This holds also when $D = 0$.

Finally a rational hermitian isotropic dual to U_* is given by the negative Hom grades and, when $D \neq 0$, the extra $\mathrm{Hom}(X', E\eta)$ in Proposition 365.1. Its pairing with U_* is perfect and the remaining space is W_*. Choose dual rational position and modulation variables in W_* by the plane removal of Lemma 385.2. The proof of Theorem 385.3 then identifies $I^{U_*}_0$ with $\theta_{W_*}Z$: a rational position outside the residual position space has a nontrivial character of $[U_*]$. This gives the asserted integral. □

403. The group action after restriction to the residual variables

Lemma 403.1. *In the Schrödinger model of Theorem 395.2, the physical finite unitary action on a hermitian space over the arithmetic field E is independent of the choice of reference lattice αM among finite adelic $\widehat{O}_E$-lattices M unimodular for the hermitian form. Here unimodular means $M = \{v : H(M, v) \subset \widehat{O}_E\}$, and α is the fixed idele in Lemma 348.1. The real action is independent of the choice of a signed orthonormal complex basis for the same hermitian space.*

Proof. Each reference lattice αM is self-dual for the alternating trace character, and its square has trivial bicharacter, by $\jmath\alpha\bar{\alpha}\widehat{O}_E = \widehat{O}_E^\vee$ in Lemma 355.2. Here the right hand side is the trace dual of the integer ring. Two such M, M' are carried to one another by a finite adelic unitary k. To see this, both have integral bases locally, using elimination over each valuation ring, and their Gram matrices J, J' are hermitian and invertible over the ring by unimodularity. At a split pair choose a basis on one branch and its dual on the other. At a nonsplit place the extension is unramified by Lemma 57.1. Modulo its real uniformizer π, the diagonalization in Lemma 345.1 makes J, J' isometric. Given an invertible A with $A^*J'A = J \bmod \pi^t$ for $t \geq 1$, put $S = A^*J'A$. Modulo π solve

$$C^*\bar{S} + \bar{S}C = [(J - S)/\pi^t] \bmod \pi,$$

where $\bar{S}$ in this display denotes residue reduction. The right side is hermitian and can be written $D + D^*$ by the trace surjectivity in that same lemma. Take $C = \bar{S}^{-1}D$. Replacing A by $A(1 + \pi^t\widetilde{C})$ for any lift $\widetilde{C}$ increases the precision by one. Completeness and invertibility of the first residue give an exact isometry. In rational coordinates the given lattices agree almost everywhere: each is contained in a common integer-denominator multiple of the standard lattice and contains a common positive integer multiple of it, by compactness and openness. Take the local isometry to be the identity there. This proves the claim about k.

Let ϕ be a unit vector on the fixed line for αM. The fixed line for $\alpha M'$ is spanned by $\mathcal{V}_k\phi$ by Proposition 391.1. For each unitary G the old operator $\mathcal{V}_G$ implements the prescribed translations. Its coefficient against the new vacuum equals the old coefficient for $k^{-1}Gk$ by the group law of Theorem 390.2. The ideal and lattice index of $k^{-1}Gk$ for αM are those of G for $\alpha M'$, since k identifies the respective intersections and ambient bases. By Theorem 375.2, the coefficient is therefore exactly $\beta(J_G)[G\alpha M' : G\alpha M' \cap \alpha M']^{-1/2}$ with the new reference. This nonzero coefficient and the intertwining law determine the operator by Lemma 394.1. This proves independence at the finite places.

At a real place a new signed basis is the image of the old one by a unitary k for the indefinite hermitian form. The Gaussian in the new coordinates, transported into the old

physical model, is proportional to $\Omega(k)\phi_0$. Indeed coordinate transport implements the change of the two symplectic bases and two unitary implementers differ by a scalar by Lemma 392.1. The scalar has modulus one. The coefficient of $\Omega(G)$ on this Gaussian is the old coefficient of $\Omega(k^{-1}Gk)$. The negative block of G in the new basis is the old negative block of $k^{-1}Gk$. The uniqueness statement of Theorem 393.1 now identifies the two operators. □

Theorem 403.2. *Let $V = U_- \oplus W \oplus U$ be a rational hermitian space over E, with U_-, U isotropic and in perfect duality, both orthogonal to the nonsingular W. Assume W has a unimodular finite lattice after division of its reference by α. Give V, W the splittings of Theorem 395.2 with reference lattices of Lemma 403.1. For an adelic isometry G preserving each of the three summands, put $A = G|_{U_-}$ and $G_W = G|_W$. Take a rational symplectic polarization of W and use U_- as the additional position variable of V. If $Zf(t) = f(0,t)$ denotes evaluation in that variable, then*

$$Z\Omega_V(G)f = \upsilon(\det_E A)^{-1}|\det_{\mathbb{Q}} A|_{\mathbb{A}}^{-1/2}\Omega_W(G_W)Zf.$$

The map Z is the product of real and finite restriction on each real-times-finite test. In particular it applies to U_*, W_* of Theorem 402.1.

Proof. In a rational E-basis of U_- choose the integral lattice M_-, and take its hermitian dual M_+ in U. The union of the two corresponding dual bases has hyperbolic Gram matrix, so $M_- \oplus M_+$ is unimodular. Use its orthogonal sum with the indicated unimodular lattice on W. Lemma 403.1 permits this choice. Also choose real signed orthonormal bases on these two orthogonal blocks separately. In a product polarization the operator on the orthogonal sum is then the product of the block operators. At finite places the reference vacuum and the index ideals factor, as does the normalized lattice sum in Definition 375.1. At infinity the normalized Gaussian factors and its coefficient is the reciprocal of the product of the two negative block determinants, by Theorem 393.1. The product implements the translations on the direct sum, so the scalar uniqueness arguments of Theorem 395.2 give the assertion in the chosen rational polarization. On the real Schwartz space the coordinate substitutions, quadratic phases and partial Fourier transforms on W act with the U_- coordinates as parameters, so they commute with evaluation there. For the Fourier transforms this follows directly from their absolutely convergent integral on Schwartz functions in Lemma 387.1.

The hyperbolic factor acts by $\upsilon(\det_E A)^{-1}\mathcal{W}_G$ on its own variables by Theorem 394.2. At zero its phase is one and the normalized substitution there multiplies by $|\det_{\mathbb{Q}} A|_{\mathbb{A}}^{-1/2}$, by Theorem 386.1. This proves the formula, including $U = 0$. The evaluation is literally a restriction of the two variables on a real-times-finite test. It remains a Schwartz function at infinity and is locally constant with compact support at finite places. For the paired application the dual was constructed at the end of Theorem 402.1. The standard residual lattice in Theorem 367.1 is of the required form: after dividing its physical coordinates by α the two scalar multiples in Definition 366.1 differ by $\bar{\alpha}/\alpha$ of norm one. Thus the resulting signed matrix lattice is unimodular. □

404. Determinant cancellation on the two definite arrows

Proposition 404.1. *In Theorem 402.1 write $X = X_j$, and write $X'' = X_{j+2}$ for $s \geq 3$, or $X'' = Y_m \oplus E(z/\pi_N)$ for $s = 1$. Let k_0 be k on the usual definite target, extended by one on the additional line if present. For g, h, k in that theorem write $A = \det_E g$, $B = \det_E h$, $C = \det_E k$. The action $Gz = (g \oplus k)zh^{-1}$ on the combined Hom space preserves U_*, W_* and their chosen negative dual. On either of the two isotropic summands its determinant is $(AC)^a B^{-2a}$, where $s = 2a + 1$. It fixes τ. Let $\mathcal{G}_0(g, h, k_0)$ on $W_* = \mathrm{Hom}(X', X^- \oplus X'')$ be the adjoint combination of the two twisted definite arrows, with their own ranks and the*

same two characters μ_l, μ_{l+1} as on the indefinite arrows. Then

$$Z\rho_*(\tau)\mathcal{G}(g,h,k) = \mathcal{G}_0(g,h,k_0)Z\rho_*(\tau).$$

Proof. The three definite unitaries fix every string vector and preserve the grades. They also fix η, ξ of Proposition 365.1 when present. The descriptions in Corollary 365.3 and Theorem 402.1 prove preservation of the three summands. By Proposition 365.2 the two components of τ vanish on X' and have image in the outside strings, so $G\tau = \tau$. Intertwining of the Heisenberg operators therefore commutes $\rho_*(\tau)$ through $\mathcal{G}$.

For $a > 0$, the middle string has a negative lines. Positive Hom entries from these lines into the outside definite blocks contribute determinant $(AC)^a$. The positive outside string lines number $a + (a-1)$. Positive Hom entries into them from X' contribute $B^{-(2a-1)}$. The additional maps from X' into $E\xi$ contribute B^{-1}. Every other positive entry has trivial action since both indices are in strings. Thus the claimed determinant holds on U_*. For $a = 0$ this space is zero and the determinant is one. By hermitian duality the determinant on the opposite summand is the inverse conjugate of that on U_*, equal to it since A, B, C have norm one. The determinant modulus over the rational adeles is consequently one by the norm calculation of Lemma 401.1.

Write n, n', n'' for the original three ranks, and d, d', d'' for the definite ones, including the additional final line. By Theorems 400.1 and 396.1, the scalar on the untwisted combined action is

$$\frac{\mu_l(B)\mu_{l+1}(C)}{\mu_l(A)\mu_{l+1}(B)}\, \upsilon_T(A)^{\lceil n'/2\rceil}\upsilon_T(C)^{\lfloor n'/2\rfloor}\upsilon_T(B)^{-\lceil n/2\rceil-\lfloor n''/2\rfloor}.$$

Indeed the first Hom action $v \mapsto hvg^{-1}$ has determinant $B^n A^{-n'}$, by its action on columns and rows, and adjunction supplies the inverse splitting character of this determinant. The remaining exponents are those in Definition 376.1. The same calculation applies to the three definite spaces: the proof of Theorem 400.1 uses just the Hom pairing and the signs of the orthonormal frames. Moreover $\det k_0 = C$ even on the enlarged target.

By Definition 348.2,

$$n = d + 2a + 1, \qquad n' = d' + 2a, \qquad n'' = d'' + 2a - 1.$$

The last equation for $a = 0$ uses $d'' = n'' + 1$. The outside ranks n, n'' differ by $2t$, and hence d, d'' have the same parity. The first two exponents involving n' increase by a upon passing from definite to original ranks. The sum $\lceil n/2\rceil + \lfloor n''/2\rfloor$ increases by $2a$: if d, d'' are even the two increases are $a+1, a-1$; if they are odd the two increases are a, a. The ratio of the original scalar to the definite one is therefore $\upsilon_T((AC)^a B^{-2a})$. Theorem 403.2 gives precisely its inverse on applying Z, proving the identity. □

Corollary 404.2. *In the notation of Proposition 404.1, let $\mathcal{J}_0$ denote the adjoint identification for the two definite Hom spaces and put*

$$v_0 = \mathcal{J}_0^{-1} Z\rho_*(\tau)u.$$

Let θ_{12} be their rational theta functional, the product of the two individual theta functionals on pure tensors. Then the integrand on the two definite quotients in Theorem 402.1 equals

$$p(gx)\,\theta_{12}\Big((\Omega_1^0(g,h)\otimes\Omega_2^0(h,k_0))v_0\Big),$$

where the superscript 0 denotes the operators on the definite arrows. For elementary real-times-finite input tests, v_0 has one real Schwartz factor and one finite factor. For the moved finite inclusion tests the latter is the shifted finite evaluation of Proposition 368.2, with the actual arrow multipliers and a nonzero scalar depending only on the fixed models. In particular, for fixed real tests and fixed infinite positions the real factor can be chosen independently of all weight-zero twists and finite positions.

Proof. Apply the proposition and the theta normalization of Theorem 400.1 on the definite pair. These identifications extend to its full real Schwartz space: the rational linear coordinate transports and partial Fourier transforms in its proof act on joint Schwartz functions, and the rational sum on the two position spaces is absolutely convergent by Proposition 388.2. This defines θ_{12} on the joint space. The finite part is the algebraic tensor product by the translated basis of Theorem 364.1.

All the coordinate identifications, the shift by τ and the restriction factor into real and finite operators by Theorems 400.1 and 403.2. In the adapted finite Schrödinger model $Z_f\rho_f(v) = Z_f$ for $v \in U_*$, since its modulation is one at zero and it is perpendicular to W_*. For $w \in W_*$, $Z_f\rho_f(w) = \rho_{W_*,f}(w)Z_f$: in the formula of Theorem 385.3 both the translation and its phase use only the residual variables at that section. This map is onto. Multiply any finite residual test by the indicator of a compact open lattice containing zero in the opposite variables to extend it. Under the unitary finite identifications of Proposition 391.1, Theorem 362.2 now identifies Z_f with a scalar multiple of its normalized quotient map in the residual lattice model. Indeed the quotient is spanned by residual Heisenberg translates of the vacuum, and a commuting endomorphism of that space is scalar by uniqueness of the fixed line. Surjectivity makes this scalar nonzero. To use the standard residual lattice, send that vacuum to the unit fixed vector of Theorem 367.1, choosing its positive overlap with the standard vacuum. The choice of phase can be transferred to the real factor while preserving the global identification.

Consequently inserting $\rho_f(\tau)$ before this map gives the shifted evaluation of Corollary 367.2. The final inverse adjoint identification is exactly the one in Proposition 368.2. The fixed scalar is independent of which vector undergoes those operations. Finally ϑ in Definition 376.1 has weight zero and hence trivial infinite factor. The two moved real tests thus depend only on the fixed infinite tests and positions, as claimed. □

405. A central translation through the successive definite pairs

Lemma 405.1. *Let π be the product of the symmetric tensor representations in Theorem 122.2 at the infinite places and let $W = \bar{\pi}$ be its conjugate representation space from Definition 13.3. Write $\mathcal{B}(\bar{v}, w) = \langle v, w\rangle$ for the complex bilinear pairing of W with π, using the tensor Hermitian product conjugate linear in the first variable. Denote the definite component of b_0 in Lemma 349.2 by b, a finite central scalar on X_0. In Theorem 350.1 use the functionals*

$$\lambda_c(v) = \mathcal{B}(\phi(h_c b^{-1}), v)$$

for a column ϕ, viewing h_c also as the corresponding definite representative. Use the right-translated stage-zero function, denoted $P_{0,\phi}$. All coordinate families of that theorem occur in this way. If $x_{0,f} \in A^{\mathrm{new}}$, then for every $a = (a_\infty, a_f) \in \mathrm{U}(X_0)(\mathbb{A})$ and its block embedding $\widehat{a}$,

$$P_{0,\phi}(\widehat{a}x_0) = \mathcal{B}(\phi(a_f), \pi(a_\infty)F_\infty(x_{0,\infty})).$$

Proof. The pairing is invariant under the simultaneous conjugate and ordinary actions because π is unitary. It identifies W linearly with the full dual of π, as is seen in an orthonormal basis. The classes of $h_c b^{-1}$ form a complete system since b is central. The freedom of evaluation in Proposition 13.5 proves the first assertion.

Right invariance allows $x_{0,f}$ to be replaced by one. In the old realization the resulting finite position is the block embedding of $a_f b$. Write $a_f b = \gamma_f h_c k$ with γ a rational definite unitary and $k \in K_c$. Its embedding lies in the corresponding old right level by Proposition 66.1. Rational left invariance and evaluation at h_c in Theorem 350.1 thus give $\lambda_c(F_\infty(\widehat{\gamma}_\infty^{-1}\widehat{a}_\infty x_{0,\infty}))$. On the definite block the string character is one by its simple-root formula in Proposition 21.2. The transformation of Theorem 122.2 therefore gives $\pi(\gamma_\infty^{-1}a_\infty)F_\infty(x_{0,\infty})$ inside λ_c. Invariance of the bilinear pairing moves γ_∞ to the first variable, acting there by the conjugate representation. Since b is central, $a_f = \gamma_f h_c b^{-1} k$, and the transformation of ϕ in Definition 13.3 gives the formula. □

Theorem 405.2. *Take the stage functions of Theorem 399.2 with the preceding seed and the indicated finite inclusion vectors. Assume $x_{0,f} \in A^{\mathrm{new}}$. Fix a split pair over N and the integral norm-one finite idele z supported on that pair which lifts the generator in Theorem 13.6. Write z_i for scalar z on X_i, embedded by identity on the string in Y_i, and $(R_z\phi)(v) = \phi(vz)$ on the initial definite group. For every even i with $2 \le i < m$, every preceding choice of positions and every $v \in \mathrm{U}(Y_i)(\mathbb{A})$,*

$$P_{i,R_z\phi}(v) = P_{i,\phi}(z_i v).$$

At stage zero one has the same identity on $v = \widehat{a}x_0$ for $a \in \mathrm{U}(X_0)(\mathbb{A})$. For every ramified column, every $0 \le i < m$, and every v with $P_{i,\phi}(v) \neq 0$, the exceptional middle-line case of Proposition 351.2 is excluded at all places.

Proof. The stage-zero identity follows from Lemma 405.1, since z has trivial real component and commutes with a_f. Prove the claim at $i = j + 2 < m$ by induction in pairs. This is a pair with $s \ge 3$, so all three residual definite spaces are X_j, X_{j+1}, X_{j+2} without an added line. On either definite arrow the simultaneous scalar at its two ends acts on the Hom space by the identity. Its two determinant values are units supported over N. Every arrow character in Definition 376.1 is trivial on these units by Theorem 376.2 because N is disjoint from the marked and twist places. The full twisted action of the simultaneous scalars on each definite arrow is therefore the identity, on every test. In particular the theta kernel $K(g, h, k)$ of Corollary 404.2 for the given positions satisfies

$$K(z_j g, z_{j+1} h, z_{j+2} k) = K(g, h, k).$$

Here the scalars in arguments of the definite groups act on the whole definite spaces. Only their embeddings into the original stages fix the strings.

Choose x, x', x'' in the paired formula with $x = x_j$, $x' = x_{j+1}$ and arbitrary $x'' = v$. Evaluate it first with $k = z_{j+2}$. Change the two integration variables to $z_j g, z_{j+1} h$. On each definite quotient this preserves the measure, since central left translation commutes with the rational left action and preserves left Haar measure. The kernel becomes $K(g, h, 1)$. The factor $P_{j,\phi}(z_j g x_j)$ becomes $P_{j,R_z\phi}(g x_j)$ by the preceding stage, using the restricted stage-zero identity when $j = 0$. The integral now gives $P_{j+2,R_z\phi}(v)$ by the same paired formula. The tests in that formula depend on the three fixed positions and are independent of ϕ. This proves the claimed central equivariance for arbitrary v.

At any positive stage i, the function $P_{i,\phi}$ in its last argument is right invariant under the whole local integral unitary group $K^{(w)}$ of Proposition 352.1. Indeed on the last input arrow $\Omega(1, k)$ fixes its inclusion test for every $k \in K^{(w)}$, by the integral case of Theorem 376.2. Right multiplication in the defining integral therefore changes no factor. If i is even then the source grades at i are even, and all the strictly grade-raising unipotents there have degree at least two. The character from Theorem 399.2 is invariant under conjugation by the embedded definite group, since its shift S_s commutes with that group by Lemma 380.1. At each fixed infinite argument all three hypotheses of Lemma 352.2 are therefore satisfied for $2 \le i < m$ even. Apply Corollary 352.3 to obtain vanishing at the exceptional points. For odd i there is no middle line. At $i = 0$ the support is inside $S_f A^{\mathrm{new}}$ by Corollary 350.2. All initial string ideals on this set are those of the reference lattice by Theorem 346.3 on the early lines. In particular its middle exponent sum is zero, since the middle generator u_0 in Lemma 353.3 has norm one. Proposition 351.2 has now excluded all remaining possibilities. □

406. Finite cuts for the periods of the ramified columns

Theorem 406.1. *For fixed arithmetic data and twist set P, use the stage functions of Theorem 405.2 for all ramified columns, all real tests, and all choices of preceding positions with $x_{0,f} \in A^{\mathrm{new}}$ whenever this position occurs. Let W_i be the union of their nonzero supports projected to the finite group of Y_i. For every $0 \le i < m$, $y \in W_i$, every finite place w and*

every string index $b < m-1-i$, $b \ge 0$, the deviation from the reference exponent satisfies

$$-i\,\mathbf{1}_P(w) \le \Delta_b(y;w) \le i\,\mathbf{1}_{\bar P}(w).$$

Let K_i denote the full integral group of the stage frame and J_i the subgroup of K_i with identity reduction at the selected places over g and at P. Put $A_0 = A^{\mathrm{new}}$, $A_i = K_i$ for $0 < i < m$, $S_i = B_{Y_i}(\mathbb{A}_f)$, and $Q = \prod_{w\in P} q_w$. There are finite cuts T_i in the sense of Definition 342.1 with $W_i/J_i \subseteq S_iT_i$. They are stable under $S_i \cap A_i$, contain representatives integral off $P \cup \bar P$, and have $m_i(t)^{-1} \in \mathbb{Z}[1/Q]$ on their saturations, where $m_i = m_{T_i}$. Every representative of T_0 lies in A^{new}. At $i > 0$ the support can be covered up to left S_i-translation by at most $(i+1)^{\lceil (m-1-i)/2\rceil |P\cup\bar P|}$ right K_i-cosets. All choices can be made independently of the column and positions.

Proof. Stage zero has its claimed support by Corollary 350.2. Right level elements preserve the reference lattice and a Bessel element acts trivially on the successive string line quotients by Theorem 399.2, giving zero deviation for the full string. Suppose the inequalities have been proved at stage i. For a nonzero period at stage $i+1$ that same theorem finds a single source point of nonzero period at rx_i which also satisfies the one-step lattice comparison with the given target. Both points have the middle-line condition by Theorem 405.2. The source finite position is one to which the inductive bound applies, since all positions in its own prior stages are the same and the current argument of the function is unrestricted. If the target string is nonempty, use the first inequality pair of Theorem 354.1 at every target index. Its upper endpoint adds at most $\mathbf{1}_{\bar P}(w)$ to a source upper bound, and its lower endpoint subtracts at most $\mathbf{1}_P(w)$ from a source lower bound. For empty string there are no inequalities. This proves the displayed conclusion simultaneously at all places for the union W_i.

At stage zero apply Corollary 350.2 with the given J_0. For each positive stage the proof of Theorem 347.1 uses just the list of possible tuples of the early deviations. The displayed one-sided bound permits at most $i+1$ values at each of the indicated places for each early index and one value elsewhere. Thus the same construction gives the indicated number of cosets. The subgroup of elements integral off $P \cup \bar P$ contains K_i and all the chosen representatives. Applying Theorem 342.3 with the local power bounds of Proposition 344.2 gives the cuts and their stated multiplicities, also with no string indices. Multiplying any representative by a member of J_i keeps it integral at the complementary places. The cut uses the whole support union, proving the asserted uniformity. □

Corollary 406.2. *Use the full product of the cuts of Theorem 406.1. At each tuple take the tensor of the finite evaluations in Corollary 404.2, with last finite position one. Divide by the fixed nonzero model scalars of that corollary, use the standard normalized residual maps, and call the resulting finite tensor $v(t)$. With D, C_* from Theorem 369.1 the tensor*

$$w = \frac{D}{C_*} \sum_{t \in \prod_{i<m} T_i} \frac{v(t)}{\prod_{i<m} m_i(t_i)}$$

is fixed by the product of the column levels, satisfies $\mathcal{S}\mathcal{P}w = w$, and has an expansion in allowed translated sums with coefficients integral away from Q. When $P = \varnothing$ the sum of the absolute values of its coefficients after every field automorphism is at most

$$D \prod_{i=1}^{m-1} \prod_{p\in\{g_0,g_1\}} \prod_{u=0}^{n_i-1} (p^{n_i} - p^u)^f \;\le\; Dg^{f\sum_{i=1}^{m-1} n_i^2} \;\le\; (g^{f n_m^2})^{2m-1}.$$

Here $n_i = r + it$ are the original stage ranks, with t the block size in this rank formula.

Proof. The finite comparison in Corollary 404.2 identifies each normalized tensor with the finite vector of Corollary 376.3. Its right level invariance there makes the choice of representatives immaterial. Apply Theorem 371.1. The embedded column levels lie in the

respective $S_i \cap A_i$ by Proposition 349.1, so Corollary 377.2 gives column invariance and the recovery identity.

In the untwisted case $J_0 = A^{\mathrm{new}} = A_0$. The first cut can be the single identity coset, of multiplicity one by Corollary 350.2. At each later stage every multiplicity is one and the cut lies in K_i/J_i by Proposition 347.2. Its size is at most the exact marked index in that proposition. Multiplying these bounds in the sum-of-absolute-values proof of Theorem 371.1 gives the first expression. Bound each marked index by $g^{fn_i^2}$ and use $D \mid (g^{fn_m^2})^m$ and $n_i \le n_m$ for the remaining two bounds. □

407. Integrating the normalized chain over the finite cuts

Lemma 407.1. *Use the groups and measures of Theorem 343.2. Suppose f satisfies its support, rational invariance and integrability assumptions and is right J-invariant in its finite argument. Choose α of Lemma 343.1. For $t \in T$ fix a representative h_t. Then*

$$\int_{D_G} f = \frac{1}{[K:J]} \sum_{t \in T} \frac{1}{m_T(t)} \int_{G_1} \alpha(x) \int_{D_U} f\big(u(x,h_t)\big)\,\mathrm{d}u\,\mathrm{d}x.$$

Each integral on the right is absolutely convergent with $|f|$ in place of f.

Proof. The cutoff computation in Theorem 343.2 applied first to $|f|$ bounds the integral of its period against $\alpha(x)\mathbf{1}_T(hJ)/m_T(hJ)$ on the full group. It equals the prescribed finite integral of $|f|$ on D_G. The same identity holds for f by positive and negative parts. Its inner period is constant in h on each h_tJ by the right invariance. Each such coset has mass $1/[K:J]$. The displayed identity and the separate finiteness follow since T is finite and its multiplicities there are positive. □

Theorem 407.2. *Use the ramified stage functions and the cuts T_i, J_i, K_i of Theorem 406.1, taking elementary tests with the indicated finite inclusion vectors. Write $G_i = \mathrm{U}(Y_i)$, $B_i = B_{Y_i}$ and $[G_i] = G_i(E)\backslash G_i(\mathbb{A})$. Use product Haar measures on $G_i(\mathbb{A})$ with $\mathrm{vol}(K_i) = 1$ at the finite places and on the Bessel groups as in Lemma 401.1. Consider continuous automorphic functions y_i, rapidly decreasing in the sense of Proposition 398.1, with*

$$y_i(v) = [K_i : J_i] \int_{[G_{i+1}]} \theta(\Omega_i(v,h) f_i) y_{i+1}(h)\,\mathrm{d}h \qquad (0 \le i < m).$$

Here $\theta(f) = F_f(0)$ for the indicated Hom space and Ω_i is its arrow operator. At the definite terminal space every continuous automorphic function meets the decay condition. Fix any complex vector space M, a vector $V(t)$ in M for each tuple $t \in \prod_{i<m} T_i$, and a linear functional $L : M \to \mathbb{C}$. Write $\xi = (\xi_0, \dots, \xi_{m-1})$ for infinite positions. Suppose a measurable scalar function $a_m(\xi)$ satisfies, for all such tuples and positions,

$$\int_{[G_m]} P_{m,\phi}(h) y_m(h)\,\mathrm{d}h = a_m(\xi) L(V(t)),$$

where the preceding stage elements have infinite components ξ_i and arbitrary fixed finite representatives of t_i. Let $\mathcal{P}_\phi$ be the full rational periodization of $P_{0,\phi}$. There is a scalar κ, depending only on a_m, the stages, measures and fixed real cutoffs, such that

$$\int_{[G_0]} \mathcal{P}_\phi(h) y_0(h)\,\mathrm{d}h = \kappa L\left(\sum_{t \in \prod_{i<m} T_i} \frac{V(t)}{\prod_{i<m} m_i(t_i)} \right).$$

Thus κ can be kept the same across twists, columns and terminal functionals whenever a_m is the same.

Proof. For a tuple of preceding positions define

$$I_i = \int_{B_i(E)\backslash G_i(\mathbb{A})} P_{i,\phi}(h) y_i(h)\,\mathrm{d}h.$$

At $i = m$ the Bessel group is G_m. For $i > 0$, absolute convergence follows by applying Theorem 398.2 to the preceding stage with fixed positions, ramified source period and rapidly decreasing target y_i. For $i = 0$ the sum of absolute seed terms in the full periodization is bounded uniformly by Proposition 204.2 as in Theorem 204.4, after summing the finitely many column coordinates. The fixed finite translation of Lemma 349.2 preserves that bound. Grouping the seed terms by rational Bessel cosets bounds the sum of the absolute values of the corresponding translates of $P_{0,\phi}$ by the same constant. Since y_0 is integrable by Proposition 398.1, partitioning the Bessel quotient by translates of a full unitary fundamental domain gives absolute convergence and the equality of I_0 with the left side in the theorem.

For every $i < m$, the defining arrows show right J_i-invariance of y_i, because $\Omega_i(j, 1)$ fixes f_i for $j \in J_i$ by Theorem 376.2. The same theorem on the preceding arrow gives right J_i-invariance of $P_{i,\phi}$ for $i > 0$, and at zero use the base right level. The indicated integrals are consequently unchanged by the representative choices in the preceding positions, where the two adjacent arrow tests both have these level invariances. At zero the seed also has this invariance. In factoring the Bessel product measure one can take the finite measure of $S_i \cap A_i$ to be one: divide the finite factor by its positive finite volume and multiply the real factor by the same number. The original product measure is preserved. These normalizations depend on A_i, independent of the twist.

Apply Lemma 407.1 to $f = P_{i,\phi} y_i$ using Theorem 406.1 for support. Write α_i for its real cutoff on $G_i(\mathbb{R})$, with the product over the real embeddings understood. For each new position $x_i = (\xi_i, h_{t_i})$ in that formula, the inner Bessel integral is

$$\int_{[B_i]} P_{i,\phi}(r x_i) y_i(r x_i) \, \mathrm{d}r = [K_i : J_i] \, I_{i+1}.$$

Substitute the given formula for y_i and apply Theorem 398.2. The point $x_{0,f}$ selected from T_0 lies in A^{new} as required at all the subsequent stages. Thus the factor $[K_i : J_i]$ cancels exactly in the cut integral.

Proceed backward in i. For each fixed preceding $\xi_0, \ldots, \xi_{i-1}$ put $a_i = \int \alpha_i(\xi_i) a_{i+1} \, \mathrm{d}\xi_i$ if absolutely convergent, and put $a_i = 0$ otherwise. These are measurable functions by Lemma 123.1, using truncation of the integrands. Inductively the value I_i is a_i times the sum of $L(V(t))$ over all the remaining indices with their remaining reciprocal multiplicities. The terminal case is the hypothesis. For the induction step insert the formula for I_{i+1} into the preceding cut integral. For each t_i, the coefficient of a_{i+1} is a finite sum independent of ξ_i. If any such coefficient is nonzero, absolute convergence for that term, supplied by the lemma at stage i, proves $\int \alpha_i |a_{i+1}| < \infty$. If they all vanish, the desired equality holds with either definition of a_i. This proves the induction without any change of order in the infinite real integrations. At $i = 0$, a_0 is a scalar depending only on the indicated real functions and measures. Take $\kappa = a_0$ and use linearity of L. □

408. Polynomial tests for a positive hermitian oscillator

Definition 408.1. At one infinite place of a positive hermitian space of complex dimension d, take the Darboux coordinates $x + il$ of Theorem 393.1. On the real position space $\mathbb{R}^d$ put $\phi_0(t) = 2^{d/4} e^{-\pi |t|^2}$. For $1 \le j \le d$ and any polynomial P in d variables put

$$C_j = \frac{2\pi t_j - \partial_{t_j}}{\sqrt{4\pi}}, \qquad A_j = \frac{2\pi t_j + \partial_{t_j}}{\sqrt{4\pi}}, \qquad H_P = P(C_1, \ldots, C_d)\phi_0.$$

Write $H_\alpha = H_{Z^\alpha}$ for a nonnegative integer multi-index α, $|\alpha| = \sum_j \alpha_j$, and $\alpha! = \prod_j \alpha_j!$. Products of these constructions are used at distinct infinite places or on two independent oscillator spaces.

Theorem 408.2. *The H_α have inner products $\langle H_\alpha, H_\beta \rangle = \alpha! \mathbf{1}_{\alpha=\beta}$. A Schwartz function orthogonal to all of them vanishes. For $u \in \mathrm{U}(d)$ the untwisted positive oscillator satisfies*

$$\Omega_\infty(u) H_P = H_{P(u^{\intercal} Z)}.$$

For every Schwartz function $f(t,t')$, with any additional real variables t', and every integer b, its projection under scalar rotations in the first factor is

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ib\theta}(\Omega_\infty(e^{i\theta}I)\otimes 1)f\ \mathrm{d}\theta = \sum_{|\alpha|=b} H_\alpha(t)\,\frac{\langle H_\alpha, f(\,\cdot\,,t')\rangle}{\alpha!}.$$

The sum is taken to be zero for $b<0$, and every coefficient on the right is Schwartz in t'.

Proof. Integration by parts on the Schwartz space gives $C_j^* = A_j$. The product rule gives $A_iC_j - C_jA_i = \delta_{ij}$ and the C_j commute with each other. Also $A_j\phi_0 = 0$. Moving the adjoints successively through the two products in an inner product proves the displayed factorial formula: a derivative beyond the available C_j's annihilates the vacuum, and the same applies with the arguments reversed. The norm of ϕ_0 is one by the product Gaussian integral of Lemma 4.2.

For a polynomial $Q(t)$, $C_j(Q\phi_0)$ is ϕ_0 times $2\sqrt{\pi}\,t_jQ - \partial_jQ/\sqrt{4\pi}$. Inducting on the total degree shows that the span of the H_α is the set of all polynomials times ϕ_0, since the leading term corresponding to α is a nonzero multiple of t^α. If a Schwartz function f_0 is orthogonal to all these, all moments of $\phi_0 f_0$ vanish. At any real Fourier frequency expand the exponential in the Fourier transform in its power series. The absolute series inside the integral is bounded by $\phi_0(t)|f_0(t)|e^{c|t|}$ for a constant depending on the frequency, which is integrable by completing the square. Thus the Fourier transform vanishes. Inversion in Lemma 392.1 and strict positivity of ϕ_0 give $f_0 = 0$.

In the symmetric gauge the two infinitesimal translations are $D_{x_j} = \partial_j$, $D_{l_j} = -2\pi i t_j$ by Lemma 392.1. Differentiating the intertwining law is valid by its difference quotient proof. Under $u = U + iV$ the real blocks on x, l are $U, -V, V, U$. It follows that conjugation carries $D_{x_j} - iD_{l_j}$ to $\sum_k u_{kj}(D_{x_k} - iD_{l_k})$. Multiplying by $-1/\sqrt{4\pi}$ gives the same transformation on C_j. The vacuum is fixed: its normalized coefficient is one by Theorem 393.1 with empty negative block, and the operator is unitary. This proves the polynomial action.

For the circle integral, real Schwartz continuity in that theorem gives convergence of Riemann sums in each Schwartz seminorm: derivatives and polynomial multiples of the integrand are uniformly continuous as functions of the angle in their uniform norms. Their limits are the derivatives and polynomial multiples of the pointwise integral, by differentiation under the integral. The operators on the first block act with t' as a parameter by the partial Fourier construction in Theorem 395.2. At each fixed t', take the Hermite inner products of the integral. Unitarity and the polynomial action multiply the coefficient at α by the mean of $e^{i(|\alpha|-b)\theta}$, equal to $\mathbf{1}_{|\alpha|=b}$ by direct integration. The claimed finite sum has just these coefficients and the orthogonality assertion gives equality. Derivatives and polynomial multiples in t' of its coefficients are integrals against a polynomial times a Gaussian of the corresponding Schwartz derivatives, giving the Schwartz bounds. □

Proposition 408.3. *For $z \in \mathbb{C}^d$ define the Schwartz test*

$$B_z(t) = \exp(2\sqrt{\pi}\,z^{\mathsf{T}}t - z^{\mathsf{T}}z/2)\phi_0(t).$$

It has Taylor coefficient $H_\alpha/\alpha!$ at z^α. Writing $v = (x,l)$ and $\zeta = \sqrt{\pi}(x+il)$, the symmetric real translation satisfies

$$\rho_a(v)\phi_0 = e^{-|\zeta|^2/2}B_{-\zeta}, \qquad \langle \phi_0, \rho_a(v)H_P\rangle = e^{-|\zeta|^2/2}P(\bar\zeta).$$

For a definite adelic Hom space take the product of the B_z over its infinite places and any fixed finite test Φ. For $c \in \mathbb{C}$ the function of all its complex variables $\mathcal{T}_{c,\Phi}(z) = \theta(B_{cz}\otimes\Phi)$ is entire, with coefficient $c^{|\alpha|}\theta(H_\alpha\otimes\Phi)/\alpha!$. In particular the value of $P(\partial)\mathcal{T}_{c,\Phi}$ at zero is $c^b\theta(H_P\otimes\Phi)$ for homogeneous P of total degree b.

Proof. The defining exponential gives $\partial B_z/\partial z_j = (2\sqrt{\pi}\,t_j - z_j)B_z = C_jB_z$. At $z=0$ this yields the coefficients by iteration. For a bounded complex z-set, all t- and z-derivatives are bounded by a polynomial in $|t|$ times $e^{-\pi|t|^2+c_1|t|+c_2}$ for fixed constants. These estimates

justify complex differentiation in every Schwartz seminorm by applying the one-variable integral formula to a difference and its derivative. Substitution of the symmetric translation of Lemma 392.1 gives the first translation formula after expanding $(t+x)^{\mathsf{T}}(t+x)$. The inverse of $\rho_a(v)$ is $\rho_a(-v)$. Move it to the first inner product variable and apply the Taylor coefficients and orthogonality above. More explicitly, $\langle B_z, H_\alpha\rangle = \bar{z}^\alpha$ by differentiating the exponential expansion under its Gaussian integral, whose Taylor series converges on every bounded set by the same Gaussian majorant. This gives the second formula for all P.

The theta functional is continuous in the real Schwartz seminorms for fixed finite test. In rational coordinates this is the lattice bound of Proposition 388.2. The changes from the real coordinates here to those coordinates are continuous in Schwartz seminorms by their linear substitutions, quadratic phases and partial Fourier transforms in Theorem 395.2. Thus complex differentiation commutes with the functional. The preceding coefficient calculation in all variables proves the last two claims. □

409. Rectangular substitutions and the definite weights

Definition 409.1. Fix $1 \le d < d'$ and nonnegative counts $\kappa(h)$ of columns of heights $1 \le h < d'$, zero for $d < h < d'$. For $a = d, d'$, use the diagram with just its columns of height $< a$. Write $b_a(Y)$ for the column of all its semistandard minor products of Definition 76.1, ordered in any fixed order, and E_a for the complex coordinate space with those indices. An empty product is one. Write $c_0 = \kappa(d)$, $b_a^* = \sum_{h<a} h\kappa(h)$, and $\xi_a^0 = b_a(I_a)$. Use $b_{d'}(Z)$ also for a d'-by-d matrix Z: all the selected minors use only the first d columns. For square a-by-a matrices U, define $S_a(U)$ by $b_a(UY) = S_a(U)b_a(Y)$. Define the rectangular matrix $S(Z) : E_d \to E_{d'}$ by

$$b_{d'}(ZY) = (\det Y)^{c_0} S(Z) b_d(Y) \qquad (Y \text{ of size } d \text{ by } d).$$

Proposition 409.2. *The matrices in Definition 409.1 exist uniquely, have integer polynomial entries, and satisfy*

$$S(AZB) = S_{d'}(A)S(Z)(\det B)^{c_0} S_d(B), \qquad b_{d'}(Z) = S(Z)\xi_d^0.$$

Here A, B are arbitrary square matrices of sizes d', d. One has $S_a(AB) = S_a(A)S_a(B)$ for matrices of size a. Each entry of $S(Z)$ is homogeneous of degree $b_{d'}^*$. For an arbitrary formal power series F in the entries of Z, the (I', I) entry of $S(\partial)F(0)$ is the coefficient of $(\det Y)^{c_0} b_d(Y)_I$ in

$$b_{d'}(\partial Y)_{I'} F(0).$$

Here the entries of ∂ are the commuting formal derivatives with respect to the corresponding entries of Z, and Y consists of independent scalar variables.

Proof. Expand each minor of ZY by multilinearity in its intermediate indices as in Proposition 76.2. A height-d minor contributes $\det Y$ times the corresponding maximal minor of Z. For each smaller height the expansion gives a sum of minors of Z times flag minors of Y. Multiply the expansions and straighten the remaining products of Y-minors by Lemma 28.1, with the matrix transpose matching its index convention. This yields integer polynomial coefficients and their claimed common degree. Independence of the standard products, also after multiplying them all by the same nonzero polynomial, gives uniqueness. For square a this argument with its diagram below a gives S_a; successive substitutions and independence give its multiplication law and $S_a(I) = I$.

For the rectangular law apply the square substitution on the left by A. It holds for rectangular arrays as well since the products here involve no columns beyond the d-th, so one can append zero columns for this calculation. Substitution of BY for Y then multiplies its determinant by $\det B$ and replaces $b_d(Y)$ by $S_d(B)b_d(Y)$. Compare coefficients to get the first identity. Putting $Y = I$ in the definition gives the second. Finally the same identities of polynomials hold after replacing every variable Z_{bi} by ∂_{bi}, since these operators commute, as can be checked on formal monomials. Applying to F at zero gives the displayed coefficient

identity. The determinants in this expression are exactly the column determinants of the differential construction in Definition 28.3, up to transposing each small determinant, which preserves its value. □

Proposition 409.3. *For the definite ranks write $d_j = k + j(t+1)$, $0 \le j \le m$, where the last rank includes the extra line in Proposition 365.1. At each place τ take the column counts of Definition 26.1 on $d_j \to d_{j+1}$. Set*

$$e_{0,\tau} = 0, \qquad e_{j,\tau} = \lfloor d_{j-1}/2 \rfloor - w_j(\tau) \quad (1 \le j \le m).$$

Take their tensor products $S(Z) = \bigotimes_\tau S_\tau(Z_\tau)$ and $E_j = \bigotimes_\tau E_{d_j,\tau}$, and define the action on E_j for $g = (g_\tau)_\tau$ in the definite real group by

$$R_j(g) = \bigotimes_\tau (\det \bar{g}_\tau)^{e_{j,\tau}} S_{d_j,\tau}(\bar{g}_\tau).$$

For every $A \in E_{j+1}^\vee$, $v \in E_j$, the polynomial $P_{A,v}(Z) = AS(Z)v$ gives the real Hermite test of Definition 408.1. For the actual twisted definite arrow the transformation is

$$\Omega^0_{j+1,\infty}(g,h) H_{P_{A,v}} = H_{P_{AR_{j+1}(h)^{-1},\, R_j(g)v}}.$$

At stage zero E_0, R_0 identify with the column representation of Definition 14.3.

Proof. At every sufficiently large arithmetic scale one has $k > 1$ and $h_{\mathrm{gap}} > t+1$, where $h_{\mathrm{gap}} = 2r^8$ denotes the integer called h in Definition 12.3, by Lemma 26.2. At $j = 0$, the new full-height count $c_\tau = \kappa_\tau(d_j)$ is $w_1(\tau) - \lfloor d_1/2 \rfloor$. At $j \ge 1$ it is $h_{\mathrm{gap}} - (t+1)$. Since $d_{j+1} - d_{j-1} = 2(t+1)$ and $w_{j+1} - w_j = h_{\mathrm{gap}}$, in both cases

$$e_{j,\tau} = \lfloor d_{j+1}/2 \rfloor - w_{j+1}(\tau) + c_\tau, \qquad e_{j+1,\tau} = \lfloor d_j/2 \rfloor - w_{j+1}(\tau).$$

There are no columns with an intermediate height, so the preceding proposition applies. At $j = 0$ its evaluation coordinates are the monomials of degree $a\mathbf{1}_A(\tau)$ in the first column, exactly those in Definition 14.3.

On the positive Hom space an orthonormal matrix unit E_{bi} transforms to $hE_{bi}g^{-1}$. The substitution $u^{\mathsf{T}}Z$ in Theorem 408.2, written with the two matrix indices, is thus $h^{\mathsf{T}}Z\bar{g}$ at each infinite place, since the definite g is unitary. The additional character of Theorem 396.1 multiplies by

$$\prod_\tau (\det h_\tau)^{\lfloor d_j/2 \rfloor - w_{j+1}(\tau)} (\det g_\tau)^{-\lfloor d_{j+1}/2 \rfloor + w_{j+1}(\tau)}$$

by Definition 376.1 and Proposition 373.2. Now use Proposition 409.2 on $S(h^{\mathsf{T}}Z\bar{g})$. Its full-height factor contributes $(\det \bar{g}_\tau)^{c_\tau}$, and $h_\tau^{\mathsf{T}} = \bar{h}_\tau^{-1}$. Combining with the two exponent identities gives exactly the stated polynomial action. □

410. The prescribed source representation in the definite oscillator

Lemma 410.1. *With $d < d'$ and κ of Definition 409.1, suppose a polynomial p on d'-by-d matrices satisfies*

$$p(Zb) = \prod_{h \le d} (b_{11} \cdots b_{hh})^{\kappa(h)} p(Z)$$

for every invertible upper triangular d-by-d matrix b. There exists a unique row $A \in E_{d'}^\vee$ with $p(Z) = Ab_{d'}(Z)$. For a product of matrix spaces at several places the analogous independent covariance in each argument gives the tensor product of these row spaces.

Proof. Extend p to square d'-by-d' matrices by using just the first d columns. On the right upper triangular group of size d' it transforms according to the same weight, since the first d columns of a product with an upper triangular matrix depend only on the first d columns. Let $\mathcal{D}$ be the space of all square-matrix polynomials with this upper triangular transformation. Right multiplication by a scalar implies that each is homogeneous of degree $\sum_{h \le d} h\kappa(h)$, by comparing powers of the scalar. Thus $\mathcal{D}$ is finite-dimensional. It is stable

under the action $f(X) \mapsto f(u^{\mathsf{T}}X)$. The span of the coordinates of $b_{d'}$ belongs to $\mathcal{D}$ and is invariant by Proposition 76.2. The entries are independent by that same proposition.

Give the homogeneous polynomials the Hermitian inner product obtained by integrating $\bar{f}g$ against the positive Gaussian density $e^{-\operatorname{tr}(X^*X)}$ on all complex matrices. All polynomial moments are finite by elementary real Gaussian integration as in Lemma 4.2, and a nonzero polynomial is nonzero on an open set, so this form is positive definite. It is invariant under the indicated left substitutions for $u \in \mathrm{U}(d')$. If the stated span were proper, its orthogonal complement inside $\mathcal{D}$ would therefore be nonzero and compact-group invariant. It is invariant under the full complex group by the differentiation and elementary shear argument in Theorem 77.2, which applies to this finite polynomial substitution representation. As in that theorem, take a lexicographically largest diagonal weight occurring in the complement. Diagonal interpolation selects a vector in that weight space, and every strictly upper elementary root raises the weight. Thus the upper unitriangular group fixes this vector under the left transpose action.

Any vector in $\mathcal{D}$ fixed under that action is determined by its value at I. On the dense set $X = lb$, where l is lower unitriangular and b upper triangular invertible, its value is the given diagonal character of b times its value at I. Existence and density of this factorization are proved by elimination of leading principal minors in Theorem 77.2. The product of the leading minors of heights h with multiplicities $\kappa(h)$ belongs to the stated span, is fixed on the left by lower unitriangular substitutions, and is one at I. A second fixed vector in the orthogonal complement is impossible. It follows that $\mathcal{D}$ equals the span. Restricting back to the first d columns, which suffice for every minor in it, proves existence and uniqueness. For multiple places expand in the monomials of all but the first matrix. Their coefficients satisfy the first-place law and hence expand in its independent minor products. The remaining coefficient polynomials satisfy the other laws by that independence. Iteration proves the product assertion. □

Theorem 410.2. *Fix an actual definite arrow $X_j \to X_{j+1}$ and its prescribed R_j of Proposition 409.3. Let L be a complex linear map from E_j to the real Schwartz space of the arrow's position variables, with optional additional real variables t' acted on trivially. Suppose*

$$L(R_j(g)v) = \Omega^0_{j+1,\infty}(g,1)L(v) \qquad (g \in \mathrm{U}(X_j)(\mathbb{R}),\ v \in E_j).$$

Here and below the real unitary group includes all archimedean factors. There is a unique row $A(t') \in E_{j+1}^{\vee}$ of Schwartz functions such that

$$L(v)(\,\cdot\,, t') = H_{P_{A(t'),v}} \qquad (v \in E_j).$$

With no additional variables A is an ordinary constant row. Conversely all these maps satisfy the displayed intertwining law.

Proof. Fix a real place, write $d = d_j, d' = d_{j+1}$, and let $e_b = \lfloor d'/2 \rfloor - w_{j+1}(\tau)$. For a scalar $g_\tau = \eta I_d$ of modulus one, the action on E_j is $\eta^{-b^*_d - de_{j,\tau}}$, by homogeneity of the basis products. Its action on Hermite degree a is η^{-a-de_b} by Theorem 408.2 and the arrow character in Proposition 409.3. The assumed equation and the circle projection of that theorem therefore force the degree in this factor to be

$$a = b^*_d + d(e_{j,\tau} - e_b) = b^*_d + d\kappa_\tau(d) = b^*_{d'}.$$

Apply this successively at every place, regarding all other variables as additional coordinates. It follows that $L(v)(\,\cdot\,, t') = H_{R(Z;t')v}$ for a uniquely determined row R of polynomials in the matrices Z_τ, with Schwartz coefficients in t'. Indeed there are only finitely many allowed multidegrees, and application to a basis of E_j determines each entry.

Fix t', suppress it, and act in one place at a time. The Hermite transformation law and its injectivity from Theorem 408.2 yield

$$R(Zu) = (\det u)^{\kappa_\tau(d)} R(Z) S_d(u), \qquad u \in \mathrm{U}(d).$$

Here $u = \bar{g}_\tau$, and the square action in the right hand side is on its own tensor factor. The identity extends to all $u \in \mathrm{GL}_d(\mathbb{C})$. For skew Hermitian D, substitute $\exp(tD)$. Both sides are entire in t and agree for real t, hence for complex t by Lemma A.3. This gives it on positive Hermitian invertible matrices by diagonalizing them and taking the logarithms of their positive eigenvalues. Every invertible matrix is a unitary times such a matrix by the diagonalization of u^*u used in Theorem 394.2. The multiplication law of Proposition 409.2 gives the extension.

Set $\xi^0 = \bigotimes_\tau b_{d_j}(I)$. For an invertible upper triangular b we have

$$S_d(b)b_d(I) = b_d(b) = \prod_{h<d} (b_{11} \cdots b_{hh})^{\kappa_\tau(h)} b_d(I).$$

Indeed the only nonzero minor in the first h columns of an upper triangular matrix with h chosen rows uses exactly the first h rows. Hence $p(Z) = R(Z)\xi^0$ satisfies all the hypotheses of Lemma 410.1. It equals $Ab_{d'}(Z)$ in the tensor product notation, for a unique row A. By Proposition 409.2 this is $AS(Z)\xi^0$. The vectors $\bigotimes_\tau S_d(u_\tau)b_d(I)$ for invertible u_τ span E_j: a linear relation against all these evaluation vectors would be a linear relation among independent products of the basis minors, by Proposition 76.2 and its polynomial continuation to singular matrices. Covariance under all these substitutions now gives $R(Z) = AS(Z)$ as rows on E_j. Finally the coefficients of A are fixed linear combinations of the finitely many coefficients of p, by solving in the independent minor basis. Restoring t' shows that they are Schwartz. Uniqueness and the converse follow from the same evaluation and Proposition 409.3. □

411. The finite class sum of the definite differential kernel

Definition 411.1. For $0 \le j \le m$ let H_j be the unitary group of the positive hermitian space X_j of Proposition 409.3. Write $H_{j,f}$ for its finite adelic group, $H_{j,\infty}$ for its product of real groups and Γ_j for its rational points over E^+. A complex algebraic modular form of weight R_j is a function $\phi : H_{j,f} \to E_j$ fixed by right translation under some compact open subgroup and satisfying $\phi(\gamma_f g) = R_j(\gamma_\infty)\phi(g)$ for $\gamma \in \Gamma_j$. Write $r_a\phi(g) = \phi(ga)$. At $j < m$ normalize finite Haar measure by $\mathrm{vol}(K_{c,j}) = 1$, using the column level in Proposition 349.1.

Fix $c \in \mathbb{C}^\times$, independent of the finite twist. Put $N_j = \sum_\tau \sum_{h<d_{j+1}} h\kappa_\tau(h)$, the total degree of the rectangular matrix $S(Z)$ on this arrow. For a finite oscillator test Φ define $J_j^c(\Phi) : E_j \to E_{j+1}$ by

$$AJ_j^c(\Phi)v = \left[P_{A,v}(\partial)\mathcal{T}_{c,\Phi}\right](0) \qquad (A \in E_{j+1}^\vee,\ v \in E_j).$$

Here $P_{A,v} = AS(Z)v$ and $\mathcal{T}_{c,\Phi}$ is the entire function in Proposition 408.3. Bilinearity in A, v defines the map by taking coordinate rows. Denote the finite factor of the twisted definite arrow by $\Omega_f(g,h)$. Its finite class operator is

$$(\mathcal{L}_{j,\Phi}^c\phi)(h) = \int_{\Gamma_j\backslash H_{j,f}} J_j^c(\Omega_f(g,h)\Phi)\phi(g)\,\mathrm{d}g, \qquad h \in H_{j+1,f}.$$

Lemma 411.2. *For any of these definite groups H, its rational points Γ embed discretely in H_f, and $\Gamma\backslash H_f/J$ is finite for every compact open $J \subset H_f$. Given a compact open K of volume one and $J \subset K$, a right J-invariant, left Γ-invariant function D with values in a finite-dimensional complex vector space satisfies*

$$\int_{\Gamma\backslash H_f} D(g)\,\mathrm{d}g = \frac{1}{[K:J]} \sum_{[g]\in\Gamma\backslash H_f/J} \frac{D(g)}{|\Gamma \cap gJg^{-1}|}.$$

The quotient measure is right invariant. For the full adelic Haar measure used on $[H] = \Gamma\backslash(H_\infty \times H_f)$ there is a number $b_H > 0$, depending only on that measure and on K, with

$$\int_{[H]} F = b_H \int_{\Gamma\backslash H_f} \int_{H_\infty} F(u,g)\,\mathrm{d}u\,\mathrm{d}g.$$

The inner measure is probability Haar measure. This last identity holds for every continuous left rational invariant function F.

Proof. For a compact subset of H_f, clear a common finite denominator of its matrix entries. All infinite coordinates of a rational unitary lie in the compact group H_∞. Bounded sets in the resulting fractional integer lattice contain only finitely many elements by Lemma 13.4. The rational group therefore meets compact subsets in finite sets, and its projection to the finite matrices is injective by injectivity of the completion of E. This gives discreteness. The same lemma gives finitely many double cosets modulo the full integral compact K_0. The intersection $K_0 \cap J$ is open in K_0, and the cosets covering the compact K_0 admit a finite subcover. Passing first to this intersection and then to J gives the asserted finiteness for J.

Construct the left Haar measure as in Proposition 388.1, and rescale it to give K volume one. Over the class of gJ, left translation identifies the quotient with $(g^{-1}\Gamma g \cap J)\backslash J$. This is the free left action of a finite group of size $|\Gamma \cap gJg^{-1}|$. Choose a measurable representative for each orbit by taking its first element in a fixed lexicographic order on the countable separating residue coordinates of J. Two distinct elements are separated by a finite congruence coordinate, so comparisons are Borel sets. All translates of the representative set have equal measure and partition J. Its measure is $\mathrm{vol}(J)/|\Gamma \cap gJg^{-1}|$, with $\mathrm{vol}(J) = 1/[K : J]$ by the left coset partition. These representative pieces together give a measurable fundamental domain D_f with compact closure and finite positive measure. Two fundamental domains give the same integral of a left invariant function, by partition into their countably many intersecting rational translates as in Lemma 21.4.

For right translation by $a \in H_f$, the measure $B \mapsto \mathrm{vol}(Ba)$ is another left invariant measure. It is a constant multiple of the original one. Indeed primitives of the form hJ', for open subgroups $J' \subset K$ of finite index, have measures determined by the respective masses of K. Principal congruence subgroups in a countable nested basis suffice. Open sets are countable unions of such cosets and decompose into disjoint such cosets by successive refinement. Agreement up to scale extends to Borel measures by the construction from these basic sets, or by applying the increasing and dominated limits of Lemma A.2 to the generating algebra on each compact coset. Now $D_f a$ is another fundamental domain. Its mass equals that of D_f, forcing the constant to be one. This also proves right invariance on the quotient.

The archimedean group is compact, so its positive smooth density in Lemma 21.4 normalizes to a probability. The full definite Haar measure is a positive constant times its product with the finite measure, by the product construction in Proposition 388.1. A fundamental domain for the full rational action is $H_\infty \times D_f$, since the finite representative is unique and its stabilizer under left multiplication at a point is trivial. Integration on this domain gives the asserted formula. Invariance of the real probability makes the inner integral left rational invariant as a function of the finite variable. All continuous integrands here are bounded on the closure of the domain. □

Theorem 411.3. *The operator in Definition 411.1 is bilinear in Φ, ϕ and produces a modular form of weight R_{j+1}. For every pair of finite elements a, b on the source and target,*

$$\mathcal{L}^c_{j,\Omega_f(a,b)\Phi}(r_a\phi) = r_b(\mathcal{L}^c_{j,\Phi}\phi).$$

For $J \subset K_{c,j}$ compact open fixing ϕ on the right and fixing Φ in the source action, its value at h is the finite sum in Lemma 411.2 with $D(g) = J^c_j(\Omega_f(g,h)\Phi)\phi(g)$.

Proof. By Proposition 409.2 and homogeneity, the defining formula for the jet is $AJ^c_j(\Phi)v = c^{N_j}\theta(H_{P_{A,v}} \otimes \Phi)$, by Proposition 408.3. For rational points γ, δ on source and target, apply Theorem 396.1 to this theta value. The real transformation in Proposition 409.3 and cancellation of the arbitrary row yield

$$J^c_j(\Omega_f(\gamma_f, \delta_f)\Phi)R_j(\gamma_\infty) = R_{j+1}(\delta_\infty)J^c_j(\Phi).$$

In particular the integrand is left invariant in g and the integral transforms on the left as prescribed in h. The smoothness of the finite operators in Theorem 395.2 gives compact open subgroups on each side fixing the finite test. Equivalently, use the finite basis of translated characteristic functions: the sufficiently small congruence subgroups fix the vacuum, each of the finitely many translations modulo its open period subgroup, and the finitely many accompanying phases. The twisting characters are locally constant on the finite group by Proposition 373.2. Intersect on the source with the right stabilizer of ϕ and the column level. This gives the asserted J and a right invariance subgroup on the target. The lemma now gives the finite sum and hence the existence and bilinearity of the operator. Finally the integrand with translated pair and form is $J_j^c(\Omega_f(ga, hb)\Phi)\phi(ga)$. Right invariance of quotient measure proves the translation identity. □

412. Matrix coefficients of a definite theta lift

Definition 412.1. For the groups and weights of Definition 411.1, a row $\lambda \in E_j^\vee$ and a modular form ϕ define

$$F_{\phi,\lambda}(u, g) = \lambda\big(R_j(u)^{-1}\phi(g)\big) \qquad (u \in H_{j,\infty},\ g \in H_{j,f}).$$

This is a continuous function on $[H_j]$. Indeed a simultaneous left translation by $\gamma \in \Gamma_j$ contributes $R_j(\gamma_\infty)^{-1}R_j(\gamma_\infty)$ between the two factors. Smoothness at the finite group and continuity of the matrix representation at infinity give continuity.

Lemma 412.2. *Let $f(t, t')$ be a real-place Schwartz test on the j-th definite arrow with possible additional real variables t'. Use probability Haar measure on $H_{j,\infty}$. Given a row $\lambda \in E_j^\vee$, there is a unique row $A(t') \in E_{j+1}^\vee$ of Schwartz functions such that*

$$\int_{H_{j,\infty}} \lambda(R_j(u)^{-1}v)\Omega_\infty(u, 1)f(\,\cdot\,, t')\,\mathrm{d}u = H_{P_{A(t'),v}} \qquad (v \in E_j).$$

Without the additional variables the row has constant entries. The row is bilinear in f, λ, continuous for the Schwartz seminorms, and independent of every finite twist with the prescribed infinite weight.

Proof. For each fixed v, the integral exists as a Schwartz function. For the first variables the action is a product of the real oscillator actions of Theorem 393.1 and of the determinant characters in Proposition 409.3. It acts only on the first block in a joint Schwartz function as in Theorem 408.2. In every Schwartz seminorm the integral is a limit of Riemann sums. Indeed the compact group admits finite measurable partitions of arbitrarily small diameter by restricting finite ball covers, and continuity gives uniform approximation of the integrand in that seminorm. Equivalently, differentiated integrals have the bounds obtained by integrating the seminorms of the integrands. Joint continuity of the action gives a finite family of input seminorms bounding each output seminorm uniformly on the compact group. To see this, at each $(u, 0)$ take a product neighborhood mapped into the unit ball of that output seminorm, take a finite subcover in u, intersect the finitely many neighborhoods of zero and use linearity. The same argument on joint Schwartz functions follows from the partial linear substitutions, phases and Fourier transforms in Theorem 395.2; polynomial factors and derivatives in t' commute through them. This proves continuity of the average.

Denote the resulting linear map in v by L. For fixed $b \in H_{j,\infty}$, the substitution $u' = bu$ gives $\Omega_\infty(b, 1)L(v) = L(R_j(b)v)$. Theorem 410.2 therefore gives the stated row and its uniqueness. Its proof recovers the coefficients by finitely many integrals against fixed Hermite functions followed by a fixed finite system of linear equations. These operations are continuous in every Schwartz seminorm of the additional variables. The averaging formula is bilinear, and the operators at infinity use only the common infinite weight, proving the remaining assertions. □

Theorem 412.3. *Let Ω be any twisted definite arrow at stage $j < m$ with the prescribed infinite weight, and let f be a real-place Schwartz test on this arrow. For every $\lambda \in E_j^\vee$ there is a row $\lambda' \in E_{j+1}^\vee$ such that*

$$\int_{[H_j]} F_{\phi,\lambda}(g)\theta\big(\Omega(g,h)(f\otimes\Phi)\big)\,\mathrm{d}g = F_{\mathcal{L}^c_{j,\Phi}\phi,\,\lambda'}(h) \qquad (h \in H_{j+1}(\mathbb{A})).$$

The same row works for all modular forms ϕ, all finite data Φ, and all finite twists of this infinite weight. More precisely $\lambda' = b_{H_j} c^{-N_j} A$, with A given by Lemma 412.2.

Proof. The integrand is rational invariant by the definition of the coefficient and Theorem 396.1. It is continuous by real Schwartz continuity, finite local constancy and the lattice bound in Proposition 388.2. Its absolute value is integrable on the compact representatives of Lemma 411.2. Write $g = (u, a)$, $h = (w, b)$. At fixed finite a, b, integrate over u using the product measure formula of that lemma. In the real action $\Omega_\infty(1, w)$ commutes with $\Omega_\infty(u, 1)$. Both this operator and theta at the fixed finite test $\Omega_f(a, b)\Phi$ are continuous in the real Schwartz topology, by the same estimates. They therefore commute with the Riemann sums and the integral in Lemma 412.2, with $v = \phi(a)$. By Proposition 409.3 the result of the real integration is

$$\theta\big(H_{P_{AR_{j+1}(w)^{-1},\,\phi(a)}} \otimes \Omega_f(a,b)\Phi\big) = c^{-N_j} A R_{j+1}(w)^{-1} J_j^c\big(\Omega_f(a,b)\Phi\big)\phi(a).$$

The equality is the homogeneous derivative identity in Theorem 411.3. Integrate in a and multiply by b_{H_j}. The row and the real representation in the right side are constant in a. The definition of $\mathcal{L}^c_{j,\Phi}$ gives the claimed coefficient at h. The formula for λ' and the last assertion of the lemma give the indicated uniformity. □

413. Composition on two consecutive definite arrows

Definition 413.1. At stages $j, j+1$, put

$$\mathcal{L}^c_{j:j+2,\,\Phi_1\otimes\Phi_2}\phi = \mathcal{L}^c_{j+1,\Phi_2}\big(\mathcal{L}^c_{j,\Phi_1}\phi\big).$$

Extend linearly in the algebraic tensor product of the two finite test spaces. This prescription is well defined by bilinearity in Theorem 411.3. On a joint real Schwartz function and a finite tensor write θ_{12} for the functional of Corollary 404.2.

Theorem 413.2. *Fix any j with $j + 2 \le m$, a real Schwartz test f on the product of the two positive oscillator spaces, a scale $c \neq 0$ and $\lambda \in E_j^\vee$. There is a row $\lambda'' \in E_{j+2}^\vee$, bilinear in f, λ and continuous in f, such that for every finite tensor y, every modular form ϕ of weight R_j, and every $k \in H_{j+2}(\mathbb{A})$,*

$$F_{\mathcal{L}^c_{j:j+2,y}\phi,\,\lambda''}(k) = \int_{[H_j]}\int_{[H_{j+1}]} F_{\phi,\lambda}(g) \times \theta_{12}\big((\Omega_1(g,h)\otimes\Omega_2(h,k))(f\otimes y)\big)\,\mathrm{d}h\,\mathrm{d}g.$$

The row depends on the fixed infinite weights and measures and works simultaneously for all their finite twists.

Proof. Linearity reduces the proof to $y = \Phi_1 \otimes \Phi_2$. Both groups have compact fundamental representatives by Lemma 411.2. Simultaneous rational invariance in the middle group follows by applying Theorem 396.1 on each arrow. For joint real tests this follows as well from the rational sum in the two position variables of Corollary 404.2. The integrand is continuous on the closure of the representatives by the lattice bound of Proposition 388.2 and local constancy of the finite tests. It is absolutely integrable, so Lemma 123.1 allows us to integrate first in g.

Write $h = (w, b)$. Average in the infinite component of g, keeping the second test variables as the additional variables in Lemma 412.2. Choose any basis $\ell_1, \dots, \ell_D$ of $E_{j+1}^\vee$. Its unique

row of functions has the form $A(t') = \sum_{\nu=1}^{D} f_\nu(t')\ell_\nu$, with each f_ν a real Schwartz test on the second arrow. Thus the averaged real vector for the finite argument $v = \phi(g_f)$ is

$$\sum_{\nu=1}^{D} H_{P_{\ell_\nu, v}} \otimes f_\nu.$$

All finite operators, the real target operator on the first arrow, and the second arrow operator commute through this first real integration. Their actions and the functional are continuous on joint Schwartz tests, by the same argument as in Theorem 412.3. The expression now contains only a finite sum of product tests, on which θ_{12} is the product of the two theta functionals.

On its first factor the real target w replaces ℓ_ν by $\ell_\nu R_{j+1}(w)^{-1}$, by Proposition 409.3. The calculation of the finite integral in Theorem 412.3 now gives

$$b_{H_j} c^{-N_j} \sum_{\nu=1}^{D} F_{\mathcal{L}^c_{j,\Phi_1}\phi,\, \ell_\nu}(h)\, \theta(\Omega_2(h,k)(f_\nu \otimes \Phi_2))$$

after integrating over $[H_j]$. Integrate each summand over $[H_{j+1}]$ using the same theorem with f_ν, ℓ_ν. If its row is λ'_ν, take $\lambda'' = b_{H_j} c^{-N_j} \sum_\nu \lambda'_\nu$. Construction by these two averages shows bilinearity and continuity by Lemma 412.2. The row construction uses neither the finite tensor, the form, the twist nor the target point. It therefore proves the result on their stated ranges, including arbitrary finite tensor sums. □

Corollary 413.3. *Equip all real tests in the successive periods of Theorem 399.2 with fixed infinite data. Write $x_j = (\xi_j, a_j)$ for the stage positions, with $a_0 \in A^{\mathrm{new}}$. Let $0 \le 2i \le m$. For each of the i pairs, take its evaluated finite tensor y_j in Corollary 404.2, where $j = 0, 2, \dots, 2i-2$. For a column ϕ put*

$$\psi_0 = \phi, \qquad \psi_{j+2} = \mathcal{L}^c_{j:j+2,\, y_j}\psi_j.$$

Then there is a row $t_{2i} \in E^\vee_{2i}$, depending only on the real positions through ξ_{2i}, the chosen real tests, c, and the fixed models and measures, such that

$$P_{2i,\phi}(\widehat{k}x_{2i}) = F_{\psi_{2i}, t_{2i}}(k_0).$$

For $2i < m$, k runs over the adelic definite group on X_{2i}, $\widehat{k}$ extends it by the identity on the string and $k_0 = k$. At $2i = m$, k runs over the adelic unitary group on Y_m, $\widehat{k} = k$, and $k_0 = k \oplus 1$ on X_m. Each row can be chosen continuously in the real positions, independently of the finite positions, twist and column.

Proof. At zero take the row $v \mapsto \mathcal{B}(v, F_\infty(\xi_0))$ of Lemma 405.1. Its invariant pairing moves $\pi(k_\infty)$ in that lemma to $R_0(k_\infty)^{-1}$ on the first argument. This gives Definition 412.1 and proves the initial case. The base's finite position hypothesis is exactly $a_0 \in A^{\mathrm{new}}$.

Suppose the formula holds at an even $j < m$. In the definite pair formula of Corollary 404.2 substitute $F_{\psi_j, t_j}(g)$ for $p(gx_j)$. The test is a joint real Schwartz function times the evaluated finite tensor y_j. A common nonzero scalar for the finite model identification, if its normalization requires one, can be included in the real test. Theorem 413.2 evaluates this period with $\phi = \psi_j, \lambda = t_j$ as the asserted coefficient at k_0. The target embedding including the last line is precisely that in Proposition 404.1. Real translation, restriction to a coordinate subspace, linear substitutions and partial Fourier transform in the pair construction are continuous on Schwartz functions by Theorems 400.1 and 395.2. The continuous dependence and twist independence of the row therefore follow at the next stage from the theorem. This proves the claim by induction. □

414. An integral operator with image the flag representation

Definition 414.1. Let $V = \mathbb{C}^n$ with its ordered orthonormal basis. A diagram α is a list $h_1 \geq \cdots \geq h_a \geq 1$ of column heights at most n, possibly empty. Put $L = \sum_i h_i$, $T_\alpha(V) = \bigotimes_i \bigwedge^{h_i} V$, with increasing basis wedges orthonormal. For an ordered array $z_1, \ldots, z_n$ of vectors in V, regarded as columns of a matrix Z, put

$$v_\alpha(Z) = \bigotimes_{i=1}^{a} (z_1 \wedge \cdots \wedge z_{h_i}), \qquad W_\alpha(V) = \operatorname{span}_{\mathbb{C}}\{v_\alpha(Z) : Z\}.$$

The empty tensor is the scalar one. Use semistandard evaluation coordinates $b_\alpha(Z)$ as in Definition 409.1, allowing columns of full height to contribute powers of $\det Z$.

Lemma 414.2. *There is a unique integer matrix I_α satisfying $I_\alpha b_\alpha(Z) = v_\alpha(Z)$. It identifies the evaluation coordinate space with $W_\alpha(V)$ and admits an integer left inverse given by coordinate selection. The squared norm on evaluation coordinates is $x^* G_\alpha x$ with $G_\alpha = I_\alpha^{\mathsf{T}} I_\alpha$. It is uniquely determined by*

$$b_\alpha(Y)^* G_\alpha b_\alpha(Z) = \prod_{i=1}^{a} \det\bigl((Y_{\leq h_i})^* Z_{\leq h_i}\bigr),$$

where $Z_{\leq h}$ consists of the first h columns. The action on $W_\alpha(V)$ by the exterior powers of g identifies in these coordinates with $S_\alpha(g)$, the matrix determined by $b_\alpha(gZ) = S_\alpha(g) b_\alpha(Z)$. It is unitary on $U(n)$ and irreducible there.

Proof. Expanding a wedge in increasing basis wedges gives exactly its row minors. Thus every tensor coordinate is a product of flag minors. Lemma 28.1 expresses each product as an integral linear combination of the semistandard ones. Each column of full height is already a determinant and may be retained as a common factor throughout straightening. Semistandard products are independent by Proposition 76.2, including this common factor. Their evaluation vectors therefore span the whole coordinate space. This proves existence and uniqueness, and selecting the semistandard coordinates among all products gives the left inverse. The induced norm consequently has the displayed integer Gram matrix. In a single wedge its polarized inner product is the determinant of the scalar inner products, by multilinear expansion in an orthonormal basis as in Proposition 76.2. Multiplication over the factors gives the displayed kernel, whose two sets of evaluation vectors span and thus determine the matrix.

Applying g to each column of Z proves the action formula. Exterior powers of a unitary matrix are unitary by the determinant inner product. Invertible Z suffice to span W_α, since each $Z + tI$ is invertible except at finitely many t and the tensor entries are polynomials in t. Thus it is the span of the general linear orbit of the standard tensor. This coincides with the span of its unitary orbit by the complexification argument of Theorem 77.2, which also proves irreducibility. Full-height columns contribute one-dimensional determinant factors to that theorem. □

Theorem 414.3. *There is an integer endomorphism D_α of the exterior tensor space $T_\alpha(V)$, in its increasing wedge bases, with the following properties. It commutes with the general linear action and is a positive scalar times orthogonal projection onto $W_\alpha(V)$. Its operator norm satisfies*

$$\|D_\alpha\| \leq L!.$$

If $z_i \in \bigwedge^{h_i} V$ are any nonzero decomposable wedges, $D_\alpha(\bigotimes_i z_i) \neq 0$. For the empty diagram $D_\alpha = 1$.

Proof. Label the L tensor slots by the boxes of the diagram. Let $I : T_\alpha(V) \to V^{\otimes L}$ send each increasing wedge to the signed sum of its ordered tensor permutations, and tensor these maps over the columns. Its Hermitian adjoint I^* sends a pure tensor to its columnwise wedges.

This follows by testing on the orthonormal basis words, including words with repeated letters in a column. Both maps have integer entries and commute with simultaneous linear changes of variables. The nonzero words in distinct increasing wedge basis images are disjoint and each occurs with coefficient of modulus one. Thus $\|I\| = \|I^*\| = (\prod_i h_i!)^{1/2}$.

Let $\mathfrak{R}$ be the subgroup of the permutations of the L slots which separately permutes each row. Set $D_\alpha = I^*(\sum_{\pi\in\mathfrak{R}} \pi)I$. The sum divided by $|\mathfrak{R}|$ is a self-adjoint idempotent, by inversion and multiplication in the finite group. Thus D_α is positive semidefinite, integral, and commutes with the diagonal general linear action. Its norm is at most $|\mathfrak{R}| \prod_i h_i!$. The column permutation group $\mathfrak{C}$ has order $\prod_i h_i!$, and $\mathfrak{R} \cap \mathfrak{C} = \{1\}$ because each row and column meet in at most one slot. The products of a row and a column permutation are therefore all distinct, giving the bound $L!$.

A symmetrized tensor in s slots is spanned by powers $z^{\otimes s}$. Indeed the sum over subsets $J \subset \{1, \dots, s\}$ of $(-1)^{s-|J|}(\sum_{j\in J} z_j)^{\otimes s}$ selects exactly the words using each z_j once, with labels counted even when the vectors coincide. Applying this argument to each row shows that the row symmetrizer has image spanned by tensors whose row r is filled by repetitions of a single vector z_r. The map I^* sends every such tensor to $v_\alpha(Z)$, so D_α has image in W_α. It is nonzero there. For the standard highest tensor $v_\alpha(I_n)$, the word with all boxes in row r filled by the r-th basis vector occurs in $Iv_\alpha(I_n)$ with coefficient one. This word is fixed by the row group, so the same coefficient in its row average is one. Hence the quadratic form of D_α on the highest tensor is positive.

By Lemma 414.2, an eigenspace of this commuting self-adjoint operator on W_α is a unitary invariant subspace of an irreducible space. Consequently D_α is a positive scalar there. Self-adjointness and the inclusion of its image in W_α make it zero on the orthogonal complement. Finally, the inner product of $\bigotimes_i z_i$ against $v_\alpha(Z)$, taken linear in $v_\alpha(Z)$, is a product of polynomials in Z. Each polynomial is nonzero since an appropriate list of basis columns detects a nonzero wedge coordinate of z_i. Their product is nonzero since polynomial rings over $\mathbb{C}$ have no zero divisors, as follows by multiplying their leading monomials. It takes a nonzero value by the polynomial identity argument of Proposition 76.2. This shows that the tensor has nonzero projection as asserted. □

415. Branching with the cost confined to the tall columns

Lemma 415.1. *Let $U \subset V$ have dimension u. Suppose every height of a diagram β is at least u, and let a_β be its number of columns. Write $d(t) = t\,\mathrm{id}_U \oplus \mathrm{id}_{U^\perp}$ for positive t. The subspace of $W_\beta(V)$ on which $d(t)$ acts by t^{ua_β} for all t is spanned by flag tensors whose column spaces all contain U. If a tensor in T_β has that weight, its image under D_β lies in this span.*

Proof. In an orthonormal basis adapted to U, the exponent on a basis wedge counts the indices from U. Thus u is the largest possible exponent in each column, and $t^{-ua_\beta} d(t)$ on the tensor space tends as $t \to \infty$ to the projection onto the stated exponent. It preserves W_β by Lemma 414.2. Since the flag tensors span W_β, their limits span its part of maximal exponent.

Consider an invertible flag array with subspaces F_h spanned by its first h columns. If the orthogonal projection of F_h onto U fails to have rank u at a used height, the maximal component of that wedge is zero, by expansion of the u-row minors. Otherwise choose a basis of F_h projecting in its first u entries to a basis of U, and take the remaining vectors in $F_h \cap U^\perp$. Its maximal component is a nonzero wedge spanning $U \oplus (F_h \cap U^\perp)$. These spaces are nested as h increases. Choosing a single ordered basis adapted to the resulting chain identifies the product of the limiting wedges with a scalar multiple of a flag tensor. Invertible arrays suffice by the preceding lemma on tensor coordinates. Conversely a flag through U already has maximal weight in each factor. The last assertion follows from equivariance and the image statement in Theorem 414.3. The argument includes the empty diagram and $u = 0$. □

Theorem 415.2. *Write $V^+ = V \oplus \mathbb{C}e_*$, with e_* a last orthonormal basis vector and $\dim V = n \geq 1$. At each height $1 \leq h \leq n$, let a source diagram α have $c_h + r_h$ columns. Retain c_h of them and shorten r_h by one box, dropping the columns of resulting height zero. The target diagram β has $c_h + r_{h+1}$ columns of height h, where $r_{n+1} = 0$. Fix $1 \leq q \leq n$ such that $c_h = 0$ for $h < q$. Define*

$$L = \sum_{h=q}^{n} h(c_h + r_{h+1}), \qquad R = \sum_{h=1}^{n} r_h.$$

There is a nonzero integer matrix B between the semistandard evaluation coordinates of α, β with $\|B\| \leq L!$ for the norms of Lemma 414.2. For every field of characteristic zero, every $g \in \mathrm{GL}_n$ over that field and every nonzero scalar s there, it satisfies

$$BS_\alpha(\mathrm{diag}(g, s)) = s^R S_\beta(g)B.$$

The tensor product of these maps over finitely many places has norm at most $(\sum_\tau L_\tau)!$.

Proof. Let $p : V^+ \to V$ forget the last coordinate, and let λ_* take that coordinate. In each retained tensor factor apply $\bigwedge^h p$. In each shortened one contract the wedge by λ_* and project its remaining factors by p. On an increasing basis wedge this second map gives zero when the last basis vector is absent, and otherwise gives the signed wedge with that vector deleted. Both operations therefore have integer matrices and norm at most one, since distinct surviving basis wedges in each factor have distinct images up to sign. The contractions at height one yield scalar factors. Permute the surviving tensor factors into a fixed order by height. Each contraction contributes s under $\mathrm{diag}(g, s)$; each other operation commutes with g. This establishes the block transformation law on the full tensor spaces before the next operation.

Call a resulting factor tall when its height is at least q, and apply $D_{\beta_{\mathrm{tall}}}$ only to their product. This integer operator commutes with g, and its norm is at most $L!$ by Theorem 414.3. It acts with the same bound in the presence of the remaining factors by expanding in an orthonormal basis there and summing squared norms. We prove that the resulting map takes $W_\alpha(V^+)$ into $W_\beta(V)$.

Take flags $F_1 \subset \cdots \subset F_n$ in V^+ for which p has full rank on F_n and λ_* is nonzero on F_1. Such flags suffice to span: these conditions are nonvanishing of polynomial coordinates of an array, and they hold, for example, on the columns $e_1 + e_*, e_2, \ldots, e_n$. They hold on a nonempty Euclidean open set of arrays, so a polynomial vanishing there is identically zero by successively varying the entries in small disks. Apply this to any linear functional on the span of flag tensors. For these flags every retained wedge spans pF_h, and every shortened wedge of positive height spans $F_h \cap V$. Indeed take the first basis vector of F_h with λ_*-value one and the others in its kernel, and contract that basis wedge. All these wedges and the height-zero scalars are nonzero.

Put $U = F_q \cap V$, of dimension $q - 1$. Every short positive-height factor comes from shortening a column with $h \leq q$, and therefore lies on the nested flag $F_h \cap V$ inside U. Every tall factor contains U. For retained ones this uses $pF_h \supset F_q \cap V$ when $h \geq q$. For shortened ones it follows from $h > q$ and nesting. The tall tensor consequently has maximal U-weight as in Lemma 415.1. Its image under $D_{\beta_{\mathrm{tall}}}$ is a linear combination of flag tensors through U. Each such flag extends the short flag inside U, so inserting the fixed short wedges produces a scalar multiple of a full flag tensor. In detail, choose a basis successively on the short subspaces, complete it in U, then in the successive tall subspaces, and finally in all of V. Each prescribed wedge differs by a scalar from its basis wedge. This proves the inclusion, also when either group of factors is empty. The image of a generic flag is nonzero: the tall wedges are nonzero and have nonzero image together by the last assertion of Theorem 414.3, and the short factors and scalar factors are all nonzero.

Precompose by I_α and postcompose by the integral coordinate selection for I_β in Lemma 414.2. These changes are isometries on the respective flag spaces in the specified norms. The resulting nonzero matrix B is integral and satisfies the claimed norm bound. On complex invertible matrices the asserted equivariance follows from the construction. The two sides have integer polynomial entries in g, s; the identities therefore hold as polynomial identities, by the open-set argument above. This proves the assertion over each stated field. Finally the tensor product norm is at most the product of the bounds by successive applications to orthonormal factor expansions. One has $\prod_\tau L_\tau! \le (\sum_\tau L_\tau)!$ by the integral multinomial coefficient. □

416. Multiplicity and the weight on the terminal subgroup

Lemma 416.1. *Let K be a compact unitary group, or a finite product of such groups, with bi-invariant probability measure, and let S, T be finite-dimensional continuous unitary representations. Write χ_S, χ_T for their traces. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_K(S,T) = \int_K \overline{\chi_S(k)}\, \chi_T(k)\, \mathrm{d}k.$$

For the flag representations of Lemma 414.2 at one place, twisted by integral powers of the determinant, the right side is one for equal highest weights and zero for distinct highest weights. The same assertion holds for independent tensor products of the indicated types.

Proof. On the space of linear maps average $L \mapsto T(k) L S(k)^{-1}$. Left invariance of Haar measure shows that its image consists of intertwiners, and every intertwiner is fixed, so the average is an idempotent with trace its image dimension. The trace of the transformation on maps is $\chi_T(k) \operatorname{tr}(S(k)^{-1})$, by expansion in matrix units. Unitarity identifies the second trace with $\overline{\chi_S(k)}$, giving the formula.

At one place the representations are irreducible by Theorem 77.2. A nonzero intertwiner between two irreducibles is an isomorphism by its kernel and image. It extends its equivariance to invertible complex matrices by differentiating along the skew Hermitian matrices, complexifying and exponentiating, as proved in that theorem. It consequently preserves the upper unitriangular fixed line and the diagonal weight on that line. The weights must coincide. Conversely equal weights have the same exterior model, including the common determinant power. Endomorphisms are scalar since any eigenspace is invariant. For the conjugate actions $g \mapsto S(\bar{g})$, replace g by $\bar{g}$ throughout, an automorphism preserving the probability by Proposition 77.3. For product groups the integrals factor by Lemma 123.1; the multiplicities are the products of the local ones. □

Proposition 416.2. *Use the diagrams of Theorem 415.2. Let $\lambda_i = \sum_{h \ge i}(c_h + r_h)$ for $1 \le i \le n+1$ and $\mu_i = \sum_{h \ge i}(c_h + r_{h+1})$ for $1 \le i \le n$, where all undefined counts are zero. Under $U(n)$ embedded by $g \mapsto \operatorname{diag}(g, 1)$, the space of intertwiners from $W_\alpha(V^+)$ to $W_\beta(V)$ has dimension one and is spanned by the integral matrix B in that theorem. This holds after twisting both representations by the same determinant exponent, conjugating the entries of both unitary arguments, and taking products of these constructions at the infinite places.*

Proof. One has $\mu_i = \lambda_i - r_i$, so $\lambda_i \ge \mu_i \ge \lambda_{i+1}$ for $1 \le i \le n$. Remove the boxes with letter $n+1$ from any semistandard tableau of shape λ. Each is the bottom box of its column and all removed boxes are rightmost in their rows. The remaining row lengths ν_i satisfy $\lambda_i \ge \nu_i \ge \lambda_{i+1}$ because there were no two removed boxes in the same column. Conversely each such diagram and each semistandard filling of it with $1, \dots, n$ uniquely extends by filling the removed boxes with $n+1$. This is the removal bijection of Lemma 76.3. By the tableau character in Theorem 80.1, evaluating at $\operatorname{diag}(g, 1)$ gives

$$\chi_\alpha(\operatorname{diag}(g,1)) = \sum_{\lambda_i \ge \nu_i \ge \lambda_{i+1}} \chi_\nu(g).$$

Indeed it holds on every diagonal unitary by the bijection, and on every unitary by orthogonal diagonalization. Full height n on the right is allowed by a determinant factor. Use the preceding lemma on this identity with χ_μ. Exactly one summand contributes. Interchanging source and target in the character integral conjugates it and hence gives the same integer multiplicity. Theorem 415.2 supplies a nonzero map in the required direction, so it spans. A common determinant power agrees on the embedded subgroup, and conjugating the arguments preserves all the identities. The product assertion follows from the last part of the lemma and nonvanishing of the tensor product of the maps. □

Definition 416.3. For the actual last space put $n = d_m - 1 = \dim_E Y_m$, identifying $X_m = Y_m \oplus E$ in the terminal definite frame of Proposition 365.1. Let $\iota(g) = \mathrm{diag}(g, 1)$ in this frame, with the ordered coordinates of Y_m used as the first n coordinates. For $1 \le h \le n$ retain

$$c_h = \begin{cases} h_2 - \lfloor (t+1)/2 \rfloor, & h = k + i(t+1), \quad 0 \le i < m, \\ 0, & \text{otherwise}, \end{cases}$$

where $t = 5r$, $h_2 = r^8$. Write $r_{\tau,h} = \kappa_\tau(h) - c_h$ and $e_\tau = e_{m,\tau}$ of Proposition 409.3. On the definite group of Y_m let R_Y be the product of $S_{\beta_\tau}(\bar{g}_\tau)(\det \bar{g}_\tau)^{e_\tau}$, where β_τ has columns of counts $c_h + r_{\tau,h+1}$ with $r_{\tau,n+1} = 0$. Denote its space by E_Y. Put

$$b_\tau = e_\tau + \sum_{h=1}^{n} r_{\tau,h}, \qquad a_N = \prod_{\tau \in \Phi} \tau(\bar{\pi}_N)^{-b_\tau}.$$

The symbol b_τ here denotes a branching exponent; the box counts in Definition 26.1 are used only in the estimates for the individual arrows. For a modular form ψ of weight R_m on X_m define

$$(\mathcal{R}\psi)(h) = a_N B \psi(\iota(h)) \qquad (h \in U(Y_m)(\mathbb{A}_f)),$$

where B is the product of the integral branching maps constructed with cutoff $q = k$.

Theorem 416.4. *For all sufficiently large scales the counts in Definition 416.3 give branching maps as stated there. The space* $\mathrm{Hom}_{U(Y_m)_\infty}(E_m, E_Y)$ *is spanned by B, and $\mathcal{R}\psi$ is a modular form of weight R_Y. One has $b_\tau \ge \kappa_\tau(1) - 2b$ at every infinite place, where b is the scale of Definition 12.3.*

Proof. For large scales, $k > 1$ and $h_2 > t + 1$, by Lemma 26.2 and $t = 5r$, $h_2 = r^8$. At k, Definition 26.1 gives $\kappa_\tau(k) = w_1(\tau) - \lfloor d_1/2 \rfloor$. Here $w_1(\tau) \ge 3h_2$ by Definition 12.3 and $d_1 = k + t + 1 = O(r)$, so this exceeds c_k. At d_i for $1 \le i < m$ the count is $2h_2 - (t+1)$, at least c_{d_i}. At every other height the retained count is zero. None of the used heights equals n: the largest possible source height apart from height one is $d_{m-1} = n - t < n$. In particular the target has no full-height columns and has determinant shift e_τ as written. Theorem 415.2 applies with $q = k$ and Proposition 416.2 gives the claimed span.

The block law at $s = 1$, including the common determinant factor, gives $BR_m(\iota(\gamma_\infty)) = R_Y(\gamma_\infty)B$ for rational γ. Therefore $\mathcal{R}\psi$ is left covariant. On the right it is invariant under the inverse image of the open stabilizer of ψ, which is open by continuity of the block inclusion and contains a compact open subgroup. This proves it is a modular form. Finally every column of height one is shortened since $k > 1$. Also $e_\tau = \lfloor d_{m-1}/2 \rfloor - w_m(\tau) \ge -2b$ by Proposition 12.4. Summing the nonnegative shortened counts proves the lower bound. □

417. The height and the finite gauge of the terminal restriction

Theorem 417.1. *Denote the operator $\mathcal{R}$ of Definition 416.3 by* Res_N. *Uniformly on the windows of Theorem 40.2, for all sufficiently large x, every field embedding $\sigma : \mathbb{C} \to \mathbb{C}$, every modular form ψ on X_m of weight R_m, and every finite argument h on Y_m, one has*

$$\left\| \sigma\big((\mathrm{Res}_N\, \psi)(h)\big) \right\|^2 \le \exp\left(x^{59} - \sum_\tau b_\tau \log N \right) \left\| \sigma(\psi(\iota(h))) \right\|^2.$$

In this formula embeddings act entrywise in the evaluation coordinates, and both norms are the Gram norms of Lemma 414.2. More exactly, x^{59} in the exponent can be replaced by $2\log L!$, where

$$L = \sum_{\tau} \sum_{s=k}^{n} s(c_s + r_{\tau,s+1}).$$

Proof. All entries of B are integers by Theorem 415.2. The entries therefore commute with every field embedding. The element π_N constructed in Proposition 12.2 is a Gaussian integer of norm N. A complex embedding of its coefficient field sends i to i or $-i$, so every $\sigma(\tau(\bar{\pi}_N))$ has modulus $\sqrt{N}$. The absolute square of the scalar a_N after embedding is thus $N^{-\sum_\tau b_\tau}$, including when some exponents are negative. The factorial norm estimate and the defining restriction give the more exact assertion.

To bound L, recall $d_i = k + i(t+1)$, $n = d_m - 1$. In the sum at one place the retained term can occur only at the m heights $d_0, \dots, d_{m-1}$, with count at most $2h_2$ there. In a shortened term $s \geq k$ implies $s + 1 > k$. The source multiplicity at such a height is zero except at $d_1, \dots, d_{m-1}$, where it is at most $2h_2$, by Definition 26.1. In particular the potentially large multiplicity at k does not enter this latter sum. Consequently $L \leq 4fmnh_2$. The estimates of Lemma 26.2 and Proposition 12.4 give $f \leq x^{12}$, $m \leq 2\sqrt{x}$, $n = O(x^{11/2})$ and $h_2 = O(x^{40})$. This proves $L = O(x^{58})$ uniformly. For $L \geq 1$, $2\log(L!) \leq 2L\log L = O(x^{58}\log x) \leq x^{59}$ eventually, using $\log x = o(x)$ by integration or the exponential series. For $L = 0$ the same bound is immediate. □

Proposition 417.2. *Let F be a characteristic-zero field with nonarchimedean absolute value and $\jmath_0 : \mathbb{C} \to F$ a field embedding. On evaluation coordinate vectors use the maximum absolute value $\|\cdot\|_F$. For a tuple $g = (g_\tau)$ of invertible n-by-n matrices over F put*

$$D_Y(g) = \bigotimes_{\tau} (\det g_\tau)^{e_\tau} S_{\beta_\tau}(g_\tau).$$

Define D_X on tuples of size $n+1$ in the same way, using the terminal source diagram and the same exponent. For scalars $s_\tau \in F^\times$ satisfying $|s_\tau| = |\jmath_0(\tau(\bar{\pi}_N))|$, put $\widetilde{g}_\tau = \operatorname{diag}(g_\tau, s_\tau)$. For every finite argument h on Y_m one has

$$\big\|D_Y(g)^{-1}\jmath_0\big((\operatorname{Res}_N \psi)(h)\big)\big\|_F \leq \big\|D_X(\widetilde{g})^{-1}\jmath_0\big(\psi(\iota(h))\big)\big\|_F.$$

For a p-adic absolute value with $p \neq N$ one may take every $s_\tau = 1$. At $p = N$ with $|N| = N^{-1}$, one may take $s_\tau = N$ at the embeddings where $\jmath_0(\tau(\bar{\pi}_N))$ is a nonunit, and one otherwise.

Proof. The block identity of Theorem 415.2 and multiplicativity of the determinant give

$$BD_X(\widetilde{g}) = \left(\prod_{\tau} s_\tau^{b_\tau}\right) D_Y(g) B.$$

Since $\jmath_0(B) = B$, the vector on the left of the proposed inequality is $\jmath_0(a_N)\prod_\tau s_\tau^{b_\tau}$ times B applied to the vector on the right. The absolute value of the scalar is one by hypothesis. Every integer is a sum of units of absolute value one or its negative, hence has absolute value at most one in F. The ultrametric inequality applied to each row of B proves the estimate. When $p \neq N$, the Gaussian integers $\pi_N, \bar{\pi}_N$ are integral at p and their product N is a unit, so both are units. At N, Lemma 13.2 shows the two factors generate coprime ideals. At any place above N one is therefore a unit and the other has absolute value $|N|$. This gives the stated choices. □

418. Separation of the terminal period

Definition 418.1. Write $H = U(Y_m)$, $K = H_\infty$, and use the embedding $\iota : H \to H_m = U(X_m)$ of Definition 416.3. Choose a unit vector $u_0 \in E_Y$ in the Gram norm of Lemma 414.2.

For a modular form $\eta : H_f \to E_Y$ of weight R_Y put

$$C_\eta(u, g) = \langle R_Y(u)u_0, \eta(g)\rangle, \qquad (C_\eta, C_\zeta)_{[H]} = \int_{[H]} \overline{C_\eta(h)} C_\zeta(h)\, \mathrm{d}h.$$

The Hermitian product is conjugate linear in its first argument. In particular the pairing is linear in ζ. These functions are continuous and rational invariant by the argument of Definition 412.1.

Theorem 418.2. *For each row $t \in E_m^\vee$ there is a scalar $s(t)$, linear in t, such that for every modular form ψ on X_m of weight R_m and η on Y_m of weight R_Y one has*

$$\int_{[H]} F_{\psi,t}(\iota(h))\, \overline{C_\eta(h)}\, \mathrm{d}h = s(t)\, (C_\eta, C_{\mathrm{Res}_N \psi})_{[H]}.$$

More precisely, let $D_Y = \dim E_Y$ and let $z(t)$ be determined by the compact average

$$\int_K R_Y(u)u_0\, {}^t R_m(\iota(u))^{-1}\, \mathrm{d}u = z(t)B$$

with probability Haar measure. Then $s(t) = D_Y z(t)/a_N$.

Proof. The integrand of the compact average is a linear map from E_m to E_Y. Left translation of u shows that its average intertwines the two group actions. Existence and uniqueness of $z(t)$ follow from the one-dimensional intertwining space and $B \neq 0$ in Theorem 416.4. In particular it is linear in t, hence continuous in the finite-dimensional row space. At fixed $g \in H_f$ integrate first over K in the left side of the assertion. Conjugating the coefficient of η gives $\langle \eta(g), R_Y(u)u_0\rangle$. Linearity in the second argument and the displayed average show that the result equals

$$z(t)\, \langle \eta(g), B\psi(\iota(g))\rangle = \frac{z(t)}{a_N} \langle \eta(g), (\mathrm{Res}_N \psi)(g)\rangle.$$

To integrate over the finite group apply Lemma 411.2 to H, with its integral compact subgroup of volume one. Both functions in the product have compact quotient representatives and all the indicated integrals are absolute.

The average on K of projections onto $R_Y(u)u_0$ commutes with K. Its commutant is one dimensional by Lemma 416.1 applied to the same product weight on both sides. Its trace is one, so the average is I/D_Y. Therefore for any two forms of weight R_Y,

$$(C_\eta, C_\zeta)_{[H]} = \frac{b_H}{D_Y} \int_{\Gamma\backslash H_f} \langle \eta(g), \zeta(g)\rangle\, \mathrm{d}g,$$

where $\Gamma = H(E^+)$. Comparing with the preceding integral gives precisely $s(t)$. □

Corollary 418.3. *Take $x_m = 1$ in Corollary 413.3 and write $\mathbf{t}$ for a tuple of preceding finite cut positions. Let $v(\mathbf{t})$ be the tensor product in order of all the evaluated finite pairs in that corollary, in the same fixed models. Write $\mathcal{L}^c_{v(\mathbf{t})}$ for the iterated class operator of Definition 413.1, extended linearly to the full tensor product. There is a continuous function $a_m(\xi)$ of the preceding infinite positions such that for every column ϕ, every terminal modular form η as above, and all cut tuples for which the initial finite argument lies in A^{new},*

$$\int_{[H]} P_{m,\phi}(h) \overline{C_\eta(h)}\, \mathrm{d}h = a_m(\xi)\, (C_\eta, C_{\mathrm{Res}_N(\mathcal{L}^c_{v(\mathbf{t})}\phi)})_{[H]}.$$

One function a_m works simultaneously for the finite twists and cut tuples, columns and terminal tests.

Proof. The even-period formula identifies $P_{m,\phi}(h)$ with $F_{\psi,t_m(\xi)}(\iota(h))$ for $\psi = \mathcal{L}^c_{v(\mathbf{t})}\phi$. Apply Theorem 418.2 and put $a_m(\xi) = s(t_m(\xi))$. The continuity and independence assertions follow from the linearity of s and those for t_m in Corollary 413.3. □

Corollary 418.4. *Let ϕ be any ramified column. Choose the finite inclusion tests and the stages of Theorem 407.2, and fix the real tests. Suppose $y_m = \overline{C_\eta}$ and its successive normalized descents y_i satisfy the continuity, automorphy and rapidity assumptions in that theorem. Then there is a scalar κ, independent of η, ϕ and of the finite twist, such that*

$$\int_{[G_0]} \mathcal{P}_\phi(g) y_0(g)\,\mathrm{d}g = \kappa\,(C_\eta, C_{\mathrm{Res}_N(\mathcal{L}^c_{v_*}\phi)})_{[H]}, \qquad v_* = \sum_{\mathbf{t}} \frac{v(\mathbf{t})}{\prod_{i<m} m_i(t_i)}.$$

The sum ranges over the actual finite cuts. For their normalized evaluations, its coefficient and support estimates are those of Corollary 406.2, with the common normalization specified there.

Proof. In Theorem 407.2 use the full finite tensor space for M, $V(\mathbf{t}) = v(\mathbf{t})$, and the linear functional $v \mapsto (C_\eta, C_{\mathrm{Res}_N(\mathcal{L}^c_v\phi)})_{[H]}$. Linearity follows from Theorem 411.3 and Definition 418.1. The required terminal separation was just proved. The descent theorem gives the identity and the dependence of κ. The finite estimates apply to the same product of cuts by Theorem 406.1 and the normalization of Corollary 404.2. □

419. The ordered parameters of the row groups

Definition 419.1. Fix an infinite place τ, suppress it from $w_l(\tau)$, and let $0 \le j \le m$. Put $\epsilon_j = (j \bmod 2)/2$. Use k, t, h_2 of Definition 416.3. The following list, with ϵ_j subtracted from *every* entry, specifies the half-parameter on Y_j. Take the $k-1$ numbers $k/2-1-i$, $0 \le i < k-1$, then the m numbers $u_l = h_2 - w_l$, $1 \le l \le m$. Immediately after each u_l with $1 \le l \le j$ insert

$$-w_l + (t-1)/2 - v, \qquad v = 0, \dots, t-1.$$

Give every entry positive sign except the u_l with $l \ge j+2$ and $l-j$ even, which have negative sign. The sign of a parameter means the signature of the coordinate carrying it. Doubled parameters are twice the displayed, shifted numbers.

Proposition 419.2. *The list in Definition 419.1 is strictly decreasing, has $r+jt$ entries and has exactly $\lfloor (m-j)/2 \rfloor$ negative signs. Any two entries of opposite signs differ by at least $2h_2$ in absolute value. At $j = m$ write $p_0 > \cdots > p_{n-1}$ for its entries. With $\beta_\tau(h) = c_h + r_{\tau,h+1}$ and e_τ as in Definition 416.3, one has*

$$2p_i = 2\left(e_\tau + \sum_{i<h<n} \beta_\tau(h)\right) + n - 1 - 2i \qquad (0 \le i < n).$$

In particular this gives the exact doubled highest-weight parameter of the terminal definite representation, including its determinant exponent.

Proof. Before insertion the list is the row list of Definition 56.1. Its consecutive u_l differ by $2h_2$ since the w_l differ by this amount. A block inserted after u_l starts $h_2 - (t-1)/2$ below that number, runs down by unit steps, and ends $h_2 - (t-1)/2$ above u_{l+1} if the latter exists. These gaps are positive by the inequalities in Theorem 416.4. The last head entry is $1 - k/2$; it lies above u_1 by Proposition 56.2. Thus the list is strictly decreasing. Its length is $k-1+m+jt = r+jt$. Its negative signs occur at $l = j+2, j+4, \dots \le m$. Their neighboring entries, when present, are unshifted-list anchors u_{l-1}, u_{l+1} before applying the common shift, since all insertions stop with the block following u_j. The distance to either neighbor is $2h_2$, giving the asserted lower bound. The count agrees with the actual signature in Theorem 348.3. In the original basis numbering a base entry i has $a = \max(0, i+1-k)$. Its sign is $(-1)^{a+\min(a,j)}$, which is positive for $a \le j$ and otherwise negative exactly when $a+j$ is odd. Substitution $a = l-1$ gives the listed signs on the anchors; the other coordinates have positive signs.

At $j = m$, m is even, so the common shift is zero. The number k is even and $t = 5r$ is odd by Definition 56.1. Write $c_* = h_2 - (t+1)/2$. Each gap from an anchor to its block or from that block to the next anchor is $1 + c_*$. The excess over one before u_1 is

$$(1 - k/2) - (h_2 - w_1) - 1 = w_1 - \lfloor (k + t + 1)/2 \rfloor - c_* = r_{\tau,k}.$$

The anchor u_l has position $d_{l-1} - 1$, counting from zero. The excesses at heights $i + 1 = d_{l-1}$ are thus c_*, and at heights $d_{l-1} - 1$ with $l \geq 2$ they are also c_*. At the latter heights this is the shortened count $r_{\tau,d_{l-1}}$, since $\kappa_\tau(d_{l-1}) = 2h_2 - (t+1) = 2c_*$. At height $k - 1$ the shortened count is the first-gap excess just computed. Every remaining gap is one. This proves $p_i - p_{i+1} = 1 + \beta_\tau(i+1)$ for $i < n-1$. The last entry is $-w_m - (t-1)/2 = e_\tau - (n-1)/2$ because $n = d_{m-1} + t$ and d_{m-1} is even. Summing the gaps back from this last entry proves the formula. The weights are $e_\tau + \sum_{i<h<n} \beta_\tau(h)$ by the diagonal action on the leading wedges in Theorem 77.2. Adding the half-sum of the standard positive roots amounts to adding $(n-1)/2 - i$: among the differences of two coordinate weights oriented from the smaller index to the larger, index i occurs $n - 1 - i$ times positively and i times negatively. This proves the last assertion with the stated convention. □

Corollary 419.3. *For the terminal tests of Definition 418.1 choose for u_0 the unit flag tensor in the basis ordered by decreasing parameter at each infinite place. Every scalar function of this terminal coefficient type and fixed finite level has the form C_η. More precisely, for the kernel*

$$K(u, v) = \dim(E_Y) \langle R_Y(u) u_0, R_Y(v) u_0 \rangle$$

the range of $F(u, a) \mapsto \int_K K(u, v) F(v, a) \, \mathrm{d}v$ on continuous automorphic functions of that level is exactly the family C_η of that level. Moreover for any such η and any fixed elementary real-times-finite last arrow test f_{m-1}, the function

$$y_{m-1}(g) = [K_{m-1} : J_{m-1}] \int_{[U(Y_m)]} \theta\big(\Omega_{m-1}(g, h) f_{m-1}\big) \overline{C_\eta(h)} \, \mathrm{d}h$$

is continuous, automorphic, of open finite level and rapidly decreasing on $U(Y_{m-1})$.

Proof. An ordered orthonormal basis gives a pure tensor of unit wedges, hence a unit vector by Lemma 414.2. For fixed a the projected function is the coefficient of

$$\eta(a) = \dim(E_Y) \int_K R_Y(v) u_0 F(v, a) \, \mathrm{d}v.$$

Rational covariance follows by a left translation in K, and the right finite level is inherited from F. Conversely the rank-one averaging identity in Theorem 418.2 gives this vector integral equal to $\eta(a)$ on C_η. This proves the range statement. Here K as an integration domain denotes the compact infinite group of Y_m.

Both Y_m and $Y_{m-1} = X_{m-1}$ are positive definite by Definition 348.2. Their quotients have compact fundamental representatives by Lemma 411.2. For g in a compact neighborhood the theta kernel is continuous and uniformly bounded on its product with the compact terminal representatives, by Theorem 395.2 and Proposition 388.2. Thus the integral exists and is continuous by dominated convergence. Rational invariance is Theorem 396.1; a right finite level fixing the source action on the finite test fixes the result, as in Theorem 411.3. Finally the reduced height in Proposition 398.1 is bounded on definite quotient representatives, as proved at the end of Theorem 398.2. A bounded continuous function on these representatives therefore satisfies every rapid decay estimate in that proposition. □

420. A rational translation seen by the finite inclusion datum

Lemma 420.1. *Fix a stage Y, its lattice Λ, and a prime ideal w of E. For each $g \in U(Y)(\mathbb{A})$ there exists $v \in L(g)$ with nonzero reduction in $(g_f \Lambda)_w / w(g_f \Lambda)_w$ and $\ell_g(v) \leq C_E \lambda(g)$. Here the lattice and lengths are those of Theorem 397.1. For any target stage T, $h \in U(T)(\mathbb{A})$,*

and prescribed residue in $(h_f\Lambda_T)_w/w(h_f\Lambda_T)_w$, there is $v' \in L(h)$ with that residue and $\ell_h(v') \le C_{T,w}H(h)$. The constants are independent of the group elements and residues.

Proof. Choose a shortest nonzero $v_1 \in L(g)$. Let e be the largest integer with $v_1 \in w^e(g_f\Lambda)_w$. It is finite since the local vector is nonzero. For $e = 0$ take $v = v_1$. For $e \ge 1$, choose $\beta \in w^{-e} \setminus w^{1-e}$ in the global fractional ideals; these ideals are strictly nested by Proposition 82.3. Subtract an element of O_E to put its infinite vector in a fixed bounded lattice parallelepiped of O_E. Since $O_E \subset w^{1-e}$, this keeps its valuation at w equal to $-e$. At every other finite place it is integral. Thus $v = \beta v_1$ belongs to $L(g)$, is primitive at w, and has length at most the fixed bound for $\max_\tau |\tau(\beta)|$ times $\lambda(g)$.

For the second assertion fix rational E-coordinates on T. For some fixed positive integer b_0, using $b(h)$ of Proposition 396.2, the finite lattice $h_f\Lambda_T$ lies between the completions of $b_0b(h)O_E^{\dim T}$ and $(b_0b(h))^{-1}O_E^{\dim T}$. This follows by applying the clearing integer to both h_f and its inverse in the fixed rational coordinates and by comparability of the fixed reference lattices. The finite completion argument of Lemma 13.4 applies between these two lattices. Concretely the intervening quotient is finite and is the product of its primary parts, so any local residue can be lifted globally, also after multiplying the lower bound by the rational prime below w. Choose one such lift u. If p is that prime, subtract a member of $pb_0b(h)O_E^{\dim T}$ to place u inside its bounded archimedean parallelepiped. This subtraction preserves membership and the required residue. The resulting v' has ordinary archimedean norm at most a fixed multiple of $b(h)$. After applying the archimedean frame h_∞^{-1} its norm is at most $C_{T,w}H(h)$. □

Theorem 420.2. *Consider the arrow $Y_j \to Y_{j+1}$ for any $j < m$. Use the initial-block inclusion a_{inc} at all the selected finite places and write $\Phi = f_{a_{\mathrm{inc}}}$ for the finite test of Corollary 357.3. Choose one marked place w over $p = g_0$. For all adelic source and target points g, h there is a rational linear map $z : Y_j \to Y_{j+1}$ of rank one with the following properties. Put $x = h^{-1}zg$ in the finite and infinite reference frames. There is a nontrivial p-th root of unity ζ such that*

$$D_{x_f}\Phi = \zeta\Phi, \qquad \|x_\infty\| \le C\,\lambda(g)\,H(h).$$

Here the norm is any of the fixed real Euclidean matrix norms, and C may depend on the fixed arithmetic data, stage and twist. The lower bound $|1-\zeta| \ge 2\sin(\pi/p) > 0$ is uniform in g, h.

Proof. In the finite source and target isometric frames Λ_j, Λ_{j+1} are standard integral lattices. Let $L_0 = \alpha\,\mathrm{Hom}(\Lambda_j, \Lambda_{j+1})$ and d_P be as in Definition 357.1. Choose once a scalar $\beta \in E^\times$ integral in the fractional ideal $d_P\alpha\widehat{O}_E$ everywhere except $\bar{w}$, where its valuation is $v_{\bar{w}}(\alpha) - 1$. For existence, take the fractional ideal with these local lower bounds. Its product by $\bar{w}$ is a proper subideal by Proposition 82.3; any element of the difference has the required properties. All but finitely many exponents in these specifications are zero.

Choose a source vector v primitive at w by Lemma 420.1. Its framed coordinate $c = g^{-1}v$ at w has nonzero reduction $\bar{c}_{\mathrm{res}}$. The inclusion sends this reduction to a nonzero vector in the target residue space. Choose a target residue at $\bar{w}$ such that its conjugate row on the w-branch pairs nontrivially with this vector. Such a row exists by choosing an index with nonzero coordinate; the target hermitian frames identify conjugate branch lattices in perfect duality. Lift the residue in $h_f\Lambda_{j+1}$ to $v' \in L(h)$ by the same lemma, using the target frame h_f^{-1} to prescribe that residue. Put

$$z(y) = \beta v'\,\langle v, y\rangle_{Y_j}.$$

This is rational of rank one because $v' \ne 0$ and the hermitian form is nondegenerate. In the infinite signed frames it is represented after moving by the analogous outer product of $h_\infty^{-1}v'$ and the signed adjoint of $g_\infty^{-1}v$, times the embeddings of β. The two vector bounds give the stated real estimate.

At the finite places the two framed vectors are integral. Hence $x_f = z_- + u$, where z_- consists solely of its component at $\bar{w}$, in $(\bar{d}_P^{-1} L_0)_{\bar{w}}$, and $u \in d_P L_0$. In the product rule for $D_{z_-} D_u$, the phase is one. Indeed only the component u_w can pair with z_-; its factor d_P cancels the single denominator in $\bar{d}_P^{-1}$ across conjugation. The remaining trace is integral even with the trace-one factor $\bar{\omega}$, by Lemma 355.2. Corollary 357.3 says that D_u fixes Φ. Translation by z_- on the Fourier sum of Proposition 357.2 reindexes its summation variable and gives the eigenvalue $\chi_{z_-}(a_{\mathrm{inc}})$.

In residue matrix coordinates this character is nontrivial. In detail, after division by the reference lattice scales, the reduction of the denominator-one-step component z_- is a unit times the outer product of the chosen target residue and the conjugate row of $\bar{c}_{\mathrm{res}}$. The exponent in the pairing with a_{inc} is the matrix trace of its conjugate transpose times the inclusion. This is a unit times the chosen nonzero row pairing. The scalar pairing of the single residue field with itself is perfect by Proposition 357.2, or by the equality with the trace dual in Lemma 355.2. The marked prime splits completely by Theorem 11.4, so its residue field is $\mathbb{F}_p$. A nontrivial additive character of this group has kernel zero, since any nonzero element generates the group. Thus the eigenvalue is a nontrivial p-th root of unity. Its distance to one is at least the chord length $2\sin(\pi/p)$. □

421. Rapid source decay of the inclusion theta kernel

Lemma 421.1. *In fixed real symplectic coordinates write D_x^{∞} for the real Heisenberg translation in the gauge of Definition 356.1. Put*

$$p_b(f) = \max_{|\nu| \le b} \sup_t (1 + \|t\|)^b |\partial^{\nu} f(t)| \qquad (b = 0, 1, 2, \ldots).$$

For every $b, B \ge 0$ integral there exist C, b' such that

$$p_b((D_x^{\infty} - I)^B f) \le C \|x\|^B p_{b'}(f) \qquad (\|x\| \le 1)$$

for every Schwartz test f. For a fixed finite inclusion test Φ, there also exist an integer $a \ge 0$, a seminorm p_b and $C_0 > 0$, independent of f, g, h, for which

$$|\theta(\Omega_j(g,h)(f \otimes \Phi))| \le C_0 H(g)^a H(h)^a p_b(f).$$

Proof. In the symmetric gauge the translation has the formula of Lemma 392.1. Passing to the given gauge multiplies the operator by a phase with real quadratic exponent in x, as in Theorem 395.2. Hence, writing $x = (a_1, l)$, the expression is a phase $\exp(iQ(x) + iL(l,t)) f(t + a_1)$, with Q quadratic and L real bilinear and fixed. For $\|x\| \le 1$ subtract $f(t)$ and use $f(t + a_1) - f(t) = \int_0^1 (a_1 \cdot \nabla) f(t + s a_1)\, \mathrm{d}s$. For the remaining phase difference use $|e^{iv} - 1| \le |v|$ from integrating ie^{iv}. Every term of a derivative striking the phase at least once contains a factor from l; its other such factors are bounded for $\|x\| \le 1$. The undifferentiated difference has a factor at most $C\|x\|(1 + \|t\|)$. The weights at $t + s a_1$ and at t differ by at most a fixed factor. These facts give $p_b((D_x^{\infty} - I) f) \le C_b \|x\| p_{b+2}(f)$. Iteration proves the first assertion.

For the second estimate inspect the seminorm bounds in Proposition 396.2. The finite supremum there depends only on Φ and the finite clearing integer of the Hom action. In the real estimate the L^2 norms of finitely many words in the position and differentiation operators are bounded by $C p_b(f)$ for one fixed b: choose b larger than their derivative orders and their polynomial degrees plus the real dimension plus one, and integrate the resulting inverse power of $1 + \|t\|$ over successive unit shells. Thus the degree in H and the seminorm index are independent of the test. Fixed changes of polarization by linear substitution, polynomial phase and partial Fourier transform preserve continuity in these seminorms by Theorem 395.2. Finally the matrix $v \mapsto hvg^{-1}$ and its inverse have clearing integers dividing a fixed integer times $b(h)b(g)$, by matrix multiplication in a rational field basis. Their real norms are bounded by a fixed multiple of the product of the two group

norm bounds. This gives the asserted estimate, increasing a to an integer if necessary. The character in Theorem 396.1 has modulus one. □

Theorem 421.2. *For an actual arrow $Y_j \to Y_{j+1}$ let $U(g) = \max(1, \lambda_{Y_j}(g)^{-1})$ and $V(h) = \max(1, \lambda_{Y_{j+1}}(h)^{-1})$. At the finite places take the inclusion test Φ of Theorem 420.2. There is an integer $a \geq 0$ such that for every integer $N_0 \geq 0$ there are $C_{N_0} > 0$ and $b_{N_0} \geq 0$ with*

$$|\theta(\Omega_j(g,h)(f \otimes \Phi))| \leq C_{N_0} p_{b_{N_0}}(f)\, U(g)^{-N_0} V(h)^{N_0+2a}$$

for every adelic g, h and every real Schwartz f. The constants may depend on the fixed stage and arithmetic data, including its finite test and twist.

Proof. Rational invariance in each group follows from Theorem 396.1. Choose the rational representatives of Theorem 397.1 so that $H(g) \leq CU(g)$ and $H(h) \leq CV(h)$, with the minima unchanged. For brevity let $T(f)$ denote the theta value at these points, with the finite test fixed. Lemma 421.1 gives $|T(f)| \leq Cp_b(f)U(g)^a V(h)^a$.

If $U(g) > 1$, use the rational z of Theorem 420.2, with $x = h^{-1}zg$. Then $\|x_\infty\| \leq C_1 V(h)/U(g)$. Rational translation by z preserves the theta value of every test: at zero the translated value of the periodized function is $F(z) = F(0)$ by Definition 356.1. Intertwining carries D_z back through $\Omega_j(g,h)$ to D_x, by Theorems 395.2 and 396.1; the character commutes with translations. In the product model D_x acts by $D^\infty_{x_\infty} \otimes D_{x_f}$. The resulting scalar identity, valid for *every* real test, is

$$T(f) = \zeta\, T(D^\infty_{x_\infty} f).$$

Consequently $T((D^\infty_{x_\infty} - I)^B f) = (\zeta^{-1} - 1)^B T(f)$ for every integer $B \geq 0$, by induction. There is no dependence on f in the choice of z.

If $U(g) \leq \max(1, C_1)V(h)$, the coarse estimate already implies the conclusion after multiplying its right side by the bound for $U(g)^{N_0+a}/V(h)^{N_0+a}$. Otherwise $U(g) > 1$ and $\|x_\infty\| \leq 1$. Apply the coarse estimate to the B-fold difference, with $B = N_0 + a$. The lower bound on $|1 - \zeta|$ in Theorem 420.2 and the real difference estimate give

$$|T(f)| \leq C'_B p_{b+2B}(f) U(g)^a V(h)^a (V(h)/U(g))^B.$$

This is the claimed inequality. Both arguments include $N_0 = 0$, and rational invariance gives the result at the original representatives. □

422. Transposition for the whole chain of inclusion kernels

Definition 422.1. For the row group $G_i = U(Y_i)$ put $U_i(g) = \max(1, \lambda_{Y_i}(g)^{-1})$ as in Theorem 421.2. Call a continuous automorphic scalar function z of moderate growth if $|z(g)| \leq CU_i(g)^s$ for some $C, s \geq 0$. The finite level of an automorphic function here is any compact open subgroup under which it is invariant on the right.

Choose real Schwartz tests $f_{i,\infty}$ and the fixed finite inclusion tests Φ_i on the consecutive arrows. Write

$$\mathcal{K}_i(g,h) = [K_i : J_i]\theta\big(\Omega_i(g,h)(f_{i,\infty} \otimes \Phi_i)\big).$$

Use the measures of Theorem 407.2. For a source function z and a target function y put

$$(\mathcal{F}_i z)(h) = \int_{[G_i]} z(g)\mathcal{K}_i(g,h)\,\mathrm{d}g, \qquad (\mathcal{D}_i y)(g) = \int_{[G_{i+1}]} \mathcal{K}_i(g,h) y(h)\,\mathrm{d}h$$

when absolutely convergent. Here $\mathcal{D}_i$ is a transpose for the bilinear integral pairing.

Theorem 422.2. *The operator $\mathcal{F}_i$ takes moderate continuous automorphic functions to moderate continuous automorphic functions, and $\mathcal{D}_i$ takes rapidly decreasing continuous automorphic functions to such functions. For every pair z, y in the respective domains both integrals below converge absolutely and*

$$\int_{[G_{i+1}]} (\mathcal{F}_i z)(h) y(h)\,\mathrm{d}h = \int_{[G_i]} z(g)(\mathcal{D}_i y)(g)\,\mathrm{d}g.$$

The outputs have open finite level.

Proof. For each of the two groups, U_i is comparable to the reduced height by Theorem 397.1. The polynomial sublevel volume bound in Proposition 398.1 implies $\int_{[G_i]} U_i^{-D} < \infty$ for sufficiently large D. Indeed on $2^j \le U_i < 2^{j+1}$ its integral is at most $C2^{j(b-D)}$ for a fixed b, and $D > b$ suffices. Similarly a rapidly decreasing function y on the target satisfies $\int_{[G_{i+1}]} |y| U_{i+1}^b < \infty$ for any fixed b.

Use the kernel bound in Theorem 421.2, including the finite constant $[K_i : J_i]$. Given a source bound CU_i^s, choose $N_0 > s + D$. Integration over the source gives a bound $C'U_{i+1}(h)^{N_0+2a}$ for the absolute integral defining $\mathcal{F}_i z$. Integrating the same bound against $|y|$ proves absolute integrability of the full product on the product of the quotients. In the reverse direction the kernel bound gives, for every N_0,

$$|(\mathcal{D}_i y)(g)| \le C_{N_0} U_i(g)^{-N_0} \int_{[G_{i+1}]} U_{i+1}(h)^{N_0+2a} |y(h)| \, \mathrm{d}h,$$

which proves rapid decrease. Fubini's formula in Lemma 123.1 yields transposition.

The kernels are jointly continuous and rationally invariant by Theorem 396.1 and the rational lattice bounds in Proposition 388.2. On compact subsets in a variable, the corresponding U_i is bounded, since $U_i \ll H_{\mathrm{red}} \le H$ and the clearing integers and real norms are bounded there. Thus the preceding majorants can be chosen uniformly there for continuity in that variable, by Lemma A.2. Rational invariance passes to the integrals. The right levels fixing the input oscillator vector on the appropriate side contain compact open subgroups by Corollary 357.3 and Theorem 376.2; their invariance passes to the outputs. □

Theorem 422.3. *For a ramified column ϕ let $z_0 = \mathcal{P}_\phi$ be its actual Poincaré series of Theorem 407.2. Recursively put $z_{i+1} = \mathcal{F}_i z_i$ for $0 \le i < m$. Let Π_Y be the compact infinite projection of Corollary 419.3, with the ordered unit flag u_0. With the finite sum v_* of Corollary 418.4, one has the equality of functions on $[U(Y_m)]$*

$$\Pi_Y z_m = \kappa \, C_{\mathrm{Res}_N(\mathcal{L}^c_{v_*}\phi)}.$$

The same constant κ works for every ramified column and every finite twist with the fixed real tests and weights. All forward and backward integrations used in this identity are absolutely convergent.

Proof. The initial Poincaré series is bounded by Proposition 204.2 and Theorem 204.4, using its finite set of column coordinates. The finite translation of Lemma 349.2 preserves boundedness. Thus the forward functions exist and have moderate growth by induction in Theorem 422.2. For every terminal test choose $y_m = \overline{C_\eta}$ and let $y_i = \mathcal{D}_i y_{i+1}$. At the definite terminal group the initial y_m is rapidly decreasing, as in Corollary 419.3. The transposition theorem now gives all the hypotheses on the y_i in Corollary 418.4 and, by its successive pairing identities,

$$(C_\eta, z_m)_{[H]} = \int_{[G_0]} z_0 y_0 = \kappa (C_\eta, C_{\mathrm{Res}_N(\mathcal{L}^c_{v_*}\phi)})_{[H]}.$$

Here $H = U(Y_m)$ and all finite representatives and tests are as in the finite descent formula.

In the compact variable the kernel of Π_Y is conjugate symmetric on interchanging its arguments. Applying Fubini with the compact representatives of Lemma 411.2 shows it is self-adjoint for $(\cdot,\cdot)_{[H]}$. It fixes every C_η by Corollary 419.3. Thus z_m in the last display can be replaced by $\Pi_Y z_m$. The difference of the two functions in the theorem belongs to the range of Π_Y and has open finite level, taking the intersection of the two levels when necessary. It is itself C_η for some η by that corollary. Testing against this difference gives its squared norm zero. A continuous automorphic function with zero squared integral vanishes everywhere: any nonzero value stays bounded away from zero in some open set, which has positive quotient measure by discreteness and the Haar construction in Lemma 411.2. This

proves the formula. The independence of κ is the independence already established in the pairing identity. □

423. Translated vacua for the projected transfer

Definition 423.1. Fix once for each definite arrow a nonzero scalar λ_i and replace its unit finite vacuum v_i by $v_i^{\circ} = \lambda_i v_i$. The scalars are independent of the finite twist. Use physical finite displacements s_i, so that the corresponding integral coordinates are measured after division by α as in Definition 368.1. For a tuple $s = (s_i)_{0 \le i < m}$ write

$$e(s) = \bigotimes_{i=0}^{m-1} D_{s_i} v_i^{\circ}.$$

For a finitely supported function Ψ of such tuples put $w_\Psi = \sum_s \Psi(s) e(s)$ and $B_\Psi \phi = \mathrm{Res}_N(\mathcal{L}^c_{w_\Psi}\phi)$, with the scale and the chain of Definition 411.1 and Definition 413.1. The finite operators $\mathcal{P}, \mathcal{S}$ on the paired factors are those of Corollary 370.2.

Theorem 423.2. *Fix the real data of Theorem 422.3 and the vacuum normalizations of Definition 423.1. There is one $b \in \mathbb{C}^\times$ such that for each finite twist with set P and $Q = \prod_{w \in P} q_w$ there is a finitely supported function Ψ with the following properties. Its values are algebraic and integral at every finite place of residue characteristic prime to Q. Each coordinate of a tuple in its support is an allowed shift of Definition 368.1 plus a shift supported above N from Proposition 368.2. The tensor w_Ψ is invariant under the simultaneous actions of $K_{c,i}$ for $0 \le i < m$, with the final group element equal to one, and it satisfies*

$$w_\Psi = \sum_s \Psi(s)\, \mathcal{S}\mathcal{P} e(s).$$

For every ramified column one has

$$b\, \Pi_Y z_m(\phi) = C_{B_\Psi \phi}.$$

When P is empty, for every field automorphism σ of $\mathbb{C}$ one has, uniformly throughout the windows and for all sufficiently large x,

$$\sum_s |\sigma \Psi(s)| \le (g^{f n_m^2})^{2m-1} N^{f((m/2)k + (t+1)(m/2)^2)} \le \exp(x^{25}).$$

Proof. Take the normalized tensor w in Corollary 406.2. It is a sum of translated auxiliary-prime sums with algebraic coefficients integral away from Q. Algebraicity here also follows directly from the finite Fourier coefficients and arrow scalars in Theorem 369.1 and Corollary 376.3. Expand every auxiliary-prime sum. On pair $j, j+1$ this gives $N^{f d_{j+1}}$ terms of unit modulus, allowing roots of unity when displacements are combined by their multiplication law. No division occurs. The support and the product of these numbers over even $j < m$ are computed in Theorem 369.1. Group terms giving the same displacement tuple, summing their coefficients; use zero off this finite set. Call the resulting function Ψ_0. With the original unit vacua its tensor is w, and with the chosen vacua it is $(\prod_i \lambda_i) w$ by multilinearity. The coefficients have not changed under this replacement. The nonarchimedean triangle inequality preserves integrality under grouping. The ordinary triangle inequality and the last estimate in Corollary 406.2 give the first bound for their total size when P is empty. The same corollary proves invariance and $\mathcal{S}\mathcal{P} w = w$, so applying the two linear operators term by term yields the displayed recovery formula also for the scaled vacua.

To get the bound uniform in the arithmetic parameters, the exponent of N is at most $f m d_m$, since it is the sum of $f d_{j+1}$ over even $j < m$. Here $d_m = n_m + 1$ by Definition 348.2. The bounds $f \le x^{12}$, $m = O(\sqrt{x})$, $n_m = O(x^{11/2})$ are in the proof of Lemma 26.2, and $1 + \log(gN) \le x$ in Corollary 23.3. The logarithm of the first bound is therefore $O_{d,M,\epsilon}(x^{49/2})$, which is at most x^{25} eventually.

The common finite model factors in Corollary 418.4 and the factor C_*/D in Corollary 406.2 convert its v_* to a scalar multiple of w, independent of the twist. Thus Theorem 422.3 gives $\Pi_Y z_m(\phi) = aC_{B_{\Psi_0}\phi}$ for one $a \in \mathbb{C}$, also incorporating the inverse of $\prod_i \lambda_i$. The coefficient is linear in the form by Definition 418.1. If $a \neq 0$, take $b = a^{-1}$ and $\Psi = \Psi_0$. If $a = 0$, the projected chain vanishes for every column and twist. Take $b = 1$ and the zero function for every Ψ. All the assertions then follow in this case as well. □

424. Evaluation on tuples of definite classes

Theorem 424.1. *Use the definite groups and measures of Definition 411.1. For $0 \leq i < m$ let $A_i \subset H_{i,f}$ represent the finite double quotient $\Gamma_i \backslash H_{i,f} / K_{c,i}$. Suppose w is a finite oscillator tensor fixed by the joint column action, with the group element at m set equal to one. Let ϕ have right level $K_{c,0}$. Fix $h \in H_{m,f}$ and put $a_m = h$. For pure tensors $e = \otimes_i e_i$ define*

$$D_{\mathbf{a}}(e) = J^c_{m-1}(\Omega_{m-1,f}(a_{m-1}, a_m)e_{m-1}) \cdots J^c_0(\Omega_{0,f}(a_0, a_1)e_0)\, \phi(a_0),$$

extending linearly to e. Here $\Omega_{i,f}$ denotes the finite action on the arrow from i to $i+1$. Then

$$(\mathcal{L}^c_w \phi)(h) = \sum_{\mathbf{a} \in \prod_{i<m} A_i} D_{\mathbf{a}}(w).$$

Each term on the right is independent of the choice of the representatives in that tuple. The number of terms is at most $\exp(mx^{25}) \leq \exp(x^{26})$ for all sufficiently large x in the windows of Corollary 23.3.

Proof. The product of linear maps is multilinear in the e_i, which gives the stated linear extension by the defining property of a tensor product. The rational covariance identity in the proof of Theorem 411.3 gives invariance of $D_{\mathbf{a}}(e)$ under the left rational groups at the m integrated variables. Indeed at an interior variable the factor $R_i(\gamma_\infty)$ on the output of one matrix cancels the inverse factor on the input of the next. At $i = 0$ it cancels against the transformation of ϕ. By the same theorem $D_{\mathbf{a}}(e)$ is continuous and has open finite levels in each integrated variable. Its integral over their product quotients is obtained by composing the individual operators. This follows by successive integration for pure tensors and by linearity for arbitrary tensors. Fubini's lemma (Lemma 123.1) applies since the finite product of the compact fundamental closures of Lemma 411.2 bounds all these continuous integrands.

For $\mathbf{k} = (k_i)_{i<m}$ in the column levels put $k_m = 1$. The finite representation law changes $D_{\mathbf{ak}}(w)$ into $D_{\mathbf{a}}((\otimes_i \Omega_{i,f}(k_i, k_{i+1}))w)$, since $\phi(a_0 k_0) = \phi(a_0)$. This equals $D_{\mathbf{a}}(w)$. Each quotient integral may now be taken at $K_{c,i}$ itself. Its stabilizers are trivial. In fact a stabilizer is finite by Lemma 411.2. At the chosen component of g_0 the level reduces to identity in the split general linear group by Proposition 349.1. The proof of Proposition 13.5 that $I + g_0 M_d(\mathbb{Z}_{g_0})$ is torsion free applies in every rank, and the rational group injects into that component. Thus the quotient measure assigns mass one to each class by Lemma 411.2. This proves the formula, including independence of representatives.

Each $K_{c,i}$ contains the principal congruence subgroup of level gN, since that subgroup reduces to identity at all specified places and satisfies the last-row congruence in Proposition 349.1. Its rank is $d_i = k + i(t+1)$, in the range of Corollary 23.3. Applying that corollary to each of the m factors proves the first counting bound. The second follows from $m = O(\sqrt{x})$ in Lemma 26.2. □

Corollary 424.2. *Take $w = w_\Psi$ as in Theorem 423.2. Embed the field generated by the coordinates being evaluated, the coefficients and the arrow matrix entries in a valued field F. At each representative and at h choose norms on the corresponding coordinate spaces over F, ultrametric when the absolute value is nonarchimedean. Suppose the embedded $\phi(a_0)$ have norm at most B_0, and that at every tuple $\mathbf{a}$ and for each s with $\Psi(s) \neq 0$, the i-th displayed jet map applied to $e_i = D_{s_i} v_i^\circ$ has operator norm at most b_i between the chosen spaces. If*

F is nonarchimedean of residue characteristic prime to Q, with absolute value extending its usual prime-adic absolute value, then the embedded value of $(\mathcal{L}_w^{\mathbf{c}}\phi)(h)$ *has norm at most* $B_0 \prod_i b_i$. *For the complex absolute value after an automorphism of* $\mathbb{C}$, *when* $P = \varnothing$ *its norm is at most*

$$\exp\left((m+1)x^{25}\right) B_0 \prod_{i<m} b_i.$$

Proof. Expand each $D_{\mathbf{a}}(w)$ of Theorem 424.1 in the tuples s. Successive applications of the operator inequalities bound each $D_{\mathbf{a}}(e(s))$ by $B_0 \prod_i b_i$. The sum is finite, so applying the embedding commutes with all its additions and multiplications. In the nonarchimedean case each embedded $\Psi(s)$ has absolute value at most one by Theorem 423.2. Taking the maximum over all terms proves the first estimate. For the complex norm the triangle inequality costs the number of class tuples times the sum of the absolute coefficient values. Apply the respective bounds from Theorems 424.1 and 423.2. □

425. The holomorphic sections of a definite arrow

Lemma 425.1. *Identify a standard positive Hom space, by its matrix entries, with* E^n. *Use* $r(\gamma)$ *of Proposition 47.2. In the real model of Lemma 392.1 the symmetric translation of a coherent vector satisfies*

$$\rho_a(\gamma_\infty) B_z = \exp\Big(H(r\gamma, z) - \|r\gamma\|^2/2\Big)\, B_{z-r\gamma}.$$

Here H in the exponent is the ordinary positive Hermitian product of Proposition 47.2. For finite physical X write $\widetilde{D}_X = \mathbf{e}_f(b(X,X)/2) D_X$, *with b from Definition 385.1. For a finite test* Φ *put* $S_\Phi(z) = \theta(B_{-z} \otimes \Phi)$. *Then* S_Φ *is entire and for rational* γ

$$S_{\widetilde{D}_{\gamma_f}\Phi} = W_{-r\gamma} S_\Phi.$$

Proof. At each real place Theorem 393.1 gives $x + il = \sqrt{2/\sqrt{d}}\,\tau(\gamma)$ in each positive coordinate. Hence $r\gamma = \sqrt{\pi}(x + il)$. Substitute the exponential of Proposition 408.3 in the formula for $\rho_a(x,l)$. The coefficient of t in the exponential multiplying $\phi_0(t)$ is $2\sqrt{\pi}(z - r\gamma)^{\mathsf{T}} t$. After extracting $B_{z-r\gamma}$ the constant exponent is $\overline{r\gamma}^{\mathsf{T}} z - \|r\gamma\|^2/2$. Multiplying over all places gives the first identity. Entireness follows from Proposition 408.3.

The simultaneous real and finite symmetric translations at γ leave the theta functional invariant. Indeed the unsymmetrized translations do so by Theorem 385.3 and the additional scalar is $\mathbf{e}(b(\gamma,\gamma)/2) = 1$ on the diagonal rational number. The intertwiners of Proposition 391.1 and Theorem 395.2 preserve this calculation in the indicated models. The inverse symmetric real translation is the translation by $-\gamma_\infty$. Consequently

$$\theta(B_{-z} \otimes \widetilde{D}_{\gamma_f}\Phi) = \theta(\rho_a(-\gamma_\infty) B_{-z} \otimes \Phi) = e^{H(r\gamma, z) - \|r\gamma\|^2/2}\, S_\Phi(z - r\gamma).$$

This is precisely the shift of Theorem 48.2. □

Theorem 425.2. *For every standard definite arrow the linear map* $\Phi \mapsto S_\Phi$ *of Lemma 425.1 is an isomorphism of the finite oscillator onto* $\mathcal{F}$ *of Definition 51.1. It sends the unit vacuum to a nonzero element of* $\mathcal{T}(\Gamma_0)$ *and intertwines* $\widetilde{D}_X$ *with* T_X *of Theorem 51.2. In particular for* $u \neq 0$, *any row A, and any column vector v of the prescribed weight,*

$$A J_j^{-u}(\Phi) v = \left[P_{A,v}(\partial)\, S_\Phi(uZ)\right]_{Z=0}.$$

For the physical translated test $D_X\Phi$, *replace* S_Φ *on the right by* $\mathbf{e}_f(-b(X,X)/2) T_X S_\Phi$. *This scalar is a root of unity.*

Proof. The finite physical reference lattice in one coordinate is $\alpha\widehat{O}_E$, the completion of $\mathcal{A}$ by Definition 44.2, Proposition 44.3, and Lemma 348.1. For the unit vacuum v_0, $D_l v_0 = v_0$ for every element l of that lattice in all n entries, by Proposition 357.2. For $\gamma \in \mathcal{A}^n$ the value of $b(\gamma,\gamma)$ is $\mathrm{Tr}_{E/K} \sum_i \bar{\gamma}_i \gamma_i$. Indeed $\jmath\bar{\omega} = 1/2 - 1/(2\sqrt{-d})$, and summation over conjugate

embeddings cancels the purely imaginary summand for this real norm. The trace is an integer by Proposition 44.3. Thus $\mathbf{e}_f(b(\gamma,\gamma)/2) = \chi_{\Gamma_0}(r\gamma)$ by Theorem 48.2. The lemma gives its period equations for S_{v_0}, since the signs are unchanged by $\gamma \mapsto -\gamma$.

Each Φ is a finite sum of translated vacua by Definition 357.1. For some positive integer h all physical translations by elements of $h\alpha\widehat{O}_E^n$ fix every summand. To see this, the commutator of a translation with a fixed displacement is a continuous additive character with open kernel by Lemma 356.2 and Proposition 18.5. Intersect these finitely many kernels with the reference lattice and take a sufficiently large integer multiple of that lattice inside the intersection. Applying the preceding sign calculation to $h\mathcal{A}^n$ now gives $S_\Phi \in \mathcal{T}(h\Gamma_0)$.

Suppose $S_\Phi = 0$ and put $f = \phi_0 \otimes \Phi$. The coherent translation formula with $z = 0$ gives $F_f^a(v_\infty, 0) = 0$ for every v_∞. The rational translation law in Theorem 385.3, multiplied by the symmetric phase as in Theorem 387.2, propagates this vanishing to all (v_∞, γ_f) for rational γ. The finite components of rational points are dense by the coordinate argument in Theorem 51.2. Continuity gives $F_f^a = 0$ everywhere. Its squared integral is the squared norm of f by Theorem 387.2. The real and finite model changes are unitary by Proposition 391.1 and Theorem 395.2. Therefore $\Phi = 0$, which proves injectivity and in particular $S_{v_0} \neq 0$.

For all rational displacements the intertwining statement is the lemma and Theorem 51.2. For fixed Φ the function $X \mapsto \widetilde{D}_X\Phi$ is locally constant by the lattice period just proved, the multiplication formula of Lemma 356.2 and local constancy of the finite phase. The action on the section is locally constant by Theorem 51.2. Density extends the identity to all finite X. The translates of any nonzero member of $\mathcal{T}(\Gamma_0)$ span $\mathcal{F}$ by that same theorem. Hence the map is onto. Finally $S_\Phi(uZ) = \mathcal{T}_{-u,\Phi}(Z)$ by the definitions. Apply Definition 411.1. The physical correction has finite order since the finite adelic character takes values in the image of $\mathbb{Q}/\mathbb{Z}$ by Definition 385.1. □

426. All complex conjugates of the reduced torsion values

Lemma 426.1. *Let λ, Λ, A, a_1 be the normalized curve data of Propositions 31.3 and 31.4. Write $s_y = (U_y\sigma_\Lambda)(0)$ and $t_y = (U_y\theta_\Lambda)(0)$. There exist $0 < c_d \le 1 \le C_d < \infty$, independent of the torsion order and of the field automorphism ν of $\mathbb{C}$, such that for every $y \in \lambda K$, with $H_\nu(y) = \max(|\nu s_y|, |\nu t_y|)$, one has*

$$c_d H_\nu(y)^4 \le H_\nu(2y) \le C_d H_\nu(y)^4, \qquad C_d^{-1/3} \le H_\nu(y) \le c_d^{-1/3}.$$

Proof. Theorem 32.4 gives $|\iota a_1|_p \le 1$ for every finite-place isomorphism. The transcendence-basis and minimal-polynomial argument in Lemma 48.1 therefore makes a_1 an algebraic integer. In particular a_1 has finitely many complex conjugates. Each conjugate A' satisfies $(A')^4 \neq 64$, since $a_1^4 - 64$ has absolute value one at every finite place by the same theorem.

After reducing the Gaussian factors, the duplication equations of Proposition 33.1 read

$$t_{2y} = t_y^4 - s_y^4, \qquad s_{2y}^2 = s_y^2 t_y^2 \left(4(t_y^4 + s_y^4) + a_1^2 s_y^2 t_y^2\right).$$

Indeed the Gaussian for $2y$ is the fourth power of that for y by Proposition 31.4. Apply ν to these two algebraic equalities. For fixed $A' = \nu(a_1)$ the maximum of the modulus of the first output and the square root of the modulus of the second output, on replacing (s_y, t_y) by (a, b), is a continuous function homogeneous of degree four in the absolute scale. On $\max(|a|, |b|) = 1$ it is strictly positive. Indeed if its first output vanishes, then $a^4 = b^4 \neq 0$. Vanishing also of the second would imply $8 + (A')^2(a/b)^2 = 0$, and squaring would give $(A')^4 = 64$. Thus its minimum on this compact set is positive and its maximum finite. Take the least of these minima and one, and the greatest of these maxima and one, over the finitely many A'. This proves the first pair of inequalities. The normalized curve was fixed with d in Proposition 31.3. Changing the fourth root used there gives only finitely many possible coefficients, which can equally be included in these extrema.

For $ny \in \Lambda$ and $l \in \Lambda$, the two reduced values at $y + l$ differ from their respective values at y by roots of unity. The sigma and theta multipliers in Propositions 31.2 and 31.4 give the

sign of l, an additional sign for theta, and the phase $\exp(i\pi\ \mathrm{Im}(\bar{l}y)/A)$; the ratio $\mathrm{Im}(\bar{l}y)/A$ belongs to $n^{-1}\mathbb{Z}$. These multipliers have unit modulus after every ν. The two values are never both zero by Theorem 33.2. Hence H_ν has positive finite values on the finite group $n^{-1}\Lambda/\Lambda$. Writing $h_j = H_\nu(2^j y)$, iteration gives

$$c_d^{(4^j-1)/3} h_0^{4^j} \ \le\ h_j \ \le C_d^{(4^j-1)/3} h_0^{4^j}.$$

If $h_0 > c_d^{-1/3}$ the lower bound tends to infinity, and if $h_0 < C_d^{-1/3}$ the upper bound tends to zero. Both possibilities contradict the finite positive range. This proves the remaining inequalities. □

Corollary 426.2. *There is $M_d \ge 1$ such that every monomial P of degree b in Lemma 48.1, every $y \in \lambda K$, and every field automorphism ν of $\mathbb{C}$ satisfy*

$$\left|\nu\big((U_y^{[b]}P)(0)\big)\right| \le M_d^b.$$

For the sections $\mathcal{S}$ of Theorem 48.2 formed with multiplicities b_i in n coordinates, all finite displacements X satisfy

$$\left|\nu\big((T_X\mathcal{S})(0)\big)\right| \le M_d^{\,n\sum_i b_i}.$$

Proof. The generators of the monomials are the sections of degrees $1, 2, 3$ in Lemma 48.1. For the first two the bounds follow directly from Lemma 426.1 on t_y and s_y^2. For the third put $g_y = (U_y^{[3]} G_\Lambda)(0)$. The last equation of Proposition 33.1, with the homogeneous Gaussian factors inserted, reads

$$g_y^2 + a_1 s_y^2 t_y g_y = s_y^6 + s_y^2 t_y^4.$$

Let K_0 be the largest modulus of the finitely many conjugates of a_1. The modulus $r = |\nu g_y|/H_\nu(y)^3$ satisfies $r^2 \le K_0 r + 2$. For example $r \le K_0 + 2$, since the opposite inequality gives $r(r - K_0) > 2$. Increasing M_d makes all three generator bounds at most M_d to the power of their degrees. The Gaussian multipliers multiply with the degrees, so the bound passes to each monomial. For any finite X, Theorem 51.2 realizes $T_X\mathcal{S}$ by a rational displacement on this section. Apply the product identity of Theorem 48.2. Its shifts all belong to λK, and the degrees in the full product sum to $n\sum_i b_i$, which proves the second inequality. □

427. An integral normalization of the vacuum in every rank

Proposition 427.1. *For the given $E = K(\zeta_{N_1})$ there is a nonzero $V^\circ \in \mathcal{T}(r(\mathcal{A}))$ in rank one such that, for every finite displacement X, $(T_X V^\circ)(0)$ is an algebraic integer and*

$$\sup_{\nu \in \mathrm{Aut}(\mathbb{C})} |\nu((T_X V^\circ)(0))| \le \exp(C_d N_1^2).$$

Here the constant is independent of the exponent s determining E in Definition 10.1.

Proof. Take $D = N_1$, $\delta = D$, and the D elements $w_j = \zeta_D^j$, $0 \le j < D$, each with multiplicity D. The geometric sum gives $\sum_j D\,\tau(w_j)\overline{\sigma(w_j)} = D^2 \mathbf{1}_{\tau=\sigma}$ since the characters of the cyclic group here have distinct exponents by Proposition 10.3. These elements belong to $\mathcal{A}$ by Proposition 44.3. Thus the length is $L = D^2 = \delta\bar{\delta}$. Use $1 \in O_K$ for the element of norm $\delta\bar{\delta}/L$ in Theorem 48.2, and take every one-variable monomial to be θ_Λ^D. The resulting section F belongs to $\mathcal{T}(\Gamma)$, where $\Gamma = D\Gamma_0$ and $\Gamma_0 = r(\mathcal{A})$, and $F(0) = 1$. Each $(T_X F)(0)$ is an algebraic integer by Corollary 51.3. All its conjugates have absolute value at most $M_d^{D^2}$ by Corollary 426.2.

For $v \in \Gamma_0$ let $U_v = \chi_{\Gamma_0}(v)^{-1} W_v$ on $\mathcal{T}(\Gamma)$. These shifts preserve the space since $\Gamma_0 \subset \Gamma^\#$ by Proposition 47.2. Their two multipliers cancel by the two multiplication equations in Theorem 49.2, making U_v an ordinary representation of Γ_0/Γ. On Γ the two sign prescriptions agree by Theorem 48.2. The invariants are exactly $\mathcal{T}(\Gamma_0)$. The unnormalized sum $P = \sum_{v \in \Gamma_0/\Gamma} U_v$ maps into the invariants by reindexing. It acts there by the nonzero

integer $|\Gamma_0/\Gamma| = D^{2f}$, since the lattice has rank $2f$ by Proposition 47.2. The invariant space has dimension one by Theorem 49.2. Since the translates W_aF, $a \in \Gamma^{\#}/\Gamma$ with any representatives, span $\mathcal{T}(\Gamma)$ by Proposition 50.1, some PW_aF is nonzero. Take this section for V°.

The shifts v, a correspond to rational displacements by Proposition 47.2. Theorem 51.2 therefore expresses $T_X V^{\circ}$ as a sum of D^{2f} single translates of F, each with a root of unity as its coefficient. Indeed each multiplication gives one value of the finite character and each χ_{Γ_0} is a sign. No reciprocal of D occurs. The values at zero are integral by the preceding construction and closure under sums and products, as in Theorem 48.2. Every absolute conjugate is bounded by $D^{2f} M_d^{D^2}$. Since $f \le D$ by Proposition 10.3, its logarithm is at most $(2 + \log M_d)D^2$, using $\log D \le D$. This gives the claim. □

Corollary 427.2. *In the standard positive space E^n the finite unit vacuum can be scaled by a nonzero scalar so that its section is*

$$V_n^{\circ}(z) = \prod_{j=1}^{n} V^{\circ}(z_{\cdot,j}).$$

In this normalization all $\theta(\phi_0 \otimes D_X v_n^{\circ})$ are algebraic integers, and all their complex conjugates have modulus at most $\exp(C_d n N_1^2)$, uniformly for finite X. Some such translated value is nonzero. These scalings can be used on every definite arrow in Theorem 423.2.

Proof. In the product the multiplier on $r(\mathcal{A}^n)$ is the product of the one-coordinate multipliers by the explicit W and sign in Theorem 48.2. Hence the nonzero product belongs to the one-dimensional vacuum space of Theorem 49.2. Its nonzero vectors correspond to precisely the nonzero multiples of the finite unit vacuum by Theorem 425.2. For rational displacements the translated product factors as the product of the translated one-coordinate sections by the same W formula. For a general finite displacement approximate all coordinates simultaneously by rational displacements, using local constancy in Theorem 51.2. The product identity persists. Apply Proposition 427.1 coordinate by coordinate. The physical correction in Theorem 425.2 is a root of unity and leaves both conclusions valid. If every translated value vanished, injectivity of evaluation on translates in Theorem 51.2 would make the product zero. Finally the scalings depend only on the field, the model identifications and the ranks, all independent of the finite twist, as required in Definition 423.1. □

428. Local expansions of the scaled definite kernels

Definition 428.1. Use λ of Proposition 31.3, $\sigma_0 = (2\pi/\sqrt{d})^{1/2}$, and s_0 of Theorem 46.1. Fix $s_1 \in \mathbb{C}^{\times}$ such that $|\iota(s_1/s_0)|_p = 1$ for every finite-place isomorphism ι. This includes the choice $s_1 = s_0$. Set $u = \sigma_0 s_1/\lambda$. For a prime p, a given $\iota : \mathbb{C} \to K_p$, $0 \ne \delta \in O_K$, and $n \ge 1$, write $\mathcal{D}_p(\delta)$ for the module over $\iota^{-1}(O)$ generated by the finite vectors $D_X v_n^{\circ}$ such that $\delta X_p \in \mathcal{A}^n \otimes \mathbb{Z}_p$. Here $O = \{z \in K_p : |z|_p \le 1\}$, X_p denotes all physical components at p, and the vacuum is normalized by Corollary 427.2. For a section use its Taylor series in all complex variables at the origin, with ordinary factorial denominators, and apply ι coefficient by coefficient when taking it over K_p.

Theorem 428.2. *Fix the data of Definition 428.1. For every $a \in \mathbb{C}$ and $\Phi \in \mathcal{D}_p(\delta)$, the transported Taylor series of $S_{\Phi}(auZ)$ is a finite O-linear combination of products of the form*

$$\prod_{i,k} F_{y_{ik}}\left(\sum_{\tau \in G} \iota(a s_1 \, \overline{\tau(w_i)}/M) Z_{\tau k} \right).$$

Here $1 \le k \le n$, F_y is from Proposition 31.4, $M \ge 1$ is prime to p, and the $w_i \in \mathcal{A}$, allowing repetitions, are a tight frame of length M^2. For each product one integer $b \ge 1$, prime to p, satisfies $b\delta y_{ik} \in \Lambda$ for all i, k. The integer M and the frame can be taken the same for every δ, Φ, a at the given p.

Proof. If $p \nmid N_1$, take the frame of Proposition 427.1 with $M = N_1$, repeating each vector according to its multiplicity. For $p \mid N_1$, choose two consecutive positive integers at least $\sqrt{24dN_1^4(1+\log_2 N_1)}+1$. At least one is prime to p. Use it for M and apply Theorem 45.2 at length M^2 and $R = 0$. Repeating vectors turns their multiplicities into ones without changing the frame identity. In both cases construct a section F in rank n by Theorem 48.2, using M for its parameter δ, 1 for the element of norm M^2/L, and θ_Λ at every repeated entry. Then $F(0) = 1$.

Apply Theorem 50.2 with $\ell(S) = S(0)$, $V_0 = V_n^\circ$, and the newly constructed F. With $N' = M^2,\ g' = fn$, it gives

$$V_n^\circ = (N')^{-g'} \sum_{v \in \mathcal{R}} (W_v V_n^\circ)(0)\, W_{-v} F.$$

The representatives v have the form $r(\xi_v)$ with $M\xi_v \in \mathcal{A}^n$. The coefficient in each summand is p-integral by Corollary 427.2 and Theorem 51.2, since $p \nmid M$. It suffices by linearity to treat one generator $D_X v_n^\circ$. Pass from the physical to the symmetric action using Theorem 425.2, and apply T_X to the sum. The multiplication rule of Theorem 51.2 replaces $T_X W_{-v} F$ by a root of unity times $T_{X+\xi_v} F$, with rational ξ_v viewed at the finite places. All extra phases are p-integral units.

There is a rational $\gamma \in E^n$ with $T_\gamma F = T_{X+\xi_v} F$ and $\gamma_p - X_p - (\xi_v)_p \in \mathcal{A}^n \otimes \mathbb{Z}_p$. Both conditions specify a neighborhood of $X + \xi_v$ by local constancy in Theorem 51.2, and rational points are dense there by its proof. Since $\delta X_p \in \mathcal{A}^n \otimes \mathbb{Z}_p,\ p \nmid M$ and $M\xi_v \in \mathcal{A}^n$, this implies $\delta\gamma_p \in \mathcal{A}^n \otimes \mathbb{Z}_p$. In a $\mathbb{Z}$-basis of $\mathcal{A}^n$ the coordinates of $\delta\gamma$ are rational numbers integral at p. A common positive denominator b_0 prime to p gives $b_0\delta\gamma \in \mathcal{A}^n$.

Use the product formula of Theorem 48.2 on this rational translate. The torsion shifts are

$$y_{ik} = -(\lambda/M)\, \mathrm{Tr}_{E/K}(\bar{w}_i \gamma_k).$$

Thus $b_0 M \delta y_{ik} \in \Lambda$ by Proposition 44.3, simultaneously in the indices. Take $b = b_0 M$. The coefficient of $Z_{\tau k}$ in the one-variable argument after scaling by au is $(\lambda/(M\sigma_0))au\,\overline{\tau(w_i)} = as_1\,\overline{\tau(w_i)}/M$. The Taylor product statement of Theorem 48.2 gives the displayed formula. All sums are finite, and in each degree the coefficient of a product or of a homogeneous linear substitution is a finite polynomial expression. Coefficientwise application of ι therefore preserves all the identities. □

429. Finite valuations of the rectangular jet matrices

Theorem 429.1. *Use the scale u of Definition 428.1. Let $\Phi \in \mathcal{D}_p(\delta)$ in any positive rank. Choose $\delta_d \in \mathbb{C}^\times$ with $\delta_d^4 = d$. Every mixed derivative at zero of*

$$S_\Phi(\bar{\delta}\,\delta_d\, uZ)$$

is integral under ι. In particular on a definite arrow, for $a \in \mathbb{C}^\times$ every entry of $J_j^{-au}(\Phi)$ satisfies

$$v_p\Big(\iota(J_j^{-au}(\Phi)_{T',T})\Big) \;\geq\; N_j\left(v_p(\iota a) - v_p(\iota\bar{\delta}) - \tfrac{1}{4}v_p(d)\right)$$

with $v_p(0) = +\infty$.

Proof. Apply Theorem 428.2 with $a = \bar{\delta}\,\delta_d$. For any product in its expansion, write $\delta' = b\delta$, with $p \nmid b$, so that $\delta' y_{ik} \in \Lambda$ for every shift. Put $t = \iota(\delta_d)$, which has valuation $v_p(d)/4$. The coefficient of $Z_{\tau k}$ in the linear argument, divided by $\iota(\overline{\delta'})t$, is

$$\frac{\iota(s_1\overline{\tau(w_i)})}{bM}.$$

This is integral. Indeed membership in $\mathcal{A}$ gives $v_p(\iota\overline{\tau(w_i)}) \geq -d_\iota$ by Definition 44.2. The valuation of ιs_1 is d_ι by Theorem 46.1 and the choice of s_1. Both integers in the denominator are p-units. Thus Theorem 39.2 gives integrality of all mixed derivatives of the product.

Evaluation of a fixed derivative commutes with coefficient transport, since it multiplies one coefficient by a product of integer factorials. It also commutes with the finite O-linear combination, proving the first assertion.

The entries of the rectangular Schur matrix are polynomials with integer coefficients homogeneous of total degree N_j by Proposition 409.2, taking the product over the places as in Definition 411.1. At scale $\bar\delta\,\delta_d\,u$ the corresponding jet is an integral linear combination of the mixed derivatives just treated. Theorem 425.2 identifies this jet with its class matrix before changing the scale. Replacing $\bar\delta\delta_d$ by a multiplies every entry by $(a/(\bar\delta\delta_d))^{N_j}$ by homogeneity. Taking valuations proves the bound. □

Theorem 429.2. *Fix a prime $p < x_1$ with $\eta_d(p) \neq 1$. On the j-th definite arrow take $\Phi \in \mathcal{D}_p(1)$ and $a \in \mathbb{C}$ with $|\iota a|_p \le 1$. For every $V \subset G$ and every charge system of Theorem 28.5, one has*

$$v_p\Big(\iota(J_j^{-au}(\Phi)_{T',T})\Big) \ge J_p(j,V) - m_{T'}$$

whenever $a \neq 0$, for all semistandard output and input indices T', T. For $a = 0$ the same inequality holds for the derivatives defining the entries at scale zero. All the quantities on the right and the charge conditions are exactly those of Definition 26.1 and Theorem 28.5, including the exponent zero at $p = d$.

Proof. Such a prime is prime to N_1 by the splitting argument of Proposition 29.2 and Lemma 24.3. The frame in Theorem 428.2 can therefore be taken to be the repeated roots of unity, with $M = N_1$. Its length M^2 is prime to p. Also $v_p(\iota s_1) = 0$ by Theorem 46.1 and Definition 44.2. Thus the scalar $\beta = \iota(a s_1/M)$ in the linear forms of that expansion belongs to O, and with $\delta = 1$ all torsion denominators in each product can be cleared by one integer prime to p. Theorem 37.3 and Theorem 30.2 give integrality through degree N_j after each of the linear substitutions in Definition 28.3. This integral coefficient condition is preserved under the finite O-linear sum of the expansion. It therefore holds for the transported series of $S_\Phi(auZ)$ itself.

Write the entries of the arrow variable as $Z_{\tau,a',b'}$ with a' the input index and b' the output index, so this is the transpose display of the rectangular variable in Definition 409.1. The determinant for a column in $F_{T'}(Y)$ in Definition 28.3 has entries $\sum_{a'} Y_{\tau,i,a'}\partial_{\tau,a',b'}$. Transposing its determinant puts the output labels in its rows and the ordered positions i in its columns. Its product over the diagram is precisely the polynomial $b_{d_{j+1}}(\partial Y^{\mathsf T})_{T'}$ across the places. Each full input-height column in a semistandard input filling has the entries $1,\dots,d_j$, in order. Removing these columns is a bijection with fillings of the shorter diagram: when they are restored on the left their entries in row i are i, and every entry in row i of a strictly increasing column is at least i. Their minor is $\det Y$ at that place. Consequently the standard coefficient indexed by the remaining T is exactly the coefficient extraction in Proposition 409.2. It gives the (T', T) jet entry by Theorem 425.2 for nonzero scale and by the same differentiation formula at zero. Theorem 28.5 now proves the asserted bound. □

430. Integral frames for a finite-place class estimate

Lemma 430.1. *Let $H = U(E^h)$ for the positive standard Hermitian form, with rational points Γ over E^+. At any rational prime p, Γ is dense in the product H_p of the groups over all completions of E^+ above p. Every finite position $g \in H_f$ therefore admits $\gamma \in \Gamma$ such that $(\gamma_f g)_p$ preserves $(O_E \otimes \mathbb{Z}_p)^h$.*

Proof. In rational matrix coordinates put $\mathfrak{u} = \{X : X^* = -X\}$. Over $\mathbb{Q}_p$ its scalar extension consists of all skew-adjoint matrices with entries in $E \otimes \mathbb{Q}_p$. Indeed the map $X \mapsto (X - X^*)/2$ is a rational projection whose image is exactly this subspace over either field, including over $\mathbb{Q}_2$. Taking a rational basis of its image, approximate each coefficient in $\mathbb{Q}_p$ by a rational

number, by truncating its p-adic expansion. Thus the rational skew-adjoint matrices are dense in the local ones.

For $h_p \in H_p$ choose a rational scalar $\eta \in K^\times$ of norm one such that $1 + \eta h_p$ is invertible in every field component. The scalars $(1 + j\sqrt{-d})/(1 - j\sqrt{-d})$, $j = 1, 2, \ldots$, are pairwise distinct in every component by cross multiplication. At a fixed component $\det(1 + Th_p)$ is a polynomial with constant term one, so it rules out only finitely many scalars. There are finitely many components by Theorem 18.2. The matrix $X = (\eta h_p - I)(\eta h_p + I)^{-1}$ is skew-adjoint, by using $(\eta h_p)^* = (\eta h_p)^{-1}$. The inverse formula is $\eta h_p = (I + X)(I - X)^{-1}$, with $I - X$ invertible. Approximate X by rational skew-adjoint matrices as above. For sufficiently good approximants the inverse in this formula exists in every component, by the determinant and adjugate formulas. It then exists over E as well. Each resulting matrix is rational unitary by taking adjoints, and after multiplication by η^{-1} they approximate h_p. Finally the integral invertible matrices form an open subgroup at p: requiring a matrix and its inverse to be integral is an open condition at every component. Approximate g_p^{-1} to obtain the last assertion. □

Proposition 430.2. *On a definite arrow let g, h be finite source and target elements preserving the standard lattices at p. Then $\Omega_{i,f}(g, h)\mathcal{D}_p(\delta) = \mathcal{D}_p(\delta)$ for every $0 \neq \delta \in O_K$.*

Proof. Put $GX = hXg^{-1}$. For the physical lattice $L_0 = \alpha M_{d_{i+1},d_i}(\widehat{O}_E)$, one has $GL_{0,p} = L_{0,p}$. The index $H = [GL_0 : GL_0 \cap L_0]$ is prime to p: the local factor at p is one and the quotient at each other rational prime l has order a power of l, by the diagonal lattice calculation in Theorem 375.2. Take representatives r for this quotient zero above p. By Theorem 358.1 the normalized intertwiner W_G sends the scaled vacuum v_n° to $H^{-1/2} \sum_r D_r v_n^\circ$. It sends $D_X v_n^\circ$ to the same sum translated by D_{GX}. Combining two displacements introduces only finite character values by Lemma 356.2. Each index $GX + r$ satisfies $\delta(GX + r)_p \in L_{0,p}$ whenever $\delta X_p \in L_{0,p}$, and $H^{-1/2}$ is a p-unit.

It remains to account for the splitting and determinant factors. Theorem 390.2 multiplies W_G by $\beta(J_G)$. The ideal J_G has valuation zero at all places above p by Definition 375.1, so its character value is a p-unit by Theorem 374.2. Both determinants in Theorem 396.1 are units above p. Theorem 372.2 gives the same conclusion for their characters, including all rank twists. Thus the entire operator preserves the module. Apply the inclusion also to the inverse pair and use the representation law to get equality. □

Theorem 430.3. *For stage i, let $D_i(g_p)$ be the matrix over K_p obtained by evaluating $\bigotimes_\tau (\det A_\tau)^{e_{i,\tau}} S_{d_i}(A_\tau)$ at the matrices of g_p under the local embeddings induced by $z \mapsto \iota(\overline{\tau(z)})$ on E. The embedding extends to its completion by limits of Cauchy sequences, with the valuation defining that place. For $v \in E_i$, the complex weight space, define*

$$|v|_{i,g} = \|D_i(g_p)^{-1} \iota(v)\|_{\max}.$$

Suppose that every $\Phi \in \mathcal{D}_p(\delta)$ satisfies $\|\iota J_i^c(\Phi)\|_{\max} \leq b_i$ for a fixed scale $c \neq 0$, where the matrix norm is its largest entry modulus. Then at all finite positions g, h,

$$|J_i^c(\Omega_{i,f}(g, h)\Phi)v|_{i+1,h} \leq b_i |v|_{i,g}.$$

Consequently, in the nonarchimedean conclusion of Corollary 424.2, these norms satisfy the required operator inequalities whenever the i-th translated vacua in the support belong to the respective $\mathcal{D}_p(\delta_i)$ and the bounds b_i hold there.

Proof. The evaluation matrices are multiplicative by Proposition 409.2 and multiplicativity of the determinant. On a rational γ one has $D_i(\gamma_p) = \iota(R_i(\gamma_\infty))$ by Proposition 409.3. Hence $|R_i(\gamma_\infty)v|_{i,\gamma_f g} = |v|_{i,g}$. For an integral unitary frame at p, both D_i and its inverse have integral entries by the integer polynomial formulas and the unit determinant. Such a matrix preserves the max norm by the ultrametric inequality applied in both directions.

Choose rational γ, ϵ on the two sides by Lemma 430.1 making $\gamma_f g, \epsilon_f h$ integral at p. The rational covariance in Theorem 411.3 identifies the prescribed inequality with that at the

new two positions, using $R_i(\gamma_\infty)v$ and the preceding isometries. There the two norms are ordinary max norms. Proposition 430.2 puts $\Omega_{i,f}(\gamma_f g, \epsilon_f h)\Phi$ in the same module as Φ. By the hypothesis all matrix entries have absolute value at most b_i, so each output coordinate has absolute value at most b_i times the input max norm. This proves the inequality. Its product over the finite set of stages gives the claimed application to the class sum. □

431. The local estimate away from marked collisions

Theorem 431.1. *Take the normalized vacua of Corollary 427.2 on all arrows. Fix s_1, u of Definition 428.1 and take $c = -\bar{\pi}_g u$, with $\pi_g \in O_K$ as in Theorem 11.4. Use the function Ψ of Theorem 423.2 at this scale. For $p \nmid NQ$ fix ι, and suppose $|\phi(g)|_{0,g} \le 1$ at all finite arguments of a column of level $K_{c,0}$, with the norms of Theorem 430.3. On $U(Y_m)$ define $|\cdot|_{Y,h}$ by the same maximum norm in the frame D_Y of Proposition 417.2, using the embeddings of that theorem on the finite matrix h_p as in Theorem 430.3. Then for every h and every sufficiently large x in the specified windows the following bounds hold.*

(1) *If $p \neq d$, then $|(B_\Psi\phi)(h)|_{Y,h} \le 1$.*
(2) *If $p < x_1$ and $\eta_d(p) \neq 1$, then*

$$|(B_\Psi\phi)(h)|_{Y,h} \le p^{-\sum_{j<m} J_p(j,\varnothing)} \le p^{-G_p + mC_p(\varnothing) + 30f\ell a/p}.$$

In particular if this prime is unserved in Definition 24.1, one has $|(B_\Psi\phi)(h)|_{Y,h} \le p^{-G_p+2e_p/3}$.

The estimates are uniform in ι, Ψ, ϕ, h for the indicated data and include the ramified columns on which the projected transfer identity holds.

Proof. The support statement of Theorem 423.2 places each translated vacuum in $\mathcal{D}_p(\pi_g)$. Indeed the displacements are zero at places above p outside the marked level unless $p \mid NQ$. At places dividing π_g their entries divided by α become integral after multiplication by $\bar{\delta}_g$ in Definition 368.1, which equals π_g at those places by Lemma 348.1. Thus the defining bound holds simultaneously above p. The first estimate now follows for the chain on X_m by taking $\delta = \pi_g$, $a = \bar{\pi}_g$ in Theorem 429.1, where a in that theorem denotes its Taylor multiplier. As $p \neq d$ its lower valuation is zero at every stage. Use Theorem 430.3 and then Corollary 424.2 with every $b_i = 1$. The condition $p \nmid Q$ ensures integrality of every coefficient in this class sum.

For the second bound, p differs from g_0, g_1, since $p < x_1$ while $\log g_i \ge 10\ell > \log x_1$ by Theorem 11.4. All the displacements in the expansion are therefore integral at p, and $|\iota(\bar{\pi}_g)|_p = 1$ because $\pi_g\bar{\pi}_g = g_0 g_1$. Apply Theorem 429.2 with $V = \varnothing$ and $\gamma_{\tau,h,b'} = 0$ for all indices. The charge condition of Theorem 28.5 is then empty, and $m_{T'} = 0$ for every output. Thus the full matrix at stage j has max norm at most $p^{-J_p(j,\varnothing)}$ on $\mathcal{D}_p(1)$. The same class and gauge arguments give the product of these bounds on X_m.

In either case evaluate the chain at $\iota_H(h) = \mathrm{diag}(h, 1)$, the terminal inclusion in Definition 416.3. The last coordinate of its evaluated frame is 1, as allowed in Proposition 417.2 because $p \neq N$. That proposition passes both bounds to $B_\Psi\phi$. Summing Theorem 26.3 over $j < m$ with $V = \varnothing$ gives the second inequality in the display. For the unserved case Proposition 24.2 gives $mC_p(\varnothing) \le e_p/3$. Also $30f\ell a/p \le e_p/3$ once $\ell \ge 30$, by Proposition 12.6. Theorem 11.4 gives this uniformly eventually. Adding the two losses proves the last bound. □

432. A one-column Fourier expansion at the auxiliary prime

Definition 432.1. Fix an embedding $\iota : \mathbb{C} \longrightarrow K_N$. Let S consist of the places defined by $z \mapsto \iota(\overline{\tau}z)$ for the f embeddings τ of the chosen CM type, and put $R = \{w : w \mid \pi_N\}$. Write $\mathfrak{o} = \prod_{w\in S} \mathbb{Z}_N$, $k_N = \prod_{w\in S} \mathbb{F}_N$, and $e_j = (0, \dots, 0, 1)^t$ in rank d_j. In each factor use the

split matrix variable T_w on the w-side, in units of α_w. The conjugate side acts by characters of this matrix variable. Normalize the measure of the integral box to one and write P_N for projection to its unit indicator on the factors at N. If $P_N v = \mathbf{1}_{\mathrm{box}} \otimes v'$, define $[F]v = F \otimes v'$ for each locally constant, compactly supported function of these matrices. This definition includes $v' = 0$. For $y \in k_N^{d_{j+1}}$ set

$$b_y(T) = \mathbf{1}_{T \in M_{d_{j+1},d_j}(\mathfrak{o})} \mathbf{1}_{(Te_j \bmod N) = y}.$$

Lemma 432.2. *The set S consists of one place in each conjugate pair above N, all of residue degree one. There exists $\delta \in O_K \setminus \{0\}$, depending on ι, such that $N^{f d_{j+1}} [b_y] D_X v^\circ \in \mathcal{D}_N(\delta)$ for every finite displacement X, every y, and every $j < m$. Moreover $v_N(\iota\bar{\delta}) = 0$.*

Proof. The prime N splits completely by Lemma 13.2. The f conjugated embeddings give the f distinct extensions of the same place of K, since the original embeddings fix K and permute its f extensions freely at a completely split prime. Conjugation gives the extensions of the other place. Put $\delta = \omega - h$, where $h \in \mathbb{Z}$ represents the residue of $\iota(\omega)$ in $\mathbb{F}_N$ and $\omega = (1 + \sqrt{-d})/2$. The original and conjugate roots are distinct since $N \nmid d$. Thus δ is divisible by N at all the places in $\bar{S}$, while $\iota\bar{\delta}$ is a unit.

In the split factor the two matrix summands, supported respectively on $S, \bar{S}$, are isotropic and dual. In fact their commutator pairs each matrix entry with its conjugate entry by the trace pairing of Lemma 355.2. Its conductor on the normalized coordinates is the integral lattice in both arguments. The proof of Proposition 391.1, applied just to these local summands, therefore sends the local vacuum to $\mathbf{1}_{\mathrm{box}}$. A displacement on S translates the argument and a displacement on $\bar{S}$ multiplies it by the corresponding character. All finite products of these actions differ from displacement by their sum by a root of unity, by the multiplication law of Theorem 358.1. The tensor description in Definition 432.1 also follows directly by taking a basis of translated vacua at N and its complementary factors. The local vacuum coefficient of $D_X v^\circ$ is zero if its normalized N-component is nonintegral: a nonintegral translation moves the support off the box, and a nontrivial character on the box has integral zero by translation invariance. Otherwise projection replaces this component by the vacuum with a root of unity. The scalar used to scale the global vacuum stays on the complementary factor in both cases.

Choose in each factor a nontrivial character of the additive group $\mathbb{F}_N$, and let ψ be their product. On integral matrices the finite geometric sum gives

$$N^{f d_{j+1}} b_y(T) = \mathbf{1}_{\mathrm{box}}(T) \sum_{t \in k_N^{d_{j+1}}} \psi\big(t^t((Te_j \bmod N) - y)\big).$$

Indeed in each differing coordinate the sum of all powers of a nontrivial N-th root of unity vanishes, and each matching coordinate contributes N. Outside the box both sides vanish. Perfect duality realizes each multiplier as a displacement supported on $\bar{S}$ in the dual of the last column, with entries in $N^{-1}\mathfrak{o}$ in normalized coordinates. To see the denominator explicitly, pairing the quotient $N^{-1}\mathbb{Z}_N/\mathbb{Z}_N$ on the dual side with $\mathbb{Z}_N/N\mathbb{Z}_N$ is injective in each argument by the conductor calculation. Both have N elements, so it realizes all residue characters. Every summand acting on the projected vacuum is consequently a translate with $\delta X'_N \in \widehat{\mathcal{A}}_N^{d_j d_{j+1}}$ and coefficient zero or a root of unity. This is precisely membership in Definition 428.1. □

Proposition 432.3. *At scale $c = -\bar{\pi}_g u$, all finite frames g, h, all displacements X, vectors v of the source weight, and residues y satisfy*

$$\big|J_j^c(\Omega_{j,f}(g,h)[b_y] D_X v^\circ) v\big|_{j+1,h} \le N^{f d_{j+1}} |v|_{j,g}.$$

In particular N^{fb} suffices for each arrow at all sufficiently large arithmetic scales.

Proof. In Theorem 429.1 use the δ of Lemma 432.2 and $a = \bar{\pi}_g$. The three valuations in its bound vanish because $N \nmid gd$ and $\iota\bar{\delta}$ is a unit. Theorem 430.3 transfers the resulting integral estimate at the identity to all frames. Apply it to $N^{fd_{j+1}}[b_y]D_X v^\circ$ and divide, using $|N|_N = N^{-1}$. Finally $d_{j+1} \le n_m + 1 = O(x^{11/2})$ by Definition 348.2 and Lemma 26.2, whereas $b = \exp(\ell)$ dominates every fixed power of x by Theorem 11.4. □

Proposition 432.4. *For $U \subseteq S$ let $\Delta_j(U)$ be the rank d_j diagonal matrix with entries $1, \dots, 1, N$ on U and identity elsewhere. Use $\psi_w(\bar{x})$ equal to the modulation at $T = 1$ of a displacement with dual coordinate $-\bar{x}/N$, lifting the residue to $\mathbb{Z}_N$. Put $\psi = \prod_w \psi_w$ and*

$$c_y(Z) = \mathbf{1}_{\mathrm{box}}(Z)\, \psi((e_{j+2}^t Z \bmod N)y) \qquad (y \in k_N^{d_{j+1}}),$$

where the second factor is evaluated only in the box. Put $A = \Delta_j(S \cap R)$, $B = \Delta_{j+2}(S \setminus R)$. On each even pair $j, j+1$ the projection followed by the sum of auxiliary translates in Corollary 370.2 takes $v \otimes v'$ to

$$\sum_{y \in k_N^{d_{j+1}}} [\, b_y(\,\cdot A)\,]v \;\otimes\; [\, c_y(B\,\cdot\,)\,]v'.$$

Proof. Projection removes all the N-components except their vacuum coefficients. Representatives in Theorem 367.1 may be changed by integral elements on their supported side: that side is isotropic and integral displacements fix the vacuum. Choose representatives with negative numerator divided by N. By Proposition 368.2 the two displacements occupy the last column and last row in consecutive matrices, their entries conjugate to each other. Under the two maps to $\mathbb{Q}_N$ related by field conjugation their numerators therefore agree. The original two definite Hom spaces have identity Gram matrices. In the same normalized entry the modulation character on either space is thus the same, by Lemma 355.2. It is nontrivial on $N^{-1}\mathbb{Z}_N/\mathbb{Z}_N$ by the conductor computation just proved.

Consider one $w \in S \cap R$, and denote the two matrix arguments by T, Z. At w the first displacement translates Te_j by $-z/N$ for a residue vector z, and the second multiplies an integral Z by $\psi_w((e_{j+2}^t Z \bmod N)z)$. All the other columns of T must be integral. Precisely one z contributes when NTe_j is integral, namely its reduction, and none otherwise. This is the claimed sum with $y_w = NTe_j \bmod N$. At $w \in S \setminus R$ the two roles are reversed. Now T must be integral, the rows of Z other than the last must be integral, and $Ne_{j+2}^t Z$ must be integral. The unique translating numerator is $Ne_{j+2}^t Z \bmod N$. Its modulation on the first factor gives precisely the indicated phase with $y_w = Te_j \bmod N$. Each computation used a pure translation and a pure modulation on different factors, so no product phase occurs. Multiplying over w proves equality on matrix arguments. Equality in the algebraic tensor product follows: in the finite-dimensional spans of the finitely many factors, point evaluations span the dual since they separate functions. The complementary components are unchanged by the projection and these displacements. □

433. Moving a residue Fourier operator across the arrows

Definition 433.1. Use $\mathfrak{o}, k_N, e_j, \psi$ of Definition 432.1 and Proposition 432.4. In rank d_j put $G_j^S = \prod_{w \in S} \mathrm{GL}_{d_j}(\mathbb{Q}_N)$, $K_j^S = \mathrm{GL}_{d_j}(\mathfrak{o})$, and $M_j = \{a \in K_j^S : \bar{a}e_j = e_j\}$, where the bar on an integral matrix is reduction modulo N. Let π_j be a complex representation of G_j^S on V_j, and $u_j = \Delta_j(S)^{-1}$ in this section. A vector is ramified here if for every $w \in S$ it belongs to the image of $1 - \pi_j(xI)$ for a unit $x \in \mathfrak{o}^\times$ equal to 1 modulo N outside w. The unit may depend on the place and the vector.

For every $y \in k_N^{d_j}$ with unit last entry y_* choose $\kappa_y \in K_j^S$ with last column y and last row y^t modulo N. Such a choice exists: if C is the identity with last column replaced by y, the matrix $y_*^{-1}CC^t$ has both prescribed boundaries and unit determinant. Lifting its entries

arbitrarily gives a matrix with unit determinant over $\mathfrak{o}$. Define

$$F_j = \pi_j(u_j) \sum_{y:\ y_* \in k_N^\times} \psi(y_*)\pi_j(\kappa_y).$$

For a function f on residue matrices identify f with its inflation to integral matrices extended by zero. Write $b_j = b_{e_{j+1}}$, $c_j = c_{e_j}$, and let r_j indicate integral matrices whose last row modulo N is e_j^t. Let $L_j[f] : V_j \to V_{j+1}$ be linear in both arguments. The two identities required below are

$$\begin{aligned} \pi_{j+1}(b)L_j[f(\bar{b}\cdot\bar{a})]\pi_j(a) &= L_j[f] && (a \in K_j^S,\ b \in K_{j+1}^S), \\ \pi_{j+1}(u_{j+1})L_j[r_j] &= \gamma_j L_j[b_j]\pi_j(u_j) && (\gamma_j \in \mathbb{C}). \end{aligned}$$

Lemma 433.2. *Under the identities of Definition 433.1, for every ramified $\phi \in V_j$,*

$$\pi_{j+1}(u_{j+1})L_j[c_j]\phi = F_{j+1}L_j[b_j]\phi = \gamma_j L_j[b_j]F_j\phi.$$

Both F_j and each $L_j[f]$ preserve ramification in their respective target representations. The operator F_j also preserves the subspace of M_j-fixed vectors.

Proof. Write $T_* = e_{j+1}^t T e_j$ on integral matrices. For $t \in k_N$ with $t_w = 0$ choose a central difference $\phi = (1 - \pi_j(xI))\xi$ at w. Multiplication of T by $\bar{x}$ preserves the indicator $1_{T_*=t}$: at w it preserves the zero residue, elsewhere it is identity. Take $a = xI, b = 1$ in the first identity of the definition. Consequently $L_j[1_{T_*=t}]\phi = 0$. Expanding $c_j(T) = \psi(T_*)$ in the indicators of T_* therefore allows restriction to unit T_*.

Now partition this restricted function by the last column y. Its weight is $\psi(y_*)$. The first identity with $a = 1, b = \kappa_y$ gives $\pi_{j+1}(\kappa_y)L_j[b_j] = L_j[b_y]$, since $\bar{\kappa}_y e_{j+1} = y$. Summing and multiplying by $\pi_{j+1}(u_{j+1})$ gives the first equality. Partition instead by the last row y^t; its weight is again $\psi(y_*)$. With $a = \kappa_y, b = 1$ the indicator of last row y^t gives $L_j[r_j]\pi_j(\kappa_y)$, as $e_j^t\bar{\kappa}_y = y^t$. The second identity of the definition gives the last equality.

Every central scalar commutes with each summand of F_j. Taking $a = x^{-1}I, b = xI$ in integral covariance gives $\pi_{j+1}(xI)L_j[f] = L_j[f]\pi_j(xI)$, so the image conditions persist. Finally take $g \in M_j$ and $k = u_j^{-1}gu_j$. In the splitting off of the last coordinate, g has integral entries, upper-right block divisible by N, and bottom-right entry 1 mod N. Thus k is integral, with integral inverse and last row e_j^t mod N. The permutation $y \mapsto z = \bar{k}y$ fixes y_*. The element $\kappa_z^{-1}k\kappa_y$ belongs to M_j because its last column reduces to e_j. For any M_j-fixed vector ξ we get $\pi_j(gu_j\kappa_y)\xi = \pi_j(u_j\kappa_z)\xi$. Summation gives the last assertion. □

Theorem 433.3. *Suppose the identities of Definition 433.1 hold along $2p$ consecutive arrows starting at stage zero. Put $B_j = L_j[b_j]$ and, for $j = 2i < 2p$, let*

$$T_i = \pi_{j+2}(u_{j+2}) \sum_{y \in k_N^{d_{j+1}}} L_{j+1}[c_y]L_j[b_y], \qquad o_i = (N^{d_{j+1}} - 1)^f.$$

For every ramified $\phi \in V_0$,

$$T_{p-1}\cdots T_0\phi = \left(\prod_{i=0}^{p-1} o_i \prod_{l=0}^{2i+1} \gamma_l\right) B_{2p-1}\cdots B_0 F_0^p\phi.$$

Empty products of operators denote the identity.

Proof. If $y_w = 0$ as an entire vector at any w, multiplication by the residue of the central scalar at w preserves the condition $Te_j = y$. The central difference argument of Lemma 433.2 gives $L_j[b_y]\xi = 0$ on every ramified ξ. Otherwise choose $k \in K_{j+1}^S$ with $\bar{k}e_{j+1} = y$: over each field factor complete the nonzero vector to a basis by adjoining standard vectors which increase its span, then lift the resulting invertible matrix. Integral covariance gives

$L_j[b_y] = \pi_{j+1}(k) B_j$. On the next arrow $c_y(T) = c_{j+1}(T\bar{k})$, so $L_{j+1}[c_y]\pi_{j+1}(k) = L_{j+1}[c_{j+1}]$. There are o_i such vectors. The lemma first at $j+1$ and then at j now gives

$$T_i\xi = o_i\, \gamma_{j+1}\gamma_j B_{j+1} B_j F_j \xi.$$

For completeness, moving a single F_s leftward in the composition order toward the initial vector gives

$$F_s B_{s-1} \cdots B_0 \eta = \left(\prod_{l<s} \gamma_l \right) B_{s-1} \cdots B_0 F_0 \eta$$

on every ramified η, by applying the second equality of the lemma successively to $B_l \cdots B_0 \eta$. All intermediate vectors remain ramified by that lemma. Induct on p, applying the pair identity to the output of the first $p-1$ pairs and the last formula with $s = 2p-2$ to $\eta = F_0^{p-1}\phi$. This gives exactly the factors and the initial power in the assertion. □

434. The auxiliary-prime estimate for the projected transfer

Definition 434.1. Keep S, R of Definition 432.1. Embed G_j^S in the finite unitary group by the indicated matrices on S, their conjugate inverse transposes on $\bar{S}$, and identity elsewhere. Denote both a matrix and its embedded element by the same letter when used in right translation r_j. Put $U = S \cap R$, $U' = S \setminus R$, $A_j = \Delta_j(U)$, and $\pi_j(a) = r_j(A_j a A_j^{-1})$ on smooth forms of the weight of stage j. For one finite vector v_j on this arrow put

$$L_j[F] = r_{j+1}(A_{j+1}) \mathcal{L}^c_{j,[F]v_j} r_j(A_j)^{-1} = \mathcal{L}^c_{j,\Omega_{j,f}(A_j, A_{j+1})[F]v_j}.$$

The equality is Theorem 411.3. Use the last-row level of Proposition 349.1 also on X_m when discussing its local subgroup at N.

Lemma 434.2. *The operators L_j satisfy both identities of Definition 433.1 with $\gamma_j = N^{f(d_{j+1}-d_j)/2}$. Define the complex character of the local general linear group, for any even rank parameter l, by*

$$D_{j,l}(a) = \mu_{j+1}(1, \det a^{\mathrm{emb}}) \prod_{w \in S} |\det a_w|_w^{-l/2},$$

where a^{emb} denotes the unitary embedding and $|N|_w = N^{-1}$. For every $V \subseteq S$,

$$\log_N |\iota D_{j,l}(\Delta_i(V))|_N = \sum_{\tau:\, w_\tau \in V} w_{j+1}(\tau) - \tfrac{1}{2} l |V|, \qquad w_\tau = \mathrm{place}(\iota \circ {}^- \circ \tau).$$

Proof. Write T for the split coordinate. On named functions the full finite action supported on the pairs at N satisfies

$$\Omega_{j,f}(a,b)[F]v_j = \frac{D_{j,d_j}(b)}{D_{j,d_{j+1}}(a)} [F(b^{-1} \cdot a)] v_j.$$

Indeed $T \mapsto bTa^{-1}$ has modulus $\prod_w |\det b_w|_w^{d_j} |\det a_w|_w^{-d_{j+1}}$: apply row operations on each column and column operations on each row. Change of variables by its inverse, with the inverse square root of this modulus, is the unitary intertwiner W_G: it has positive vacuum overlap, and this property uniquely specifies the intertwiner by Theorem 358.1. For the remaining phase, the pair of determinant ideals at $w, \bar{w}$ contributes $\beta(\mathfrak{p}_w)^{-v_w(\det G_w)}$ by the diagonal calculation of Theorem 375.2. The rank twists contribute its inverse, exactly as in the proof of Theorem 376.2: the possible leftover exponents are the two rank parities, and $d_i = k + i(t+1)$ is even since k is even and t odd by Definition 56.1. The remaining character is the ratio of the two μ_{j+1}-values. All these operators act on the designated split factor and leave the complementary test in the formula fixed.

For integral a, b their character factors are one by the finite unit character in Theorem 376.2 and $N \nmid Qg$. Applying Theorem 411.3 and conjugating the two outer translations by A_j, A_{j+1} gives integral covariance in precisely the stated order. For $a = u_j, b = u_{j+1}$, observe

$r_j^{\mathrm{row}}(u_{j+1}^{-1}Tu_j) = b_j(T)$, where r_j^{row} is the last-row indicator denoted r_j in Definition 433.1. In block form the argument scales its upper-right entries by N^{-1}, its lower-left entries by N, leaving its lower-right entry fixed. Integrality with the last-row condition after this change is exactly integrality with the last-column condition before it. Both determinants have the same norm-one embedding at the finite places, so the μ_{j+1} factors cancel. The positive modulus factor is γ_j, giving diagonal covariance.

For the valuation formula put $z = \det \Delta_i(V)^{\mathrm{emb}}$. Its valuations are 1 on V, -1 on $\bar{V}$ and zero elsewhere. Theorem 372.2 gives z^h/a integral unit everywhere with $a\bar{a} = 1$ and $\mu_{j+1}(1,z)^h = \prod_\tau \tau(a)^{w_{j+1}(\tau)}$ for some $h \geq 1$, since the weight is $-w_{j+1}$. The place induced by $\iota\tau$ is $\bar{w}_\tau$. Hence $v_N(\iota\tau(a)) = -h$ when $w_\tau \in V$ and zero otherwise. Since $v_N(N) = 1$, including the positive factor $N^{l|V|/2}$ gives exactly the displayed equality. □

Proposition 434.3. *Let v_j be arbitrary finite vectors. On ramified columns the definite chain of the recovered pairs $\mathcal{SP}(\otimes_{j<m} v_j)$ is*

$$t_N L_{m-1}[b_{m-1}] \cdots L_0[b_0] F_0^{m/2}, \qquad |\,\iota(t_N)\,|_N \leq N^{mfb}.$$

The scalar t_N is independent of the v_j and the column. The operator F_0 preserves the column level.

Proof. The local level at N in the embedded coordinates equals $A_j M_j A_j^{-1}$. On U the original last row fixes e_j^t modulo N, so its lower-left block is divisible by N and its bottom-right entry is 1 mod N. Conjugation by A_j^{-1} on the left and A_j on the right gives exactly the last-column subgroup. On U' the condition is already on the column, by inverse transpose at the conjugate place. This proves the equality, including the reverse inclusion, since the block calculations reverse. By Theorem 13.6, at each split pair a ramified column lies in the image of $1 - r_0(z)$ for a supported central unit. It is also a scalar in the present coordinates and commutes with A_0, so it witnesses ramification for Definition 433.1. Lemma 433.2 shows F_0 preserves the level at N, and its summands commute with translations at all other primes.

At an even $j = 2i$ put $B = \Delta_{j+2}(U')$. In the pair formula of Proposition 432.4 move A_j from the first argument and B from the second by the change of variables in Lemma 434.2 and translation covariance. Before summing in y the pair is

$$d_i'\, r_{j+2}(B^{-1})\, \mathcal{L}^c_{j+1,[c_y]v_{j+1}}\, \mathcal{L}^c_{j,[b_y]v_j}\, r_j(A_j)^{-1}, \quad d_i' = D_{j,d_{j+1}}(A_j) D_{j+1,d_{j+1}}(B).$$

Indeed the first change uses $(a,b) = (A_j, 1)$ and the second $(1, B^{-1})$. Since $B^{-1} = A_{j+2}u_{j+2}$, their sum is $d_i' T_i$ of Theorem 433.3 with the present L_j, π_j. That theorem yields t_N as the product of d_i', of the positive integers o_i, and of the factors γ_l occurring there. Each o_i is prime to N; every γ_l has N-adic absolute value at most one since $d_{l+1} \geq d_l$. Lemma 434.2 bounds $\log_N |\iota d_i'|_N$ by $2b(|U| + |U'|) = 2fb$, using Proposition 12.4. Multiplication over $m/2$ pairs proves the assertion. □

Theorem 434.4. *Use the scale, vacua and Ψ of Theorem 431.1. For a ramified column ϕ suppose the shifted column $\xi = F_0^{m/2}\phi$ satisfies $|\xi(g)|_{0,gA_0} \leq 1$ at every g. Then for every terminal argument h*

$$|(B_\Psi \phi)(h)|_{Y,h} \leq N^{mfb+f\sum_{j<m} d_{j+1}} \leq N^{2mfb}$$

at all sufficiently large scales, uniformly in the twist.

Proof. Apply Proposition 434.3 to the expansion of $\mathcal{SP} w_\Psi = w_\Psi$ in Theorem 423.2. Each summand of the resulting chain has modified arrow $\Phi \mapsto T_j\Phi = \Omega_{j,f}(A_j, A_{j+1})[b_j]\Phi$. The tensor $(\otimes T_j)w_\Psi$ is jointly invariant for the same column level. To verify this assertion at N, each original level element is integral, acts on the box by an integral change of variables with modulus one and trivial character, and hence fixes the box and its coefficient projection. After conjugation by A_j, A_{j+1} the two matrices are in M_j, M_{j+1}; their left and right action fixes b_j since each fixes the last basis vector modulo N. This proves that T_j intertwines the

two level actions at N. At the complementary places the operations commute because they concern separate oscillator factors. Tensoring shows that the original invariance implies the assertion.

By Proposition 434.3, ξ has the required column level. Theorem 424.1 therefore expresses this modified chain at $h' \in H_{m,f}$ as the sum over tuples of classes. On each arrow at g, g' the test is $\Omega_{j,f}(gA_j, g'A_{j+1})[b_j]D_X v^\circ$. Proposition 432.3 bounds its operator norm in exactly these shifted frames by $N^{fd_{j+1}}$. The coefficients of Ψ are integral at N since $N \nmid Q$. The ultrametric inequality gives the product of these bounds times the bound for t_N, uniformly in h'. Finally take $h' = \mathrm{diag}(h, 1)$. In the frame $h'A_m$ the last entries at w_τ are N exactly when $w_\tau \mid \pi_N$, equivalently when $\iota\tau(\bar{\pi}_N)$ is a nonunit. Proposition 417.2 now gives the indicated norm on Y_m. Proposition 432.3 also proves $d_{j+1} \le b$ eventually, which gives the last inequality. □

435. The index of the initial Fourier condition

Definition 435.1. Fix an embedding ι at N and use $O_\iota = \{z \in \mathbb{C} : |\iota z|_N \le 1\}$. For a finite frame C in the initial unitary group put

$$\mathcal{E}(C) = \{\phi \in \mathrm{Col} : |\phi(g)|_{0,gC} \le 1 \text{ at every finite } g\}.$$

Write $c_0 = \dim_{\mathbb{C}} \mathrm{Col}$ and let A_{top} be the set A of the U nontrivial symmetric-power factors in Definition 13.3. For $V \subseteq S$ set

$$h(V) = \#\{\tau \in A_{\mathrm{top}} : w_\tau \in V\}, \qquad t(V) = \frac{a}{k}\, h(V) c_0.$$

Use $A_0 = \Delta_0(S \cap R)$ from Definition 434.1.

Proposition 435.2. *For every $V \subseteq S$, set $D = \Delta_0(V)$. The module $\mathcal{E}(A_0)$ is contained in $\mathcal{E}(A_0 D^{-1})$. In suitable common class coordinates these two modules are respectively $\bigoplus_\lambda N^{s_\lambda(V)} O_\iota$ and $O_\iota^{c_0}$. The index exponents are*

$$s_\lambda(V) = \sum_{\substack{\tau \in A_{\mathrm{top}} \\ w_\tau \in V}} m_{\tau,k}, \qquad \sum_\lambda s_\lambda(V) = t(V) \in \mathbb{Z}_{\ge 0},$$

where λ runs over a class and a tensor monomial $\prod_{\tau,i} e_{\tau,i}^{m_{\tau,i}}$ of degree a in each top factor. Consequently the least integer $s \ge 0$ for which $\mathcal{I}(\mathcal{E}(A_0 D^{-1}), \mathcal{E}(A_0); N^s)$ holds is $t(V)$.

Proof. In the S-coordinates the level conjugated by A_0^{-1} is M_0 by Proposition 434.3. The level conjugated by $(A_0 D^{-1})^{-1}$ is $D M_0 D^{-1}$. It too is integral: on V a matrix in M_0 has an upper-right block divisible by N; conjugation divides that block by N and multiplies the lower-left block by N. Apply the same argument to its inverse. Hence for either frame C the gauge at gC is unchanged by replacing g by gk', $k' \in K_c$, on a column: the frame then has an integral unit right factor. Left rational invariance of the gauge is the calculation of Theorem 430.3. Testing one representative in each class therefore suffices.

For each class choose a representative g so that $(gA_0 D^{-1})_N$ belongs to the intersection of the maximal integral unitary subgroup K_N^0 with $D K_N^0 D^{-1}$. Indeed this intersection is an open neighborhood of the identity, and Lemma 430.1 allows left multiplication of any representative by a rational element to enter it. Then $(D^{-1} g A_0)_N$ also is integral unitary. Thus in these representatives the $A_0 D^{-1}$ condition is the ordinary integral condition. The A_0 condition is membership in the image under the weight action of D of the ordinary integral module, since gA_0 is D times an integral frame at N. For arbitrary complex coordinate vectors this says precisely that their entries at λ belong to $N^{s_\lambda(V)} O_\iota$: the weight action at zero has determinant exponent zero and D multiplies just the selected last letters by N. These coordinate vectors run independently through every complex tuple by Proposition 13.5.

For each fixed τ permutations of the k letters give bijections on the monomial indices. The sum of $m_{\tau,i}$ over all classes and monomials is therefore the same for each i; its sum

over i is ac_0. This evaluates the exponent sum as claimed, including when V is empty. The ring O_ι is a valuation ring because one of z, z^{-1} has absolute value at most one for any $0 \neq z \in \mathbb{C}$. Lemma 25.1 now gives the assertion about the least exponent. □

Theorem 435.3. *For $p_0 = m/2$ set*

$$E_N = t(S \cap R) + p_0 t(S), \qquad \mathcal{E}_* = \{\phi \in \mathrm{Col} : F_0^{p_0}\phi \in \mathcal{E}(A_0)\}.$$

Then $\mathcal{I}(\mathcal{E}(1), \mathcal{E}_*; N^{E_N})$ holds and, uniformly at the prescribed scales,

$$E_N \ \leq \ (p_0+1)\frac{aU}{k}c_0 \ \ll \ x^{-4}mfb\,c_0.$$

For every finite list of ramified columns in $\mathcal{E}(1)$ one can therefore recombine the list by a matrix over O_ι of determinant dividing N^{E_N} so that every resulting column satisfies the bound in Theorem 434.4. The same matrix works for every permitted finite twist at this embedding.

Proof. Write $F = F_0$ and $u = \Delta_0(S)^{-1}$. For $\phi \in \mathcal{E}(A_0)$ evaluate

$$(F\phi)(g) = \sum_{y:\, y_* \in k_N^\times} \psi(y_*)\phi(gA_0u\kappa_y A_0^{-1}).$$

The input frame for each displayed value is $gA_0u\kappa_y$. Right multiplication by κ_y preserves its gauge because κ_y is integral with integral inverse. The coefficients are roots of unity. Thus $F\phi$ is integral in the frame gA_0u, and it has column level by Proposition 434.3. We have proved $F\mathcal{E}(A_0) \subseteq \mathcal{E}(A_0u)$.

For any finite list in $\mathcal{E}(1)$, first apply Proposition 435.2 with $V = S \cap R$ to obtain a list in $\mathcal{E}(A_0)$ at cost $N^{t(S\cap R)}$. Given a current list whose i-th Fourier powers lie in $\mathcal{E}(A_0)$, their next powers lie in $\mathcal{E}(A_0u)$. Apply that same proposition with $V = S$ to these next powers; its recombination matrix also acts on the original list by linearity of F. The newly recombined next powers belong to $\mathcal{E}(A_0)$ and the cost divides $N^{t(S)}$. Repeating p_0 times and multiplying determinants gives the property of Definition 15.5. We used no inverse to F. By Definition 435.1, $t(S \cap R) \leq t(S) = aUc_0/k$. Theorem 11.4 and Definition 12.3 give $a \leq 2xb$, $k \gg x^5$, $U \leq f$ for sufficiently large x, which gives the second bound. Recombination by complex scalars preserves the ramified subspace. The preceding level-prime theorem therefore applies to every output. The index calculation uses only the fixed initial representation and the prime N, so the same recombination gives all the twists as asserted. □

436. Split Hecke polynomials on a definite theta chain

Lemma 436.1. *Let k_q be a field of q elements. Write $b(n,i)$ for the number of codimension-i subspaces of k_q^n, with value zero outside $0 \leq i \leq n$. Then*

$$\sum_i b(n,i)q^{i(i-1)/2}Z^i = \prod_{\ell=0}^{n-1}(1+q^\ell Z).$$

The number of codimension-i subspaces containing a given h-dimensional subspace is $b(n-h,i)$.

Proof. Subspaces containing a fixed subspace correspond bijectively under quotient to subspaces of the quotient, with codimension preserved. Split $k_q^n = H \oplus k_q e$, where H is a hyperplane. A codimension-i subspace in H has codimension $i-1$ there. One outside H is specified by its codimension-i intersection W in H and the class modulo W of an h such that $e+h$ lies in the subspace. This gives $b(n,i) = b(n-1,i-1) + q^i b(n-1,i)$. For the weighted polynomial P_n it says $P_n(Z) = (1+Z)P_{n-1}(qZ)$, including the end coefficients. Induction from $P_0 = 1$ proves the product. □

Proposition 436.2. *Let p be served, choose $\lambda \in \Lambda$ as in Theorem 24.4 for the residue embeddings induced by $\iota\tau$, and assume $p \nmid Q$. Its residue cardinality is $q = p^L$ and p generates the local maximal ideal. Write $T_{j,i}$ for the operators of Definition 230.1 on X_j at λ, and*

$$P_j(X) = \sum_{i=0}^{d_j} (-X)^i q^{i(i-1)/2} T_{j,i}.$$

Write z_j for the value of μ_{j+1} on the norm-one finite idele with component p at λ, p^{-1} at $\bar{\lambda}$, identity elsewhere. On the arrow $j \to j+1$ put $\mathcal{L}_j = \mathcal{L}^c_{j,\Phi_j}$ for any translate $\Phi_j = D_{s_j} v_j^\circ$ occurring in Theorem 423.2. Then $|\iota z_j|_p = 1$, and on all inputs fixed by the local integral group at λ one has the polynomial identity

$$P_{j+1}(X)\mathcal{L}_j = \prod_{\ell=0}^{d_{j+1}-d_j-1} (1 - z_j q^{d_j/2+\ell} X)\, \mathcal{L}_j P_j(q^{(d_{j+1}-d_j)/2} X).$$

The output of $\mathcal{L}_j$ is fixed by the local integral group.

Proof. In Lemma 24.3 the residue field is generated by a root of degree two and a primitive N_1-th root. The first is fixed by the even powers of Frobenius, the second exactly by powers s with $p^s = 1 \bmod N_1$. The least positive even such s equals L, since L is even by Theorem 24.4 and $p^s = (-p)^s$ for even s. These conditions compute the degree: in a finite field of p^a elements the a-th Frobenius power is identity by the multiplicative group order, and no smaller positive power can fix all its elements by the root bound for $X^{p^s} - X$. Also $p \nmid N_1 d$, and the two defining polynomials for O_E/pO_E have simple roots over an algebraic closure. They are the quadratic polynomial for ω and the cyclotomic polynomial in Proposition 44.1. Factoring the first over $\mathbb{F}_p$ and then the second over each factor expresses O_E/pO_E as a product of fields. The ideal factorization of Proposition 82.3 therefore gives valuation one for p at each factor.

The two sets of embeddings $\iota\tau$ at $\lambda, \bar{\lambda}$ are the two fibres of equal size in the h-orbit of Theorem 24.4. This orbit avoids the distinguished set D, so its nominal weight is constant a/x_1; the rounded w_{j+1} is constant on it by Definition 12.3. The two contributions of opposite valuations cancel in the character power formula of Theorem 372.2. This proves the unit assertion. By $p \nmid QgN$ and the displacement description of Theorem 423.2, the local component of Φ_j is a scalar multiple of the vacuum. In the split matrix realization of Proposition 391.1 its local function is the integral-box indicator. The same entrywise duality proof as in Lemma 432.2 works over the valuation ring O_λ, since the normalized lattice is self-dual. The character on integral stage determinants is trivial as $p \nmid Qg$. Thus the box, and hence the output, is target-invariant by Theorem 411.3.

For a rank a, the right cosets of $K \operatorname{diag}(pI_i, I_{a-i}) K$, $K = \mathrm{GL}_a(O_\lambda)$, are classified by their column lattices. They are exactly the inverse images of codimension-i subspaces U in the residue space. In fact lift a basis of U completed to a residue basis to $k \in K$, whose first $a-i$ columns give the chosen basis of U. The symmetric representative $g_U = k \operatorname{diag}(I_{a-i}, pI_i)\, k^t$ has this column lattice and also the same row lattice after transpose. Two matrices with the same column lattice differ on the right by an element of K, by applying both inverse basis changes. This proves the claim and gives representatives with both required properties.

In the target Hecke sum, translation by g_U on the output changes the box to the indicator of $g_U^{-1} T$ integral, multiplied by $z_j^i q^{id_j/2}$. In the source sum translation by g_U on the input changes the box to the indicator of $T g_U^{-1}$ integral, multiplied by $z_j^i q^{id_{j+1}/2}$. These equalities follow by Theorem 411.3 with target g_U, respectively source g_U^{-1}, and the two determinant/Jacobian factors proved in Lemma 434.2. The latter calculation uses only split coordinates, even ranks and the unramified unit character at the chosen pair, so it gives these factors with residue cardinality q and local uniformizer p here.

Both conditions imply T integral. For an integral T of residue rank h, the target condition requires U to contain its column space, and the source condition requires U to contain its

transposed row space. The two counts in degree i are $b(d_{j+1}-h,i)$ and $b(d_j-h,i)$. By Lemma 436.1 the target polynomial on the box is $\prod_{0\le \ell<d_{j+1}-h}(1-z_j q^{d_j/2+\ell}X)$. Removing its first $d_{j+1}-d_j$ factors leaves exactly the source polynomial at $q^{(d_{j+1}-d_j)/2}X$. Outside the box all terms vanish. Equality of local tests gives the asserted identity by linearity of the lift. □

Corollary 436.3. *Let $1\le J\le m$, $\mathcal{C}_J=\mathcal{L}_{J-1}\cdots\mathcal{L}_0$, and $i_0=\lceil \log_p m\rceil$. At every such λ there are $a_i\in O_\iota=\{z:|\iota z|_p\le 1\}$, independent of all displacements in the support of Ψ, with*

$$p^{\Delta_J}\mathcal{C}_J T_{0,i_0}=\sum_{i=0}^{i_0} a_i T_{J,i}\mathcal{C}_J,\qquad \Delta_J=L\left(\frac{i_0(d_J-d_0)}{2}+\binom{i_0}{2}\right)\le\frac{\delta_p}{2}$$

on λ-spherical inputs, where $\delta_p=i_0 f(n_m+i_0)$ as in Proposition 26.4.

Proof. Iterating Proposition 436.2 gives $P_J(X)\mathcal{C}_J=H(X)\mathcal{C}_J P_0(sX)$ with $s=q^{(d_J-d_0)/2}$. Here H is a product of the displayed linear factors at arguments $q^{(d_J-d_{l+1})/2}X$ for stages $l<J$. Its coefficients are in O_ι by the unit assertion, and $H(0)=1$. Its inverse exists in $O_\iota[[X]]$: the constant is one, and each successive coefficient is minus a finite sum of products of earlier ones with positive-degree coefficients of H. Denote these inverse coefficients by h_n. Compare coefficients of X^{i_0} after multiplying by H^{-1}. This gives the assertion with $a_i=(-1)^{i+i_0}h_{i_0-i}q^{i(i-1)/2}$, since $s^{i_0}q^{i_0(i_0-1)/2}=p^{\Delta_J}$. Here $i_0\le k$ eventually by $m=O(\sqrt{x})$ and $k\gg x^5$. Finally $L\le f=|G|$, $d_J-d_0\le n_m$ since $d_m=n_m+1$ and $d_0=k\ge 1$. These inequalities give the stated bound. □

437. The jet matrix at marked neighbours

Definition 437.1. Let p,ι,Λ,V have the meanings of Proposition 436.2 and Theorem 24.4. Set

$$A_\lambda=a\,\#\{\tau\in A:\mathfrak{p}_\tau=\lambda\},\qquad Y_\lambda=p^{A_\lambda}T_{0,i_0}\quad\text{at }\lambda,\quad Y=\prod_{\lambda\in\Lambda}Y_\lambda.$$

Hecke operators in different factors commute on the common spherical space: each is a finite sum, and each pair of terms has disjoint local support. Thus the product on columns may be taken in any order. For $0\le j<m$ write $r_\tau(j)$ for the number r_τ of columns of height greater than one in Definition 26.1.

Lemma 437.2. *Each Y_λ carries columns integral in the ordinary p-adic gauge to columns integral in that gauge.*

Proof. At this pair K_c contains the full integral group by Definition 13.3 since $p\nmid gN$. A left integral translation permutes the right cosets of a double coset, and the resulting right unit factors fix the vector. Thus the output retains this local level, and all other level actions commute with the sum. For an individual representative $s=k_1dk_2$, the local diagonal d has p on i_0 entries at λ and the inverse on the conjugate entries, all other coordinates being one. For ϕ integral,

$$D_0(g_p)^{-1}\iota\phi(gs)=D_0(s_p)\,\big(D_0((gs)_p)^{-1}\iota\phi(gs)\big).$$

Both unit factors of $D_0(s_p)$ act isometrically. The negative powers in $D_0(d)$ come only from the tops for which $\mathfrak{p}_\tau=\lambda$, as the weight evaluates at the conjugated embedding in Theorem 430.3. Each such factor is a symmetric power of degree a with determinant exponent zero, so no entry of its diagonal has valuation below $-a$. The entire loss is at most A_λ. Multiplying by p^{A_λ} cancels it, and summing uses the ultrametric inequality. □

Theorem 437.3. *For each $\lambda\in\Lambda$, let s_λ be any representative in a basic double coset of degree $0\le i_\lambda\le i_0$ in rank d_{j+1}. Embed $s=\prod_\lambda s_\lambda$ in the target group with its inverse transpose at the conjugate factors. Let $\Phi\in\mathcal{D}_p(1)$, $c=-\bar{\pi}_g u$. For all finite frames g,h and every coefficient vector v,*

$$\big|J_j^c(\Omega_{j,f}(g,hs)\Phi)v\big|_{j+1,h}\le p^{\sum_\lambda A_\lambda+i_0\sum_{\tau\in V}r_\tau(j)-J_p(j,V)}|v|_{j,g}.$$

Proof. Write the p-component $s = k'dk''$ with integral unitary k', k'', where d has p on coordinates I_λ of size i_λ at λ and p^{-1} on those coordinates at $\bar{\lambda}$. Use the identity at the other factors. These decompositions are the defining basic double cosets. Choose a rational source element γ with γg integral at p. Choose a rational target element ϵ with $v_0 = (\epsilon h)_p k' d$ in $K_p^0 \cap d^{-1} K_p^0 d$ at the target rank. Both choices are possible by Lemma 430.1. Then $(\epsilon h s)_p = v_0 k''$ is integral, and $(\epsilon h)_p^{-1} = k' d v_0^{-1}$ lies in $K_p^0 d$. Rational covariance as in Theorem 430.3 therefore reduces the desired bound to the max norm of $D_{j+1}(d) \iota J_j^c(\Phi')$, where $\Phi' = \Omega_{j,f}(\gamma_f g, \epsilon_f h s)\Phi$ belongs to $\mathcal{D}_p(1)$ by Proposition 430.2. The transformed input has ordinary max norm $|v|_{j,g}$.

For a target tableau T' write $n_\tau(I)$ for the number of its entries at τ in I. The determinant part of $D_{j+1}(d)$ has valuation zero. Indeed the two fibres above each pair have equal size and the same value of $d_j/2 - w_{j+1}(\tau)$, as proved for the weights in Proposition 436.2. The tableau part is diagonal on T': a flag minor acquires one factor from each selected row, by the definition in Proposition 409.2. Its valuation is

$$\sum_{\lambda \in \Lambda} \left(\sum_{\bar{\mathfrak{p}}_\tau = \lambda} n_\tau(I_\lambda) - \sum_{\mathfrak{p}_\tau = \lambda} n_\tau(I_\lambda) \right),$$

with the signs as in Lemma 437.2.

Choose the charge $\gamma_{\tau,h',b'}$ of Theorem 28.5 as follows. When $\mathfrak{p}_\tau = \lambda \in \Lambda$, assign one at height $h' = 1$ to each $b' \notin I_\lambda$ and zero elsewhere. When $\bar{\mathfrak{p}}_\tau = \lambda \in \Lambda$, assign one to $b' \in I_\lambda$ at every height and zero elsewhere. Give zero at all other places. For any relevant odd e, $\sigma = (-p^e)\tau$ induces the conjugate prime of τ by Lemma 24.3. If $\tau \in V$, the two charges at height one on its side and at any height on the other side cover every b'. If $\sigma \in V$ and its height is one, the same two sets in reverse order cover every b'. These are all the charge conditions since p is inert.

Apply Theorem 429.2 with $a = \bar{\pi}_g$, a unit at p. In the valuation of the scaled row T' subtract its charge $m_{T'}$ and add the displayed diagonal valuation. All terms from the conjugates of the selected fibres cancel. At $\tau \in V$ each height-one entry now costs exactly one whether or not it is in I_λ, giving $a\mathbf{1}_A(\tau)$ in total. A taller column costs one for each of its entries in I_λ. The entries increase strictly down the column, so this costs at most $|I_\lambda| \le i_0$. Summing over columns and places gives the stated valuation for every matrix entry and therefore the operator inequality. □

438. Divisibility on the marked Hecke image

Theorem 438.1. *Let $p < x_1$ be served and fix ι, Λ, V as in Definition 437.1. Write $\mathcal{E}_p$ for the columns integral in the ordinary gauge at p, and put*

$$s_p = \lceil 2mfa/p \rceil, \qquad \mathcal{E}'_p = Y^m \mathcal{E}_p + p^{s_p} \mathcal{E}_p.$$

Then $\mathcal{E}'_p \subseteq \mathcal{E}_p$. For every twist with $p \nmid Q$, every coefficient function Ψ of Theorem 423.2 with the scale and normalization of Theorem 431.1, and every $\xi \in \mathcal{E}'_p$, one has

$$|(B_\Psi \xi)(h)|_{Y,h} \le p^{-G_p + (30\ell+3)fa/p}$$

at every terminal argument h, for all sufficiently large x.

Proof. The module inclusion follows by Lemma 437.2. Consider a tuple s in the support of Ψ and put $\Phi_j = D_{s_j} v_j^\circ$. Each such datum is in $\mathcal{D}_p(1)$ at this prime by Theorem 423.2 since $p \nmid gNQ$. For a marked λ and output rank d_{j+1} use the local oscillator representation $b \mapsto \Omega_{j,f}(1, b)$. Denote by $T^{\mathrm{osc}}_{j+1,i,\lambda}$ the sum in Definition 230.1 using this representation. Take any common choices of right coset representatives. On all the local fixed vectors the

operator is independent of these choices. At each stage put

$$\mathcal{T}_j = \prod_{\lambda\in\Lambda}\left(p^{A_\lambda-\Delta_{j+1,\lambda}}\sum_{i=0}^{i_0} a_{i,j,\lambda}T^{\mathrm{osc}}_{j+1,i,\lambda}\right),$$

where the coefficients and $\Delta_{j+1,\lambda}$ come from Corollary 436.3, with $J=j+1$. They are independent of the tuple s. A prefix of the original chain will be denoted by $\mathcal{C}_j$ with $\mathcal{C}_0=1$. Translation covariance of Theorem 411.3 gives

$$\mathcal{L}^c_{j,\mathcal{T}_j\Phi_j}\mathcal{C}_j\eta = \mathcal{C}_{j+1}Y\eta$$

for every initial column η. In detail, moving a single target Hecke sum outside the last arrow gives the same sum on the output of $\mathcal{C}_{j+1}$. The corollary then moves it to $p^{A_\lambda}T_{0,i_0,\lambda}$ on the initial column after the displayed scalar is included. Operators at the remaining primes commute with this move, and the resulting initial column is spherical there. Iteration proves the identity. Induction on prefix length consequently identifies the chain with every datum replaced by $\mathcal{T}_j\Phi_j$, applied to η, with the original chain applied to $Y^m\eta$.

We justify evaluation of this replacement at the original class level. Every Φ_j in the support is spherical under both integral actions at each marked pair, since its local component is a multiple of the box by Proposition 436.2. Source translations commute with the target sum. A target integral translation permutes its right cosets with integral right factors; these fix each spherical vector. Thus $\mathcal{T}_j$ commutes with the level actions on the spherical subspace. Translations of the rest of the level at other pairs commute term by term. It follows from the joint invariance in Theorem 423.2 that $(\otimes_j\mathcal{T}_j)w_\Psi$ is jointly invariant, so Theorem 424.1 applies to it.

For a tuple of representatives in this sum, expand $\mathcal{T}_j$ on each individual Φ_j. Its Hecke translations range precisely over the cases in Theorem 437.3. The coefficients $a_{i,j,\lambda}$ are p-integral and multiplication by $p^{\sum_\lambda(A_\lambda-\Delta_{j+1,\lambda})}$ absorbs the $\sum_\lambda A_\lambda$ in that theorem. The resulting operator bound on this stage is

$$p^{\sum_\lambda\Delta_{j+1,\lambda}+i_0\sum_{\tau\in V}r_\tau(j)-J_p(j,V)}.$$

All sums in the integral class formula are finite and each $\Psi(s)$ is integral at p. The product of these bounds therefore bounds the output on $\eta\in\mathcal{E}_p$ by the ultrametric inequality. Since $\Delta_{j+1,\lambda}\le\delta_p$, Proposition 26.4 gives the exponent $-G_p+(30\ell+3)fa/p$. Proposition 417.2 passes this estimate to the restricted chain since $p\neq N$.

For the second summand of $\mathcal{E}'_p$, use Theorem 431.1. By Definition 24.1 and geometric summation, $mC_p(\varnothing)\le mUa/(p-1)\le 2mfa/p$ for $p\ge 2$. Multiplication by p^{s_p} cancels this loss, leaving exponent at most $-G_p+30f\ell a/p$. Both bounds pass to the sum of these modules by linearity and the ultrametric inequality. □

Corollary 438.2. *Let $\beta_p=\lceil G_p-e_p\rceil$ with the conventions of Proposition 25.3. At every prime p there is an explicit O_ι-module of columns on which $|(B_\Psi\phi)(h)|_{Y,h}\le p^{-\beta_p}$ for all ramified ϕ in the module and all twists with $p\nmid Q$. At a served prime take $\mathcal{E}'_p$, at N take $\mathcal{E}(1)\cap\mathcal{E}_*$ of Theorem 435.3, and elsewhere take $\mathcal{E}_p$. The determinant-index assertion*

$$\mathcal{I}(\mathcal{E}_p,\mathcal{E}'_p;p^{l_p})\quad\text{for some integer } 0\le l_p\le e_pc_0$$

at each served p suffices for the index condition at every prime in Proposition 25.3 and its application to the local recombinations.

Proof. At a served prime $e_p=3fa\ell^2/p$; $(30\ell+3)fa/p\le e_p-1$ eventually because $fa/p\ge fa/x_1$ tends to infinity by Theorem 11.4 and the scale bounds in Lemma 26.2. Theorem 438.1 now gives the stated rounding. At an unserved small nonsplit prime Theorem 431.1 costs at most $2e_p/3\le e_p-1$ eventually by the same bounds. At N one has $G_N=0$, $e_N=4mfb$ by Proposition 12.6. Theorem 434.4 costs at most $2mfb\le e_N-1$; the determinant exponent of Theorem 435.3 is at most e_Nc_0. Its recombinations of integral columns also stay in

$\mathcal{E}(1)$, giving the asserted intersection. At all other primes $G_p = e_p = 0$; integrality is Theorem 431.1 since d is in the small nonsplit range eventually. Module equality gives index exponent zero at those primes and at the unserved ones. Definition 15.5 then gives the asserted sufficiency. □

439. The class matrix of one marked Hecke operator

Definition 439.1. Fix p, λ as in Definition 437.1 and an embedding ι at p. To avoid confusion with the global auxiliary level, write $D = L/o$, $P = p^o$, where o, L are the orders of $-p$ in Definition 24.1. Choose a top τ_0 with $\mathfrak{p}_{\tau_0} = \lambda$. For $s \in \mathbb{Z}/D\mathbb{Z}$, put $\tau_s = (-p)^{os}\tau_0 \bmod N_1$. Let $W = (O_E)_\lambda$, the fractions of elements of O_E with denominator outside λ. Set $O = \{z \in K_p : |z|_p \le 1\}$. Define maps $W \to O$ by

$$\sigma_s(x) = \begin{cases} \iota\tau_s(x) & s \text{ even}, \\ \iota\tau_s(\bar{x}) & s \text{ odd}. \end{cases}$$

On k-by-k invertible matrices over W define the representation in the integral evaluation coordinates of Definition 14.3 by

$$R_\lambda(g) = \bigotimes_{s \text{ even}} S(\sigma_s(g^{-t})) \otimes \bigotimes_{s \text{ odd}} S(\sigma_s(g)), \qquad g^{-t} = (g^{-1})^t,$$

with an arbitrary fixed ordering of the factors. Here the matrix entries and their images in K_p come from algebraic numbers, so matrices over O in this formula are also matrices over $\iota(O_\iota)$.

Lemma 439.2. *The D embeddings listed are exactly the tops inducing λ or $\bar{\lambda}$. The first prime is induced for even s and the second for odd s, and D is even. The ring W has residue field of cardinality P^D and uniformizer p. The σ_s are ring homomorphisms and $\sigma_{s+1}(x) - \sigma_s(x)^P \in pO$ for every $x \in W$, including the index crossing $s = D - 1$.*

Proof. By Theorem 24.4 the orbit returns to the tops exactly at the multiples of o; its full period is L, with $m \mid L/o$. Since o is odd each return exchanges the two prime fibres by Lemma 24.3. These facts prove the enumeration and the parity assertion. Each σ_s uses λ on E by this description, so every denominator in W maps to a unit. The assertion about W follows from the calculation of the residue degree and unramifiedness in Proposition 436.2.

To prove the congruence, compare the two induced maps to O/pO. On ζ_{N_1} the two exponents agree even before reduction: the additional minus in passing to τ_{s+1} is canceled by the change of conjugation. On ω they agree since $X^2 - X + (1+d)/4$ is irreducible in the inert residue field of K, where odd powers of Frobenius exchange the two roots. These root identities hold modulo pO: the maps on O_E factor there through the field O_E/λ, by the uniformizer calculation, with nonzero elements mapping to units. Since O_E is generated over $\mathbb{Z}$ by ω, ζ_{N_1} by Proposition 44.1, this proves equality on O_E. It extends to W by inversion of the indicated units. □

Theorem 439.3. *Let $r_0 = i_0 = \lceil \log_p m \rceil$, $t = \operatorname{diag}(pI_{r_0}, I_{k-r_0})$, and $K = \mathrm{GL}_k(O_\lambda)$. Put*

$$\mathcal{P} = K \cap tKt^{-1}, \qquad \mathcal{P}' = K \cap t^{-1}Kt.$$

Let $C = \Gamma\backslash H_{0,f}/K_c$ and H_{rest} be the tensor product of the evaluation coordinate spaces of the tops outside this pair. Give it the integral lattice in these coordinates. There are class representatives giving integral coordinates for $\mathcal{E}_p$, and an edge set $\mathcal{B}$ with two bijections

$$e \longmapsto (c(e),\, b_e\mathcal{P}), \qquad e \longmapsto (c'(e),\, k_e\mathcal{P}')$$

onto $C \times K/\mathcal{P}$ and $C \times K/\mathcal{P}'$ respectively, with $b_e, k_e \in \mathrm{GL}_k(W)$. In these coordinates the class matrix is

$$\iota(Y_\lambda\phi)(c) = \sum_{e:\, c(e)=c} \Big(R_\lambda(b_e)\operatorname{diag}_\beta(p^{d(\beta)})\, R_\lambda(k_e^{-1}) \otimes u_e\Big)\, \iota\phi(c'(e)),$$

where u_e is an invertible integral matrix on H_{rest}. Here $\beta=(\beta_{b,s})$ ranges over nonnegative arrays with each $\sum_{1\le b\le k}\beta_{b,s}=a$, and

$$d(\beta)=\sum_{s\ \mathrm{even}}\sum_{b>r_0}\beta_{b,s}+\sum_{s\ \mathrm{odd}}\sum_{b\le r_0}\beta_{b,s}.$$

Proof. Choose representatives g_c integral at all the places above p by Lemma 430.1. Since K_c is maximal integral above this p, multiply on the right by the inverse of the p-component supported above p. We can thus assume $(g_c)_p=1$ in every class. The transformation law and Proposition 13.5 now identify $\mathcal{E}_p$ with all O_ι-valued class coordinate tuples: both right level and rational left changes preserve the gauge by Theorem 430.3.

Embed t in the unitary group at the split pair. Let K' agree with K_c off the pair and with $\mathcal{P}$ there, and set $\mathcal{B}=\Gamma\backslash H_{0,f}/K'$. For a fixed c its fibre consists of g_cb_eK' with $b_e\mathcal{P}$ running once through $K/\mathcal{P}$, since $\Gamma\cap g_cK_cg_c^{-1}=1$ by Proposition 13.5. We can choose b_e in $\mathrm{GL}_k(W)$: the condition defining $\mathcal{P}$ is divisibility of the upper-right block by p, so the coset is determined by reduction. Lift an invertible residue matrix from O_E/λ; its determinant lies outside λ. Right multiplication by t gives a bijection of the edge set onto the quotient for $t^{-1}K't$, with inverse right multiplication by t^{-1}. The new level is $\mathcal{P}'$ at the pair. The same fibre calculation on this side gives the second bijection.

In particular $g_cb_et=\gamma_eg_{c'}k'_e$ for $\gamma_e\in\Gamma$, $k'_e\in K_c$. At λ write $k_e=(k'_e)_\lambda$. Using the identity p-components of the representatives gives $(\gamma_e)_\lambda=b_etk_e^{-1}$, and therefore $k_e\in\mathrm{GL}_k(W)$: all its entries and those of its inverse are rational over E, integral at λ. At other pairs over p, γ_e and its inverse are integral. Thus on all the other tops its weight action is an integral automorphism u_e.

The terms of $T_{0,r_0,\lambda}$ in a given class are $\phi(g_cb_et)=R_0(\gamma_{e,\infty})\phi(g_{c'})$; b_etK runs once over the desired cosets by the definition of $\mathcal{P}$. On the even tops, where $\iota\tau_s$ induces λ, conjugating the rational unitary matrix γ_e gives γ_e^{-t}. On the odd tops its ordinary local matrix at λ is already evaluated by the conjugated embedding. This proves the action $R_\lambda(b_e)R_\lambda(t)R_\lambda(k_e^{-1})$ on the linked factors, interpreting R_λ on t over the fraction field. In an even factor t contributes minus the number of letters in the first block; in an odd factor it contributes plus this number. Multiplication by p^{A_λ}, with $A_\lambda=aD/2$ by Lemma 439.2, replaces the negative count at each even factor by the positive count in the second block. This gives exactly $p^{d(\beta)}$ on each monomial and proves the formula. □

440. Evaluation coordinates and depth in all integral frames

Lemma 440.1. *For a matrix g of size k, let $M_a(g)$ denote the coefficient matrix of the substitution $x_j\mapsto\sum_i g_{ij}x_i$ on monomials of degree a; the input monomial indexes its column. In the integral evaluation coordinates of Definition 14.3,*

$$S(g)=M_a(g^t)^t.$$

In particular, with the ordinary polynomial representation

$$P_\lambda(g)=\bigotimes_{s\ \mathrm{even}}M_a(\sigma_s(g))\ \otimes\ \bigotimes_{s\ \mathrm{odd}}M_a(\sigma_s(g^{-t})),$$

one has $R_\lambda(g)=P_\lambda(g^{-1})^t$. The linked factor of each edge in Theorem 439.3 is exactly

$$\Big(P_\lambda(k_e)\,\mathrm{diag}_\beta(p^{d(\beta)})P_\lambda(b_e^{-1})\Big)^t$$

in the gauge used for the local index.

Proof. Expand the defining identity $b(gx)=S(g)b(x)$. In row β it reads

$$\prod_j\Big(\sum_i g_{ji}x_i\Big)^{\beta_j}=\sum_\alpha S(g)_{\beta,\alpha}x^\alpha.$$

The coefficient of x^{α} on the left is $M_a(g^t)_{\alpha,\beta}$. This proof uses integral polynomial identities, including when the coefficient characteristic divides a multinomial. Apply the identity at each even and odd factor of Definition 439.1. Transposing the resulting matrix product reverses the factors; the middle diagonal is fixed by transpose. This proves both remaining claims. □

Definition 440.2. Use the cycle in Lemma 439.2. Choose $\varrho \in K_p$ with $\varrho^2 = p$ and set $\mathfrak{m} = \{z \in O : |z|_p < 1\}$, $\mathbf{k} = O/\mathfrak{m}$. Write $\mathcal{V}$ for the free O-module of coefficients in the linked polynomial representation P_λ, and $\bar{\mathcal{V}}$ for its reduction to $\mathbf{k}$. The depth of β is $d(\beta)$ in Theorem 439.3. Set $D_\varrho = \operatorname{diag}_\beta(\varrho^{d(\beta)})$ and $B = \max_\beta d(\beta)$. For an integer $t \ge 0$ define the subspace $V_t \subset \bar{\mathcal{V}}$ of polynomials whose transform by every $\overline{P_\lambda(g)}$, $g \in \mathrm{GL}_k(W)$, has zero coefficients at every monomial of depth less than t. The same constructions work over $\iota(O_\iota)$ and its residue field by using ϱ in the image of ι.

Proposition 440.3. *Let C' be a finite set, let $s : E' \to C'$ have nonempty finite fibres, and take $h_e \in \mathrm{GL}_k(W)$ so that in each fibre the left cosets $\mathcal{P}h_e$ cover $\mathrm{GL}_k(W)$ viewed inside K. Consider the matrix*

$$J : \mathcal{V}^{C'} \longrightarrow \mathcal{V}^{E'}, \qquad (JF)_e = D_\varrho P_\lambda(h_e) F_{s(e)}.$$

It factors as $J = QA$ over O, where A is square of the source rank and Q has an integral left inverse. Its determinant satisfies

$$2v_p(\det A) \le |C'| \sum_{t=1}^{B} \dim_{\mathbf{k}} V_t.$$

The assertion applies both to $s(e) = c(e), h_e = b_e^{-1}$ and to $s(e) = c'(e), h_e = k_e^t$ for the actual edges of Theorem 439.3.

Proof. For $u \in \mathcal{P} \cap \mathrm{GL}_k(W)$, its upper-right block is divisible by p. On an even letter the deep variables are in the second block. Rescale each deep variable by ϱ. An entry mapping a deep variable into a shallow one is then divided by ϱ; this entry lies in $pO = \varrho^2 O$. On an odd letter the deep variables are in the first block. The lower-left block of u^{-t} is now divisible by p, giving the same integrality statement. By monomial substitution and tensor product, this proves $D_\varrho P_\lambda(u) D_\varrho^{-1}$ is integral. The same calculation for u^{-1} proves it is invertible over O. Consequently if JF is divisible over O by a scalar z, then $D_\varrho P_\lambda(g) F_c$ is divisible by z for every c, g: choose e above c with $g h_e^{-1} \in \mathcal{P}$. The element $g h_e^{-1}$ also belongs to $\mathrm{GL}_k(W)$.

The matrix J has full column rank since each fibre contains an invertible block over the fraction field. The pivot argument of Lemma 25.1, selecting the entry of least valuation in the remaining rectangle, gives $J = U \binom{D'}{0} V$ with integral invertible U, V and nonzero diagonal D'. Take Q to be the first columns of U, $A = D'V$; the corresponding rows of U^{-1} are a left inverse. A column of V^{-1} is primitive and has a primitive component in at least one fibre of the source. Applying any $P_\lambda(h_e)$ there preserves primitivity. Multiplication by D_ϱ then leaves an entry of valuation at most $B/2$. Thus every diagonal entry a_l of D' has $0 \le 2v_p(a_l) \le B$.

For an integer $0 \le j < B$ take the columns of V^{-1} for which $2v_p(a_l) > j$. Their reductions are independent, since V is invertible over O. For each of these columns F, JF is divisible by a_l; applying the first paragraph shows that the coordinates of $P_\lambda(g)F_c$ at all depths at most j have positive valuation. Hence their reductions lie in $V_{j+1}^{C'}$. There can be at most $|C'| \dim V_{j+1}$ such columns. For any real $0 \le x \le B$, $x \le \sum_{j=0}^{B-1} \mathbf{1}_{x>j}$. Apply this to $2v_p(a_l)$ and sum. For the two applications the first coset coverage follows by inverting $b_e\mathcal{P}$; transposing $k_e\mathcal{P}'$ gives $\mathcal{P}k_e^t$, since $(\mathcal{P}')^t = \mathcal{P}$ by the two block descriptions. The bijections of the class-matrix theorem give the two coverings. All operations on this matrix can also be taken over $\iota(O_\iota)$, a valuation ring, giving the last variant of the definition. □

441. Weighted polynomial interpolation with multiplicity

Definition 441.1. Let K_0 be a field, $n \geq 1$, and k_i positive integers for $1 \leq i \leq n$. The weight of a monomial X^α in $K_0[X_1, \ldots, X_n]$ is $\sum_i \alpha_i/k_i$. At a point $z \in K_0^n$, a polynomial vanishes to order at least $d \in \mathbb{Z}_{\geq 0}$ if its expansion in the translated variables $X_i - z_i$ has zero coefficients in total degrees below d.

Theorem 441.2. *Suppose a finite set $Z \subseteq K_0^n$ has at least $\prod_i k_i$ points. For each i suppose g_i is a polynomial all of whose monomials have weight less than one, with $g_i(z) = z_i^{k_i}$ on Z. Then Z has exactly $\prod_i k_i$ points. For each $d \geq 1$ the ideal of polynomials vanishing to order at least d on Z equals the d-th power of $(X_i^{k_i} - g_i : 1 \leq i \leq n)$. In particular each nonzero such polynomial has a monomial of weight at least d.*

Proof. Set $h_i = X_i^{k_i} - g_i$. The polynomials

$$X^\alpha h^\gamma, \qquad 0 \leq \alpha_i < k_i, \quad \gamma \in \mathbb{Z}_{\geq 0}^n, \qquad h^\gamma = \prod_i h_i^{\gamma_i},$$

form a basis of the polynomial ring as a vector space. Indeed each has a unique highest-weight monomial $X^{\alpha + (k_i \gamma_i)_i}$ with coefficient one. These leading monomials are distinct and cover all monomials by Euclidean division in each exponent. Comparing the terms of greatest weight proves independence in any finite linear combination. For spanning, subtract the corresponding basis element from a monomial and induct on the weight multiplied by $\prod_i k_i$; this is a nonnegative integer and every remaining monomial has strictly smaller weight.

Write $I = (h_i)_i$. The basis elements with $|\gamma| \geq d$ span I^d. One inclusion is by divisibility; for the reverse one, expand an arbitrary polynomial multiplying any product h^η, $|\eta| = d$, in the displayed basis. The product just adds η to its index γ. Hence the codimension of I^d is $(\prod_i k_i) \# \{\gamma : |\gamma| < d\}$. For a point z, its ideal $\mathfrak{m}_z = (X_i - z_i)_i$ satisfies $f \in \mathfrak{m}_z^d$ exactly under the vanishing condition of Definition 441.1, by expansion in the translated monomials. Its codimension is $\# \{\gamma : |\gamma| < d\}$. For $z \neq z'$ a differing coordinate gives $a \in \mathfrak{m}_z$, $1 - a \in \mathfrak{m}_{z'}$ by affine rescaling. Expanding $1 = (a + (1-a))^{2d-1}$ gives $1 \in \mathfrak{m}_z^d + \mathfrak{m}_{z'}^d$. For each fixed z, multiply, over the other points, polynomials equal to 1 mod $\mathfrak{m}_z^d$ and to 0 mod $\mathfrak{m}_{z'}^d$ at that other point. These products interpolate arbitrary classes in all the quotients by multiplying individual representative polynomials and adding. Thus the codimension of $\bigcap_z \mathfrak{m}_z^d$ is $|Z| \# \{\gamma : |\gamma| < d\}$. Since I^d is contained in this intersection, the two codimensions and the assumed point count force equality of both the cardinalities and the ideals.

For nonzero $f \in I^d$ use its finite basis expansion. Every index there has $|\gamma| \geq d$. At the largest weight of these terms a leading monomial survives by independence of the distinct leading monomials. Its weight is $\sum_i \alpha_i/k_i + |\gamma| \geq d$, as required. □

442. Interpolation in the linked symmetric powers

Theorem 442.1. *Use D, P, a, k and $\mathbf{k}$ of Definition 439.1 and Definition 440.2. Write $r = i_0$ and number the k coordinates $0, \ldots, k-1$ in this section. Assume $1 \leq 2r \leq k$, $2r < D$, as holds eventually by $D \geq m$, $r \leq 1 + \log_2 m$, $k \gg m$. A coordinate (b, s) is deep if $b \geq r$ for even s or $b < r$ for odd s. For a monomial β define*

$$C(\beta) = \frac{1}{P} \sum_{\substack{(b,s) \text{ deep} \\ b < 2r}} \beta_{b,s} + \frac{1}{P^{2r}} \sum_{\substack{(b,s) \text{ deep} \\ b \geq 2r}} \beta_{b,s}.$$

Every nonzero $F \in V_d$, $d \geq 1$, has a monomial of nonzero coefficient and $C(\beta) \geq d$. In particular, for $M_0 = \binom{a+k-1}{k-1}^D$ one has

$$\sum_{d \geq 1} \dim_{\mathbf{k}} V_d \leq M_0 \frac{Da}{k} \left(\frac{r}{P} + \frac{k - 2r}{2P^{2r}} \right).$$

Proof. Let $k_q = W/pW$ of size $q = P^D$. Each σ_s reduces to an embedding $k_q \to \mathbf{k}$, denoted $\bar{\sigma}_s$; by Lemma 439.2 its successor is its P-th power. Write $y^{(s)} = \bar{\sigma}_s(y)$. Replace all variables at shallow coordinates by independent indeterminates $x_{b,s}$ and let $K_0 = \mathbf{k}(x_{b,s} : (b,s) \text{ shallow})$. Keep the deep coordinates as polynomial variables $Z_{b,s}$ over K_0. This substitution is injective, by collecting coefficients of the deep monomials in the independent shallow variables.

Take a k-by-k matrix A_y over k_q supported on the following positions with independent entries y_{bc}:

$$(b, r+b)\ (0 \le b < r), \qquad (b,c)\ (0 \le b < r,\ 2r \le c < k).$$

It is square-zero, and may be lifted with the same support to W, where its square is still zero. In the action of $1 + A_y$, the shallow variables are fixed. In each even letter, a deep variable in column c gains $\sum_b x_{b,s} y_{bc}^{(s)}$; in each odd letter, a deep variable indexed by $b < r$ gains $-\sum_c x_{c,s} y_{bc}^{(s)}$. Entries outside the chosen support are read as zero. The second sign comes from $(1 + A_y)^{-t} = 1 - A_y^t$. Call the vector of these gains z_y. Since the coefficients of the transform of F at every depth below d are zero, the polynomial obtained from F after the substitution vanishes at z_y to order at least d: its translate $F(Z + z_y)$ has no terms of ordinary deep degree less than d.

Assign costs $1/P$ to the deep indices $b < 2r$ and P^{-2r} to all other deep indices. We show for each i that all $y_{bc}^{(i)}$, as functions on the points, are represented by polynomials in Z of weight at most $1/P$. For $c \ge 2r$ choose the r integers j in $0 \le j < 2r$ for which $i - j$ is even. Their letters $i - j$ are distinct modulo D since $D > 2r$. The equations at those even letters give

$$(z_y)_{c,i-j}^{P^j} = \sum_{b<r} x_{b,i-j}^{P^j} y_{bc}^{(i)}.$$

Here iterating Frobenius carries the residue value to the letter i, and taking powers distributes over sums in characteristic p. The r-by-r coefficient matrix is invertible over K_0: the terms of its determinant are distinct monomials in independent variables indexed by the distinct row letters and column indices. Its inverse expresses the unknowns as linear combinations of $Z_{c,i-j}^{P^j}$, each of weight at most $1/P$. For an even i, the coordinate $r + b$ gives $y_{b,r+b}^{(i)} = (z_y)_{r+b,i}/x_{b,i}$. For an odd i, use the coordinate b. Its value is $-x_{r+b,i} y_{b,r+b}^{(i)} - \sum_{c \ge 2r} x_{c,i} y_{bc}^{(i)}$; the latter entries have already been recovered. In both cases division is by a nonzero indeterminate of K_0 and the claimed weight bound follows. These formulas also apply when there are no columns $c \ge 2r$, by the empty sum.

Recovery at one letter proves the z_y distinct. There are $q^{r(k-2r+1)}$ of them. For each deep coordinate v write its cost as $1/k_v$. At every letter there are exactly r deep indices below $2r$, and in the $D/2$ even letters there are $k - 2r$ more. Hence $\prod_v k_v = P^{Dr + Dr(k-2r)} = q^{r(k-2r+1)}$. Furthermore the function $(z_y)_v^{k_v}$ has a polynomial of weight less than one on these points. Indeed k_v is P^j with $j = 1$ or $2r$; raise the defining sum for $(z_y)_v$ to this power. It becomes a K_0-linear combination of entries $y_{bc}^{(s+j)}$. The polynomials just obtained have weight at most $1/P < 1$. Theorem 441.2 therefore gives a monomial of the substituted F with weight at least d. Its nonzero coefficient comes from at least one original monomial with the same deep exponents, hence the same cost.

Restriction to the coefficients of monomials of cost at least d is thus injective on V_d. Sum these dimension bounds over positive integers d. Each monomial contributes at most $\lfloor C(\beta) \rfloor \le C(\beta)$. There are M_0 monomials. In each letter their average exponent in any specified coordinate is a/k, by permuting coordinates and summing all k exponents. The counts of deep indices just computed now give the claimed sum. □

Corollary 442.2. *In either half of Proposition 440.3, with the actual classes, set* $z_\lambda = (Da/k)(r/P + (k-2r)/(2P^{2r}))$. *Then*

$$2v_p(\det A) \le |C| M_0\, z_\lambda, \qquad z_\lambda \le \frac{128 A_\lambda \ell}{mp}$$

for all sufficiently large arithmetic scales at every served prime. Tensoring the half matrix with the identity of H_{rest} *multiplies its determinant cost bound by* $\dim H_{\mathrm{rest}}$.

Proof. Use Theorem 442.1 in Proposition 440.3. For the numerical bound note $P = p^o \geq p$, $p^r \geq m$, and $r \geq 1$, so $P^{2r} \geq mp$. Also $r \leq 1+\log_2 m$, $mr/k \leq 1$ eventually from the scales in Lemma 26.2 and Definition 12.3. It follows that $z_\lambda \leq Da(1/(mp)+1/(2mp)) = 3A_\lambda/(mp)$, since $Da = 2A_\lambda$. This even implies the asserted bound as $\ell \to \infty$. The tensor statement follows by reordering the basis into one copy of every matrix for each basis element of the fixed factor. Determinants multiply over block diagonal matrices. □

443. The volume of cyclic interlacing inequalities

Definition 443.1. Fix integers $k, D \geq 1$, a real number $P > 1$, and a sign $\epsilon_s \in \{1, -1\}$ for each $s \in \mathbb{Z}/D\mathbb{Z}$. Put $q = P^D$. Number the rows $0, \ldots, k-1$. For permutations σ_s of the rows define

$$(L_\sigma \nu)_{b,s} = \epsilon_s (P\nu_{b,s+1} - \nu_{\sigma_s(b),s}).$$

Write L_1 for the identity permutations. Define Ω to be the image under L_1 of the following inequalities for all $b < k-1, s$:

$$\begin{cases} P\nu_{b+1,s+1} \leq \nu_{b,s}, & \epsilon_s = 1, \\ \nu_{b+1,s} \leq P\nu_{b,s+1}, & \epsilon_s = -1. \end{cases}$$

For $A \in \mathbb{R}^D$, let $\Sigma(A)$ consist of the arrays $\beta \geq 0$ with $A_s - 1 \leq \sum_b \beta_{b,s} \leq A_s$ for all s. Volume is Euclidean volume in the kD coordinates.

Lemma 443.2. *For strictly decreasing real* k*-tuples* x, y*, assume* $x_b \neq y_c$ *for all* b, c*. The matrix* $(\mathbf{1}_{x_b > y_c})_{b,c}$ *has determinant one when* $x_0 > y_0 > x_1 > y_1 > \cdots > x_{k-1} > y_{k-1}$*. It has determinant zero otherwise.*

Proof. Each row consists of some zeros followed by ones. The numbers of ones decrease weakly down the rows. A zero row or two equal counts give a zero determinant. In the remaining case the counts are the k distinct positive integers $k, k-1, \ldots, 1$. This gives exactly the displayed inequalities, and the matrix is upper triangular with ones on the diagonal. □

Theorem 443.3. *With the notation of Definition 443.1,*

$$\mathrm{vol}(\Omega \cap \Sigma(A)) = \left(\prod_{j=1}^{k} \frac{q^j - 1}{q - 1} \right)^{-1} \mathrm{vol}(\Sigma(A)).$$

When q *is the cardinality of a finite field the denominator counts its complete flags in dimension* k*.*

Proof. First compute the Jacobians. Write S_s for the permutation matrix taking $(u_b)_b$ to $(u_{\sigma_s(b)})_b$. Up to signs and rearrangement of coordinates, the new variables are the residuals $P\nu_{s+1} - S_s\nu_s$. Successively replacing $\nu_1, \ldots, \nu_{D-1}$ by the first $D-1$ residuals, retaining ν_0, has absolute determinant $P^{k(D-1)}$ by triangularity. Holding these residuals fixed, the last residual $P\nu_0 - S_{D-1}\nu_{D-1}$ has linear part in ν_0 equal to $P^{1-D}(qI - S_{D-1} \cdots S_0)$. This also describes the one-letter case. Therefore

$$|\det L_\sigma| = |\det(qI - S_{D-1} \cdots S_0)|.$$

It is positive since a permutation preserves the maximum norm and $q > 1$. In particular $|\det L_1| = (q-1)^k$.

In integrating indicators below one can discard the finitely many equality hyperplanes between two entries being compared, or between two different entries within one letter. Each is a proper hyperplane, including when $D = 1$ since $P > 1$. For clarity, the Euclidean volume rules needed here follow by slicing bounded polyhedra in coordinates. An affine hyperplane

has zero volume by slicing in any coordinate with nonzero coefficient. The same conclusion follows by covering its bounded portion with grid boxes of side δ: over each choice of the other coordinates it meets a bounded number of layers of boxes, so their volume tends to zero. Thus inner and outer grid sums of a bounded polyhedron agree in the limit. They also give the iterated integrals of its interval slices by squeezing. Permuting coordinates preserves volume, translation or shear in one coordinate translates the corresponding slices, and scaling a coordinate multiplies slice length by its absolute scale. Elimination writes each invertible linear change as a product of these elementary changes, which proves the absolute determinant rule. All sets integrated below are bounded or finite unions of bounded polyhedra, by invertibility and boundedness of $\Sigma(A)$.

For a fixed ν, impose the column-sum intervals on $L_1\nu$. Their indicator $I(\nu)$ is unchanged by independently permuting the rows at each letter, and the column sums of $L_\sigma\nu$ are the same. Form the matrices

$$(H_s)_{bc} = \mathbf{1}_{\epsilon_s(P\nu_{c,s+1}-\nu_{b,s})>0}.$$

Expanding and multiplying their determinants gives

$$I(\nu)\prod_s \det H_s = \sum_\sigma \left(\prod_s \operatorname{sgn}\sigma_s\right)\mathbf{1}_{L_\sigma\nu\in\Sigma(A)}$$

off the equality walls. Independently sort the entries in each letter of ν in decreasing order. This permutes the rows of one matrix and the columns of its predecessor by the same permutation, preserving the product of determinants. For the sorted array, apply Lemma 443.2 to H_s^t at a positive sign and to H_s at a negative sign. The product is one exactly when the sorted array has positive L_1-coordinates and satisfies the defining interlacings. Conversely these strict interlacings and the positive coordinates force decreasing order in each letter: in either orientation, two successive entries of that letter enclose an entry of its neighbor. Thus $I(\nu)\prod_s \det H_s$ is the sum over the $(k!)^D$ choices of row permutations $\tau = (\tau_s)_s$ of the indicators $\mathbf{1}_{L_1(\tau\nu)\in\Omega\cap\Sigma(A)}$.

Integrate these two evaluations. Every product permutation $S = S_{D-1}\cdots S_0$ occurs $(k!)^{D-1}$ times and its sign is the product of the individual signs. The result is

$$\operatorname{vol}(\Omega\cap\Sigma(A)) = K_k(q)\operatorname{vol}(\Sigma(A)), \qquad K_k(q) = \frac{(q-1)^k}{k!}\sum_{S\in\mathfrak{S}_k}\frac{\operatorname{sgn}(S)}{|\det(qI-S)|}.$$

Only k, q occur on the right in the expression for $K_k(q)$.

Evaluate $K_k(q)$ with one positive letter, $P = q$ and $A = 1$. Here $\beta = (q-1)\nu$, and the inequalities are $\beta_b \geq q\beta_{b+1}$. Put $\beta_k = 0$ and $d_j = (q^j-1)/(q-1)$. The change

$$z_b = d_{b+1}(\beta_b - q\beta_{b+1}), \qquad 0 \leq b < k,$$

maps this part of the slab bijectively onto the whole slab. Back substitution from $b = k-1$ shows that $z \geq 0$ forces all entries of β nonnegative, and $d_{b+1} - qd_b = 1$ gives $\sum_b z_b = \sum_b \beta_b$. The determinant is $\prod_{j=1}^k d_j$. This slab has positive volume, since it contains $[0, 1/k]^k$, so its volume cancels to give the asserted factor. Finally a line in an h-dimensional space over the field of order q can be chosen in $(q^h-1)/(q-1)$ ways by counting its nonzero vectors. After choosing a line, the rest of a flag is a flag in the quotient. Induction from dimension zero gives the stated product. □

444. Thickened inequalities at the boundary between two blocks

Definition 444.1. Use the notation of Definition 443.1 with D even and $\epsilon_s = 1$ at even s. For $i, s \in \mathbb{Z}/D\mathbb{Z}$, let $[s-i]_D$ be the representative in $\{0, \ldots, D-1\}$ and put $M_s = P^{[s-i]_D}$. For $0 \leq b < k-1$, denote the two variable indices in rows $b, b+1$ at letter s by y_s, x_s, where y_s is the index in row $b+1$ at even s, and in row b at odd s. Define the linear form

$$E_{i,b}(\beta) = (1-q^{-1})\beta_{y_i} + q^{-1}\sum_s M_s(\beta_{y_s} - \beta_{x_s}).$$

For $1 \le r < k$ and real H let $\Omega_H^{(r)}$ be the set of β with $E_{i,b}(\beta) \le H\mathbf{1}_{b+1=r}$ for every i, b.

Lemma 444.2. *One has $\Omega_0^{(r)} = \Omega$. For $H \ge 0$ there is a single nonnegative translation taking*

$$\Omega_H^{(r)} \cap \Sigma(A) \text{ into } \Omega \cap \Sigma(A'), \qquad A'_s = A_s + H\mathbf{1}_{s \text{ even}} \frac{q(P^{2r}-1)}{q-1}.$$

Proof. For $\beta = L_1\nu$ put $\Delta_s = \nu_{b,s} - \nu_{b+1,s}$. In both parities $\beta_{y_s} - \beta_{x_s} = \Delta_s - P\Delta_{s+1}$. In $\sum_s M_s(\Delta_s - P\Delta_{s+1})$ every term cancels except at i, since $PM_{s-1} = M_s$ there except when $s = i$, where the two values are $q, 1$. Hence

$$E_{i,b}(L_1\nu) = (1-q^{-1})(\beta_{y_i} - \Delta_i).$$

The nonpositivity conditions are precisely the interlacings in Definition 443.1. Invertibility of L_1 was proved in Theorem 443.3.

Set $v_b = (1-q^{-1})^{-1}P^{2(r-1-b)}$ for $b < r$ and $v_b = 0$ otherwise. Take $\nu_{b,s} = v_b$ at even letters and $\nu_{b,s} = Pv_b$ at odd ones. Then $\xi = L_1\nu$ is $(P^2-1)v_b$ at even letters and zero at odd ones. At even i the excess on ξ is $(1-q^{-1})(P^2v_{b+1} - v_b)$. This equals $-\mathbf{1}_{b+1=r}$. At odd i the excess is $-(1-q^{-1})P(v_b - v_{b+1})$, which is nonpositive and equals $-P$ at the cut. Thus addition of $H\xi$ takes the indicated cone into Ω. Every entry of ξ is nonnegative, and summing its entries at each even letter gives $(1-q^{-1})^{-1}(P^{2r}-1)$ by the finite geometric sum. Addition translates both endpoints of each column-sum interval by that same column total multiplied by H. □

Proposition 444.3. *Take $a \ge 1$ integral, $H \ge 0$, and $A_s = a + k + 1$. Write*

$$F_k(q) = \prod_{j=1}^{k} \frac{q^j - 1}{q-1}, \qquad C_r = \frac{P^{2r}-1}{1-q^{-1}}, \qquad u = \frac{D(k-1)}{a}\left(k + 1 + \frac{HC_r}{2}\right).$$

Then

$$F_k(q)\,\mathrm{vol}(\Omega_H^{(r)} \cap \Sigma(A)) \le \exp(u)\binom{a+k-1}{k-1}^D.$$

In Definition 439.1 take $r = i_0$, $h = x^{40}$, $H = h + D(k+1)$. Uniformly at all served primes, for sufficiently large x the factor $\exp(u)$ is at most $1 + x^{-14}$.

Proof. Translate by Lemma 444.2 and apply Theorem 443.3 at the new endpoints. In k nonnegative coordinates the simplex of sum at most $z \ge 0$ has volume $z^k/k!$: successively integrate the last coordinate from 0 to z, starting with length in dimension one. For $z \ge 1$, subtracting the volume at $z-1$ gives at most $z^{k-1}/(k-1)!$ by integrating the derivative over that unit interval. The slabs are independent in the letters, so their volumes multiply by iterated integration. Divide each resulting bound by $\binom{a+k-1}{k-1} = \prod_{j=1}^{k-1}(a+j)/j \ge a^{k-1}/(k-1)!$. The binomial expression counts the placements of $k-1$ separators among $a+k-1$ slots, equivalently monomials of degree a in k variables. The inequality $1 + t \le \exp(t)$ for $t \ge 0$ follows from the exponential series. Use it at $t = (k + 1 + HC_r\mathbf{1}_{s \text{ even}})/a$ and sum the exponents, noting that exactly $D/2$ letters are even.

For the arithmetic assertion, $D = L/o \le f \le x^{12}$: the orbit length L is at most the order f of the group modulo N_1, as in Lemma 439.2. Also $k = O(x^5)$ and $a \ge x\exp(\ell)$, with $\ell = x^\epsilon$, by Lemma 26.2 and its proof. At $p \ge m$, $o = r = 1$ by Proposition 26.4 and its proof, and $\log p \le (10^6 \log x)^2$. At $p < m$, both $o \le r$ and $r \le 1 + \log_2 m$ by Definition 24.1, while $m \le 2\sqrt{x}$. In either case $\log C_r = O((1+\log x)^3)$, using $q \ge 2$. The other factors in the numerator of u grow at most as a fixed power of x times C_r. Hence $u \le x^{-14}/2$ eventually: every fixed power of $\log x$ is $o(x^\epsilon)$, as seen from the exponential series with argument $\epsilon \log x$. For $0 \le u \le 1/2$ the series also gives $\exp(u) - 1 \le \sum_{n \ge 1} u^n \le 2u$. This proves the uniform estimate. □

445. A polynomial detecting an extremal exponent

Definition 445.1. Use $W, O, p, P, q, D, \sigma_s$ of Definition 439.1, and the row pair and letter i of Definition 444.1. Use coefficients $B = O$, or $B = \mathbf{k}$ with the reduced maps σ_s of Definition 440.2. For $z \neq 0$ put $v(z) = \lfloor 2v_p(z) \rfloor$ over O, where $v_p(p) = 1$, and $v(z) = 0$ over $\mathbf{k}$. Set $v(0) = +\infty$. For $g \in \mathrm{GL}_k(W)$ let gF denote the polynomial action of Lemma 440.1, also on polynomials of arbitrary multidegrees by the same linear substitutions. For real coordinate weights c, the cost of a nonzero term zX^α is $v(z) + c \cdot \alpha$.

Theorem 445.2. *Suppose $P \geq 5$ and $2/5 \leq c(y_s) - c(x_s) \leq 8/5$ for every s. Let $G \subseteq \mathrm{GL}_k(W)$ be a subgroup containing the invertible matrices on the chosen two rows acting as the identity on the complementary rows. Let F be a polynomial over B with nonzero coefficient at α of order l. Suppose every term of gF, for every $g \in G$, has cost at least $l + c \cdot \alpha$. Then $E_{i,b}(\alpha) \leq 0$.*

Proof. Write $\delta_s = c(y_s) - c(x_s)$, $\omega = \delta_i/q$, and $M_s = P^{[s-i]_D}$. Use lower-case x_s, y_s for polynomial variables as well as their indices. Make new variables X_s, Y_s for $s \neq i$ and only X_i at i. Variables outside the pair are retained. Write $\pi\alpha$ for the exponents after merging x_i, y_i into X_i and relabelling the rest. Define integers

$$m(\alpha) = q \sum_s M_s \alpha_{x_s}, \qquad e(\alpha) = q\alpha_{y_i} + \sum_{s \neq i} M_s \alpha_{y_s}.$$

Let $\eta_s = -1$ for s of the same parity as i, and $\eta_s = 1$ otherwise. With one additional variable T use the substitution Φ given by

	x_s	y_s
$s = i$	$X_i T^q$	X_i
$s \neq i$	$X_s T^{qM_s} - \eta_s Y_s T^{M_s}$	$Y_s - \eta_s X_s$.

It fixes every variable outside the pair. Put $I_l = \{z : v(z) > l\}$. This is an ideal: the order of a sum is at least the minimum and the order of a product is at least the sum, by the valuation inequality and $\lfloor u + v \rfloor \geq \lfloor u \rfloor + \lfloor v \rfloor$. Let $A(T)$ be the coefficient at $\pi\alpha$ in ΦF over B/I_l, treating T as scalar. We prove that its degree is $m(\alpha)$ and that it vanishes to order $e(\alpha)$ at all q residues.

At infinity give T weight $-\omega$, X_i weight $c(y_i)$, and, for $s \neq i$, give X_s, Y_s weights $c(x_s) + qM_s\omega$, $c(y_s)$, respectively. The first summand of each substitution has the original weight, including both entries at i. At $s \neq i$ each second summand has strictly greater weight. The two increments are $\delta_s - M_s\omega > 0$ and $M_s\delta_i - \delta_s > 0$, since $5 \leq M_s \leq q/5$. Fixed variables keep their weights. Every product obtained by expansion therefore has at least its original cost and gains strictly if it uses a second summand. The new variables with exponent $\pi\alpha$ alone have weight $c \cdot \alpha + \omega m(\alpha)$. Consequently coefficients of T^j there with $j > m(\alpha)$ have order strictly greater than l. At $j = m(\alpha)$ all terms using a second summand also have order greater than l. Using just first summands gives exactly F_α: from $\pi\alpha$ one knows every exponent except the split of $\alpha_{x_i} + \alpha_{y_i}$, and $m(\alpha)$ determines that split. Thus $A \neq 0$ and its degree is $m(\alpha)$.

Fix $t \in W$, let $u = \sigma_i(t)$ and $u_s = \sigma_s(t)$. Take the matrix on rows $b, b+1$ which is their swap times the upper transvection of parameter $-t$ for even i, t for odd i. In the order x_s, y_s it acts on variables by

$$gx_s = y_s, \qquad gy_s = x_s + \eta_s u_s y_s.$$

Indeed the even action has columns of the swap times an upper transvection. The odd action takes the inverse transpose and interchanges the row names, reversing the sign of the parameter. Use $F = g(g^{-1}F)$ before substituting, and replace T by $T + u$. For $h = M_s$ or qM_s write

$$D_{s,h}(T) = (T + u)^h - u_s - T^h.$$

All its coefficients have order at least two. For positive intermediate exponents the binomial coefficients are divisible by p, since successive p-th powers distribute over sums modulo p. The constant coefficient has this same divisibility by iterating the congruence of Lemma 439.2 through $[s-i]_D$ steps or through $D+[s-i]_D$ steps. In the residue field these error polynomials vanish. In particular this holds at $s = i, h = q$.

At this finite point give T weight ω, X_i weight $c(x_i)$, and for $s \neq i$ give X_s, Y_s weights $c(x_s)$, $c(y_s) - M_s\omega$, respectively. At i the substituted images of gx_i, gy_i are X_i and $X_i(T^q + D_{i,q})$. Their terms have at least the weights of x_i, y_i, since $q\omega = \delta_i < 2$. At $s \neq i$ the corresponding images are

$$Y_s - \eta_s X_s, \qquad X_s(T^{qM_s} + D_{s,qM_s}) - \eta_s Y_s(T^{M_s} + D_{s,M_s}).$$

The first has weight at least $c(x_s)$ because $M_s\omega < \delta_s$. For the second, its two pure powers have weights $c(x_s) + M_s\delta_i \geq c(y_s)$ and $c(y_s)$. Its errors have cost at least $c(x_s) + 2$ and $c(y_s) - M_s\omega + 2$, both at least $c(y_s)$. Thus every substituted term of $g^{-1}F$ respects its cost bound. The finite-point weight of the variables at $\pi\alpha$ alone is $c \cdot \alpha - \omega e(\alpha)$. For $j < e(\alpha)$, the coefficient at that monomial and T^j therefore has order at least $l + \omega(e(\alpha) - j) > l$. It vanishes in B/I_l, proving $T^{e(\alpha)} \mid A(T + u)$.

Take q lifts t of the distinct residues of W. Their images have pairwise unit differences by Lemma 439.2. The factors $T - u$ for different lifts thus generate the unit ideal in pairs, by subtracting them. Their positive $e(\alpha)$-th powers are also pairwise comaximal: expand $1 = (v + w)^{2e(\alpha)-1}$ when $v + w = 1$ comes from the two ideals. For comaximal ideals I, J, writing $1 = v+w$ in the same way gives $I \cap J = IJ$. Products of ideals comaximal with a fixed ideal remain comaximal with it, by multiplying representatives of one in the quotient. Hence the product of the indicated monic powers divides A. Multiplying a nonzero polynomial by a monic polynomial adds their degrees even over B/I_l. It follows that $qe(\alpha) \leq m(\alpha)$, also true for $e(\alpha) = 0$. As $E_{i,b}(\alpha) = (qe(\alpha) - m(\alpha))/q^2$, this is the asserted inequality. □

446. Divisibility in every frame preserving the two blocks

Definition 446.1. Put $r = i_0$ in the cycle of Definition 439.1, with $1 \leq r < k$. Let $L_r(W)$ be the block diagonal subgroup with block sizes $r, k - r$. For a real h let $U(h) \subset \bar{\mathcal{V}}$ be the subspace of residue polynomials F of degree a in each letter for which every monomial β in every gF, $g \in L_r(W)$, satisfies $E_{i,r-1}(\beta) \leq h$ for all i, with $E_{i,r-1}$ as in Definition 444.1. Here and below a monomial in a polynomial means one with nonzero coefficient. The subspace condition follows since vanishing of each specified coefficient after a fixed linear action is linear.

Theorem 446.2. *In either factorization $J = QA$ of Proposition 440.3, put $F_e = (Qj)_e$ for any integral column j. For every $g \in L_r(W)$, every exponent β and every letter i,*

$$v((gF_e)_\beta) \;\geq\; \tfrac{1}{2}E_{i,r-1}(\beta)$$

with the coefficient order of Definition 445.1. In particular these coefficients belong to $\varrho^c O$ when $E_{i,r-1}(\beta) > 2c$ for any nonnegative integer c.

Proof. The prime is inert in the quadratic field. The primes $2, 3$ divide M and split there by Proposition 10.3. Thus $p \geq 5$ and $P = p^o \geq 5$. Take an integer $b_0 \geq 0$ such that $\varrho^{b_0}A^{-1}$ is integral. This is possible because A is finite and invertible over the fraction field. Put $F^* = \varrho^{b_0}A^{-1}j$ and fix one source component f of F^*. Every entry of $JF^* = \varrho^{b_0}Qj$ is divisible by ϱ^{b_0}. By the coset coverage and integrality argument of Proposition 440.3, in every full frame $g' \in \mathrm{GL}_k(W)$ one therefore has

$$v((g'f)_\beta) + d(\beta) \geq b_0.$$

Here $v(\varrho^n z) = n + v(z)$ for integral $z \neq 0$ and integer n whenever the product is integral, directly from the order definition.

Fix i and abbreviate $E = E_{i,r-1}$. Write d_z for the zero or one depth of the individual variable z and E_z for its coefficient in E. Use coordinate weights $c_z = d_z - E_z/2$. For the boundary pair, y_s is deep and x_s shallow at both parities. Hence

$$c_{y_s} - c_{x_s} = \begin{cases} (1-q^{-1})/2, & s = i, \\ 1 - M_s/q, & s \neq i. \end{cases}$$

These numbers belong to $[2/5, 8/5]$, since $q \geq P \geq 5$ and $M_s/q \leq 1/P$ for $s \neq i$.

If $f = 0$ the component contributes zero. Otherwise minimize order plus c-weight among all terms of all $g'f$. A minimum is attained: there are finitely many exponent vectors of the fixed multidegree, and for each occurring exponent the nonnegative integral orders have a least member. Choose an attaining frame and exponent α. Every translate of the polynomial in that frame is still a translate of f. Thus Theorem 445.2 gives $E(\alpha) \leq 0$. The minimum is at least b_0 by the displayed depth bound, since subtracting $E(\alpha)/2$ increases it. This proves, for every g', β,

$$v((g'f)_\beta) + d(\beta) - b_0 \;\geq\; E(\beta)/2.$$

At an edge from this source component, $\varrho^{b_0} F_e = D_\varrho P_\lambda(h_e) f$. The action of a block diagonal g commutes with D_ϱ: it sends each variable to a sum of variables of the same depth. Apply g and use the preceding estimate at $g' = gh_e$. This gives the coefficient inequality, and order at least c implies divisibility by ϱ^c. □

447. Counting residue polynomials by flags

Theorem 447.1. *Let $p, \lambda, r, D, P, q, k, a$ and $U(h)$ be as in Definition 446.1, with $P \geq 5$ and $h \geq 0$. Let $d_\mathcal{P} = [K : \mathcal{P}]$ be the index of the two-block parahoric of Theorem 439.3. For $H = h + D(k+1)$ and $A_s = a + k + 1$,*

$$d_\mathcal{P} \dim_{\mathbf{k}} U(h) \;\leq\; F_k(q) \operatorname{vol}(\Omega_H^{(r)} \cap \Sigma(A)).$$

In particular, with $h = x^{40}$, uniformly at every served prime for all sufficiently large x,

$$d_\mathcal{P} \dim_{\mathbf{k}} U(x^{40}) \leq (1 + x^{-14}) \binom{a+k-1}{k-1}^D.$$

Proof. Write $k_q = W/pW$ and $L = \mathrm{GL}_r(k_q) \times \mathrm{GL}_{k-r}(k_q)$. Any element of L lifts to $L_r(W)$ by lifting entries, since a determinant with nonzero residue is a unit. The action on residue polynomials factors through L, by the successive residue embeddings of Lemma 439.2. Let $B_L \subset L$ be the two lower triangular subgroups and let $\mathcal{T}$ be representatives of $B_L \backslash L$ including the identity.

For each integer $j \geq 1$ put $m' = q^j$. Take one polynomial $F_u \in U(h)$ for each array of residues $0 \leq u_{b,s} < m'$. Increase $u_{r-1,s}$ by $m'(k - \lfloor \sum_b u_{b,s}/m' \rfloor)$ at each letter s, and write $\gamma(u)$ for the resulting array. All its entries are nonnegative and in each letter its sum lies in $[km', (k+1)m')$. Form

$$B(X) = \sum_u X^{\gamma(u)} F_u(X^{m'}), \qquad X^{m'} = (X_{b,s}^{m'})_{b,s}.$$

This is linear over $\mathbf{k}$, including when its scalars are outside k_q. Substitution changes only the variables. It is injective on the tuple $(F_u)_u$: distinct summands have distinct exponent arrays modulo m', and substitution is injective on each summand. Every residue matrix entry under the polynomial action is fixed by raising to m'. Indeed the entries of each letter lie in an image of k_q and $z^q = z$ in every such image. Taking m'-th powers distributes over sums of variables in characteristic p. It follows by substitution on the variables that for $g \in L$

$$gB = \sum_u (gX^{\gamma(u)})(gF_u)(X^{m'}).$$

Give $X_{b,s}$ weight b at even s, and $-b$ at odd s. Take a nonzero tuple and choose a term α of least weight among the polynomials $g_0 B,\ g_0 \in \mathcal{T}$. This attains the least weight in all $gB,\ g \in L$. Indeed left multiplication by an element of B_L replaces a variable in an even letter by a sum from its own and lower rows. In an odd letter its inverse transpose uses its own and upper rows. Both operations preserve or increase the weight of every term. They give every g by coset coverage. For all adjacent pairs within one block, the weights satisfy $c(y_s) - c(x_s) = 1$. For each such b, the two-row subgroup lies in $L_r(W)$. Apply Theorem 445.2 to $g_0 B$ in the residue field. The simultaneous conclusions are $E_{i,b}(\alpha) \le 0$ for all i and $b + 1 \ne r$.

At the boundary, the nonzero coefficient at α is a sum with at least one nonzero contributing term $\alpha = m'\delta + \gamma'$, where δ occurs in $g_0 F_u$ and γ' occurs in $g_0 X^{\gamma(u)}$. Each letter of γ' has the same total degree as that of $\gamma(u)$ by the linear substitutions within letters. The coefficients of $E_{i,r-1}$ are at most one by Definition 444.1. Therefore $E_{i,r-1}(\gamma') \le D(k+1)m'$. By Definition 446.1 the other contribution satisfies $E_{i,r-1}(\delta) \le h$. Consequently $\alpha/m' \in \Omega_H^{(r)}$. The multidegrees also give $\alpha/m' \in \Sigma(A)$, because δ has degree a at each letter.

Put $S = \Omega_H^{(r)} \cap \Sigma(A)$. Restricting the coefficients of each $g_0 B$, $g_0 \in \mathcal{T}$, to $\alpha \in m'S \cap \mathbb{Z}^{kD}$ thus detects every nonzero tuple. The dimension inequality for this injective linear map is

$$(m')^{kD} \dim U(h) \ \le\ |\mathcal{T}|\, \#(m'S \cap \mathbb{Z}^{kD}).$$

The ratio of the lattice count to $(m')^{kD}$ tends to $\mathrm{vol}(S)$ as $j \to \infty$. Indeed the cubes of side $1/m'$ with a grid vertex in S, using that vertex as the lower corner, agree with S up to cubes meeting its finite union of boundary hyperplanes in a fixed bounded box. Their total volume tends to zero by the grid argument of Theorem 443.3.

Reduction of K modulo p is onto $\mathrm{GL}_k(k_q)$ by the same unit-determinant lifting argument. The lower block parabolic stabilizes the subspace spanned by the last $k - r$ coordinate vectors, and $\mathcal{P}$ is its inverse image. Thus $d_{\mathcal{P}}$ counts the $(k-r)$-dimensional subspaces: every basis of one can be extended to a basis of the whole space. For each such subspace, a full flag through it is determined by a full flag in it and one in its r-dimensional quotient. The lower triangular groups are the stabilizers of the corresponding standard flags, and transitivity follows by taking a basis adapted to a flag. Hence the number of flags extending each subspace is $|\mathcal{T}|$. By Theorem 443.3, $d_{\mathcal{P}}|\mathcal{T}| = F_k(q)$. Multiplication of the limit inequality by $d_{\mathcal{P}}$ proves the first claim. Proposition 444.3 proves the second one. □

448. The index of a pairing of two matrices

Lemma 448.1. *Let O be a valuation ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{k}$. Let $V = O^n,\ W' = O^N$, let $Q, M : V \to W'$, $\beta : W' \to V$ be integral linear maps, and suppose $RQ = I_n$ for an integral $R : W' \to V$. Let $a, z \in O \setminus \{0\}$ and $w \in \mathfrak{m} \setminus \{0\}$. Suppose*

$$\beta M \equiv a I_n \bmod aw, \qquad Qu - Mv = awF \quad \Longrightarrow \quad \bar{F} \in S$$

for all $u, v \in V,\ F \in W'$, with one fixed subspace $S \subset W'/\mathfrak{m}W'$. There is an integer $0 \le b \le \dim_{\mathbf{k}} S - n$ such that

$$\mathcal{I}(V, \beta Q V + zV;\ a^{n-b} z^b).$$

Here $\mathcal{I}$ is the divisibility of determinant indices in Definition 15.5.

Proof. Write $\beta M = a(I + wB)$. The second factor has determinant one in the residue field and is invertible over O by the adjugate formula. Right multiplication of M by its inverse preserves the difference hypothesis and gives $\beta M = aI$. Write $W' = QV \oplus W_0$, where $W_0 = \ker R$. This is a finite free summand: it is generated by the columns of $I - QR$, and the rectangular pivot argument of Lemma 25.1 gives a basis of their span by clearing the row and column of each successive nonzero pivot. Write $M = QM_1 + M_0$ accordingly. By the same argument there is a basis $v_1, \dots, v_n$ of V with $M_0 v_j = t_j u_j$ at its nonzero pivots, where those u_j form part of a basis of W_0. Use $t_j = 0$ on the kernel.

Call an index exceptional when $t_j \notin awO$, and let b count them. Then $t_j \neq 0$ and $aw/t_j \in O$ by total divisibility. With $v = (aw/t_j)v_j,\ u = M_1 v$ the difference is $-awu_j$, so $\bar{u}_j \in S$. Also $\bar{Q}V \subset S$ by taking $u \in awV,\ v = 0$. The b reductions are independent and independent of the n basis images of Q, by the direct-sum bases of W'. This proves $n + b \leq \dim S$.

For the other indices $M_0 v_j = awu'_j$ with integral u'_j, including zero. Therefore

$$\beta Q M_1 v_j = a(v_j - w\beta u'_j).$$

Replace these v_j by $v'_j = v_j - w\beta u'_j$ and leave the exceptional basis elements fixed. Their residues still form a basis, so their determinant is a unit and the resulting vectors are a basis of V. In this basis $\beta QV + zV$ contains av'_j in each of the $n - b$ indicated directions and zv_j in each exceptional direction. Applying Lemma 25.1 to this diagonal submodule gives the claimed determinant divisibility. □

449. Weighted traces on the tensor powers of a Levi subgroup

Lemma 449.1. *For p a prime and $f_0 \geq 1$, let γ be a vector of $u \geq 1$ nonnegative exponents of sum $d < p^{f_0}$. Set $B_\gamma = p^{uf_0} \prod_j \gamma_j!/d!$. Then B_γ and p^{uf_0}/B_γ are integral at p. With M_d as in Lemma 440.1,*

$$B_\alpha M_d(g)_{\alpha\beta} = B_\beta M_d(g^t)_{\beta\alpha}$$

for any u-by-u matrix g in characteristic zero. For the representation P_λ take $f_0 = oD$ and $a < q = p^{f_0}$. The integral diagonal matrix B whose entry at β is $\prod_s B_{\beta_s}$, using $u = k, d = a$, satisfies

$$BP_\lambda(g^t) = P_\lambda(g)^t B, \qquad p^{kf_0 D} B^{-1} \text{ is integral.}$$

Proof. The reciprocal quotient $d!/\prod_j \gamma_j!$ is the number of words with the indicated multiplicities. The number of factors of p in $d!$ is $\sum_{e \geq 1} \lfloor d/p^e \rfloor$ by counting one contribution from each multiple of p^e. Subtracting the corresponding sums for the $\gamma_j!$ gives, for each e, at most $u - 1$ by adding the remainders modulo p^e. The contributions for $e \geq f_0$ are zero. This proves integrality. An array (n_{ij}) of nonnegative integers with row sums α, column sums β contributes to $M_d(g)_{\alpha\beta}$ the product of $g_{ij}^{n_{ij}}$ times $\prod_j \beta_j!/\prod_{i,j} n_{ij}!$ by expansion of the input monomial. Multiplication by B_α makes the factor symmetric in the two sets of marginals. Transpose the array and sum. The tensor identity follows letter by letter since transpose commutes with taking the inverse transpose, and the diagonal denominators multiply. □

Theorem 449.2. *Take $W, O, p, q = p^{f_0}$ and one embedding $\sigma : W \to O$ of Definition 439.1. Let $L = L_r(W)$, let $X = (X_{bc})$ be a generic k-by-k matrix, and put $F^g(X) = F(\sigma(g)^{-1} X \sigma(g))$. Give the rows and columns real weights x_b. For a monomial X^α put*

$$v_R(\alpha) = \sum_{bc} x_b \alpha_{bc}, \qquad v_C(\alpha) = \sum_{bc} x_c \alpha_{bc}.$$

Let $J \subseteq \mathbb{R}$, and let $d_* \geq 0$ be an integer satisfying $d_* + k < q$. There is an integral linear functional κ_J on $O[X]$ with the following properties.

1. *$\kappa_J(F^g) = \kappa_J(F)$ for all $g \in L$. It vanishes on degrees above d_* and on monomials whose row weight differs from their column weight.*
2. *If $v_R(\alpha) \in J$ and $|\alpha| \leq d_*$, then $\kappa_J(X^\alpha) = p^{k^2 f_0}(X^\alpha)(I_k)$.*
3. *It is a finite integral linear combination of functionals $F \mapsto l(F^g)$, with $g \in L$ and $l(X^\alpha) = 0$ whenever $v_R(\alpha) \notin J$. One may take each l supported in a single degree at most d_*.*

Proof. Work first in fixed degree $d \leq d_*$. Let $T = (O^k)^{\otimes d}$ with basis e_t indexed by words of length d in the row indices, including the empty word at $d = 0$. Write $A(g) = \sigma(g)^{\otimes d}$. Associate to X^α the endomorphism $K(X^\alpha)$ whose entry at output word s and input word t is B_α from Lemma 449.1 with $u = k^2$, if both of the following hold: the ordered pairs (t_j, s_j)

have multiplicities α, and the two entries of each pair belong to the same block. All other entries are zero. Extend K linearly. It is integral. It satisfies

$$K(F^g) = A(g)K(F)A(g)^{-1}.$$

Indeed deleting monomials involving off-block variables amounts to first restricting X to a block diagonal matrix, which commutes with the conjugation. In the fraction field, write a monomial as $\prod_{j=1}^{d} X_{b_j c_j}$. Its matrix K on the block variables is $p^{k^2 f_0}/d!$ times the sum of all permutations of the tensor factors of $\bigotimes_j E_{c_j b_j}$, with E_{cb} the elementary matrix. Each allowed entry occurs $\prod_{bc} \alpha_{bc}!$ times. In degree one the substitution corresponds to $E_{cb} \mapsto \sigma(g) E_{cb} \sigma(g)^{-1}$, by matrix multiplication. Tensoring, permuting and summing gives the identity in every degree, hence also over O.

We need an integral span assertion. For a block diagonal matrix Y, the entries of $Y^{\otimes d}$ are either zero or the degree-d monomials in its block entries. The evaluations of the complete list of these monomials on invertible residue block matrices span the coordinate space over the residue field of O. In fact a linear dependence annihilating all these evaluations gives a polynomial P_0. Multiply it by the product of the two block determinants. The product vanishes on all tuples of block entries from $\bar{\sigma}(k_q)$, since a noninvertible tuple has zero determinant. Its degree in each variable is at most $d + k < q$. A univariate polynomial of degree less than q vanishing at q distinct points is zero by division by its linear factors. Applying this in one variable at a time proves the same assertion for this grid, even with coefficients in the larger residue field. The determinant polynomial is nonzero, so $P_0 = 0$. This proves spanning by ordinary row reduction. Select a square minor with nonzero residue determinant, lift its evaluation matrices to L as in Theorem 447.1, and invert the minor over O by its unit determinant. The coefficient matrices of $Y^{\otimes d}$ consequently lie in the integral span of a finite list of $A(g)$. These coefficient matrices have ones at a specified set of word pairs and zero elsewhere. Both all $A(g)$ and the $K(X^\alpha)$ are integral linear combinations of them, taking account of the transposed indexing of K.

Let T_J be the span of $A(g)e_t$ for all $g \in L$ and words t with $\sum_j x_{t_j} \in J$. It has a finite generating set by the preceding finite span assertion. The pivot proof of Lemma 25.1 therefore gives a free basis in its span. It is stable under all $A(g)$ by the group law and under all $K(F)$ by that same span assertion. Define $\kappa_J(F)$ in this degree as the trace of $K(F)$ on T_J, the sum of the diagonal entries in a basis. It is integral. Trace is invariant under change of basis because $\operatorname{tr}(UV) = \operatorname{tr}(VU)$, by expanding the double sum of entries. The covariance identity proves conjugation invariance. A matrix $K(X^\alpha)$ acts only on words of weight $v_R(\alpha)$ and its outputs have weight $v_C(\alpha)$. At distinct weights its square is zero, so its trace on T_J vanishes: over the fraction field take a basis of its image, extend to its kernel and then to the entire space. Its diagonal in this basis is zero.

If $v_R(\alpha) \in J$, every basis word on which $K(X^\alpha)$ can act is already in T_J. Its image on all of T is therefore in T_J. Extending a basis of the latter over the fraction field gives zero on the quotient, so its trace on T_J equals its trace on T. No diagonal entry is possible for a monomial using an off-diagonal variable. For a diagonal monomial the number of words giving a diagonal entry is $d!/\prod_b \alpha_{bb}!$, and each entry is B_α. This gives $p^{k^2 f_0}$ exactly as in the stated evaluation.

Finally choose a basis v_j of T_J with coordinate maps v_j^*, and write each v_j as an integral sum of $A(g)e_t$ at the chosen weights. In the trace formula $\sum_j v_j^*(K(F)v_j)$ each summand becomes a combination of

$$v_j^*(A(g)K(F^{g^{-1}})e_t).$$

The map $l : H \mapsto v_j^*(A(g)K(H)e_t)$ is integral and kills monomials of row weight outside J by the word description. This gives the required conjugate expression in degree d, also when $T_J = 0$. Sum the finitely many functionals over $0 \le d \le d_*$, extending each by zero on other degrees. Conjugation preserves degree, which proves all three properties on $O[X]$. $\square$

450. Frobenius products and the weight of a matrix coefficient

Definition 450.1. Use Definition 446.1. Set $t = D/2$, $B_0 = P^t$ and $x_b = (\mathbf{1}_{b=r} - \mathbf{1}_{b=r-1})/B_0$. Let X_s be independent generic k-by-k matrices. For $H \in L_r(W)$ write $H_s = \sigma_s(H)$. With entrywise powers denoted by brackets, put, for $0 \le d \le t$,

$$\Psi^H_{s,d} = H_s (H_{s-d}^{-1} X_{s-d} H_{s-d})^{[P^d]} H_s^{-1}, \qquad Z^H_{s,j} = \Psi^H_{s,j-1} \cdots \Psi^H_{s,0}, \quad Z^H_{s,0} = I.$$

Set $j_b = t + d_{b,s}$ with $d_{b,s} \in \{0,1\}$ the depth of the variable in row b of letter s. Define A^H_s by taking its row b from Z^H_{s,j_b} at even s and from $\mathrm{adj}(Z^H_{s,j_b})^t$ at odd s. Put

$$\mathcal{F}_H(X) = \bigotimes_s M_a(A^H_s), \qquad \mathcal{F} = \mathcal{F}_1.$$

The adjugate is the transpose of the matrix of signed minors. For any monomial in all entries let ν_s denote its exponent array in X_s. Use the row and column weights v_R, v_C of Theorem 449.2 with the above x_b.

Lemma 450.2. *The matrix $\mathcal{F}_H$ equals $P_\lambda(H)\mathcal{F}((H_s^{-1}X_sH_s)_s)P_\lambda(H)^{-1}$, and $\mathcal{F}((I)_s) = I$. Each matrix entry has degree, in every X_s, at most $d_* = Dak(t+2)B_0$. At all sufficiently large served scales one has $Dak(t+2) + a + k < B_0$, and hence $d_* + k < q$.*

Proof. In a fixed product the conjugations by H_s telescope. Taking the transpose of an adjugate changes conjugation by H_s to conjugation by H_s^{-t}. For invertible matrices this follows from $\mathrm{adj}(Z) = \det(Z)Z^{-1}$ by the cofactor expansion. For arbitrary matrices apply the identity to $Z + uI$ over the field of rational functions in u and specialize the polynomial identity at $u = 0$. Rows at each length form a block preserved by both H_s and H_s^{-t}, so selecting rows commutes with the respective left change. Right changes are common to the whole letter. The coefficient action M_a is multiplicative by composition of the substitutions. This proves covariance. Setting every matrix equal to I gives the stated specialization. Each $\Psi^H_{s,d}$ is homogeneous of degree $P^d \le B_0$. A product of at most $t+1$ of these matrices has degree at most $(t+1)B_0$; its minors of size $k-1$ have degree at most $(k-1)(t+1)B_0$. The tensor touches Da entries counting repetitions. This proves the asserted degree bound. At a served prime $D \ge m$ by Lemma 439.2, so $B_0 \ge 2^{m/2}$. By Proposition 10.5, $m \ge \sqrt{x}$ up to a fixed positive factor and $0 < \epsilon < 1/100$. The bounds $D \le x^{12}, k = O(x^5)$ of Proposition 444.3 and $a \le 2x\exp(x^\epsilon)$ of Proposition 12.2 therefore give $Dak(t+2) + a + k < B_0$ eventually. Multiply by $B_0 \ge 1$ and use $B_0^2 = q$ for the last assertion. □

Theorem 450.3. *Assume $Dak(t+2) \le B_0$. Fix a letter i and a monomial ν with $v_R(\nu_j) = v_C(\nu_j)$ for all $j \ne i$. For any $H \in L_r(W)$ and exponent vectors β, γ of the coefficient representation, let $z = [X^\nu](\mathcal{F}_H)_{\beta\gamma}$. Then, with the order of Definition 445.1,*

$$2v(z) + 1 \;\ge\; \left| E_{i+t,r-1}(\beta) - v_R(\nu_i) \right|.$$

Proof. Fix a sign $\epsilon = \pm 1$ and put

$$f_n(b) = \frac{\epsilon x_b P^n}{2q}, \qquad u = \frac{1}{2B_0}.$$

Give the variable $(X_j)_{bc}$ the weight $f_D(b)$ if $j = i$, and $f_{[j-i]_D}(b) - f_{[j-i]_D}(c)$ otherwise. Its absolute weight is at most $1/B_0$. For a fixed output letter s put $\lambda_d(b) = f_{d+[s-d-i]_D}(b)$. Each λ_d has sum zero over b. At $d = 0$ its absolute value is at most u, and for $0 \le d \le t+1$ it is at most $1/2$, since $d + [s-d-i]_D \le D + t$.

Every term of entry bc of $\Psi^H_{s,d}$ has cost at least $\lambda_{d+1}(b) - \lambda_d(c) - u$, where cost is coefficient order plus the weight of its variables. To see this, reduce its formula modulo p. By Lemma 439.2, conjugation and the entrywise power now commute with the change of embedding from $s-d$ to s. The matrix reduces to $X_{s-d}^{[P^d]}$, because raising to P^d is additive and multiplicative in characteristic p. Thus the difference has coefficients of order at least two. Its terms have degree P^d, with variable weight at least $-P^d/B_0 \ge -1$; their costs

are at least one, which bounds the claimed difference of the λ's. For the pure entry put $j = s - d$. If $j \neq i$, then $[j-1-i]_D = [j-i]_D - 1$; multiplying the variable weight by P^d gives exactly $\lambda_{d+1}(b) - \lambda_d(c)$. If $j = i$, the weight after multiplication is $P^d f_D(b) = \lambda_{d+1}(b)$ and $|\lambda_d(c)| = |f_d(c)| \le u$. This proves the entry estimate, also if terms combine by addition.

Multiplication adds the lower bounds for cost. Summing along a matrix-product path telescopes its intermediate weights: entry bc at length j has cost at least $\lambda_j(b) - \lambda_0(c) - ju$. At an odd letter the signed minor deletes row b and column c, so summing the same bounds over its permutations gives at least $-\lambda_j(b) + \lambda_0(c) - (k-1)ju$, using the two zero sums. With $j = t + d_{b,s}$ both parity cases give the bound

$$(-1)^s \lambda_{t+d_{b,s}}(b) - k(t+2)u$$

on entry bc of A_s^H. The first term here equals $\epsilon E_{i+t,r-1}(\mathbf{e}_{b,s})/2$, where $\mathbf{e}_{b,s}$ is the unit exponent vector. Indeed the numerator is zero outside the boundary rows. At the shallow one its signed weight is $-P^{[s-i-t]_D}/q$, and at the deep one it is $P^{1+[s-1-i-t]_D}/q$, before multiplication by $\epsilon/2$. These are precisely the coefficients in Definition 444.1.

In an entry of $M_a(A_s^H)$ with output β_s, each term is a product of a entries using row b exactly $\beta_{b,s}$ times, by polynomial expansion. The full tensor thus has cost at least $\epsilon E_{i+t,r-1}(\beta)/2 - Dak(t+2)u$. By hypothesis the subtracted term is at most $1/2$. For the specified monomial the variable weight outside i vanishes, since each letter contributes a scalar times its row weight minus its column weight. The remaining weight is $\epsilon v_R(\nu_i)/2$. Subtract it from the cost bound and use both signs to obtain the claimed absolute value inequality. □

451. The integral operator selecting the permitted weights

Definition 451.1. For $c \ge 1$ integral and $\theta \in \mathbb{R}$, let $\mathcal{U}(\theta, c) \subset \mathcal{V}$ be the integral polynomials for which every coefficient at β of gF, $g \in L_r(W)$, lies in $\varrho^c O$ whenever $E_{i,r-1}(\beta) > \theta$ for some i. This is an O-submodule since the coefficient tests and the action are integral linear. The group law makes it stable under $L_r(W)$.

Theorem 451.2. *Use the arithmetic range of Lemma 450.2, put $f_0 = oD$, $K_0 = (k^2+k)f_0 D$, and let B be the diagonal matrix of Lemma 449.1. For every $c \ge 1$ integral and real θ there is an integral matrix Z such that*

$$Z\mathcal{V} \subset \mathcal{U}(\theta, c), \qquad y^t B Z \equiv p^{K_0} y^t \pmod{\varrho^c} \quad (y \in \mathcal{U}(\theta - 4c - 2, c)).$$

The congruence is entrywise in the row.

Proof. Use the degree bound d_* and variable weights of Definition 450.1 and Lemma 450.2. At each letter s, take the two functionals from Theorem 449.2 for the map σ_s, with weight sets $\mathbb{R}$ and $(\theta - 2c - 1, \infty)$. Call them a_s and b_s, and set $l_s = a_s - b_s$. All three are conjugation invariant and annihilate unbalanced monomials, meaning those with $v_R \neq v_C$. Also l_s vanishes whenever $v_R > \theta - 2c - 1$, by their common scaled evaluation in the allowed degrees and common zero above the cutoff.

Given a functional u_s at each letter, denote by $[u_s]_s F$ the result of applying their tensor product entrywise to a polynomial matrix F. Explicitly a monomial X^ν is replaced by $\prod_s u_s(X_s^{\nu_s})$ and the result is extended by linearity. This also shows multilinearity in the functionals and compatibility with constant matrices on each side. Put

$$Z_0 = [l_s]_s \mathcal{F}, \qquad B' = p^{kf_0 D} B^{-1}, \qquad Z = Z_0 B'.$$

All these entries are integral by Lemma 449.1.

For $H \in L_r(W)$, invariance in all letters and covariance in Lemma 450.2 give $P_\lambda(H) Z_0 = ([l_s]_s \mathcal{F}_H) P_\lambda(H)$. Fix a row β for which $E_{i+t,r-1}(\beta) > \theta$. Every monomial surviving the functionals is balanced outside i and has $v_R(\nu_i) \le \theta - 2c - 1$. Theorem 450.3 gives order at least c for its coefficient. Consequently the indicated row of $P_\lambda(H) Z_0 B'$ is divisible by ϱ^c. All letters occur as $i + t$, proving the image assertion.

Evaluation with $[a_s]_s$ gives $p^{k^2 f_0 D} I$ on $\mathcal{F}$, since every letter has degree at most d_* and evaluation at $(I)_s$ gives I by Lemma 450.2. Expand $[a_s]_s - [l_s]_s$ by replacing one factor at a time in any fixed order. Each difference has $b_i = a_i - l_i$ at the replaced letter, with all other factors invariant and annihilating unbalanced monomials. In the finite expansion of b_i supplied by Theorem 449.2, each term applies an integral functional killing $v_R(\nu_i) \le \theta - 2c - 1$, after a conjugation by some $H \in L_r(W)$. Invariance at the other letters permits the same conjugation everywhere. Covariance therefore writes the corresponding matrix as

$$P_\lambda(H)^{-1} A P_\lambda(H),$$

where A is obtained by applying the untransported functionals to $\mathcal{F}_H$. In its entries every surviving monomial is balanced away from i and has $v_R(\nu_i) > \theta - 2c - 1$.

For y in the stated submodule let $y' = P_\lambda(H^{-t})y$. The matrix H^{-t} is block diagonal, so y' lies in the same submodule. By Lemma 449.1, $y^t B P_\lambda(H)^{-1} = (y')^t B$. For any row β of A with $E_{i+t,r-1}(\beta) > \theta - 4c - 2$, the factor y'_β is divisible by ϱ^c. In every other row the surviving monomials have $v_R(\nu_i) - E_{i+t,r-1}(\beta) > 2c + 1$. Their coefficients are divisible by ϱ^c by Theorem 450.3. Hence the product of $y^t B$ with this entire conjugate matrix is divisible by ϱ^c. Sum over the telescoping expansion and multiply by the integral B'. Since $BB' = p^{k f_0 D} I$, the contribution from $[a_s]_s \mathcal{F}$ is exactly $p^{K_0} y^t$. This proves the row congruence. □

452. The elementary divisors of a single marked operator

Definition 452.1. For an integral endomorphism T of a free module V of finite rank over a valuation ring with valuation $v_p(p) = 1$, and $s \in \mathbb{Z}_{\ge 0}$, let

$$d_s(T) = \sum_j \min\{s, v_p(t_j)\}.$$

Here t_j are the diagonal entries in any Smith reduction of T, and $v_p(0) = +\infty$. By Lemma 25.1, $\mathcal{I}(V, TV + p^s V; z)$ holds precisely when $v_p(z) \ge d_s(T)$ for $z \ne 0$ in the ring. Indeed each nonzero generator of $t_j O + p^s O$ has the valuation in the sum, and the product of such generators is in the ring. This also shows independence of the reduction.

Theorem 452.2. *For each sufficiently large arithmetic scale, each served p, each $\lambda \in \Lambda$, and every $s \in \mathbb{Z}_{\ge 0}$, the endomorphism Y_λ of the integral columns $\mathcal{E}_p$ in Definition 437.1 satisfies*

$$d_s(Y_\lambda) \;\le\; c_0 \left(\frac{3 A_\lambda}{mp} + x^{40} + \frac{s}{x^{14}} \right), \qquad c_0 = \dim \mathrm{Col}.$$

Proof. Use the two edge matrices J_b, J_k of Proposition 440.3, with frames b_e^{-1}, k_e^t and sources $c(e), c'(e)$, respectively. Tensor them with the identity on H_{rest} and reorder the factors in the same way on each side. Write V for the class-coordinate module of rank c_0 over O, and W' for the edge-coordinate module. Each edge has M_0 linked coordinates with $M_0 = \binom{a+k-1}{k-1}^D$, and $h' = \dim H_{\mathrm{rest}}$ remaining coordinates, so $c_0 = |C| h' M_0$ by Theorem 439.3. Write $J_j = Q_j A_j$, $R_j Q_j = I$ for $j = b, k$ by tensoring the displayed factorizations of the proposition in these same coordinates. Theorem 446.2 applies on every edge and each fixed remaining coordinate.

Extend the diagonal B of Lemma 449.1 to each class or edge independently, writing $B_V, B_{W'}$ for the two extensions. Let U act on each edge by its u_e on the remaining factor. This is an integral automorphism and commutes with $B_{W'}$. In the chosen gauges,

$$Y_\lambda B_V = J_b^t \, U B_{W'} \, J_k = A_b^t \, (Q_b^t U B_{W'} Q_k) \, A_k.$$

Indeed on edge e the linked factor on the right is

$$P_\lambda(b_e^{-1})^t D_\varrho B D_\varrho P_\lambda(k_e^t) = P_\lambda(b_e^{-1})^t D_\varrho^2 P_\lambda(k_e)^t B.$$

Here B commutes with D_ϱ. By Lemma 440.1 this is the edge term of Y_λ times B. The transposed restriction sums over precisely the edges entering the indicated output coordinate, as in Theorem 439.3.

Set $K_0 = (k^2 + k)(oD)D,\ c = 2K_0 + 2,\ \theta = x^{40}$. One has $6c + 2 \le \theta$ eventually, since $oD = L \le f \le x^{12},\ D \le x^{12},\ k = O(x^5)$ by Proposition 444.3 and its proof. Take Z of Theorem 451.2 and extend it to $Z_{W'}$ on the linked coordinate at every edge. Apply Lemma 448.1 with

$$Q = Q_k, \qquad \beta = Q_b^t U B_{W'}, \qquad M = U^{-1} Z_{W'} R_b^t, \qquad a_0 = p^{K_0}, \quad w = \varrho, \quad z = p^s,$$

where a_0 is the scalar denoted a in that lemma. Each edge slice of each column of Q_b belongs to $\mathcal{U}(2c, c)$ by Theorem 446.2, and $2c \le \theta - 4c - 2$. Hence the row congruence of Theorem 451.2 gives

$$\beta M = Q_b^t B_{W'} Z_{W'} R_b^t \equiv a_0 Q_b^t R_b^t = a_0 I \bmod \varrho^c.$$

Since $a_0 w = \varrho^{c-1}$, this implies the required congruence.

For a difference $Q_k u - M v = a_0 w F$, each fixed edge slice on the left lies in $\mathcal{U}(\theta, c)$. For Q_k use Theorem 446.2 again. For M use the image assertion of the projector theorem and the fact that U^{-1} makes only integral linear combinations among the remaining coordinates of a single edge. Thus division by ϱ^{c-1} leaves each tested coefficient divisible by ϱ. Its reduction vanishes, proving that $\bar{F}$ lies in the direct sum of one $U(\theta)$ per edge and remaining coordinate. This subspace has dimension

$$|C|\, d_{\mathcal{P}}\, h' \dim U(\theta) \le (1 + x^{-14}) c_0$$

by Theorem 447.1. The split inclusion lemma now bounds $d_s(\beta Q_k)$ by $c_0(K_0 + s x^{-14})$, since its exceptional count is at most $x^{-14} c_0$.

For the outside square factors the determinants have combined valuation at most $3c_0 A_\lambda/(mp)$ by the stronger inequality proved in Corollary 442.2. Each full determinant bounds its own truncated cost by Definition 452.1. Apply Lemma 25.2 at p^s to the three factors, using a generator of the diagonal index for each, to add their costs. The integral multiplier B_V gives $Y_\lambda B_V V + p^s V \subseteq Y_\lambda V + p^s V$, so the same bound holds for Y_λ. The size estimates above give $K_0 \le x^{40}$ uniformly.

Finally the entries of Y_λ in the indicated class basis lie in $\iota(O_\iota)$ by Theorem 439.3. Perform Smith reduction there. The same invertible changes, diagonal entries and their divisibilities give a Smith reduction over O, since the smaller ring is the intersection of its fraction field with O. Consequently the computed truncated valuation is exactly the one on the original integral columns. This proves the assertion in that lattice. □

453. The local index and divisibility of the transferred columns

Theorem 453.1. *For all sufficiently large arithmetic scales, every served prime p and every embedding ι above it, put*

$$s_p = \lceil 2mfa/p \rceil, \qquad \mathcal{E}'_p = Y^m \mathcal{E}_p + p^{s_p} \mathcal{E}_p$$

as in Theorem 438.1. There is an integer l_p such that

$$\mathcal{I}(\mathcal{E}_p, \mathcal{E}'_p; p^{l_p}), \qquad 0 \le l_p \le 3c_0 fa/p.$$

Proof. All the Y_λ preserve $\mathcal{E}_p$ by Lemma 437.2. Apply Lemma 25.2 at the single modulus p^{s_p} to the m copies of the list of operators in Y. By Definition 452.1, the sum of their truncated costs bounds that of the whole product. Theorem 452.2 gives

$$\frac{d_{s_p}(Y^m)}{c_0} \le \frac{3}{p} \sum_{\lambda \in \Lambda} A_\lambda + m|\Lambda| \left(x^{40} + \frac{s_p}{x^{14}} \right).$$

Here $c_0 > 0$ since the identity class contributes the nonzero coefficient representation in Proposition 13.5. Theorem 24.4 gives $\sum_\lambda A_\lambda = Ua/2,\ m|\Lambda| \le U \le f$. The right side is thus bounded by

$$\frac{3fa}{2p} + Ux^{40} + \frac{U}{x^{14}}\left(\frac{2mfa}{p} + 1\right).$$

The last two contributions together are $o(fa/p)$, uniformly. Indeed a/p dominates every fixed power of x by Lemma 26.2 and its proof. Also $2mU/x^{14} \le 4x^{-3/2}$ since $U \le f \le x^{12}$ and $m \le 2\sqrt{x}$. Eventually $d_{s_p}(Y^m) \le 2c_0 fa/p$. Take $l_p = \lceil d_{s_p}(Y^m)\rceil$. The cost characterization gives the claimed index and the rounding costs at most one, absorbed by $c_0 fa/p \to \infty$. □

Corollary 453.2. *In Proposition 25.3 the local divisibility and index assertions for B_Ψ hold at every prime with the modules specified in Corollary 438.2. They also hold with the smaller losses*

$$\widetilde{e}_p = \begin{cases} (30\ell + 4)fa/p, & p \text{ served}, \\ e_p, & \text{otherwise}. \end{cases}$$

In particular with $\widetilde{\beta}_p = \lceil G_p - \widetilde{e}_p\rceil$, on these modules $|(B_\Psi\phi)(h)|_{Y,h} \le p^{-\widetilde{\beta}_p}$ for the ramified columns and twists with $p \nmid Q$ of that corollary. The exponents of the local index witnesses are at most $c_0\widetilde{e}_p$, and $\sum_p \widetilde{e}_p \log p = o(fa\ell^3)$.

Proof. At a served prime Theorem 453.1 gives the index with $3fa/p$ per coordinate. The valuation cost in Theorem 438.1 is $(30\ell + 3)fa/p$. Its difference from $\widetilde{e}_p$ is $fa/p \ge 1$ eventually, so the same estimate implies the one with the integer ceiling. Also $3fa/p \le \widetilde{e}_p$. Corollary 438.2 proves the two assertions at all other primes. Finally $0 \le \widetilde{e}_p \le e_p$ at every prime eventually by $\ell \to \infty$ and $e_p = 3fa\ell^2/p$ for served primes. Proposition 25.3 gives $\sum_p e_p \log p = o(fa\ell^3)$. Decreasing the losses preserves this upper bound and replacing them by e_p again weakens both the local valuation and the index requirements. This also proves the original assertions. □

454. Quadratic ideal classes and normalized period lattices

Lemma 454.1. *Let $K = \mathbb{Q}(\sqrt{-d})$ with d a prime congruent to $7 \bmod 8$, and $O_K = \mathbb{Z}[\omega]$ as in Lemma 11.3. Its ideal class group is finite and squaring is a bijection on it. It has fractional ideal representatives I_c, one per class, with $I_1 = O_K$, $I_{c^{-1}} = I_c^{-1}$ and $\prod_c I_c = O_K$. For a fractional ideal I, define $NI = \operatorname{area}(I)/\operatorname{area}(O_K)$ in the fixed complex embedding. Then $I\bar{I} = (NI)O_K$.*

Proof. Use ideal factorization and multiplicativity of the norm from Proposition 82.3, and finiteness from Lemma 82.5. The area formula for integral ideals is Lemma 82.2; clearing a rational integer denominator extends it multiplicatively to fractional ideals. For an odd prime p, the quotient of O_K by p is the algebra of $X^2 - X + (d+1)/4$ over $\mathbf{F}_p$. It is a quadratic field or a product of two copies of $\mathbf{F}_p$ when $p \ne d$, by the discriminant $-d$; at $p = d$ the ramified prime is $(\sqrt{-d})$, whose square is (d) and whose norm is d. At two the polynomial has two distinct roots. Thus on a split pair conjugation exchanges the primes of norm p, at an inert prime (p) the norm is p^2, and at the ramified prime the square and norm are as just computed. Multiplying the prime factorizations gives $I\bar{I} = (NI)O_K$.

If $[I]^2 = 1$, then $\bar{I} = aI$ for some $a \in K^\times$. Comparing areas gives $a\bar{a} = 1$. Choose $b = 1 + a$ if $a \ne -1$, and $b = \sqrt{-d}$ otherwise. In both cases $b/\bar{b} = a$, so bI equals its conjugate. Its two exponents at a split pair are equal and give a power of the rational principal ideal (p). All inert primes and the unique ramified prime are principal. Thus bI and I are principal. The squaring map on the finite abelian class group is now injective and hence surjective. Its only self-inverse class is the identity. Take arbitrary representatives from one class in each inverse pair and the inverse ideals in the other. Their product is as claimed. □

Definition 454.2. For a lattice Γ homothetic to a fractional O_K-ideal, let $t \in (\frac{1}{2}\Gamma)/\Gamma$ be the mixed nonzero class of Lemma 32.1. Call (Γ, t, a_1) normalized if $\wp''_\Gamma(t) = 2,\ a_1^2 = 12\wp_\Gamma(t)$. Use $\sigma_\Gamma, \theta_\Gamma, \xi_\Gamma, \phi_\Gamma, G_\Gamma$ of Propositions 31.2, 31.4, 31.3 and 33.1 with this lattice and half-period. In particular $\theta_\Gamma(0) = 1,\ \phi_\Gamma(0) = 0,\ \phi'_\Gamma(0) = 1$. As in Proposition 31.4, changing the lift of t multiplies numerator and denominator of the normalized translate by the same phase after canceling the z-linear factors.

Proposition 454.3. *Every such ideal lattice admits a nonzero homothety making it normalized. For a normalized lattice, a_1 is an algebraic integer and $\Delta_\Gamma = g_2^3 - 27g_3^2 = a_1^4 - 64$ is an algebraic unit. If $a \neq 0$ and $a\Gamma \subseteq \Gamma'$ for two normalized lattices, a is an algebraic integer.*

Proof. The simple zero of $\wp'$ at the chosen half-period in the proof of Proposition 31.3 gives $\wp''(t) \neq 0$. Under multiplication of the lattice and t by $z \neq 0$, this value changes by z^{-4}, by the defining series for $\wp$. Take a fourth root of $\wp''(t)/2$ and choose either permitted a_1. With $e = \wp_\Gamma(t)$ the differential equation in that proposition gives $g_2 = 12e^2 - 4,\ g_3 = 4e^3 - g_2 e$. Substitution gives $\Delta_\Gamma = 144e^2 - 64 = a_1^4 - 64$. The index-two operation of Lemma 32.1 multiplies the lattice by $O_K + (\omega/2)O_K$. Some power of this ideal is principal by Lemma 454.1, so every initial ideal lattice returns up to homothety. The mixed half-period is intrinsic and is also carried back by that homothety. Proposition 32.2 and Lemma 32.3 therefore apply to this cycle. The calculation in Theorem 32.4 identifies its initial D with $a_1^4 - 64$, using $(e_t - e_T)(e_t - e_S) = \wp''(t)/2 = 1$. It gives $|\iota\Delta_\Gamma|_p = 1,\ |\iota a_1|_p \leq 1$ for every finite place isomorphism. The argument at the end of Lemma 48.1, applied also to Δ_Γ^{-1}, gives the algebraic assertions.

Fix a finite place isomorphism and write $H_\Gamma(z) = \max(|\iota\sigma_\Gamma(z)|_p, |\iota\theta_\Gamma(z)|_p)$. The addition and duplication identities of Proposition 33.1 use only the simple half-period, its normalization and the common sigma multiplier, so hold on Γ, Γ'. With the unit just proved, the proof of Theorem 33.2 gives $H_\Gamma(2z) = H_\Gamma(z)^4 > 0$ and $|\iota G_\Gamma(z)|_p \leq H_\Gamma(z)^3$ for both curves. Write $n = [\Gamma' : a\Gamma] = |a|^2 \mathrm{area}(\Gamma)/\mathrm{area}(\Gamma')$. For $f = \sigma_{\Gamma'}, \theta_{\Gamma'}$, the quotients $f(az)^2/\theta_\Gamma(z)^{2n}$ are periodic. Indeed the exponential multipliers agree by the area formula and all signs disappear on squaring. In the proof of Lemma 34.1 the pole orders two and three of ξ_Γ, ϕ_Γ give the same polynomial expression and the same bound on Γ. Applying it to both numerators shows $H_{\Gamma'}(az)/H_\Gamma(z)^n$ is bounded by a constant independent of z. Doubling raises this ratio to its fourth power, so it is at most one everywhere.

Apply Proposition 34.3 on Γ to $\theta_{\Gamma'}(az)^2$ in degree $2n$ and $G_{\Gamma'}(az)^2$ in degree $6n$. Its proof applies because the monic quadratic for ϕ_Γ, ξ_Γ, the local parameter $\phi_\Gamma = z + O(z^2)$, and the unit discriminant are precisely those used in that proposition. The periodicity in the second case follows from $G_{\Gamma'} = \phi_{\Gamma'}\theta_{\Gamma'}^3$, and the two height inequalities follow from the preceding paragraph. After division by the respective powers of θ_Γ, both series belong to $O[[\Phi_\Gamma]]$, where Φ_Γ is the transported Taylor series of ϕ_Γ. The first series has constant term one, so divide the second by its cube. This proves that the transported $\phi_{\Gamma'}(az)^2$ lies in $O[[\Phi_\Gamma]]$. Its coefficient at Φ_Γ^2 is $\iota(a)^2$, since both local parameters have derivative one and constant term zero. Hence a is integral at every such place, and Lemma 48.1 again gives algebraic integrality. □

455. Conjugating a normalized elliptic differential

Proposition 455.1. *Let $C = (\Gamma, t, a_1)$ be as in Definition 454.2. On the cubic with coordinates $X = \wp_\Gamma,\ Y = \wp'_\Gamma$ mark the origin at infinity, the point $(a_1^2/12, 0)$, and the differential $\eta = dX/Y$. For $\nu \in \mathrm{Aut}(\mathbb{C}/K)$ there is a normalized lattice $C^\nu = (\Gamma^\nu, t^\nu, \nu a_1)$ such that the conjugate cubic, points, group law and differential are given by these same formulas on C^ν. A rational function regular at the origin has the Taylor series of its conjugated function on C^ν obtained by applying ν to the coefficients in the period variable z. For normalized C, D one has*

$$a\Gamma_C \subseteq \Gamma_D \quad \Longleftrightarrow \quad (\nu a)\Gamma_{C^\nu} \subseteq \Gamma_{D^\nu}.$$

The induced maps of torsion are conjugated by the same marked group identifications.

Proof. By Lemma 304.2 the smooth cubic, group operations and complex multiplications are algebraic. In these coordinates the equation is $Y^2 = 4X^3 - g_2X - g_3$ and η pulls back to dz, so it is regular, nowhere vanishing and invariant under translations. These statements are algebraic: regularity and nonvanishing are tested in regular local coordinates, and invariance follows from the identity for the pullback under addition, namely the sum of the two pullbacks to the product. They hold after applying ν to the coefficients. The new cubic is smooth and irreducible since $\nu\Delta_\Gamma \neq 0$ and the cubic polynomial in X has three distinct roots. The transported group remains projective and commutative. Use its invariant differential to identify its tangent line with $\mathbb{C}$, and use Lemma 284.1 to obtain a lattice Γ^ν with pullback of the conjugate differential equal to dz. The transported scalar action of O_K has the same tangent scalars, because ν fixes K. Hence the lattice is stable under O_K and is homothetic to a fractional ideal by the rational-span proof of Proposition 304.3.

In this variable $X' = Y$. At the identity X has a double pole and is even, since inversion on the cubic is $(X, -Y)$. Its leading term is z^{-2} by comparing the leading coefficients in $X'^2 = 4X^3 - \nu(g_2)X - \nu(g_3)$. The coefficient of z^{-4} in that equation makes its constant term zero. There are no other poles, so $X - \wp_{\Gamma^\nu}$ is entire elliptic with constant term zero and vanishes by Lemma 31.1. This also identifies Y. The marked point is the mixed half-period because its order and its two nonzero images under $\omega, \bar{\omega}$ are preserved by conjugation. At that point $X = (\nu a_1)^2/12$. Since $\nu g_2 = 12(\nu(a_1^2/12))^2 - 4$, the second derivative of X there is two. This gives exactly the asserted normalized lattice.

For a regular rational function f near a point P, define the rational derivative $\partial f = df/\eta$. It is regular wherever f is regular, since η is nowhere vanishing. In the period variable it is d/dz. Evaluation at P and algebraic differentiation commute with conjugation, which thus carries each $(\partial^j f)(P)/j!$ to the corresponding coefficient at the conjugate point. Finally multiplication by a under the left inclusion is a holomorphic group homomorphism and therefore algebraic by Lemma 304.2. Its pullback of the target differential is a times the source differential. Conjugation makes the derivative νa; the lifted map in Lemma 284.1 is multiplication by that derivative. Conversely, undo the construction by ν^{-1}. Applying it twice restores the original group and differential, and the period uniformization with a fixed differential is unique by the same lemma. This proves the equivalence and its compatibility with torsion. □

Theorem 455.2. *Write $\Theta_C(Z)$ for the ordinary Taylor series of θ_Γ at zero in the period variable and let ν act on series coefficientwise. Then*

$$\nu\Theta_C = \Theta_{C^\nu} \qquad (\nu \in \mathrm{Aut}(\mathbb{C}/K)).$$

Proof. The logarithmic second derivative $-(\theta_\Gamma'/\theta_\Gamma)'$ equals $\xi_\Gamma + c_C$ for a constant c_C: its multiplier gives periodicity after two logarithmic derivatives, and its double pole at t has principal part $(z-t)^{-2}$, the same as ξ_Γ. Subtract and use Lemma 31.1, as in Proposition 35.2. Now $\xi_\Gamma = \wp_\Gamma(z+t) - a_1^2/12$ is a rational function on the cubic by its algebraic translation law. It transports to ξ_{Γ^ν} by Proposition 455.1. Both theta series have constant term one and are even by Proposition 31.4, whose translation argument uses only $2t \in \Gamma$. The preceding identity, applied to formal logarithms at zero, now shows that $\Theta_{C^\nu}/(\nu\Theta_C) = \exp(cZ^2)$ for some $c \in \mathbb{C}$. Indeed all coefficients of degree greater than two in its logarithm vanish, as do its constant and linear terms.

Take $\alpha = \sqrt{-d}$, with $\alpha\bar{\alpha} = d$. The function $H_\alpha(z) = \theta_\Gamma(\alpha z)/\theta_\Gamma(z)^d$ is elliptic. In fact $\alpha = 2\omega - 1$ acts identically on $\Gamma/2\Gamma$. Both theta signs depend only on that quotient by the multiplier in Proposition 31.4, and d is odd. The exponential factors cancel since $|\alpha|^2 = d$. The quotient is rational on the cubic: it is a polynomial in ξ_Γ, ϕ_Γ by subtracting its possible poles at t as in Lemma 34.1. Its divisor consists of the inverse images of t under $[\alpha]$, with multiplicity one since the derivative is nonzero, less d copies of t. Its value at the origin is one. The conjugate function on C^ν has these same descriptions using t^ν, by Proposition 455.1.

Thus it equals the corresponding H_α there: the quotient of two functions with this divisor is a holomorphic nowhere-zero function on the compact torus, hence is constant, and the value at zero fixes it. Comparing their Taylor series now gives $\exp(c(\alpha^2 - d)Z^2) = 1$. The coefficient of Z^2, with $\alpha^2 = -d \neq d$, forces $c = 0$. □

456. The phases of conjugate torsion translates

Lemma 456.1. *Let $C, D = C^\nu$ be as in Proposition 455.1. Write $[z]_C$ for the point represented by z modulo the period lattice, and U_y for the translate in Proposition 31.4 with the area of its own lattice. Let $n \geq 1$, $ny \in \Gamma_C$, $ny' \in \Gamma_D$. If $\nu[-y/(2n)]_C = [-y'/(2n)]_D$, then the Taylor series at zero satisfy*

$$\nu \operatorname{Tay}_0(U_y \theta_C) = \operatorname{Tay}_0(U_{y'} \theta_D).$$

Proof. Put $\epsilon_C(l) = s(l) \exp(2\pi i \Im(\bar{l}t)/A_C)$ for $l \in \Gamma_C$, with $s(l)$ as in Proposition 31.2. Since $2t \in \Gamma_C$, this sign depends only on l mod $2\Gamma_C$ and $\epsilon_C(nl) = \epsilon_C(l)^{n^2}$. Define

$$H_n(z) = \frac{\theta_C(nz)}{\theta_C(z)^{n^2}}, \qquad P_{y,n}(z) = \frac{(U_y \theta_C)(nz)}{\theta_C(z)^{n^2}}.$$

Their numerators and denominators have the same multiplier under $z \mapsto z + l$. For $P_{y,n}$ the extra factor is $\exp(2\pi i \Im(\bar{l}ny)/A_C) = 1$, since the oriented area of two lattice vectors is an integral multiple of A_C. Both quotients are rational on C by the pole subtraction in Lemma 34.1. Denote the analogous functions on D by primes. The divisor of H_n is $[n]^*[t] - n^2[t]$, where $[n]^*$ sums the preimages of the indicated point. All preimages are simple since $[n]$ has nonzero derivative. This divisor transports to that of H'_n. The quotient of the two functions with that divisor is constant by Lemma 31.1, and their values at zero are one. Thus $\nu H_n = H'_n$.

Likewise the divisor of $P_{y,n}$ is $[n]^*[t - y] - n^2[t]$. By the hypothesis, multiplication by $-2n$ on the two curves identifies $\nu[y]_C = [y']_D$. Thus $\nu P_{y,n}$ and $P'_{y',n}$ have identical divisors. If $y \notin \Gamma_C$, put $z_0 = -y/(2n)$. The denominator is nonzero there, since $[z_0]_C = t$ would give $[y]_C = 0$. The value of $\theta_C(nz_0) = \theta_C(-y/2)$ is also nonzero for the same reason. As $nz_0 = -y/2$, the exponent defining U_y vanishes there and evenness gives $P_{y,n}(z_0) = H_n(z_0)$. These facts hold at $z'_0 = -y'/(2n)$ on D too. The conjugate point and both values of H_n correspond, proving $\nu P_{y,n} = P'_{y',n}$.

If $y \in \Gamma_C$, the translate is $\epsilon_C(y)\theta_C$. The sign is negative exactly when $[y/2]_C = t$. Indeed $s(y) = 1$ on the zero class modulo $2\Gamma_C$ and is negative on each of its three nonzero classes. The alternating area pairing modulo two is nonzero precisely on distinct nonzero vectors, as is seen in an oriented lattice basis. This proves the sign rule by its defining formula. Conjugation carries $[y/2]_C$ to $[y'/2]_D$, and t to t^ν. Hence the two signs agree, and again $\nu P_{y,n} = P'_{y',n}$. Take Taylor coefficients at zero by Proposition 455.1 and multiply by $\operatorname{Tay}_0(\theta_D)^{n^2}$ using Theorem 455.2. Substituting Z/n for Z proves the claim. □

Theorem 456.2. *Suppose $\Gamma_C = \lambda O_K$, and $\nu \in \operatorname{Aut}(\mathbb{C}/K)$. There exist $\lambda_\nu \in \mathbb{C}^\times$, a fractional ideal I of norm one, and an invertible finite adele b_ν of K such that*

$$\Gamma_{C^\nu} = \lambda_\nu I, \qquad |\lambda_\nu|^2 = 2 \operatorname{area}(\Gamma_{C^\nu})/\sqrt{d}, \qquad b_\nu \widehat{O}_K = \widehat{I}.$$

Here $\widehat{I} = \varprojlim_m I/mI$, with m running through positive integers under divisibility. In particular $e\lambda_\nu \in \Gamma_{C^\nu}$ for some positive integer e. For every finite $S \subset K$, every integer $n \geq 1$ with $nS \subset O_K$, and every $a \in I$ congruent to b_ν modulo $2n^2\widehat{I}$, one has

$$\nu \operatorname{Tay}_0(U_{\lambda x} \theta_C) = \operatorname{Tay}_0(U_{\lambda_\nu a x} \theta_{C^\nu}) \qquad (x \in S).$$

Such a can at the same time be chosen in any prescribed finite-adelic neighbourhood of b_ν.

Proof. By Proposition 455.1 the new lattice is homothetic to a fractional ideal. By Lemma 454.1 its class is a square. Write $\Gamma_{C^\nu} = vJ^2$ with J a fractional ideal and $v \neq 0$. Take $\lambda_\nu = vNJ$, $I = J^2/NJ = J/\bar{J}$. Then $NI = 1$. The area identity follows from $\operatorname{area}(O_K) = \sqrt{d}/2$ in the basis $1, \omega$. Clearing denominators in a basis of I gives an integer $e \in I$, for I spans K over $\mathbb{Q}$.

For each integer $m \geq 1$, identify I/mI with the m-torsion of C^ν by $v_0 \mapsto [\lambda_\nu v_0/m]_{C^\nu}$, and similarly identify O_K/mO_K with that of C. Algebraicity of multiplication and the tangent scalar computation of Proposition 455.1 make conjugation an O_K-linear isomorphism between these two finite modules. It sends 1 to a residue b_m. For $m \mid M$, multiplying the points by M/m gives the reduction of b_M to b_m. Hence $b = (b_m)_m \in \widehat{I}$ and multiplication by b gives an isomorphism $\widehat{O}_K \to \widehat{I}$. Indeed the finite-level inverse maps commute with the same reductions. The Chinese remainder calculation of Proposition 82.3 identifies these completions with the products of the local integer rings and the completed ideals, respectively: factor each mO_K into prime powers, split its quotient by these powers, and pass to the inverse limit. Each local component of b generates the corresponding nonzero local ideal, and outside the finitely many primes dividing I it is a unit. Thus $b_\nu = b$ is a finite idele.

Choose a in the residue b_{2n^2}. The transported point of $\lambda/(2n^2)$ is then represented by $\lambda_\nu a/(2n^2)$. Multiplication by $-nx \in O_K$ gives the hypothesis of Lemma 456.1 for $y = \lambda x$, $y' = \lambda_\nu a x$. This proves the formula simultaneously for all x. A neighbourhood of b_ν contains $b_\nu + m\widehat{I}$ for some m: only finitely many local congruences need tightening. Choose a representative in I of b_{2mn^2} to meet both conditions. □

457. The absolute discriminant of a period lattice

Lemma 457.1. *Let $\Gamma = \lambda(\mathbb{Z} + \tau\mathbb{Z})$ with $y = \Im\tau > 0$, $A = |\lambda|^2 y$ and $Q = \exp(2\pi i\tau)$. Then*

$$\sigma_\Gamma(\lambda z) = \frac{\lambda}{\pi} e^{\pi z^2/(2y)} \sin(\pi z) \prod_{j \geq 1} \frac{(1 - Q^j e^{2\pi i z})(1 - Q^j e^{-2\pi i z})}{(1 - Q^j)^2}.$$

The function $S(w) = |\sigma_\Gamma(w)| \exp(-\pi|w|^2/(2A))$ is Γ-periodic. In the modulus of the preceding product the net exponential for $S(\lambda z)$ is $\exp(-\pi(\Im z)^2/y)$.

Proof. On each compact set $e^{\pm 2\pi i z}$ are bounded, so the factors of the product differ from one by $O(|Q|^j)$ uniformly. For all sufficiently large j the logarithms obtained by the power series of $\log(1 + w)$ converge uniformly and absolutely when summed over j. Exponentiation gives a locally uniform product and a nonvanishing tail. The disk formula of Lemma A.3 makes the result holomorphic. The product is even, and has value one at zero. The displayed expression is odd with derivative one in $w = \lambda z$.

Put $W = e^{2\pi i z}$, and write $Q^{1/2} = e^{\pi i \tau}$. For translation by 1, the sine changes sign and the product stays fixed. Under $z \mapsto z + \tau$ the ratio of the sines is $Q^{-1/2}(QW - 1)/(W - 1)$. The ratio of the products is $(1 - W^{-1})/(1 - QW)$, by cancellation in the finite products followed by a limit. Multiplication gives $-Q^{-1/2}W^{-1}$. The identities extend through zeros by continuity. With the Gaussian these are exactly the two generating multipliers in Proposition 31.2, as $\tau - 2iy = \bar{\tau}$. Its uniqueness proves the formula. Taking absolute values in that proposition cancels the Gaussian in S under every period. Finally $\Re(z^2) - |z|^2 = -2(\Im z)^2$, giving the stated exponential. □

Proposition 457.2. *In Lemma 457.1 set $\Delta_\Gamma = g_2^3 - 27g_3^2$, with g_2, g_3 from the differential equation of Proposition 31.3. Then*

$$|\Delta_\Gamma| = (2\pi/|\lambda|)^{12} |Q| \prod_{j \geq 1} |1 - Q^j|^{24}.$$

Proof. Let u, v, w represent the three distinct nonzero half periods and put $e_h = \wp_\Gamma(h)$. They are the three distinct roots of $4X^3 - g_2 X - g_3$: the zeros of $\wp'$ at the three half periods

are simple by the proof of Proposition 31.3, and an equal value at two of them would give four zeros counted with multiplicities for an elliptic function $\wp - e_h$ with just one double pole. The addition identity in the proof of Proposition 33.1 used only the sigma multiplier, its simple zero and oddness. Taking moduli in it, the Gaussians cancel by the parallelogram identity. Since $u+v$ and $u-v$ both represent w, this gives

$$|e_u - e_v| = \frac{S(w)^2}{S(u)^2 S(v)^2}.$$

The discriminant identity $\Delta_\Gamma = 16\prod_{h<k}(e_h - e_k)^2$, with any ordering of the roots, follows by expansion: for the monic polynomial $X^3 + pX + r$ the squared product of differences is $-4p^3 - 27r^2$, on substituting $e_w = -e_u - e_v$. Thus $|\Delta_\Gamma| = 16/(S(u)S(v)S(w))^4$.

Use $u = \lambda/2,\ v = \lambda\tau/2,\ w = \lambda(1+\tau)/2$. In the product of the three formulas of Lemma 457.1, the three sines and Gaussians together have modulus $|Q|^{-1/4}|1-Q|/4$. Indeed the first sine is one. For the other two write the sine as $e^{-\pi iz}(e^{2\pi iz}-1)/(2i)$. Their two arguments $e^{2\pi iz}$ are $Q^{1/2}, -Q^{1/2}$, and their Gaussian factor together is $|Q|^{1/4}$. The numerators of the infinite products give

$$\prod_{j\ge 1} |1+Q^j|^2\,|1-Q^{2j+1}|\,|1-Q^{2j-1}|.$$

Multiplying this by $|1-Q|$ gives one, since $\prod_{j\ge1}(1+Q^j) = 1/\prod_{j\ge1}(1-Q^{2j-1})$ by factoring $1-Q^{2j}$. All rearrangements are permitted by the absolute summability of logarithms of the tails. We obtain

$$S(u)S(v)S(w) = \frac{(|\lambda|/\pi)^3 |Q|^{-1/4}}{4\prod_{j\ge1}|1-Q^j|^6}.$$

Substitution in the discriminant gives the assertion. □

458. A first limit formula for lattice sums

Lemma 458.1. *Let $\tau = u+iy$, $y>0$, and $Q = e^{2\pi i\tau}$. For $n\ge1,\ 1\le s\le 3/2$ put*

$$J(s) = \int_{\mathbb{R}} (1+x^2)^{-s}\,\mathrm{d}x, \qquad D_n(s) = \sum_{m\in\mathbb{Z}} ((m+nu)^2 + (ny)^2)^{-s} - (ny)^{1-2s}J(s).$$

Then $\sum_{n\ge1} D_n(s)$ converges uniformly on this interval and is continuous there. Its value at $s=1$ is

$$-\frac{2\pi}{y}\sum_{j\ge1}\log|1-Q^j|.$$

Also $J(1)=\pi,\ J'(1) = -2\pi\log 2$.

Proof. Write $b = ny$ and $f(x) = (x^2+b^2)^{-s}$. On $[t,t+1]$ the absolute difference between $f(t)$ and the integral of f is at most $\int_t^{t+1}|f'(x)|\,\mathrm{d}x$ by the fundamental theorem of calculus. Sum over the consecutive intervals starting at $t = m+nu$. The total variation is $2b^{-2s}$, because f increases to $f(0)$ and decreases afterwards. The integral is $b^{1-2s}J(s)$ by substitution. For $ny\ge1$ the variation is at most $2/(ny)^2$. For each fixed n the series over m has a uniform summable tail bounded by a constant times $\sum_{m\ne0} m^{-2}$. These estimates and Lemma A.2 prove continuity and uniform convergence.

For $b>0$, the periodization of $1/(x^2+b^2)$ has Fourier coefficients $\pi e^{-2\pi b|j|}/b$ at $j\in\mathbb{Z}$. Indeed absolute uniform convergence on a period allows unfolding. For $t\ge0$ the integral of $e^{itx}/(x^2+b^2)$ over the real axis is $\pi e^{-bt}/b$. On closing the contour in the upper half-plane, the pole at ib has residue $e^{-bt}/(2ib)$ and the semicircle of radius $R>b$ costs at most $\pi R/(R^2-b^2)$. Use the contour formula of Lemma A.3 and let $R\to\infty$. Complex conjugation gives the integral for negative t. The Fejér kernel argument in Lemma 4.2 now identifies the

continuous periodization with its absolutely summable Fourier series. Evaluation at nu and subtraction of the constant term give

$$D_n(1) = \frac{2\pi}{ny} \sum_{j \geq 1} \Re(Q^{nj}).$$

The double series with weights $|Q|^{nj}/n$ is bounded by $\sum_n |Q|^n/(n(1-|Q|))$. Summing first over n gives the asserted logarithms by the power series of $-\log(1-z)$.

Differentiation of J near one is justified by the integrable majorant $(1+x^2)^{-3/4}(1+\log(1+x^2))$. Set $x = \tan v$. This gives $J(1) = \pi$ and $J'(1) = 4\int_0^{\pi/2} \log(\cos v)\,\mathrm{d}v$. The last integral equals $I = \int_0^{\pi/2} \log(\sin v)\,\mathrm{d}v$ by reflection. Both converge since $\sin v/v \to 1$ at zero. Adding the two integrals and using $\sin(2v) = 2\sin v \cos v$ gives $2I = I - (\pi/2)\log 2$ by changing $2v$ to the integration variable and reflecting about $\pi/2$. This proves the derivative. □

Theorem 458.2. *Let Γ be a rank-two complex lattice of area A, and $k > 0$. For real $s > 1$ put*

$$Z_\Gamma(s;k) = \sum_{0 \neq l \in \Gamma} (|l|^2/(kA))^{-s}.$$

The sum converges. With γ as in Definition 12.3, as $s \downarrow 1$ it has the expansion

$$Z_\Gamma(s;k) = \frac{\pi k}{s-1} + \pi k \left(2\gamma - \log\frac{A}{k\pi^2} - \frac{1}{6}\log|\Delta_\Gamma|\right) + o(1).$$

Proof. Take a positively oriented lattice basis $\lambda, \lambda\tau$ and the notation of Lemma 458.1. The comparison with the maximum norm in the proof of Proposition 31.3 bounds the tail, for fixed $s > 1$, by a constant times $\sum_{j\geq 1} j^{1-2s}$. Separating $n = 0$, $n > 0$, $n < 0$ in $l = \lambda(m + n\tau)$ gives for $1 < s \leq 3/2$

$$Z_\Gamma(s;k) = 2(ky)^s \left(\zeta(2s) + \sum_{n \geq 1} D_n(s) + y^{1-2s} J(s)\zeta(2s-1)\right).$$

For use at $2s = 2$, the Fourier cosine coefficients of the continuous 2π-periodic extension of x^2 on $[-\pi, \pi]$ are $4(-1)^j/j^2$ for $j \geq 1$, by two integrations by parts. The constant term is $\pi^2/3$. The same Fejér kernel argument as in Lemma 4.2 recovers the function from its absolutely summable series. Setting $x = \pi$ gives $\zeta(2) = \pi^2/6$. By Proposition 3.3 and the computation of the regularized derivative in Proposition 12.4, $\zeta(1+\delta) = 1/\delta + \gamma + o(1)$ for $\delta \downarrow 0$.

Put $h = s - 1$ and expand the last summand including its prefactor. Since $k^s y^{1-s} = k(1 + h\log(k/y) + O(h^2))$, Lemma 458.1 gives its contribution divided by πk as $h^{-1} + 2\gamma + \log(k/(4y)) + o(1)$. For the first two summands the contributions divided by πk tend respectively to $\pi y/3 = -(\log|Q|)/6$ and $-4\sum_{j\geq 1}\log|1-Q^j|$. Finally $A = |\lambda|^2 y$, and Proposition 457.2 gives

$$\log\frac{k}{4y} - \frac{1}{6}\log|Q| - 4\sum_{j\geq 1}\log|1-Q^j| = -\log\frac{A}{k\pi^2} - \frac{1}{6}\log|\Delta_\Gamma|.$$

This is the desired constant term. □

459. A common period correction over the quadratic field

Proposition 459.1. *Let $K = \mathbb{Q}(\sqrt{-d})$ as in Lemma 454.1, let $h = |\mathrm{Cl}(O_K)|$, $\eta_d(m) = (m/d)$, and $k = 2/\sqrt{d}$. For each ideal class choose one complex lattice Γ_c homothetic to a fractional ideal of that class, and let A_c be its area. Then*

$$\left(\prod_c A_c |\Delta_{\Gamma_c}|^{1/6}\right)^{1/h} = \frac{2\pi^2}{\sqrt{d}} \exp\left(\gamma - \frac{L'(1,\eta_d)}{L(1,\eta_d)}\right).$$

Proof. Write $\Gamma_c = v_c J_c$. For $0 \neq b \in J_c$,

$$\frac{|v_c b|^2}{kA_c} = \frac{b\bar{b}}{NJ_c} = N((b)J_c^{-1}),$$

because $\operatorname{area}(O_K) = 1/k$. The ideal on the right is integral and has class c^{-1}. Conversely, for each integral ideal $B \neq 0$ in that class, BJ_c is principal and contained in J_c. Its nonzero generators differ by precisely the units of O_K. There are finitely many units, since their norms equal one and the lattice O_K has only finitely many points in the closed unit disk. Put $w = |O_K^\times|$. Thus for real $s > 1$,

$$\sum_c Z_{\Gamma_c}(s;k) = w \sum_{0 \neq B \subset O_K} (NB)^{-s} = w\zeta(s)L(s,\eta_d),$$

where the middle sum is over integral ideals. Indeed by Proposition 82.3 ideals are unique products of prime ideals. At rational p the local factors are $(1-p^{-s})^{-2}$ for a split prime, $(1-p^{-2s})^{-1}$ for an inert prime, and $(1-p^{-s})^{-1}$ for the ramified prime d. The splitting calculation of Lemma 454.1 and Lemma 9.1 identify the three cases with $\eta_d(p) = 1, -1, 0$, including $p = 2$ in the first case. All products converge absolutely since the positive ideal factors are at most those of $\zeta(s)^2$. This also justifies the multiplication of the two Dirichlet Euler products by finite factorization and passage to the limit.

The character η_d is nonprincipal since exactly half the nonzero residues are squares, as shown in Lemma 9.1. Proposition 3.3 therefore gives its Taylor expansion at one. Compare the coefficient of $1/(s-1)$ in the preceding identity by Theorem 458.2. This gives $h\pi k = wL(1,\eta_d) > 0$. Comparing the constants, using the expansion of ζ from the same theorem's proof, and dividing by $h\pi k$, gives

$$2\gamma + \log(k\pi^2) - \frac{1}{h}\sum_c \log(A_c|\Delta_{\Gamma_c}|^{1/6}) = \gamma + \frac{L'(1,\eta_d)}{L(1,\eta_d)}.$$

Exponentiation proves the formula. □

Theorem 459.2. *Let C be a normalized lattice with $\Gamma = \lambda O_K$. There exists an algebraic unit $U \neq 0$, depending on C, such that for every $\nu \in \operatorname{Aut}(\mathbb{C}/K)$,*

$$\operatorname{area}(\Gamma_{C^\nu})|\nu U|^2 = \frac{2\pi^2}{\sqrt{d}} \exp\left(\gamma - \frac{L'(1,\eta_d)}{L(1,\eta_d)}\right).$$

Proof. Take the inverse-paired fractional ideals I_c of Lemma 454.1. Normalize the lattice $I_c\Gamma$ by a complex factor $u_c \neq 0$ using Proposition 454.3. Call this curve C_c, so $\Gamma_c = u_c I_c \Gamma$, where multiplication by a fractional ideal means the sums of products with lattice vectors. For all $z \in I_c$ and $b \in I_c^{-1}$ one has $(u_c z)\Gamma \subseteq \Gamma_c$ and $(u_c^{-1}b)\Gamma_c \subseteq \Gamma$. Thus $u_c z, u_c^{-1}b$ are algebraic integers by that proposition, with zero included. Taking a nonzero rational integer $z \in I_c$ proves u_c algebraic. Set $u_* = \prod_c u_c$. Since $\prod_c I_c = O_K$, there is an expression $1 = \sum_j \prod_c z_{cj}$, with $z_{cj} \in I_c$, by the definition of ideal multiplication. Multiply by u_* to express it as a sum of products of algebraic integers. The same argument for $\prod_c I_c^{-1} = O_K$ makes u_*^{-1} integral. The ring property follows by the finite-module argument in the proof of Theorem 48.2, which applies to sums as well as products. Thus u_* is a unit.

Choose any $12h$-th root

$$U^{12h} = u_*^{12} \prod_c \Delta_{\Gamma_c}.$$

All factors on the right are algebraic units by Proposition 454.3. If a power of a complex number is an algebraic integer, substitution of that power into a monic integer polynomial gives a monic integer polynomial for the number itself. Applying this also to U^{-1} shows U is an algebraic unit.

Fix ν as in the statement and write $\Gamma^\nu = \Gamma_{C^\nu}$. By Proposition 455.1, the two inclusions for $u_c z, u_c^{-1}b$ become the corresponding inclusions for $(\nu u_c)z, (\nu u_c)^{-1}b$ on the conjugate curves. Summing over z gives $(\nu u_c)I_c\Gamma^\nu \subseteq \Gamma_{C_c^\nu}$. The reverse follows from the second family

by multiplying by $(\nu u_c)I_c$ and using $I_c I_c^{-1} = O_K$. Consequently the lattices $\Gamma_{C_c^\nu}$ run once through all ideal classes up to homothety: multiplication by the fixed class of Γ^ν permutes the classes. Their areas are $|\nu u_c|^2 N I_c \operatorname{area}(\Gamma^\nu)$. Moreover $\Delta_{\Gamma_{C_c^\nu}} = \nu \Delta_{\Gamma_c}$ by the conjugation of the smooth cubic. Multiply the areas times the sixth roots of the absolute discriminants and take their geometric mean. Since $\prod_c N I_c = 1$, the result is $\operatorname{area}(\Gamma^\nu)|\nu U|^2$ by the defining equation for U. Apply Proposition 459.1 to these conjugate lattices. □

Corollary 459.3. *In Definition 428.1 one may take $s_1 = s_0/U$ simultaneously at every place. For the period λ_ν of Theorem 456.2 put $t_\nu = \nu(s_1)\sigma_0/\lambda_\nu$. Then*

$$|t_\nu|^2 = \Omega^{-1}.$$

Proof. Every finite place gives U and U^{-1} absolute values at most one by their monic equations. Hence $|\iota(s_1/s_0)|_p = 1$, as required in Definition 428.1. Using $\sigma_0^2 = 2\pi/\sqrt{d}$, the period-area identity of Theorem 456.2 gives $|t_\nu|^2 = \pi|\nu s_0|^2/(\operatorname{area}(\Gamma_{C^\nu})|\nu U|^2)$. Theorem 46.1 gives $|\nu s_0|^2 = \mathrm{rd}_E/\sqrt{d}$. Now substitute Theorem 459.2 and Definition 12.3. □

460. Conjugate jets of the cyclotomic frame

Definition 460.1. For the embeddings $G = \operatorname{Hom}_K(E, \mathbb{C})$ of Proposition 10.3 and $\nu \in \operatorname{Aut}(\mathbb{C}/K)$, write $\mathcal{C}_\nu$ for coefficientwise application of ν to a series followed by the variable permutation $Z_{\tau k} \mapsto Z_{\nu\circ\tau,k}$. This is an invertible additive and multiplicative map on formal series, semilinear over ν. For $a \in \mathbb{C}^\times$ and $S \in \mathcal{F}$ write $J_a S = \operatorname{Tay}_0(S(aZ))$ in these variables, as in Definition 428.1.

Theorem 460.2. *Take $D = N_1$, $w_j = \zeta_D^j$ for $0 \le j < D$, and $u = \sigma_0 s_1/\lambda$ with $s_1 = s_0/U$ from Corollary 459.3. For every standard positive rank $n \ge 1$ there is an $F \in \mathcal{F}$ with $F(0) = 1$ whose finite translates span $\mathcal{F}$. For every $\nu \in \operatorname{Aut}(\mathbb{C}/K)$ there is $F_\nu \in \mathcal{F}$ with $F_\nu(0) = 1$, also cyclic, such that for all $X \in V_f$,*

$$\mathcal{C}_\nu(J_u T_X F) = J_{t_\nu} T_{b_\nu X} F_\nu, \qquad |t_\nu|^2 = \Omega^{-1}.$$

The scalar b_ν in Theorem 456.2 acts through the diagonal inclusion of the finite adeles of K in those of E, on all coordinates. It is independent of n.

Proof. Use the frame of Proposition 427.1 in each of the n coordinates. For either $(C_*, \lambda_*) = (C, \lambda)$ or $(C_*, \lambda_*) = (C^\nu, \lambda_\nu)$ put

$$B_{jk}^*(Z) = \frac{\lambda_*}{D\sigma_0} \sum_{\tau \in G} \overline{\tau(w_j)} Z_{\tau k}, \qquad F_*(Z) = \prod_{k=1}^{n} \prod_{j=0}^{D-1} \theta_{C_*}(B_{jk}^*(Z))^D.$$

In the first case this is the section F of Theorem 48.2, with $\delta = D$, $\alpha = 1$. In the second case use $\operatorname{area}(\Gamma_{C^\nu}) = |\lambda_\nu|^2\sqrt{d}/2$. The frame identity $\sum_j D\,\tau(w_j)\overline{\tau'(w_j)} = D^2 \mathbf{1}_{\tau=\tau'}$ gives in both cases

$$\frac{\pi}{\operatorname{area}(\Gamma_{C_*})} \sum_{jk} D\,\overline{B_{jk}^*(v)} B_{jk}^*(z) = H(v, z).$$

Multiplying the elementary exponential factors in Proposition 31.4 thus expresses each $W_v F_*$ as a product with every theta factor translated by $B_{jk}^*(v)$. Both functions have value one at zero.

Choose $e \ge 1$ with $e\lambda_\nu \in \Gamma_{C^\nu}$ as in Theorem 456.2. For $h \in 2De\,\mathcal{A}^n$, the displacement $B_{jk}^\nu(rh) = (\lambda_\nu/D)\mathrm{Tr}_{E/K}(h_k \bar{w}_j)$ belongs to $2\Gamma_{C^\nu}$ by Proposition 44.3 and O_K-stability. Here B^ν denotes the second choice of the displayed forms. All theta signs are one on twice a period by Lemma 456.1. The prescribed sign $\chi_{\Gamma_0}(rh)$ is also one, since h is twice an integral frame vector and its norm trace is a multiple of four. Thus $F_\nu = F_*$ in the second case belongs to $\mathcal{T}(2De\Gamma_0)$. For either F_*, choose any level divisible by its own. It is a nonzero vector at that level, and its translates by the dual lattice span the whole level by Proposition 50.1. These are finite translations by Theorem 51.2 since the dual lattice is rational in the coordinates of

Proposition 47.2. The eligible levels are cofinal in Definition 51.1. This proves both cyclicity assertions.

First take $X = \gamma \in E^n$, so $T_\gamma = W_{-r\gamma}$. Put $x_{jk} = -\mathrm{Tr}_{E/K}(\gamma_k \bar{w}_j)/D \in K$. The source translation has one-variable shifts λx_{jk}. By local constancy of $T_Y F_\nu$ in Theorem 51.2, all finite scalars sufficiently close to b_ν act on γ with the same resulting translate of F_ν. Multiplication by γ is continuous coordinatewise in the finite adeles. Use Theorem 456.2 for the finite set of x_{jk}, choosing $a \in I \subset K$ in that neighbourhood. The target action equals $T_{a\gamma}$ on F_ν and has shifts $\lambda_\nu a x_{jk}$. Thus the translated factors have exactly conjugate one-variable Taylor series. Under $\mathcal{C}_\nu$ the linear form $B_{jk}(uZ)$ becomes $B^\nu_{jk}(t_\nu Z)$: the coefficient before renaming is $\nu(s_1)\overline{(\nu \circ \tau)(w_j)}/D$. Here $\overline{w_j} = w_j^{-1}$, and $t_\nu = \sigma_0 \nu(s_1)/\lambda_\nu$. Finite products and substitution of linear forms with zero constant term commute with coefficientwise conjugation in each fixed degree. This proves the identity for rational γ. Both sides are locally constant functions of X into the series ring taken as a set, since the sections themselves are locally constant. Rational points are dense by Theorem 51.2, which proves the identity everywhere. The modulus of t_ν is Corollary 459.3. □

461. A semilinear bijection on the finite oscillator

Theorem 461.1. *With the notation of Theorem 460.2, there is a unique ν-semilinear bijection $\mathcal{L}_\nu : \mathcal{F} \to \mathcal{F}$ satisfying*

$$J_{t_\nu} \mathcal{L}_\nu S = \mathcal{C}_\nu(J_u S) \qquad (S \in \mathcal{F}).$$

For all $X, Y \in V_f$,

$$\nu c(X, Y) = c(b_\nu X, b_\nu Y), \qquad \mathcal{L}_\nu T_X = T_{b_\nu X} \mathcal{L}_\nu.$$

If $\kappa_\nu \in \widehat{\mathbb{Z}}^\times$ is the exponent by which ν acts on all roots of unity, then $b_\nu \bar{b}_\nu = \kappa_\nu$ in the rational finite ideles.

Proof. The map J_t on entire functions is injective for $t \neq 0$. Indeed vanishing of all Taylor coefficients implies vanishing on a polydisk by the successive disk formula of Lemma A.3, then on every complex line through zero by the one-variable identity principle. The scaled Taylor map in the assertion has $t_\nu \neq 0$ by Corollary 459.3. On the spanning family $T_X F$, its prescribed preimages are $T_{b_\nu X} F_\nu$ by Theorem 460.2. Every linear relation between the first family gives the relation with conjugated coefficients between the second by this identity and injectivity of J_{t_ν}. Extending by semilinearity defines the unique map. Its kernel is zero by injectivity of J_u and $\mathcal{C}_\nu$. Its image contains all finite translates of F_ν, since b_ν is an idele, so the same theorem gives surjectivity.

By local constancy in Theorem 51.2 there is an open subgroup H of V_f such that $T_x F = F$ and $T_{b_\nu x} F_\nu = F_\nu$ for $x \in H$. For any $y \in V_f$, $x \in H$, write the product $T_y T_x F$ as $c(y, x) T_{y+x} F$. Apply $\mathcal{L}_\nu$ to both expressions. Since $T_{b_\nu y} F_\nu \neq 0$, the target addition law gives $\nu c(y, x) = c(b_\nu y, b_\nu x)$. For arbitrary x choose an integer $m \geq 1$ with $mx \in H$, using the lattice neighbourhoods of Definition 51.1. Replace (y, x) by $(y/m, mx)$. The bilinear trace in the defining character gives the same values on both sides, proving the identity for all y, x. Apply it to $T_y T_x F$ again, now with arbitrary x. Cyclicity proves the operator identity.

For the last assertion write $\mathbf{e}_f(h) = \prod_p \mathbf{e}_p(h_p)$ on the rational finite adeles as in Proposition 18.5. The trace definition in Proposition 47.2 gives $\mathcal{E}_p(b_\nu X, b_\nu Y) = (b_\nu \bar{b}_\nu)_p \mathcal{E}_p(X, Y)$, with the product in the fixed scalar field $\mathbb{Q}_p$. The pairing $\mathcal{E}(X, Y)/2$ takes every finite rational adelic value. In fact choose two integral symplectic basis vectors with pairing one from Lemma 49.1 at level one, and multiply the second by twice the desired finite adele. Thus

$$\nu \mathbf{e}_f(h) = \mathbf{e}_f((b_\nu \bar{b}_\nu) h) \qquad (h \in \mathbb{A}_{\mathbb{Q},f}).$$

On the p^j-th roots of unity ν acts by an invertible exponent modulo p^j, by applying it to a primitive root. The exponents are compatible under powering, giving $\kappa_{\nu,p} \in \mathbb{Z}_p^\times$. As $\mathbf{e}_p$ depends only on the finite polar part modulo $\mathbb{Z}_p$, its values are p-power roots and $\nu \mathbf{e}_p(v) = \mathbf{e}_p(\kappa_{\nu,p} v)$. A scalar $a \in \mathbb{Q}_p$ with $\mathbf{e}_p(av) = 1$ for all v must be zero: otherwise set $v = 1/(pa)$. This character at $1/p$ is nontrivial by Proposition 18.5. Test h supported at p to conclude $(b_\nu \bar{b}_\nu)_p = \kappa_{\nu,p}$ for every p. □

Corollary 461.2. *Identify the finite oscillator $\mathcal{H}_f$ of a standard positive space with the sections $\mathcal{F}$ by Theorem 425.2, and denote the transported $\mathcal{L}_\nu$ by the same symbol. For the physical translations of Definition 356.1 one has*

$$\mathcal{L}_\nu D_X = D_{b_\nu X} \mathcal{L}_\nu \qquad (X \in V_f).$$

Proof. The scalar correcting T_X to D_X in Theorem 425.2 is $\mathbf{e}_f(-b(X,X)/2)$, with $b(X,X)$ the rational finite trace of $\jmath\bar{\omega} H(X,X)$ in Definition 356.1. Multiplying X by b_ν scales this trace by $b_\nu \bar{b}_\nu$. The additive character identity of Theorem 461.1 therefore conjugates the correction to the target correction. Combine this with that theorem's intertwining of T_X. □

462. Conjugate vacuum lines and the finite Weil operator

Theorem 462.1. *Use the physical realization of Corollary 461.2. Let R be any finite set of places of E with $R \cap \bar{R} = \varnothing$, let $v_0 = \mathbf{1}_{L_0}$ be the unit reference vacuum, and let $\mathcal{V}_G$ be the finite Weil representation of Theorem 390.2. There exist a scalar z in the finite ideles of E with $z\bar{z} = 1$, an integral unit idele a, and $c \in \mathbb{C}^\times$ such that*

$$b_\nu = za, \qquad a_w = 1 \ (w \in R), \qquad \mathcal{L}_\nu v_0 = c \mathcal{V}_z v_0, \qquad \mathcal{L}_\nu \mathcal{V}_G = \frac{\nu \Delta(G)}{\Delta(G)} \mathcal{V}_G \mathcal{L}_\nu$$

for every finite E-linear isometry G, with $\Delta(G)$ as in Definition 375.1. The last quotient has complex modulus one. Let $[f]_R$ denote a vector given by a locally constant compactly supported function f in the coordinates at R of Theorem 360.1, and the reference vacuum off $R \cup \bar{R}$. Then

$$\mathcal{L}_\nu [f]_R = c \, \mathcal{V}_z [\nu \circ f]_R.$$

Proof. By Theorem 461.1, $b_\nu \bar{b}_\nu = \kappa_\nu \in \widehat{\mathbb{Z}}^\times$, also after extending to E. At a place fixed by conjugation this forces valuation zero. At each pair $w, \bar{w}$ meeting R, choose $w \in R$ and put $z_w = (b_\nu)_w$, $z_{\bar{w}} = \overline{(b_\nu)_w}^{-1}$, where the bar identifies the two completions. Do the same, with either orientation, at the finitely many other pairs where b_ν has nonzero valuation. Give z value one elsewhere. Then $z\bar{z} = 1$, and $a = b_\nu / z$ is a unit at every place with $a_w = 1$ on R. In particular $aL_0 = L_0$, $a\bar{a} = \kappa_\nu$.

Set $A = \mathcal{V}_z^{-1} \mathcal{L}_\nu$, viewing z as the scalar isometry. By Corollary 461.2 and Theorem 358.1, $A D_X = D_{aX} A$. Since v_0 is fixed by D_l for all $l \in L_0$, so is $A v_0$. The joint fixed space is the line spanned by v_0: the translates indexed by V_f / L_0 span and have distinct characters under L_0, by that same lattice transport theorem. Bijectivity gives $A v_0 = c v_0$ for $c \neq 0$. Thus $K_\nu = c^{-1} A$ is ν-semilinear, takes $D_X v_0$ to $D_{aX} v_0$, and fixes v_0.

Put $h = [GL_0 : GL_0 \cap L_0] = |\Delta(G)|^2$. By Definition 375.1 the value of $\mathcal{V}_G$ on v_0 is the sum of the $D_r v_0$ over this quotient multiplied by $\Delta(G)/h$. The scalar a commutes with G, so it permutes this quotient. Changing a representative inside its coset leaves $D_r v_0$ unchanged by Theorem 358.1. Hence

$$K_\nu \mathcal{V}_G v_0 = (\nu \Delta(G) / \Delta(G)) \mathcal{V}_G v_0.$$

The identical identity with v_0 replaced by $D_X v_0$ and K_ν applied on the right follows by moving the translation through the operators, since $aGX = GaX$. These vectors span. Undo $K_\nu = c^{-1} \mathcal{V}_z^{-1} \mathcal{L}_\nu$ and use that the representation sends commuting isometries z, G to commuting operators. The factor $\Delta(G) = \beta(J_G)\sqrt{h}$ has conjugates of modulus $\sqrt{h}$: Theorem 374.2 gives modulus one for every conjugate of $\beta(J_G)$, and $\nu\sqrt{h} = \pm\sqrt{h}$.

In the split coordinates, D_x, supported at R, translates a function by x, which is fixed by a. A displacement D_y supported at $\bar{R}$ multiplies by the character $t \mapsto \psi_f(\jmath H(y,t))$. Conjugating this character replaces y by ay, since $\bar{a}_{\bar{w}}$, transported to $w \in R$, equals $\kappa_{\nu,p}$ for $w \mid p$ by $a\bar{a} = \kappa_\nu,\ a_w = 1$. This uses the additive character identity in Theorem 461.1 after taking the local rational trace. Pointwise conjugation fixes the box indicator. It therefore agrees with K_ν on translates and character multiples of that indicator, since both actions have the same semilinearity and intertwine these two sets of generators. They span the stated functions by the finite Fourier construction of Theorem 360.1. Undoing K_ν proves the last formula. □

463. A mean-value estimate on the moved period torus

Lemma 463.1. *Let $n \ge 1,\ g = fn,\ \Gamma_0 = r(\mathcal{A}^n)$ as in Proposition 47.2, and let $M_0 = \widehat{\mathcal{A}}^n$ in physical finite coordinates. For a finite E-linear isometry G of the standard positive form let $M = GM_0$ and $\Gamma_G = r(E^n \cap M)$. In the ordinary Euclidean metric on $\mathbb{C}^g$ one has*

$$\operatorname{vol}(\mathbb{C}^g/\Gamma_G) = \pi^g, \qquad \|l\|^2 \in \sigma_0^2\mathbb{Z}_{>0} \quad (0 \ne l \in \Gamma_G).$$

Proof. Each of M, M_0 is compact open, and their intersection has finite index in both. The rational-point quotients by this intersection have the same indices as the adelic lattices: every coset is open and contains a rational vector by the density in Theorem 51.2. The rational points of M_0 are $\mathcal{A}^n$, by that theorem's coordinate argument. Thus Γ_G is commensurable with Γ_0. The two finite indices are equal, since the isometry preserves additive volume locally. In fact G preserves the nondegenerate rational alternating matrix of Proposition 47.2, so at each rational prime its rational determinant squared is one. The local change of additive measure is the absolute determinant by the elimination argument of Theorem 360.1 applied over $\mathbb{Q}_p$. Consequently the two real covolumes are equal. In ordinary real coordinates on $\mathbb{C}^g$, the matrix of E_H of Proposition 47.2 has determinant π^{-2g}, from its g alternating two by two blocks. In a lattice basis of Γ_0 it has determinant one by Lemma 49.1 at level one. Change of basis gives the square of the real covolume as π^{2g}.

For $l = r(v)$ put $q(v) = \operatorname{Tr}_{E/K} \sum_i \bar{v}_i v_i$. It is rational, since it belongs to K and is fixed by its nontrivial conjugation. On the reference lattice $\mathcal{A}^n$ it is integral by Proposition 44.3. It remains in $\mathbb{Z}_p$ on the completion at p: approximate all entries by lattice vectors and use continuity of this quadratic polynomial and closedness of $\mathbb{Z}_p$. The finite isometry preserves the hermitian norm, hence preserves its local trace. Thus for rational $v \in M$ the number $q(v)$ lies in every $\mathbb{Z}_p$ and is an integer by rational prime factorization. In the positive real coordinates $\|r(v)\|^2 = \sigma_0^2 q(v)$ by Proposition 47.2. It is positive when $v \ne 0$, proving the second assertion. □

Theorem 463.2. *Let n, g, G be as in Lemma 463.1, let X be any finite displacement and Φ a scalar multiple of $D_X\mathcal{V}_G v_0$. Write $\|\Phi\|_2$ for the finite oscillator norm of Theorem 358.1. For its holomorphic section S_Φ put $A_\Phi(z) = |S_\Phi(z)|e^{-\|z\|^2/2}$. Then A_Φ is periodic under Γ_G and*

$$\frac{1}{\pi^g}\int_{\mathbb{C}^g/\Gamma_G} A_\Phi(z)^2\,\mathrm{d}z = \|\Phi\|_2^2, \qquad \sup_{z\in\mathbb{C}^g} A_\Phi(z) \le e^{r_d^2/2}\, r_d^{-g}\sqrt{g!}\,\|\Phi\|_2, \qquad r_d = \min(1, \sigma_0/3).$$

Here the integral uses ordinary Euclidean volume on the period torus.

Proof. For $l \in M = GM_0$, $D_l\mathcal{V}_G v_0 = \mathcal{V}_G v_0$ by intertwining and the reference vacuum condition. The displacement D_X adds only its unit-modulus commutator from Lemma 356.2. Thus every D_l, $l \in M$, acts on Φ by a phase.

Consider the modulus of the adelic function $F^a_{\phi_0\otimes\Phi}$ in Theorem 387.2. Use the real and finite unitary model changes of Theorem 395.2 and Proposition 391.1, as in Theorem 425.2. The modulus is rationally periodic, and it is unchanged under addition of $(0, l)$, $l \in M$, by the phase property and the Heisenberg product law. On the slice with finite coordinate zero this modulus is $A_\Phi(rx_\infty)$ by Lemma 425.1 at coherent parameter zero: the translated real

Gaussian is $e^{-\|rx_\infty\|^2/2}B_{-rx_\infty}$. The real formula there extends from rational coordinates to all real coordinates by continuity. The slice maps onto the quotient by rational vectors and $(0, M)$, since any finite vector can be placed in M by subtracting a rational one, by Theorem 51.2. The kernel on the slice is precisely $E^n \cap M$ in these coordinates. This proves the asserted periodicity and identifies that quotient with $\mathbb{C}^g/\Gamma_G$. Indeed the induced map from this compact torus is a continuous bijection, and the target is Hausdorff since the image of the compact group M in the compact adelic quotient is closed.

The probability Haar measure of the rational adelic quotient pushes to normalized volume on this torus by translation invariance. Uniqueness follows by averaging any continuous test function against both probability measures in either order, using Lemma A.2. The squared integral of $F^a_{\phi_0\otimes\Phi}$ on the rational quotient equals $\|\phi_0 \otimes \Phi\|_2^2$ by the character orthogonality calculation in Theorem 387.2. The Gaussian ϕ_0 of Lemma 392.1 has norm one by the Gaussian integral in Lemma 4.2. Now use the covolume in Lemma 463.1 to obtain the integral identity.

For fixed z the entire function $W_z S_\Phi(w)$ has absolute value $A_\Phi(z)$ at $w = 0$, and modulus $e^{\|w\|^2/2}A_\Phi(z + w)$ at every w, by the defining formula in Theorem 48.2. Its squared modulus at zero is at most its average squared modulus on the ball $\|w\| < r_d$. To see this, first average the function on $e^{i\theta}w$ over $0 \le \theta \le 2\pi$: this gives its value at zero by the circle formula of Lemma A.3. Cauchy–Schwarz and integration over the rotation-invariant ball give the squared inequality. The ball about z embeds in the torus by the period lower bound in Lemma 463.1, since $2r_d < \sigma_0$. On it the squared Gaussian factor costs at most $e^{r_d^2}$. Its Euclidean volume is $\pi^g r_d^{2g}/g!$: for the unit ball integrate the last complex coordinate in polar coordinates, giving volume $2\pi \int_0^1 t(1 - t^2)^{g-1}\,\mathrm{d}t = \pi/g$ times the preceding unit volume, starting from dimension zero. Scaling gives the general radius. The integral over the embedded ball is at most the full integral of A_Φ^2. Substitution and a square root complete the proof. □

464. An absolute bound for conjugates of differential kernels

Theorem 464.1. *In the notation of Theorems 462.1 and 463.2, use v_n° of Corollary 427.2. Choose $C_d \ge 0$ as in that corollary and put*

$$H_n = \exp(C_d n N_1^2)\, e^{r_d^2/2} r_d^{-fn} \sqrt{(fn)!}.$$

For a finite sum $\Phi = \sum_j c_j D_{X_j} \mathcal{V}_{G_j} v_n^\circ$ and every $\nu \in \mathrm{Aut}(\mathbb{C}/K)$ one has

$$\sup_{z\in\mathbb{C}^{fn}} e^{-\|z\|^2/2} |S_{\mathcal{L}_\nu\Phi}(z)| \le H_n \sum_j |\nu c_j|.$$

There is C_d', independent of the exponent determining the field E, for which $\log H_n \le C_d'(N_1 n + 1)^2$.

Proof. Write $S = S_{\mathcal{L}_\nu v_n^\circ}$. For every finite X, the evaluation $(T_{b_\nu X} S)(0)$ is $\nu((T_X S_{v_n^\circ})(0))$ by Theorem 461.1 and its series identity at zero. The correcting phase for the physical D_X is a root of unity by Theorem 425.2. Corollary 427.2 bounds the modulus of this value by $B_n = \exp(C_d n N_1^2)$. The finite scalar b_ν is invertible, so this holds at every finite displacement. At a rational displacement γ the modulus in question is $e^{-\|r\gamma\|^2/2}|S(-r\gamma)|$ by Theorem 51.2. The vectors $r(E^n)$ are dense in the real space $\mathbb{C}^{fn}$, since r extends to a real isomorphism in Proposition 47.2 and rational coordinates are dense in real coordinates. Continuity gives $e^{-\|z\|^2/2}|S(z)| \le B_n$ for every z.

By Theorem 462.1 the finite vector $\mathcal{L}_\nu v_n^\circ$ is a multiple of $\mathcal{V}_z v_0$ for a norm-one finite scalar z. The mean identity of Theorem 463.2 therefore gives $\|\mathcal{L}_\nu v_n^\circ\|_2 \le B_n$. By that same conjugation theorem and Corollary 461.2, $\mathcal{L}_\nu D_X \mathcal{V}_G v_n^\circ$ is obtained from $\mathcal{L}_\nu v_n^\circ$ by a finite isometry, a translation, and a scalar of modulus one. The finite representation and the translations preserve Hilbert norm by Theorems 390.2 and 358.1. It is still a scalar multiple

of a single displaced moved vacuum, so Theorem 463.2 bounds its weighted section by H_n. Semilinearity and the triangle inequality give the bound for the sum. Finally $f \le N_1$ by Proposition 10.3, $\log(g!) \le g \log g \le g^2$ for $g = fn \ge 1$, and r_d depends only on d. These inequalities give the stated constant. □

Lemma 464.2. *Let S be entire on $\mathbb{C}^g$, $g \ge 1$, and $|S(z)| \le Be^{\|z\|^2/2}$ everywhere for some $B \ge 0$. For every polynomial P in g variables and $a \in \mathbb{C}$,*

$$[P(\partial)S(aZ)]_{Z=0} = \pi^{-g} \int_{\mathbb{C}^g} S(z)P(a\bar{z})e^{-\|z\|^2}\,\mathrm{d}z.$$

If $L \ge 0,\ A \ge 0,\ w \in \mathbb{R}$ and $|P(a\bar{z})| \le A\exp(w + L\|z\|^2/(2L+2))$ for all z, the absolute value of this integral with its prefactor is at most $BAe^w(2L+2)^g$.

Proof. By linearity take $P(Z) = Z^\alpha$ for the identity. Use polar coordinates in each complex variable. On each product of circles of fixed radii the Taylor series of S converges absolutely and uniformly by the successive disk formulas of Lemma A.3. Integration over angles retains exactly its coefficient of z^α. In one complex dimension $\pi^{-1}\int_{\mathbb{C}} |z|^{2j}e^{-|z|^2}\,\mathrm{d}z = j!$, by the change $t = |z|^2$ and j integrations by parts. Multiplying these moments yields the factorial converting that coefficient to the derivative. Absolute integrability for changing the order of integrations follows from the Gaussian bound times the polynomial and $e^{-\|z\|^2}$, by Lemma A.2. For the estimate, insert both absolute bounds. The remaining exponent is $-\|z\|^2/(2L+2)$. The product of the one-variable Gaussian integrals, divided by π^g, is $(2L+2)^g$. □

Corollary 464.3. *Use Φ, ν, H_n of Theorem 464.1. For $a \in \mathbb{C}$, P a polynomial, put $P_\nu = \mathcal{C}_\nu P$ and $\alpha = (\nu a)t_\nu$. Suppose $L \ge 0,\ A \ge 0,\ w \in \mathbb{R}$ are such that for all real $R \ge 0$ and all $y \in \mathbb{C}^{fn}$,*

$$\|y\|^2 \le \frac{|\nu a|^2}{\Omega}(2L+2)R \quad \Longrightarrow \quad |P_\nu(y)| \le Ae^{w+LR}.$$

Then

$$|\nu([P(\partial)S_\Phi(auZ)]_{Z=0})| \le H_n\left(\sum_j |\nu c_j|\right)Ae^w(2L+2)^{fn}.$$

To bound the modulus under an arbitrary automorphism of $\mathbb{C}$, one may use either that automorphism or its composition on the left with complex conjugation, choosing the one fixing K.

Proof. The derivative at zero is a finite sum of products of polynomial coefficients, factorials and Taylor coefficients. Hence by Theorem 461.1 its conjugate is $[P_\nu(\partial)S_{\mathcal{L}_\nu\Phi}(\alpha Z)]_{Z=0}$. The same permutation of variable indices in both series and polynomial leaves this contraction unchanged. Set $R = \|z\|^2/(2L+2)$, $y = \alpha\bar{z}$, and use $|t_\nu|^2 = 1/\Omega$ from Corollary 459.3. The bound follows from Lemma 464.2 and Theorem 464.1. Finally an automorphism preserves the set of the two roots of $X^2 + d$, so its action on K is the identity or complex conjugation. Left composition with complex conjugation preserves the absolute value of every number. □

465. Rectangular differential matrices in the tensor norm

Lemma 465.1. *Use the rectangular matrix $S(Z)$ of Definition 409.1 in a single block of size n'-by-n, $1 \le n < n'$, with $\kappa(h) = 0$ for $n < h < n'$. Define $\lambda_i = \sum_{i \le h < n'} \kappa(h)$ for $1 \le i \le n$. Use the tensor norms from Lemma 414.2. If $s_1 \ge \cdots \ge s_n \ge 0$ are the singular values of Z, then*

$$\|S(Z)\|_{\mathrm{op}} \le \prod_{i=1}^{n} s_i^{\lambda_i}, \qquad \sum_i s_i^2 = \sum_{bi} |Z_{bi}|^2$$

with $s_i^0 = 1$, including $s_i = 0$.

Proof. For every height h of the target diagram take one $\bigwedge^h Z$ per column. On the source tensor space adjoin, for each full-height column $h = n$, an extra factor $e_1 \wedge \cdots \wedge e_n$ of norm one. On evaluating an invertible array Y, the source flag vector with these extra factors equals the full flag tensor at Y divided by $(\det Y)^{\kappa(n)}$. The product of the exterior maps therefore agrees with $S(Z)$ on these spanning vectors, by Proposition 409.2 and Lemma 414.2. They span since invertible arrays are dense and their evaluations span the same finite-dimensional space as those of all arrays. In particular the identification is isometric at both ends.

The Hermitian matrix Z^*Z has an orthonormal eigenbasis with nonnegative eigenvalues. For completeness maximize v^*Z^*Zv on the compact unit sphere. Varying the maximizer along each perpendicular unit vector, and along its multiple by i, shows that the perpendicular component of Z^*Zv vanishes. Its perpendicular complement is invariant by the Hermitian identity. Induct there. In the resulting basis Ze_i are orthogonal with lengths s_i, the nonnegative square roots of the eigenvalues, ordered decreasingly. Their squared sum is the trace of Z^*Z, which gives the stated entry sum by direct multiplication. The image of an increasing orthonormal basis wedge of size h has length the product of its s_i, and distinct nonzero images are orthogonal. On the tensor product of these bases the same orthogonality persists. Hence the norm of the full tensor map is bounded by $\prod_h (s_1 \cdots s_h)^{\kappa(h)}$. Restriction to the source flag subspace proves the claim. □

Theorem 465.2. *Take one such rectangular block at each $\tau \in G$, with the same n, n' and possibly different column counts. Use their tensor product $S(Z)$, put $b = nn'$, $g = fb$, $N = \sum_{\tau,i} \lambda_{\tau i}$, and set $E(0) = 0$, $E(k) = k \log k - k$ for $k \geq 1$. Let $\Phi = \sum_j c_j D_{X_j} \mathcal{V}_{G_j} v_b^\circ$, and $a \neq 0$. For any $\nu \in \mathrm{Aut}(\mathbb{C}/K)$ the entrywise conjugate differential matrix, between its tensor norm spaces, satisfies*

$$\begin{aligned} \left\| \nu([S(\partial) S_\Phi(auZ)]_{Z=0}) \right\|_{\mathrm{op}} \;&\leq\; H_b \left(\sum_j |\nu c_j| \right) \left(e(2 + N/g) \right)^g \\ &\quad \cdot \exp\left(\frac{1}{2} \sum_{\tau,i} \left\{ E(\lambda_{\tau i}) + \lambda_{\tau i} \log \frac{|\nu a|^2}{\Omega} \right\} \right). \end{aligned}$$

When $N = 0$ the factor $(e(2 + N/g))^g$ can be replaced by 2^g. In every case $E(k) \leq \log(k!)$.

Proof. The coefficients of S are integers by Proposition 409.2. Renaming the blocks by $\nu \circ \tau$ only permutes the input variables of the polynomial. Let A, v be arbitrary target row and source column with $\|\nu A\|_{\mathrm{dual}} \leq 1$, $\|\nu v\| \leq 1$; all complex rows and columns arise after applying ν. The polynomial $\mathcal{C}_\nu(AS(Z)v)$ at y therefore has absolute value at most $\prod_{\tau,i} s_{\tau i}^{\lambda_{\tau i}}$, with singular values taken in the correspondingly renamed blocks, by Lemma 465.1. Tensor operator norms multiply, as follows either by using the orthogonal tensor bases in that lemma or by diagonalizing the positive Gram matrix of each operator in an orthonormal basis.

Put $\theta = |\nu a|^2 / \Omega$. For $L > 0$ and $\sum_{\tau,i} s_{\tau i}^2 = \|y\|^2 \leq \theta(2L + 2)R$, set $v_L = \theta(1 + 1/L) > 0$. For all $x \geq 0$, $k \geq 0$ integral, $x^k e^{-x} \leq e^{E(k)}$. For $k = 0$ this follows from $e^{-x} \leq 1$, and for $k > 0$ differentiation gives the maximum at $x = k$. Apply the inequality with $x = s_{\tau i}^2 / v_L$ and take square roots. Multiplication bounds the polynomial by

$$e^{LR} (1 + 1/L)^{N/2} \exp\left(\frac{1}{2} \sum_{\tau,i} \{ E(\lambda_{\tau i}) + \lambda_{\tau i} \log \theta \} \right),$$

because $\sum x/2 \leq LR$. Use this bound in Corollary 464.3. For $N > 0$ choose $L = N/(2g)$. This minimizes $(2L+2)^g (1+1/L)^{N/2}$, since the logarithmic derivative is $(g - N/(2L))/(L+1)$. Its value is at most $(2 + N/g)^g e^g$ by $\log(1 + x) \leq x$. If $N = 0$ the polynomial is constant of modulus at most one by the same exterior estimate, and $L = 0$, $w = 0$ in the corollary give the sharper assertion. Taking the supremum over the two unit balls now gives the

operator norm: a row of dual norm one attains the length of any fixed nonzero image by the Hermitian inner product. Finally $\int_1^k \log t\,\mathrm{d}t \le \sum_{j=1}^k \log j$ for $k \ge 1$ by monotonicity, and $k = 0$ gives equality separately. □

466. Cancellation of the terminal weight and the first-row overlap

Proposition 466.1. *For all sufficiently large scales, put $X_\tau = a\mathbf{1}_A(\tau)$, $t_1 = 5r$, and $d_i = k + i(t_1+1)$ as in Definition 26.1. The terminal exponent of Definition 416.3 is*

$$b_\tau = X_\tau - mh_2.$$

Consequently the logarithmic contribution of the squared scalar $|\sigma a_N|^2$ at every complex automorphism is $-Ua\log N + fmh_2\log N$.

Proof. There is one group of X_τ height-one columns, all shortened. The number of tall columns in the last diagram is $w_1(\tau) - \lfloor d_1/2\rfloor + (m-1)(2h_2 - (t_1+1))$. Of these, exactly $m(h_2 - \lfloor (t_1+1)/2\rfloor)$ are retained by Definition 416.3. Add the determinant exponent $\lfloor d_{m-1}/2\rfloor - w_m(\tau)$ of Proposition 409.3. By Definition 12.3, $w_m = w_1 + (m-1)2h_2$. Also t_1+1 is even as r is a power of three. Thus $\lfloor d_{m-1}/2\rfloor - \lfloor d_1/2\rfloor = (m-2)(t_1+1)/2$. The total coefficient of t_1+1 is $(m-2)/2-(m-1)+m/2 = 0$, leaving $X_\tau - mh_2$. Each conjugate of $\bar\pi_N$ has modulus $\sqrt{N}$ by Theorem 417.1. Summing over τ, with $|A| = U$, $|G| = f$, proves the claim. □

Lemma 466.2. *Use J of Definition 12.3 and let $\lambda_{j,\tau i}$ be the length of the i-th row, counting from 1, on the j-th arrow of Definition 26.1. Throughout the windows of Theorem 40.2 one has, for all sufficiently large x,*

$$\sum_{j<m}\sum_{\tau}\sum_{i=1}^{d_j} J(\lambda_{j,\tau i}) \le mk\sum_{\tau} J(w_0(\tau)) + mUJ(a) + x^{75} + \frac{2mUa}{x_1}(1+\log(1+x_1)).$$

Proof. Let $r_{j,\tau}$ count the tall columns on this arrow, and $h = 2h_2$. Then its first row has length $X_\tau + r_{j,\tau}$, its next $k-1$ rows all have length $r_{j,\tau}$, and each row below the k-th has length at most jh. Indeed the columns at height k stop there, while the higher columns have total multiplicity $j(h - t_1 - 1)$. The factorial identity gives exactly

$$J(X_\tau + r_{j,\tau}) = J(X_\tau) + J(r_{j,\tau}) + \log\binom{X_\tau + r_{j,\tau}}{r_{j,\tau}}.$$

By Lemma 26.2, $r_{j,\tau} \le w(\tau) + x^{49} \le w_0(\tau) + x^{49}$ and $r_{j,\tau} \ge w(\tau)$. Put $\theta = g_0g_1/\Omega$. Theorem 11.4, Lemma 12.1 and Definition 12.3 give $1 \le \theta$ eventually and $\log\theta = O_{d,M,\epsilon}(\ell)$: the L-value and logarithmic derivative are fixed, $b^{20} \le g_0g_1 \le b^{22}$ and $1 \le \mathrm{rd}_E \ll_d f \le O_d(x^{12})$. Thus J is increasing on nonnegative integers and each increment up to $2a+1$ costs at most $\log(2a+1) + \log\theta = O_{d,M,\epsilon}(\ell)$. All row lengths and w_0 in this comparison are in this range by Proposition 12.4 and Lemma 26.2, as $a \ge xb$, $w \le a/x$, $b = e^\ell$. For the first k rows the change from $J(r_{j,\tau})$ to $J(w_0(\tau))$ therefore costs at most $O_{d,M,\epsilon}(kx^{49}\ell)$ per (j,τ). The rows below k together cost $O_{d,M,\epsilon}(d_jjh\ell)$ there. The sum of these two errors is at most x^{75} eventually, since $m \le 2\sqrt{x}$, $d_m = O(x^{11/2})$, $h = O(x^{40})$, $f \le x^{12}$, $\ell = x^\epsilon$, $\epsilon < 1/100$ by Proposition 10.5 and Lemma 26.2.

The binomial term vanishes off A. At $\tau \in A$ one has $w(\tau) = a/x_1$: A reduces to 1 mod M and D to -1 mod M, distinct since $16 \mid M$, by Definition 24.1. Throughout the windows $v = \log x_1 \le \ell/10^6$ by Theorem 11.4, so $a/x_1 \ge x\exp((1-10^{-6})\ell) \ge x^{49}$ eventually. It follows that $a/x_1 \le r_{j,\tau} \le 2a/x_1$. For a positive integer r' one has $\binom{a+r'}{r'} \le (e(a+r')/r')^{r'}$, using the numerator bound $(a+r')^{r'}$ and $\log(r'!) \ge r'\log r' - r'$ from Theorem 465.2. Thus the binomial term is at most $2a(1+\log(1+x_1))/x_1$ at each of the mU occurrences. Summing proves the inequality. □

467. The archimedean estimate for the untwisted transfer

Theorem 467.1. *Take the trivial twist in Theorem 423.2, the normalized vacua of Corollary 427.2, and $c = -\bar{\pi}_g u$ as in Theorem 429.1 and Corollary 459.3, with $\pi_g \bar{\pi}_g = g_0 g_1$. Write B_Ψ for the projected definite class operator with terminal restriction in Definition 423.1. Fix $\sigma \in \mathrm{Aut}(\mathbb{C})$ and a column ϕ of level $K_{c,0}$. Suppose $\|\sigma\phi(y)\|^2 \le e^V$ for every finite argument y, for some real V. Then throughout the windows of Theorem 40.2, for all sufficiently large x, at every finite argument h,*

$$\|\sigma((B_\Psi\phi)(h))\|^2 \le \exp\left(V + h_0 + 2\sum_p G_p \log p + E_x\right),$$

$$E_x = x^{76} + \frac{2mUa}{x_1}(1 + \log(1 + x_1)) = o_{d,M,\epsilon}(fa\ell^3).$$

The norms are the tensor norms of Lemma 414.2. In particular for the spanning columns of Corollary 14.5 this gives the bound $\exp(h_0 + 2fa\ell^3 + 2\sum_p G_p \log p)$ required in Proposition 25.4.

Proof. First replace σ, if necessary, by $\nu = \mathfrak{c} \circ \sigma$ fixing K, as in Corollary 464.3. Entrywise complex conjugation preserves these norms, since their Gram matrices are real integer matrices by Lemma 414.2. For a fixed tuple of definite representatives on arrow j write $n = d_j,\ n' = d_{j+1},\ b_j = nn',\ g_j = fb_j$. The finite operator on the physical test $D_{s_j} v^\circ_{b_j}$ is, up to a scalar all of whose complex conjugates have modulus one, a single translation of $\mathcal{V}_G v^\circ_{b_j}$. Indeed G acts on Hom matrices by multiplication on the left and right by the stage isometries. The finite arrow in Definition 376.1 multiplies its Weil operator by a quotient of determinant characters. Their values at finite norm-one ideles have all conjugates of absolute value one by Theorems 372.2 and 374.2. The Weil action on the full finite group is Theorem 390.2. Its translation intertwining moves s_j to Gs_j.

By Theorem 425.2, the jet at $c = -\bar{\pi}_g u$ in this arrow is the rectangular differential at scale $\bar{\pi}_g u$. Apply Theorem 465.2 with a there equal to $\bar{\pi}_g$. Its squared modulus under ν is $g_0 g_1$, so its logarithmic weight is $\log((g_0 g_1)/\Omega)$. Since $E(k) \le \log(k!)$ for every row length, the logarithm of the squared operator norm is at most

$$2\log H_{b_j} + 2g_j \log(e(2 + N_j/g_j)) + \sum_{\tau,i} J(\lambda_{j,\tau i}),$$

with the same inequality for a zero operator interpreted as a direct exponential upper bound. Use Corollary 424.2 and then Theorem 417.1. The logarithm of the squared bound for the whole transfer is thus at most V plus the sum over these arrows, plus $2(m+1)x^{25} + x^{59} - \sum_\tau b_\tau \log N$.

Here $\sum_{j<m}(2\log H_{b_j} + 2g_j \log(e(2 + N_j/g_j))) \le x^{59}$ for sufficiently large x. In fact Theorem 464.1 gives $\log H_{b_j} \le C'_d(N_1 b_j + 1)^2$; $N_1 \le x^{12}$, $f \le x^{12}$, $d_m \ll x^{11/2}$, $m \le 2\sqrt{x}$, and $\log N_j \le 3\ell$ by Corollary 23.3 and Lemma 26.2. Proposition 466.1 gives the terminal term $-Ua \log N + fmh_2 \log N$. The added positive quantity is at most x^{59} eventually, since $h_2 = r^8 \ll x^{40}$, $\log N \le 30m\ell$ by Propositions 10.5 and 12.2. Combining these polynomial errors with Lemma 466.2 gives the stated E_x, because $x^{75} + 3x^{59} + 2(m+1)x^{25} \le x^{76}$ eventually.

It remains to identify the three main terms from the row weights and the level. Proposition 12.5 gives

$$2\sum_p G_p \log p = m\sum_{\tau \in A} P(a,\tau) + mk\sum_\tau P(w(\tau),\tau).$$

Substitution in Definition 12.3 yields exactly

$$mk\sum_\tau J(w_0(\tau)) + mUJ(a) - Ua\log N = h_0 + 2\sum_p G_p \log p.$$

For the error term, $fa\ell^3 \geq xe^{\ell}$ and $\ell = x^{\epsilon}$. Also $U \leq f$, $m \leq 2\sqrt{x}$ and $v = \log x_1 \geq 10^6 \log x$. For $v \geq 1$, $(1 + \log(1 + e^v))e^{-v} \leq (2 + v)e^{-v}$, a decreasing upper bound. These inequalities give $E_x/(fa\ell^3) \to 0$ uniformly. Finally take $V = fa\ell^3$ for Corollary 14.5 and $E_x \leq fa\ell^3$ eventually to obtain the last assertion. □

468. The finite right action on the projected row chain

Definition 468.1. Fix a twist ϑ with set P and put $M_g = \{w : w \mid \bar{\pi}_g\}$ in the notation of Definition 357.1. At stage i write K_i for the stabilizer of the integral lattice of Definition 348.2. Use its isometric frame from Theorem 348.3 to reduce at the places of $M_g \cup P$. The first block has size r; the next i blocks have size t. Let $K_0(g) \subset K_0$ be the subgroup reducing to the identity at M_g. On right $K_0(g)$-invariant functions the element $s \in \mathcal{G}$ acts by $R_0(s)z(h) = z(hk_0)$, where $k_0 \in K_0$ has marked reduction s. At each later stage and for each element k of its finite adelic group use the right-translation convention $R(k)F(h) = F(hk)$. For a bounded continuous automorphic z at level $K_0(g)$ set

$$C_{U_\vartheta z} = \Pi_Y \mathcal{F}_{m-1} \cdots \mathcal{F}_0 z,$$

using Definition 422.1 with the real tests of Theorem 422.3. Theorem 422.2 gives the successive absolutely convergent integrals and an open right level. Corollary 419.3 gives a unique vector-valued coefficient $U_\vartheta z$, recovered by its explicit compact integral. In particular U_ϑ is linear.

Theorem 468.2. *Suppose $k_m \in K_m$ is block upper triangular after reduction at every $w \in M_g \cup P$. Let $A_{\nu,w}$ denote diagonal block ν of its reduction, numbered from 0 to m, and set $s = (A_{0,w})_{w \in M_g} \in \mathcal{G}$. For every z in Definition 468.1 and finite terminal argument h,*

$$(U_\vartheta z)(hk_m) = \left(\prod_{i=1}^{m} \prod_{w \in M_g \cup P} (\omega_w \vartheta_w)^i (\det A_{i,w}) \right) (U_\vartheta R_0(s) z)(h).$$

In particular $U_1 z$ is invariant under the terminal integral subgroup reducing to the identity at M_g.

Proof. Let $G_{i,w}$ be the upper left block submatrix up to and including block i of the reduction of k_m. The determinant of an upper block triangular matrix is the product of the block determinants: expanding in permutations, the last block of rows must use the last block of columns, and induction removes one block at a time. Thus all $G_{i,w}$ are invertible. For $i < m$ lift these matrices to one $k_i \in K_i$. At a chosen split place lift each entry to the local integers. The determinant is a unit by its nonzero residue, so the inverse given by the adjugate is integral. In the conjugate component use the inverse adjoint, and take the identity at the remaining pairs and nonsplit places. This is an integral isometry in the chosen frames since the sets of chosen places and their conjugates are disjoint by Definition 357.1. At $i = 0$ this also proves that the marked reduction is onto. Its kernel $K_0(g)$ is normal, so its action through $R_0(s)$ is independent of the lift and is a representation.

For $1 \leq i \leq m$, block triangularity gives $G_{i,w}\binom{I}{0} = \binom{I}{0} G_{i-1,w}$. Theorem 376.2 multiplies the inclusion test on arrow $i - 1$ by $d_i = \prod_w (\omega_w \vartheta_w)^i (\det A_{i,w})$, since $\det G_{i,w} / \det G_{i-1,w} = \det A_{i,w}$. The real test is unchanged by these finite elements. The representation law in Theorem 396.1 therefore gives $\mathcal{K}_{i-1}(ak_{i-1}, hk_i) = d_i \mathcal{K}_{i-1}(a, h)$. In the forward integral at hk_i replace its source variable by ak_{i-1}. Right invariance of the quotient measure gives

$$R(k_i)\mathcal{F}_{i-1} y = d_i \, \mathcal{F}_{i-1} R(k_{i-1}) y.$$

All these changes of variables on the successive forward images are legitimate by Theorem 422.2; equivalently the covariance identity bounds the translated integrals by the original absolute integrals. Iterating gives the displayed product. The infinite compact projection and the recovery integral of Corollary 419.3 commute with translation in the independent

finite variable. For the trivial twist $P = \varnothing$, and identity reductions make every $d_i = 1$ and $s = 1$. □

Corollary 468.3. *For $s \in \mathcal{G}$ choose $k_\vartheta(s) \in K_m$ having marked reduction* $\operatorname{diag}(s, I_{mt})$ *and identity reduction at P. Let $e = \sum_w a_w[w]$ be any sum of distinct central idempotents of Lemma 22.1. For a ramified column ϕ put*

$$Z_s\phi = |\mathcal{G}| R_0(e) R_0(s)(b\mathcal{P}_\phi)$$

with the common $b \neq 0$ of Theorem 423.2. Then for every twist and every finite h,

$$(U_\vartheta Z_s\phi)(h) = |\mathcal{G}| \sum_{w \in \mathcal{G}} a_w (B_{\Psi_\vartheta}\phi)(h k_\vartheta(ws)).$$

Proof. The local lifts exist by the matrix construction in Theorem 468.2. On these lifts every block after the first has determinant one, so their action is $U_\vartheta R_0(s) = R(k_\vartheta(s)) U_\vartheta$. The boundedness and initial principal level of $\mathcal{P}_\phi$ follow from Theorem 204.4 and Lemma 349.2. By recovery of coefficients the identity in Theorem 423.2 reads $U_\vartheta(b\mathcal{P}_\phi) = B_{\Psi_\vartheta}\phi$. Finally $R_0(w)R_0(s) = R_0(ws)$ by the right-translation convention. Expanding $R_0(e)$ gives the asserted equality. □

469. Short rows and their simultaneous local index

Lemma 469.1. *In the notation of Corollary 468.3 let $T_{\vartheta,s}\eta = |\mathcal{G}| \sum_w a_w R(k_\vartheta(ws))\eta$. It acts on coefficient forms invariant under the subgroup $J_\vartheta^0 \subset K_m$ reducing to the identity at $M_g \cup P$. For every prime p and embedding ι of Definition 16.3 it preserves the module given by $|\eta(h)|_{Y,h} \leq 1$ for all h, with the gauges of Theorem 431.1.*

For $P = \varnothing$ take $J = J_1^0$, write $\Gamma = U(Y_m)(E^+)$, where E^+ is the real subfield of E, and use the terminal class norm

$$\|\eta\|_{\sigma,J}^2 = \sum_{h \in \Gamma \backslash U(Y_m)_f / J} \frac{\|\sigma(\eta(h))\|^2}{|\Gamma \cap hJh^{-1}|}.$$

Then $\|T_{1,s}\eta\|_{\sigma,J}^2 \leq |\mathcal{G}|^2 \|\eta\|_{\sigma,J}^2$ for every $\sigma \in \operatorname{Aut}(\mathbb{C})$.

Proof. The kernel J_ϑ^0 is normal in K_m, and the prescribed residues $(\operatorname{diag}(s, I), 1)$ multiply as s does. Thus the right translations give a representation of $\mathcal{G}$ on this space. All $U_\vartheta z$ belong to it by Theorem 468.2. At p the weight matrices $D_Y(k_p)^{\pm 1}$ of any $k \in K_m$ have integral entries. Indeed the terminal frame is the standard one by Definition 348.2, and these matrices are products of determinant units and integral evaluation polynomials, as in Theorem 430.3 and Proposition 417.2. Multiplicativity gives $|v|_{Y,hk} = |v|_{Y,h}$ by applying the ultrametric inequality to both matrices. Each coefficient $|\mathcal{G}|a_w$ is integral by Proposition 22.2, so the same inequality applied to the sum proves preservation.

For the complex assertion every $\gamma \in \Gamma$ acts by a unitary matrix at every embedding of E, including after application of σ: the embeddings on E commute with complex conjugation as shown in Proposition 14.4. Their actions on exterior powers, determinant powers and subspaces of tensor products preserve the induced inner products by Lemma 414.2. Thus the norm is independent of the rational and level choices of representatives. The stabilizers are finite by Lemma 411.2. Right multiplication by $k_1(s)$ permutes the classes and preserves their stabilizers up to rational conjugacy, since it normalizes J. It acts unitarily also on the space of entrywise σ-transported forms with this norm. Now $\sigma(e)$ is a self-adjoint idempotent by Proposition 22.2. In this unitary action its norm is at most one by orthogonal decomposition. The remaining translation is an isometry, proving the asserted squared factor. □

Theorem 469.2. *Let $B = fa\ell^3$, $\ell = \log b_{\mathrm{sc}}$, where b_{sc} denotes the scale b of Theorem 11.4. Use the rational $c > 0$ of Proposition 25.3 and the maps Z_s of Corollary 468.3. For all*

sufficiently large x on the windows of Theorem 40.2 there is a spanning family of everywhere integral ramified columns such that, for every member ϕ, every $s \in \mathcal{G}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\|U_1(cZ_s\phi)\|_{\sigma,J}^2 \le \exp(H - m\mathcal{R} + o_{d,M,\epsilon}(B)),$$

with uniform error. In addition, take any *finite list $y_i = cZ_{s_i}\phi_i$ of everywhere integral ramified columns so mapped, with at most u distinct s_i. For each p, ι there is one square matrix A_ι of the size of the list over $O_\iota = \{z : |\iota z|_p \le 1\}$ satisfying*

$$|\iota(\det A_\iota)|_p \ge e^{-uc_0 e_p \log p}, \qquad \left| \left(U_\vartheta \sum_j (A_\iota)_{ij} y_j \right)(h) \right|_{Y,h} \le 1$$

for every i, h and every allowed twist with $p \nmid Q$. Here c_0 is the dimension of the column space and the e_p are the losses of Proposition 12.6.

Proof. Take the family constructed in Corollary 14.5. Its proof gives an upper bound $V = f \log 4 + 2aU \log(kaC_0)$ for the logarithm of its squared pointwise norm at every conjugate. Since $U \le f$, $\log a \le 2\ell$ and $\log(kC_0) = O_{d,M}(1 + \log x)$ there, $V = O_{d,M,\epsilon}(fa\ell)$ uniformly. For the terminal group the rank $r + mt = k + m(t+1) - 1$ is less than $2n_m$ in Corollary 23.3. Moreover J contains the principal level modulo gN: at the marked places reduction modulo gN implies identity modulo the residue prime. Hence there are at most $\exp(x^{25})$ classes and each of their weights is at most one. Theorem 467.1 bounds the pointwise squared norm of the raw definite image at every representative. Apply Lemma 469.1 and the identity of Corollary 468.3. After also multiplying by c, the logarithmic upper bound is

$$V + h_0 + E_x + 2L_{\mathrm{loc}} + x^{25} + 2\log|\mathcal{G}|,$$

because $2\log c + 2\sum_p G_p \log p \le 2L_{\mathrm{loc}}$ by Proposition 25.3. The proof of Proposition 12.4 gives $h_0 - H + m\mathcal{R} \le U\{m(1 + \log a) + a(G_0 + 15\ell/2)\} = O(fa\ell)$, since $m \le a$, $G_0 \le 23\ell$. By Proposition 15.2 and Theorem 11.4, $\log|\mathcal{G}| \le 22fr^2\ell$ and $r = O(x^5)$. The polynomial costs are $o(B)$ by $a \ge x\exp(x^\epsilon)$. The same holds for $fa\ell$, for L_{loc} by Proposition 25.3, and for E_x by Theorem 467.1. These bounds give the first assertion.

For the second assertion take the column modules and their index witnesses of Corollary 453.2. For the embedding ι these witnesses depend on the column module and on p; its guaranteed gain holds simultaneously for all the indicated B_{Ψ_ϑ}. On the ramified submodule the identity of Corollary 468.3 reads $U_\vartheta Z_s = T_{\vartheta,s} B_{\Psi_\vartheta}$, with $T_{\vartheta,s}$ preserving the integral module by Lemma 469.1. For the common target of the Z_s take the linear span of their images as bounded continuous functions of principal level. Its boundedness follows from Theorem 204.4. Proposition 25.3 now supplies the single matrix for the entire list and all allowed twists. For the empty list the empty matrix has determinant one. □

470. Independent short rows

Proposition 470.1. *Let W, D, R, Q_s be as in Definition 218.1. Identify old and new marked row groups by transport of the initial lattice in Lemma 349.2. Use the transported good central idempotent for e in Corollary 468.3. Write $S_{\mathrm{row}} z(h) = z(hb_0)$ for the same initial change of frame. For every subset T of the marked group, identifying its elements by this transport, one has*

$$\sum_{s \in T} cZ_s(\mathrm{Col}_{\mathrm{ram}}) = S_{\mathrm{row}} \left(\sum_{s \in T} RQ_s(V_{\mathrm{ram}}) \right) \subseteq S_{\mathrm{row}} W.$$

Here c is the positive rational of Theorem 469.2.

Proof. Let b_X be the central finite definite component of b_0. The substitution $\phi \mapsto [h \mapsto \phi(hb_X^{-1})]$ is an automorphism of the column space: it commutes with the left rational covariance and, by centrality, with the right level. Its inverse uses b_X. It commutes with each central difference operator at N in Theorem 13.6, hence carries the ramified space

onto itself. Use an orthonormal basis of the complex weight space of the real group and its conjugate basis for the coefficient representation. The bilinear pairing of Lemma 405.1 sends the latter basis to the dual of the former. In particular $\lambda_d(v) = \mathcal{B}(\phi(h_d b_X^{-1}), v)$ has as row of coordinates exactly the coordinates of the translated column at h_d. This describes the entire space V_{ram} in Definition 218.1 when ϕ varies over the ramified columns.

By Theorem 350.1 the rational periodization with these rows of coordinates is their linear combination of the original full columns. By Definition 215.1 that sum is $\sqrt{\mathfrak{v}}\,Qv$ in the old frame for the resulting coordinate vector v; the normalizing volume is positive. Lemma 349.2 identifies its right translate by b_0 with $\mathcal{P}_\phi$. If k is an old integral representative, then

$$R(b_0 k b_0^{-1}) S_{\mathrm{row}} = S_{\mathrm{row}} R(k).$$

The lattice transport sends the full stabilizer to the full stabilizer since the basis transport there differs from b_0 only by a central scalar. It also identifies their principal kernels, so this equality descends to both marked group actions. The idempotent inducing R is the product of the marked projections of Theorem 217.3. Their product is a sum of distinct central projections: the regular unitary action decomposes into irreducible spaces by Lemma 22.1; on each irreducible both commuting central projections act by 0 or 1, as does their product. Transport therefore gives an allowed choice of e. Multiplication by $c|\mathcal{G}|b\sqrt{\mathfrak{v}}$ is by a nonzero scalar, by Theorem 423.2 and Proposition 25.3. Taking spans proves the equality and Theorem 218.2 gives the inclusion. □

Corollary 470.2. *Write b_{sc} for the arithmetic scale. There is a set T with $|T|c_0 \le 3D\log(1+b_{\mathrm{sc}}^2)$ and an independent family $y_i = cZ_{s_i}\phi_i$ of length $n \ge (1-4/b_{\mathrm{sc}}^2)D$, with $s_i \in T$ and short everywhere integral ϕ_i, for which*

$$\|U_1 y_i\|_{\sigma,J}^2 \le \exp(H - m\mathcal{R} + o_{d,M,\epsilon}(B)).$$

For every p, ι a simultaneous integral recombination for all twists prime to p has determinant with absolute value at least $\exp(-|T|c_0 e_p \log p)$. The total logarithmic cost is at most $3D\log(1+b_{\mathrm{sc}}^2)L_{\mathrm{loc}}$. If $\dim W \le D$, the same assertions hold with D replaced throughout this corollary by $\dim W$.

Proof. Apply Proposition 218.3 to the old column spaces and transport its set by Proposition 470.1. The resulting sum has the same dimension, since the inverse translation uses b_0^{-1}. By Corollary 14.5 the short columns span the entire ramified source. Their images under the selected maps therefore span this sum. Extract a basis from those images by adjoining a vector outside the current span until its full dimension has been reached. This uses only spanning by short integral columns in the new domain. Theorem 469.2 gives both bounds for the chosen rows. Sum the logarithmic costs using $L_{\mathrm{loc}} = \sum_p e_p \log p$. Under the dimension hypothesis use Corollary 218.4 to choose the old translates with its improved count and dimension. The same transport and extraction apply. □

471. The terminal level and its inner product

Theorem 471.1. *At the marked places M_g require identity reduction on Y_m; at the twist places P require upper block triangular reduction of type $(r, t, \dots, t)$, with m blocks of size t. Let J_ϑ be this subgroup of K_m. Write $q_w = |\mathbf{F}_w|$, $Q = \prod_{w\in P} q_w$, $g_h(q) = \prod_{a=0}^{h-1}(q^h - q^a)$, and set*

$$A_h = \prod_{w\in P} g_h(q_w), \qquad W_i(P) = \prod_{w\in M_g\cup P} g_{r+it}(q_w), \qquad k_m = mrt + \binom{m}{2}t^2.$$

Empty products are one. If $I_m(P) = [K_m : J_\vartheta]$, then

$$I_m(P) = \frac{W_m(P)}{A_r A_t^m Q^{k_m}}.$$

At rank $r+t$ let $I_{\mathrm{aux}}(P)$ be the integral level index where at each marked place the first r coordinate vectors are fixed individually, and at each twist place their complementary t-plane is preserved. Put

$$\delta(P)=\frac{I_{\mathrm{aux}}(P)A_r^2Q^{rt}}{W_0(P)},\qquad C=W_0(\varnothing)\left(\frac{W_1(\varnothing)}{\delta(\varnothing)}\right)^m.$$

Then $\delta(P)>0$ and the normalized index is

$$I_m(P)=CQ^{-k_m}\frac{\delta(P)^mW_m(P)}{W_0(P)W_1(P)^m}.$$

Proof. The frame and split lifts in Theorem 468.2 show that reduction is onto the full product in every rank in question. The number of invertible h-by-h matrices over $\mathbf{F}_q$ is $g_h(q)$ by choosing successive columns outside their predecessors' span, as in Proposition 15.2. An upper block triangular matrix is invertible precisely when all its diagonal blocks are invertible. All entries strictly above the diagonal blocks are free. For the r-block and m successive t-blocks these provide $mrt+\binom{m}{2}t^2$ entries. Multiplying the orders at P, with trivial subgroup at M_g, gives $A_rA_t^mQ^{k_m}$. Division gives the first formula.

Put $V_g=\prod_{w\in M_g}g_t(q_w)q_w^{rt}$. At a marked place of the auxiliary level the matrices have diagonal blocks I_r, D, lower left block zero and arbitrary upper right block. At a twist place the upper right block is zero and both diagonal blocks are arbitrary invertibles, with all rt lower left entries free. The orders therefore give

$$I_{\mathrm{aux}}(P)=\frac{W_1(P)}{V_gA_rA_tQ^{rt}},\qquad \frac{\delta(P)}{W_1(P)}=\frac{1}{V_gW_0(\varnothing)A_t},$$

using $W_0(P)=W_0(\varnothing)A_r$. All factors are positive. At $P=\varnothing$ the second ratio equals $1/(V_gW_0(\varnothing))$. Dividing by that value and substituting in the first formula for $I_m(P)$ proves the identity. □

Proposition 471.2. *Use the terminal coefficient space E_Y of Definition 418.1, put $d_Y=\dim E_Y$, and use $b_H>0$ of Lemma 411.2 with finite K_m-volume one. Let z,z' satisfy Definition 468.1. Set $\eta=U_\vartheta z$, $\eta'=U_\vartheta z'$, and write F,F' for their scalar forward chains before terminal projection. Then*

$$\sum_{h\in\Gamma\backslash U(Y_m)_f/J_\vartheta}\frac{\langle\eta(h),\eta'(h)\rangle}{|\Gamma\cap hJ_\vartheta h^{-1}|}=\frac{I_m(P)d_Y}{b_H}\,(F,\Pi_YF')_{[H]}.$$

Proof. At $k\in J_\vartheta$ the two coefficients transform by the same character by Theorem 468.2. It has modulus one since each tame factor has finite image. Rational left multipliers are unitary at infinity by Lemma 469.1. Hence their pointwise Hermitian product is a function on the indicated double quotient. Compact averaging in Theorem 418.2 gives $\int_{H_\infty}\overline{C_\eta(u,h)}C_{\eta'}(u,h)\,\mathrm{d}u=\langle\eta(h),\eta'(h)\rangle/d_Y$, with probability Haar measure. Indeed the averaged rank-one orthogonal projection commutes with the irreducible compact action and has trace one, which determines its scalar. Apply both integral formulas of Lemma 411.2 and the index of Theorem 471.1. Finally $C_\eta=\Pi_YF$, $C_{\eta'}=\Pi_YF'$ and Π_Y is a self-adjoint idempotent by Corollary 419.3 and the compact-integral argument of Theorem 422.3. This gives the displayed pairing. □

472. A real test for the projected terminal coefficient

Proposition 472.1. *Let $K=U(Y_m)_\infty$, $d_Y=\dim E_Y$, and take the unit vector u_0 of Corollary 419.3. For the last arrow denote its action at infinity by Ω_∞. If f is its real Schwartz test, set*

$$f^\natural=d_Y\int_K\langle u_0,R_Y(k)u_0\rangle\,\Omega_\infty(1,k)f\;\mathrm{d}k,$$

with probability Haar measure. This is a Schwartz test independent of the finite twist. For any moderate automorphic function y at stage $m-1$, the last forward integral with real factor $f^\natural$ equals $\Pi_Y \mathcal{F}_{m-1} y$ with original real factor f. Consequently replacing the last test by $f^\natural$ realizes $C_{U_\vartheta z}$ directly as the forward chain, for every z of Definition 468.1.

Proof. The real operator on the consecutive spaces is the real Weil representation multiplied by determinant characters, as in Theorem 396.1. Theorem 393.1 gives continuity of the Weil action in every Schwartz seminorm. The determinant characters on real isometries are determined by their infinity weights, which are independent of the finite twist by Theorem 376.2. Their scalar functions are continuous and have modulus one. Thus on the compact group K the seminorms p_b of Lemma 421.1 of the integrand are uniformly bounded for each b. Define $f^\natural$ by the pointwise integral. On each compact real coordinate box, derivatives pass through the integral: a coordinate difference quotient is an average of the next derivative along a segment, which has a uniform bound there, so apply Lemma A.2 and iterate. Multiplying each resulting derivative by its defining power of $1+\|x\|$ gives its required uniform seminorm estimate. This proves the Schwartz and independence claims.

Writing $v = uk$ in the projection kernel of Corollary 419.3 gives, for every terminal function F,

$$(\Pi_Y F)(u,h) = d_Y \int_K \langle u_0, R_Y(k) u_0 \rangle F(uk,h)\, \mathrm{d}k.$$

By the representation law of Theorem 396.1 the substitution $u \mapsto uk$ in the last forward kernel simply acts on its Schwartz test by $\Omega_\infty(1,k)$. For justification of the interchange, $Y_{m-1} = X_{m-1}$ and Y_m are both definite by Definition 348.2. The source quotient has compact fundamental representatives by Lemma 411.2, and y is bounded there by continuity. With u, h fixed the kernel on these representatives times K is uniformly bounded by the seminorm estimate in Lemma 421.1. Integration in k commutes with evaluation of the theta functional on the test: the seminorm estimate gives continuity of this linear functional; finite partitions of compact K with representative in each part approximate the continuous integrand uniformly in that seminorm. Such partitions follow from a finite open cover on which its variation is arbitrarily small, made disjoint by successive differences. The corresponding sums converge to the pointwise test integral in the seminorm by the derivative estimates above. Finally exchange the source and compact integrals by Lemma 123.1. This gives precisely the kernel with real test $f^\natural$. Applying the assertion to the preceding moderate forward image, given by Theorem 422.2, proves the conclusion. □

Corollary 472.2. *Let $z_1, \dots, z_d$ be bounded continuous automorphic functions of initial principal level. Write $F_i^\natural$ for the scalar forward image with last test as in Proposition 472.1. The determinant of the d-by-d class Gram matrix of $U_\vartheta z_i$ at level J_ϑ equals*

$$\left(\frac{d_Y C}{b_H} Q^{-k_m} \frac{\delta(P)^m W_m(P)}{W_0(P) W_1(P)^m} \right)^d \det\left((F_i^\natural, F_j^\natural)_{[H]} \right)_{i,j}.$$

For $d \geq 1$ it is positive exactly when the functions $F_i^\natural$ are linearly independent. In particular this criterion applies to the independent rows of Corollary 470.2.

Proof. Proposition 472.1 gives $F_i^\natural = C_{U_\vartheta z_i}$, fixed by the self-adjoint terminal projection. Apply Proposition 471.2 entrywise and use Theorem 471.1. Factoring out the positive scalar from each row gives its d-th power. The integral of the squared modulus of any nonzero continuous function is positive, as proved at the end of Theorem 422.3. Thus on the finite span of these functions the Petersson form is positive definite. Expressing the family in an orthonormal basis, the kernel of its Gram matrix is the kernel of its column map, because the associated quadratic form is the squared norm of the linear combination. All Gram eigenvalues are nonnegative, so positivity of its determinant is equivalent to independence. □

473. Injectivity from descent along the triangular array

Definition 473.1. Fix $m \geq 1$, a complex vector space V and vector spaces $F_{i,n}$ for $i, n \geq 0,\ i+n \leq m$. For $i+n < m$ let

$$a_{i,n} : F_{i,n} \longrightarrow F_{i,n+1}, \qquad d_{i,n} : F_{i,n+1} \longrightarrow F_{i+1,n}$$

be linear. Write $A_{i,0} = 1_{F_{i,0}}$ and $A_{i,n+1} = a_{i,n}A_{i,n}$. At level zero take injections $j_i : V \longrightarrow F_{i,0}$ for $0 \leq i \leq m$. Write W for any subspace of V. For $i < m$ suppose there is an injective $H_i : W \longrightarrow W$ with

$$d_{i,0}a_{i,0}j_i x = j_{i+1}H_i x \qquad (x \in W).$$

Theorem 473.2. *In Definition 473.1, suppose for each $i+n+2 \leq m$ there is a scalar $t_{i,n}$ with*

$$a_{i+1,n}d_{i,n}y = t_{i,n}\, d_{i,n+1}a_{i,n+1}y \qquad \big(y \in A_{i,n+1}j_i(W)\big).$$

Then for all $i+n+1 \leq m$ and $x \in W$,

$$A_{i+1,n}j_{i+1}H_i x = \left(\prod_{u=0}^{n-1} t_{i,u}\right) d_{i,n}A_{i,n+1}j_i x.$$

For every $i+n \leq m$, the restriction of $A_{i,n}j_i$ to W is injective. The assertions also hold with the arrows in any row multiplied by individually prescribed nonzero scalar weights, in the sense that their successive composites have the same kernels on W.

Proof. For $n = 0$ the identity is the relation defining H_i. If it holds at n, apply $a_{i+1,n}$ on the left of both sides. On the right the vector to which $a_{i+1,n}d_{i,n}$ is applied is exactly $A_{i,n+1}j_i x$. The square relation in the statement may therefore be used at that vector. Its new source after applying $a_{i,n+1}$ is $A_{i,n+2}j_i x$, proving the identity at $n+1$.

Prove injectivity simultaneously in all rows by induction on the length. For length zero it follows from injectivity of j_i. If $A_{i,n+1}j_i x = 0$, the displayed identity gives $A_{i+1,n}j_{i+1}H_i x = 0$, irrespective of the value of the scalar product in that identity. By the assertion for length n and $H_i x \in W$ we have $H_i x = 0$; then $x = 0$. Multiplying individual arrows by nonzero scalars multiplies their composite by their product, which preserves its kernel. □

Corollary 473.3. *For one twist take any finite-dimensional space W of bounded initial principal-level forms as in Definition 468.1. Suppose there is an array as in Theorem 473.2 for this W such that its weighted first-row composite on W is the scalar forward chain with the last test $f^\natural$ of Proposition 472.1. Then U_ϑ is injective on W and its class Gram determinant in every basis is strictly positive.*

Proof. Any vanishing linear combination of the forward images is the image of the same linear combination in W. Its injectivity follows from Theorem 473.2. The equivalence of independence with positivity of the determinant is Corollary 472.2. Coefficient recovery in Definition 468.1 then gives the assertion for U_ϑ. □

474. Gram forms along an interval of the triangular array

Theorem 474.1. *Endow the spaces of Definition 473.1 with complex inner products conjugate linear in the first variable. Let $W \neq 0$ be finite dimensional and suppose the j_i are isometries on W. Choose positive weights $w_0, \dots, w_m$. Suppose the square multipliers of Theorem 473.2 are $t_{i,n} = w_{n+2}/(w_n\overline{\beta_{i,n}})$ with $\beta_{i,n} \neq 0$. Impose the following two pairing identities. For every $x \in W,\ z \in F_{i,1}$,*

$$\langle z, a_{i,0}j_i x\rangle = \langle d_{i,0}z, j_{i+1}x\rangle.$$

For each $i+n+2 \leq m$ and $z \in F_{i,n+2}$, suppose there is $z' \in F_{i,n+1}$ such that for all $v \in F_{i,n+1}$ and $y \in F_{i+1,n}$,

$$\langle z, a_{i,n+1}v\rangle = \langle z', v\rangle, \qquad \langle d_{i,n}z', y\rangle = \kappa_{i,n}\langle d_{i,n+1}z, a_{i+1,n}y\rangle,$$

where $\kappa_{i,n}$ is independent of z, v, y. Write $S_i = w_0 H_i$ on W and $X_{i,n} = (\prod_{u<n} w_u) A_{i,n} j_i|_W$, where products over $u < n$ start at zero. Then S_i is positive definite and self-adjoint. With the operators $S_i, \dots, S_{i+n-1}$ in this order, for $i + n \le m$ one has

$$\langle X_{i,n}x, X_{i,n}y\rangle = L_{i,n}\frac{w_0 w_1^n}{w_n}\langle x, S_i \cdots S_{i+n-1}y\rangle,$$
$$L_{i,n} = \prod_{h=0}^{n-2}\prod_{u=0}^{n-h-2} \kappa_{i+h,u}\beta_{i+h,u}.$$

The scalar product $L_{i,n}$ has exactly $\binom{n}{2}$ factors of the form $\kappa\beta$. All empty products are identities.

Proof. Putting $z = a_{i,0} j_i x$ in the first pairing identity gives $\langle a_{i,0} j_i x, a_{i,0} j_i y\rangle = \langle H_i x, y\rangle$. Its conjugate symmetry gives self-adjointness of H_i. If its quadratic form is zero then $a_{i,0} j_i x = 0$, hence $H_i x = 0$ by Definition 473.1; thus $x = 0$. Multiplication by $w_0 > 0$ proves the assertion about S_i.

For $i + n + 1 \le m$, $x \in W$ and $z \in F_{i,n+1}$ one has

$$\langle z, A_{i,n+1} j_i x\rangle = \left(\prod_{u<n} \kappa_{i,u}\right)\langle d_{i,n} z, A_{i+1,n} j_{i+1} x\rangle.$$

The assertion at $n = 0$ is the first pairing identity. For the induction apply the first equality involving z' to $v = A_{i,n+1} j_i x$, then apply the formula for n with z', and use the second equality involving z' on $A_{i+1,n} j_{i+1} x$. This gives the formula with factor $\kappa_{i,n}$ appended.

Put $z = A_{i,n+1} j_i x$ and use y for the argument in the second slot. By Theorem 473.2, $d_{i,n} z = (\prod_{u<n} t_{i,u})^{-1} A_{i+1,n} j_{i+1} H_i x$. The conjugate of the inverse product, from the first slot, is $(w_0 w_1)/(w_n w_{n+1}) \prod_{u<n} \beta_{i,u}$; this follows by cancellation of consecutive positive weights, including $n = 0$. Multiplying by the squared weights of $X_{i,n+1}$, and accounting for those already in $X_{i+1,n}$, gives

$$\langle X_{i,n+1}x, X_{i,n+1}y\rangle = \frac{w_1 w_n}{w_{n+1}}\left(\prod_{u<n} \kappa_{i,u}\beta_{i,u}\right)\langle X_{i+1,n} S_i x, X_{i+1,n} y\rangle.$$

Here the factor $1/w_0$ from $H_i = S_i/w_0$ is included. For length zero the norm identity is that j_i is isometric. For positive length iterate this recurrence, moving S_i from the first slot to the second by its self-adjointness after applying the formula for the shorter interval. The positive weights telescope to $w_0 w_1^n / w_n$, and the numbers of new square factors are $n - 1, n - 2, \dots, 1$. Their sum is $\binom{n}{2}$. □

Corollary 474.2. *Under the hypotheses of Theorem 474.1 all S_i commute pairwise and $L_{i,n} > 0$. In an orthonormal basis of W, with $d = \dim W$,*

$$\det(X_{i,n}^* X_{i,n}) = \left(L_{i,n}\frac{w_0 w_1^n}{w_n}\right)^d \prod_{h=0}^{n-1} \det S_{i+h}.$$

Proof. The Gram form on the left of the theorem is positive definite by Theorem 473.2. We prove commutation of S_i, S_j by induction on $j - i > 0$. The intermediate product $M = S_{i+1} \cdots S_{j-1}$ (the identity for adjacent indices) consists of commuting positive definite matrices by the shorter distances. Each factor commutes with both endpoints by the same induction. The product $S_i M$ is positive definite: simultaneously diagonalize its commuting factors by decomposing into the eigenspaces of one Hermitian matrix and repeating on each preserved eigenspace, as proved in Lemma 79.2. In an orthonormal eigenbasis of $S_i M$ its positive eigenvalues multiply the strictly positive diagonal entries of S_j in the trace of $S_i M S_j$. This trace is real and positive. The interval Gram matrix is a scalar multiple of this matrix; its trace is also real and positive. Its scalar, and thus L on this interval, is positive real. The same matrix is consequently self-adjoint, giving $S_i M S_j = S_j M S_i$. Cancel

the invertible commuting M to obtain $S_iS_j = S_jS_i$. This proves the induction; at lengths zero and one the empty scalar product is one. Taking determinants in Theorem 474.1 now gives the formula. □

475. Cancellation in the comparison of chain determinants

Proposition 475.1. *Fix one twist and a nonzero finite-dimensional Hilbert space W of bounded initial forms. Suppose its scalar forward map with terminal test $f^\natural$ in Proposition 472.1 is $X_{0,m}$ of Theorem 474.1 as a map with the indicated terminal Petersson inner product and $w_n = W_n(P)$ of Theorem 471.1. With $d = \dim W$, let Z_ϑ be the determinant of the terminal class Gram matrix of U_ϑ in an orthonormal basis of W. Then*

$$Z_\vartheta = \Big(\frac{d_Y C}{b_H} Q^{-k_m}\delta(P)^m L_{0,m}\Big)^d \prod_{i=0}^{m-1} \det S_i > 0.$$

Suppose also that W_i^+ has dimension d, that $v_{i,\vartheta} : W_i^+ \to H_i^+$ is a linear map to a Hilbert space, and that with nonzero complex c_i, a_i and endomorphisms T_i, T_i^+ of W, W_i^+, respectively,

$$S_i = c_iT_i, \qquad v_{i,\vartheta}^* v_{i,\vartheta} = a_i\delta(P)T_i^+, \qquad \det T_i = \det T_i^+.$$

Writing $\lambda_i(\vartheta) = \det(v_{i,\vartheta}^* v_{i,\vartheta})$, one has $\lambda_i(\vartheta) > 0$ and

$$Z_\vartheta = |c(\vartheta)|^d \prod_{i=0}^{m-1} \lambda_i(\vartheta), \qquad |c(\vartheta)| = \frac{d_Y C}{b_H} Q^{-k_m} L_{0,m} \prod_{i=0}^{m-1} |c_i/a_i|.$$

Proof. By Corollary 472.2 the class Gram determinant equals $(d_Y I_m(P)/b_H)^d$ times the Petersson Gram determinant. Use Corollary 474.2 to compute the second determinant. By Theorem 471.1,

$$I_m(P)\frac{W_0(P)W_1(P)^m}{W_m(P)} = CQ^{-k_m}\delta(P)^m.$$

Each S_i is positive definite, and $L_{0,m} > 0$, proving the first equality and positivity. In the second assertion $\det T_i \neq 0$ because S_i is invertible. Hence $v_{i,\vartheta}^* v_{i,\vartheta}$ is a positive semidefinite matrix with nonzero determinant, so it is positive definite. Taking absolute determinants in the hypotheses gives

$$\delta(P)^d \det S_i = |c_i/a_i|^d \lambda_i(\vartheta).$$

Substitute in the first equality. The determinant equality hypothesis follows in particular from equality of characteristic polynomials by taking constant coefficients, even when those operators act on different spaces. □

Corollary 475.2. *Suppose the data of Proposition 475.1 are given for the trivial twist and for twists ϑ_ι indexed by the places $\iota = (p, j)$ of Definition 16.3. Assume $p \nmid Q(\vartheta_\iota)$, and that d_Y, C, b_H and the product of the absolute ratios $\prod_i |c_i/a_i|$ are common to these twists. For each square of Theorem 474.1 set $D_{h,u}(\vartheta) = (\kappa_{h,u}\beta_{h,u})(\vartheta)$. Suppose*

$$\mathrm{ht}\Big(|D_{h,u}(\vartheta_\iota)/D_{h,u}(1)|^{-2}\Big) \le E_{h,u} \qquad (h + u + 2 \le m).$$

Then the scalar family in the volume factorization satisfies

$$\mathrm{ht}\Big(|c(\vartheta_\iota)/c(1)|^{-2}\Big) \le \sum_{h+u+2\le m} E_{h,u},$$

where $h, u \ge 0$. In particular a uniform nonnegative per-square bound E_* gives $\binom{m}{2}E_*$.

Proof. All square factors in $L_{0,m}$ are nonzero by Corollary 474.2. The trivial twist has $Q = 1$. Substitute the formula of Proposition 475.1; after division the common factors cancel. The inverse squared modulus is the product of the indicated families and $Q(\vartheta_\iota)^{2k_m}$. This rational power is a p-unit at $\iota = (p, j)$ by the coprimality hypothesis. The constant witnesses $h = 1, A = 0, e_p = 0$ give that family height at most zero in Definition 16.3. Apply Proposition 16.4 to all factors and count the square indices as in Theorem 474.1. □

476. Finite matrix coefficients for the first arrows

Theorem 476.1. *Use a hermitian source and target of ranks r, h with finite isometric frames for the standard lattice, $h \geq r$, and the finite arrow of Theorem 396.1. Write χ_H for the character in its denominator on the source determinant. Suppose its unit character is a product of residue characters at $M = M_g \cup P$. Set $D_M = \prod_{w \in M} q_w$. In the split coordinates divided by α at M choose a complex function F on $\prod_{w \in M} M_{h,r}(\mathbf{F}_w)$. Take its extension by $F(\bar{X})$ on integral matrices and by zero elsewhere. Take the unit vacuum at all other pairs and nonsplit places. Denote the resulting finite vector by v_F. Assume that for each $w \in M$ a fixed r-by-h residue matrix L_w satisfies $L_w \bar{X}_w = I_r$ on the support of F.*

For g in the finite source group let g_* be g away from $M \cup \bar{M}$ and the identity on these pairs. In the source frame let $\mathfrak{a}(g_*)$ be the integral ideal whose valuation at w is the determinant valuation of the row lattice

$$\{v \in O_w^{1 \times r} : v(g_*)_w^{-1} \in O_w^{1 \times r}\}.$$

Define

$$\rho_h(g_*) = \frac{\beta(\mathfrak{a}(g_*))^h}{\chi_H(1, \det g_*)\,(N\mathfrak{a}(g_*))^{h/2}}.$$

The matrix coefficient $\langle v_F, \Omega_f(g,1) v_F \rangle$ vanishes unless $g_w \in I_r + \mathfrak{p}_w M_r(O_w)$ at every $w \in M$. When this condition holds it equals

$$\rho_h(g_*)\,\|v_F\|^2, \qquad \|v_F\|^2 = D_M^{-hr} \sum_{\bar{X}} |F(\bar{X})|^2.$$

Proof. The isometry of Theorem 360.1 gives the coordinate description at the split pairs, with integral lattice of mass one. Reduction there has D_M^{hr} equally measured fibers and the other vacuum factors have norm one, proving the stated norm. The row and column action in Corollary 360.2 changes F on this set to a scalar times the function evaluated at $X g_w$ in component w. The coordinates here include all the chosen places simultaneously; no factorization of F among them is needed. If the two supports intersect, X_w and $X_w g_w$ are integral with their indicated reductions. Lift L_w entrywise to $\widetilde{L}_w$ over O_w. Both $\widetilde{L}_w X_w$ and $\widetilde{L}_w X_w g_w$ are integral and congruent to I_r. The adjugate formula gives an integral inverse to the first and hence $g_w \equiv I_r \bmod \mathfrak{p}_w$. This proves disjointness of the supports if the condition fails.

When the condition holds, the action at the chosen pairs fixes the entire vector: the matrix substitution and its inverse preserve the support and its reduction, the Jacobian is one, the ideal multiplier has valuation zero there, and the determinant unit characters are trivial on principal units. These statements follow from Theorems 360.1 and 390.2 and Definition 375.1. For g_* the lattice map is right multiplication by g_*^{-1} on h rows. Its saturation lattice $S_w(G)$ in that definition is the direct sum of h copies of the displayed row lattice. Hence $J_G = \mathfrak{a}(g_*)^h$. Also $[GL_0 : GL_0 \cap L_0] = [L_0 : L_0 \cap G^{-1} L_0] = N J_G$: locally diagonalize the integral sublattice over the discrete valuation ring as in Lemma 111.1, then multiply the residue cardinalities raised to the diagonal valuations. In the vacuum average of Theorem 358.1 only the identity coset pairs with the vacuum, by disjoint support. Its coefficient is $(N J_G)^{-1/2}$. The full finite action adds $\beta(J_G)$ and the inverse determinant character by Theorem 396.1. The chosen pairs and their complement split as tensor factors because the lattice, pairing and actions split there, as in Theorem 360.1. Multiplication gives the desired formula. □

Corollary 476.2. *For $h = r + t$ let F_{row} be the inclusion indicator at all $w \in M$. Let F_{aux} equal A_r times the indicator of matrices reducing to the inclusion at M_g and having first r rows I_r at P. The two norm squares are, respectively,*

$$D_M^{-hr}, \qquad D_M^{-hr} A_r^2 Q^{rt}.$$

Both finite matrix coefficients on the subgroup integral off $M \cup \bar{M}$ are supported on the principal source level, and on it they equal their respective norm squares. The ratio of the auxiliary to row finite masses, each obtained by dividing the norm square by the principal source level index $W_0(P)$, is $A_r^2 Q^{rt}$. Multiplying the auxiliary norm square divided by $W_0(P)$ by its target index $I_{\mathrm{aux}}(P)$ gives $D_M^{-hr}\delta(P)$.

Proof. Use the projection onto the top r rows for the common left inverse. At M_g there is a single permitted matrix. At each $w \in P$ the auxiliary lower t-by-r block is arbitrary, giving Q^{rt} permitted tuples. The norm and coefficient assertions follow from Theorem 476.1; for an element integral off the chosen pairs the complementary ideal is the unit ideal and the complementary determinant character is trivial by its prescribed unit parts. Reduction onto the full product on the source was proved in Theorem 468.2, so the index is $W_0(P)$. The last claim is the definition of $\delta(P)$ in Theorem 471.1. □

477. The nonidentity finite cosets of a first arrow

Lemma 477.1. *Let O be the integer ring of a nonarchimedean local field L with uniformizer π and residue size q, and $M = O^{1\times r}$. For another lattice M' of rank r write $q^a = [M' : M\cap M']$, $q^b = [M : M \cap M']$. For fixed $a, b \geq 0$ the number of possibilities for M' is at most $q^{r(a+b)}$. The lattices Mg^{-1} parametrize right cosets $g\,\mathrm{GL}_r(O)$ in $\mathrm{GL}_r(L)$. Replacing g by its inverse transpose interchanges its two lengths a, b; applying an isomorphism of valued fields to all the entries preserves them.*

Proof. Diagonal reduction over O, after clearing a denominator as in Lemma 111.1, gives elementary powers $\pi^{u_1}, \dots, \pi^{u_r}$ for g^{-1}, up to integral invertible factors. Thus $a = \sum \max(0, -u_i)$ and $b = \sum \max(0, u_i)$. Negating all the u_i interchanges the lengths. Transposition preserves the powers by transposing the diagonalization. A valued-field isomorphism preserves uniformizers up to units, proving the last claims. Every lattice has a basis over O by the same diagonal reduction, so is Mg^{-1}. Two matrices give the same row lattice precisely when they differ on the left by an integral invertible row operation, which is right coset equivalence on their inverses.

Every sublattice of length b is reached through b successive index-q sublattices: in diagonal coordinates increase the nonnegative exponents of the desired sublattice one unit at a time. At each step the choices are the hyperplanes modulo π, because a quotient of size q of O-length one is its residue field. Equivalently each step in this construction is the kernel of a nonzero residue linear form. There are $(q^r - 1)/(q - 1) \leq q^r$ such hyperplanes, counting the linear forms up to nonzero scalar. Every superlattice of length a is likewise reached through a steps. The superlattices of length one of M_0 correspond to lines in $\pi^{-1}M_0/M_0$, with the same bound. Choose first the intersection with M, then a superlattice of this intersection. The product of the two bounds is $q^{r(a+b)}$. □

Theorem 477.2. *Use the standard finite unitary source frames of Theorem 476.1. Let G_* be the restricted product of its local unitary groups off the chosen pairs $M \cup \bar{M}$ and K_* their product of integral stabilizers. Use the function ρ_h of that theorem on G_*, with χ_H trivial on its determinant units, and put $a_0 = h/2 - 2r$. If $a_0 \geq 2$, then*

$$\sum_{gK_* \in G_*/K_*,\ gK_* \neq K_*} |\rho_h(g)| \leq \exp(24 f\, 2^{-a_0}) - 1.$$

In particular, for the first arrows with ranks $r, h = r + t = 6r$ on the arithmetic windows this is $O(f\, 2^{-r}) = o(1)$, uniformly in the twist.

Proof. By Theorems 476.1, 374.2 and 372.2, the absolute weight is $(N\mathfrak{a}(g))^{-h/2}$. The ideal and this weight are constant on integral double cosets by the row lattice definition and integral changes of basis. Each local coset determines the lattice of Lemma 477.1. At a split pair the group is the full general linear group on one branch. The other matrix is its

conjugate inverse transpose in the isometric frame, so its length a is the first branch's b. The contribution of this pair to $N\mathfrak{a}(g)$ is q^{a+b}, where q is the common residue size. The sum of absolute local weights, including the identity coset, is at most $(1-q^{-(h/2-r)})^{-2}$, by summing two geometric series.

At a nonsplit place let q_E denote the residue size in E. The map from the unitary cosets to row lattices is injective, since an integral general linear matrix belonging to the unitary group lies in its integral stabilizer. The determinant of a unitary has valuation zero: its product with its conjugate is one and the place is fixed by conjugation. Hence $a = b$ by the elementary-power calculation in Lemma 477.1. The local contribution to $N\mathfrak{a}(g)$ is now q_E^{a}. Summing the bound $q_E^{2ra}q_E^{-ha/2}$ gives at most $(1-q_E^{-a_0})^{-1}$.

Above a rational prime p there are at most $2f = [E:\mathbb{Q}]$ places of E, by the prime decomposition and degree formula of Proposition 82.3, or by counting the factors of O_E/pO_E, an algebra of dimension $2f$ over $\mathbf{F}_p$. Each residue size is at least p. For $0 \le y \le 1/4$, $-\log(1-y) = \int_0^y (1-u)^{-1}\,\mathrm{d}u \le 2y$. Both local logarithmic bounds are therefore at most $4p^{-a_0}$. To sum over all rational primes use

$$\sum_{p} p^{-a_0} \le 2^{-a_0} + \int_2^{\infty} u^{-a_0}\,\mathrm{d}u \le 3\,2^{-a_0}.$$

Multiplying the local bounds over finite sets of places gives at most $\exp(24f\,2^{-a_0})$. Every coset has nonidentity components at finitely many places, and the summands are nonnegative. Increasing the finite sets exhausts their sum, whose identity term is one. This proves the asserted tail bound. For $h = 6r$ apply Proposition 10.5: r grows on the order of x^5, whereas $f = O_M(x^{12})$. Thus $f2^{-r} \to 0$, and the exponential series gives the final estimate uniformly. □

478. Uniform positivity of the finite first-arrow operator

Theorem 478.1. *Use G_*, K_*, h, r of Theorem 477.2. Let π be a strongly continuous unitary representation of G_* on a complex Hilbert space $\mathcal{H}$. For K_*-fixed y put*

$$T_h y = \sum_{gK_* \in G_*/K_*} \rho_h(g)\pi(g)y, \qquad \eta_h = \exp(24f\,2^{-h/2+2r}) - 1.$$

If $h/2 - 2r \ge 2$, the series converges absolutely in Hilbert norm. It gives a self-adjoint operator on $\mathcal{H}^{K_*}$ with $\|T_h - I\| \le \eta_h$. For $\eta_h < 1$ one has

$$(1-\eta_h)\|y\|^2 \le \langle y, T_h y\rangle \le (1+\eta_h)\|y\|^2, \qquad \|T_h^{-1}\| \le (1-\eta_h)^{-1}.$$

If A is a d-dimensional subspace invariant under T_h, then $(1-\eta_h)^d \le \det(T_h|_A) \le (1+\eta_h)^d$ in this range.

Proof. With the vacuum of norm one at the omitted complements in Theorem 476.1, ρ_h on G_* is precisely the matrix coefficient of the vacuum away from the chosen pairs, evaluated in the full unitary finite action. The vacuum is fixed by K_*, since the matrices preserve its lattice and the determinant unit characters are trivial there. It follows that ρ_h is bi-K_*-invariant, $\rho_h(1) = 1$, and $\rho_h(g^{-1}) = \overline{\rho_h(g)}$ by unitarity. The vector summands are well defined on the stated right cosets. Reindexing by $g \mapsto kg$ for $k \in K_*$ fixes their sum. The sum of the norms off the identity coset is at most $\eta_h\|y\|$ by Theorem 477.2. This gives absolute convergence, invariance, and the asserted norm estimate.

Give Haar measure on G_* mass one on K_*. It is invariant under inversion: on each local unitary factor, conjugation has Jacobian of modulus one on the unitary Lie algebra by the trace-pairing and local-coordinate argument of Proposition 110.2. The assertion passes to finite products and then the restricted product by integration of compactly supported functions on finite sets of factors times the remaining integral groups. On fixed vectors the sum equals integration against $\rho_h(g)$, by its absolute convergence and the coset volumes.

Inverting g, using unitarity of π and the preceding identity for ρ_h, interchanges the two arguments of the associated Hermitian pairing with conjugation. This is self-adjointness.

Cauchy–Schwarz bounds $|\langle y,(T_h-I)y\rangle|\le \eta_h\|y\|^2$. For $\eta_h<1$, the geometric sum $\sum_{n\ge0}(I-T_h)^n$ converges in operator norm by summation of η_h^n; multiplication on either side by T_h and telescoping gives the identity. The inverse estimate follows. On a finite-dimensional invariant subspace diagonalize the self-adjoint restriction by Lemma 79.2 and use the bounds on each unit eigenvector. Their product is its determinant. □

Corollary 478.2. *At $h=6r$ and all sufficiently large arithmetic scales let $W\subset \mathcal{H}^{K_*}$ be a nonzero subspace and $v:W\to H'$ a linear map to a complex Hilbert space. If*

$$\langle vx,vy\rangle = A\langle x,T_hy\rangle \qquad (x,y\in W),\qquad A\in\mathbb{C}^\times,$$

then $A>0$ and v is bounded below by $\sqrt{A(1-\eta_h)}$ on W. If two finite-dimensional spaces of this kind have equal characteristic polynomials for every pair of corresponding finite double-coset convolutions and their finite linear combinations, their T_h have the same characteristic polynomial, provided the same ρ_h is used and both spaces are invariant under those convolutions.

Proof. Theorem 477.2 gives $\eta_h<1$ eventually. At any nonzero y, the quadratic form of T_h is then strictly positive by Theorem 478.1. The identity at $x=y$ and $A\ne0$ imply that $\|vy\|^2>0$ and $A>0$. The lower bound follows from the same identity.

Each integral double coset is a finite union of right cosets: $K_*\cap gK_*g^{-1}$ is open in the compact group K_*, so it has finite index by an open coset cover. Take finite unions of double cosets exhausting G_*; one may enumerate the cosets because local lattices of bounded indices are finite by Lemma 477.1, the places are countable, and only finitely many local components are nonintegral. The truncated sums converge in operator norm on both fixed spaces by the same absolute tail estimate. In a fixed basis each coefficient of a characteristic polynomial is a polynomial in the entries, by determinant expansion, hence continuous. Passing to the limit in the polynomial equality gives the claim. □

479. Graph terms of the doubled theta kernel

Definition 479.1. Let S,T be nonsingular hermitian E-spaces of positive ranks r,h, and put $V=\mathrm{Hom}_E(S,T)$ with the trace hermitian pairing of Definition 355.1. Set $D=S\oplus S^-$, using the negative form on the second copy. For an r-dimensional totally isotropic rational E-subspace $A\subset D$ put

$$\mathcal{N}_A=\{X\in\mathrm{Hom}_E(D,T): X|_A=0\}\subset V\oplus V^-.$$

For Schwartz tests f_1,f_2 in the rational Schrödinger realization of Theorem 385.3 write F_{f_i} for the corresponding theta sections. The zero term of their doubled tensor along A is

$$Z_A(f_2,f_1)=\int_{\mathcal{N}_A(E)\backslash\mathcal{N}_A(\mathbb{A}_E)} F_{f_1}(v_1)\overline{F_{f_2}(v_2)}\,\mathrm{d}(v_1,v_2),\qquad X=(v_1,v_2).$$

The quotient has its additive probability measure.

Lemma 479.2. *The space $\mathcal{N}_A$ is maximal totally isotropic in $\mathrm{Hom}(D,T)$ and the defining integral exists. If $A=\{(x,-\gamma x):x\in S\}$ for a rational isometry γ of S, then*

$$Z_A(f_2,f_1)=\langle f_2,\Omega(\gamma,1)f_1\rangle$$

with the oscillator inner product of the rational Schrödinger model.

Proof. The hermitian identification of D with the conjugate dual carries $A=A^\perp$ onto the linear covectors killing A; indeed $A\subset A^\perp$ and both have dimension r by nonsingularity. Thus $\mathcal{N}_A$ is spanned by maps $z\mapsto b\langle a,z\rangle_D$, $a\in A, b\in T$. Two such maps have trace pairing a product with the factor $\langle a,a'\rangle_D$, hence zero. Its dimension is hr, half that of $\mathrm{Hom}(D,T)$, so it is maximal. On it the two bicharacters in the product section cancel. The

rational translation law of Definition 356.1 then gives an ordinary continuous function on the compact additive quotient of Lemma 356.2. This proves existence.

For the displayed graph the condition on X reads $v_1 = v_2\gamma$. The additive rational parametrization $v \mapsto (v\gamma, v)$ identifies quotient probabilities. The rational covariance of Theorem 395.2 gives $F_{f_1}(v\gamma) = F_{\Omega(\gamma,1)f_1}(v)$: the Hom action of $(\gamma, 1)$ is right multiplication by γ^{-1}, and rational determinant characters are one by Theorem 396.1. The norm identity in the proof of Theorem 387.2 identifies the integral of $|F_f|^2$ on the additive quotient with $\|f\|^2$. Polarization gives the pairing identity, since the map $f \mapsto F_f$ is complex linear. □

Theorem 479.3. *If S is definite at some infinite place, every rational A in Definition 479.1 is the graph of a unique isometry in the convention of Lemma 479.2. Without the definiteness assumption, suppose the first finite test at a split place w in the realization of Theorem 360.1 is supported on injective h-by-r matrices over E_w. Then $Z_A(f_2, f_1) = 0$ for every nongraph A. The assertion applies to either first-arrow residue test of Corollary 476.2, also after arbitrary source and target adelic isometries act on the tests.*

Proof. If $A \cap (S \oplus 0) = 0$, the second projection is a bijection onto S by dimension. Thus $A = \{(cy, y) : y \in S\}$. Isotropy gives $\langle cy, cy'\rangle_S = \langle y, y'\rangle_S$. Nonsingularity makes c injective and hence bijective, showing that A is the indicated graph with $\gamma = -c^{-1}$. The graph determines γ uniquely. If S is definite at one place, every rational $(x, 0) \in A$ has $\langle x, x\rangle_S = 0$ there and hence $x = 0$. This proves the first assertion.

For a nongraph A choose $x \neq 0$ with $(x, 0) \in A$. At the chosen split place choose a matrix C of a basis of A_w, with first column $(x, 0)$. In the local matrix model write $X = (X_1, X_2)$. The compact support of a local component of the doubled test has $XC \neq 0$, since its first column is $X_1 x \neq 0$. We explain why its zero term annihilates such a test. Under $\mathcal{N}_A(\mathbb{A}_E)$-translations the zero term is invariant, by translation on its additive quotient and triviality of the bicharacter on this subspace, exactly as in Lemma 356.2. At $\bar{w}$ these translations act by all the characters of XC. Indeed the pure opposite-branch action is modulation by the perfect trace pairing on the matrix space at w, by Theorem 360.1, with its sign reversed on the conjugated second tensor. The local subspace $(\mathcal{N}_A)_{\bar{w}}$ pairs trivially with $\{X : XC = 0\}$ by total isotropy. Its dimension hr equals the codimension of this kernel, since restriction to a basis of A_w is surjective. It is therefore the whole annihilator. The trace pairing identifies its parameters with all linear residue characters in the entries of XC. The factor α in the split coordinates may be absorbed in the dual parameters by its invertibility.

The image XC of the compact support misses a sufficiently small open lattice D' around zero: otherwise points of the compact image would approach zero, so that zero belonged to it. All values lie in a larger compact open lattice D''. Average the modulations associated with the characters of the finite group D''/D', extending them by the local trace characters as in Proposition 361.1. The average multiplies the test by the indicator of $XC \in D'$ on its support, by finite character orthogonality. It is zero, whereas every translation in the average gives the same zero term. This proves vanishing. A test with finitely many local tensor components follows by linearity, or by taking the union of their compact supports. The top r rows of both named first-arrow tests reduce to I_r, so those matrices are injective. Isometries on the source and target act locally by invertible right and left substitutions with scalar factors, as in Corollary 360.2, preserving injectivity of all matrices in the moved support. □

480. Absolute integrability at the real places

Lemma 480.1. *Let $p, q \geq 0$, $r = p+q > 0$, $G = U(p, q)$, $K = U(p) \times U(q)$, for the diagonal form $(I_p, -I_q)$. Write $\mathbb{B} = \{Z \in M_{p,q}(\mathbb{C}) : I_q - Z^*Z > 0\}$. Then $G/K = \mathbb{B}$, and, up to a fixed positive scalar, Haar integration on G is integration of $g_Z k$ against*

$$\det(I_q - Z^*Z)^{-r}\,\mathrm{d}Z\,\mathrm{d}k,$$

with Lebesgue measure in real and imaginary entries and probability Haar measure on K. If s_j are the singular values of Z, $1 \le j \le \min(p,q)$, and $\tanh u_j = s_j$, $u_j \ge 0$, then $g_Z k$ is a product of two compact factors surrounding the simultaneous real two-by-two boosts with entries $\cosh u_j, \sinh u_j$ on paired positive and negative lines. The other lines are fixed by the middle factor.

Proof. Every negative q-plane projects isomorphically to the negative coordinate summand by definiteness of the other summand and the dimension. Its graph has columns $\binom{Z}{I}$, and is negative precisely under the displayed matrix inequality. Put $D_Z = (I - Z^*Z)^{-1/2}$, $A_Z = (I - ZZ^*)^{-1/2}$. The matrices $\binom{I}{Z^*}A_Z$, $\binom{Z}{I}D_Z$ give orthonormal positive and negative columns orthogonal to each other. Their concatenation $g_Z \in G$ sends 0 to Z. The positive square roots exist by diagonalization as in Lemma 79.2; their dependence on Z is continuous, for example by the power series for $(1-u)^{-1/2}$ converging uniformly on compact subintervals of $[0,1)$. The stabilizer of the negative coordinate plane preserves its orthogonal complement, and is K. Singular-value coordinates follow by diagonalizing Z^*Z, taking Zv/s_j for the corresponding positive-side unit vectors when $s_j > 0$, and completing both orthonormal bases. The section on each pair has diagonal $1/\sqrt{1-s_j^2}$ and off-diagonal $s_j/\sqrt{1-s_j^2}$, as required.

For $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the transformed graph has coordinate $Y = (AZ+B)(CZ+D)^{-1}$; the denominator is invertible by the negative-plane projection argument. Taking the forms on the graph columns gives $I - Y^*Y = (CZ+D)^{-*}(I - Z^*Z)(CZ+D)^{-1}$, where $L^{-*} = (L^*)^{-1}$. Differentiation gives $\mathrm{d}Y = (A - YC)\,\mathrm{d}Z(CZ+D)^{-1}$ as a linear differential on the matrix variable. Block elimination in $g\left(\begin{smallmatrix} I & Z \\ 0 & I \end{smallmatrix}\right)$ gives $\det(A-YC)\det(CZ+D) = \det g$ and $|\det g| = 1$. Taking the determinants of left multiplication on each column and right multiplication on each row gives the real Jacobian $|\det(CZ+D)|^{-2r}$. Hence the proposed density on the ball is invariant by change of variables. In the coordinates $g_Z k$, left translation changes k by a left multiplier depending on Z. Its Haar probability is preserved. This proves left invariance and the Haar formula up to the chosen normalization, as in Lemma 21.4. When either rank is zero the ball has one point with mass one and all the formulas reduce to compact Haar integration. □

Theorem 480.2. *At an infinite place let the source have signature (p,q) and the target have dimension $h > 0$ and any hermitian signature. For any two Schwartz oscillator tests f_1, f_2, the coefficient of the actual unitary pair satisfies*

$$|\langle f_2, \Omega_\infty(g,1) f_1\rangle| \le C(f_1,f_2)\det(I - Z^*Z)^{h/2}, \qquad g = g_Z k.$$

For the two unit Gaussian vacua of Theorem 393.1 the two sides are equal with $C = 1$. In particular the coefficient is integrable on the source when $h \ge 2r$. The same holds on the product of the infinite groups, including Schwartz tests on their direct sum.

Proof. Absorb the two compact factors of Lemma 480.1 into the two tests. Their Schwartz seminorms are uniformly bounded on those compact groups by Theorem 393.1, also after a fixed real change of polarization by Proposition 392.2. The additional determinant characters have modulus one by Theorem 396.1.

On a paired source plane the sum and difference of its two basis vectors diagonalize the boost with eigenvalues e^{u_j} and e^{-u_j}. Each eigenline is hermitian isotropic and they pair perfectly with each other. On the corresponding Hom block the action is right multiplication by the inverse source matrix. Take its h complex coordinates expanding by e^{u_j} as real configuration coordinates, with the opposite h complex coordinates as their momentum duals. Their alternating pairing is the real trace of the hermitian pairing as in Lemma 385.2. This choice depends only on the fixed coordinate pairs and is also valid at $u_j = 0$. On the remaining fixed symplectic block take any complementary real Lagrangians, obtained by successively splitting off paired basis vectors as in that lemma. The boost in these

coordinates is $(A, A^{-\mathsf{T}})$ on position and momentum, with $|\det_{\mathbb{R}} A| = \exp(2h\sum_j u_j)$. In the symmetric gauge its unitary implementer on the positions is $f(x) \mapsto |\det A|^{-1/2} f(A^{-1}x)$ up to a scalar of absolute value one, by direct Heisenberg substitution and uniqueness in Proposition 392.2. The absolute pairing is therefore bounded by $\|f_2\|_1 \|f_1\|_\infty \exp(-h\sum_j u_j)$, with the compactly moved tests in these norms. Their bounds are uniform since a sufficiently high Schwartz seminorm bounds the L^1-norm by integration of $(1 + \|x\|)^{-b}$ for b greater than the real dimension. As $e^{-u_j} \le (\cosh u_j)^{-1}$, the upper bound follows.

For the Gaussian at each compact factor the negative-to-negative block is unitary. Its vacuum coefficient has modulus one by Theorem 393.1; a unit vector whose inner product with another unit vector has modulus one differs by a unit scalar. On the boost its negative block in the signed Hom coordinates has one factor $\cosh u_j$ per paired source plane and per target row of either sign; the unpaired coordinates give ones. The coefficient in that theorem therefore has absolute value $\prod_j (\cosh u_j)^{-h} = \det(I - Z^*Z)^{h/2}$.

Multiplication by the density of Lemma 480.1 leaves $\det(I - Z^*Z)^{h/2-r} \le 1$ when $h \ge 2r$. The ball lies in a bounded Euclidean box by $\|Z\|_{\mathrm{op}} < 1$; hence its Lebesgue measure is finite. For the product of infinite places choose the direct sum of the preceding real polarizations. The determinant of its dilation is the product of their determinants. The same L^1-supremum bound for the two Schwartz functions on this sum yields the product of the determinant bounds, even if the tests are not pure tensors. Product integration finishes the proof. □

481. Unfolding the graph contribution to the doubled kernel

Theorem 481.1. *Let $G = U(S)(\mathbb{A}_{E^+})$, $\Gamma = U(S)(E^+)$, $[G] = \Gamma\backslash G$, and use compatible Haar and quotient measures, with the finite integral stabilizer of mass one. Take ranks r, h with $h/2 - 2r \ge 2$ and $M_g \ne \varnothing$. Use the same nonzero residue test v_F of Theorem 476.1 on both sides, and arbitrary real Schwartz factors f_∞, f'_∞. Write $f = f_\infty \otimes v_F$, $f' = f'_\infty \otimes v_F$ and*

$$\Phi(g) = \langle f', \Omega(g,1) f \rangle, \qquad \Phi_\infty(a) = \langle f'_\infty, \Omega_\infty(a,1) f_\infty \rangle.$$

Then $\Phi \in L^1(G)$. For $y, y' \in L^2([G])$ the integral of the sum

$$\int_{[G]^2} \overline{y'(g')} y(g) \sum_{\gamma \in \Gamma} \Phi(g'^{-1}\gamma g)\, \mathrm{d}g\, \mathrm{d}g' = \int_G \Phi(u) \langle y', R(u) y \rangle_{[G]}\, \mathrm{d}u$$

is absolutely convergent, including termwise absolute values on the left. The sum inside the pairing is the sum of the rational zero terms of Definition 479.1 on $\Omega(g',1)f', \Omega(g,1)f$, with equality wherever the absolute graph series converges.

If y is fixed at finite places by the principal source level at M and the integral level elsewhere, the right side also equals

$$\frac{\|v_F\|^2}{W_0(P)} \int_{G_\infty} \Phi_\infty(a)\, \langle y', R(a) T_h y \rangle_{[G]}\, \mathrm{d}a.$$

Here T_h is the operator of Theorem 478.1 on the finite factors off $M \cup \bar{M}$.

Proof. The real and finite actions on tensor tests factor by Theorem 396.1. The real coefficient is integrable by Theorem 480.2. At the finite places the coefficient is supported on the principal subgroup of the chosen pairs, with value $\|v_F\|^2 \rho_h(g_*)$ there, by Theorem 476.1. Its absolute integral is finite by summation of integral cosets in Theorem 477.2. This proves absolute integrability of their product.

Take a measurable fundamental domain for Γ on G. The construction of Lemma 21.4 applies to the adelic group: a compact set has bounded denominators at finite places and bounded real matrix entries, so meets only finitely many rational matrices by Lemma 82.2; remove earlier rational translates from a countable cover by neighborhoods with disjoint nontrivial translates. For the nonnegative majorant with $|\Phi|, |y|, |y'|$, sum the translates

of the inner domain by γ. Rational invariance of y gives integration on G, and $v = g'u$ converts the result to

$$\int_G |\Phi(u)| \int_{[G]} |y'(g')|\,|y(g'u)|\,\mathrm{d}g'\,\mathrm{d}u \le \|\Phi\|_1 \|y'\|_2 \|y\|_2.$$

Here nonnegative integration is justified by Lemma 123.1. Cauchy–Schwarz and right invariance of quotient measure give the inequality, using Proposition 110.2 for unimodularity. Thus the identical unfolding with the complex factors is valid. Theorem 479.3 removes every nongraph rational subspace on the moved tests. For a graph the zero term of Lemma 479.2 is $\langle \Omega(g',1)f', \Omega(\gamma g,1)f\rangle = \Phi(g'^{-1}\gamma g)$ by the unitary representation law. This proves the identification.

Finally integrate on the chosen principal subgroup, of mass $1/W_0(P)$ by Corollary 476.2. Off these pairs each right integral coset has mass one and y is invariant under its integral subgroup. The remaining sum acts on y by T_h, commuting with the real operators by their separate group variables. The sum and integral interchange by the established absolute bound, or by norm convergence of T_h together with the integrability of Φ_∞. □

Corollary 481.2. *In Theorem 481.1 use $f' = f$ and $h = 6r$. Put $H = U(T)$, $\Theta(g,h') = \theta(\Omega(g,h')f)$ and $vy(h') = \int_{[G]} y(g)\Theta(g,h')\,\mathrm{d}g$. Let $W \neq 0$ be a space of initial forms with the stipulated finite level. Suppose the following assertions hold on these forms. The target pairing of the integral lifts expands by product integration, and for almost all g, g' on the two source quotients the target mean satisfies*

$$\frac{1}{\operatorname{vol}[H]} \int_{[H]} \overline{\Theta(g',h')}\Theta(g,h')\,\mathrm{d}h' = \sum_{\gamma\in\Gamma} \Phi(g'^{-1}\gamma g).$$

Suppose the weak real convolution $L_\infty = \int_{G_\infty} \Phi_\infty(a)R(a)\,\mathrm{d}a$ acts by a nonzero scalar α on W. Then on W the Gram form of v is

$$\langle vx, vy\rangle_{[H]} = \operatorname{vol}[H]\,\alpha\,\frac{\|v_F\|^2}{W_0(P)}\langle x, T_h y\rangle_{[G]}.$$

Thus v is injective for all sufficiently large scales.

Proof. The asserted expansion and target mean reduce the Gram pairing to the graph integral in Theorem 481.1. Absolute integrability of the real coefficient makes its weak integral bounded by $\|\Phi_\infty\|_1$ on unit vectors. The finite operator commutes with this convolution: every partial sum of its right translations commutes with every real operator, and the tail is bounded in norm on the fixed space by Theorem 478.1. Consequently the integrand paired after real integration gives $\langle x, T_h L_\infty y\rangle = \alpha\langle x, T_h y\rangle$. The target quotient volume is positive and finite in the stated mean, and $\|v_F\| > 0$; apply Corollary 478.2 to obtain injectivity. □

482. A height series over rational subspaces

Definition 482.1. Let E be a CM field, O its ring of integers, Σ one embedding per conjugate pair, and $b = 2|\Sigma|$. Give $E_\infty^n = \prod_{\tau\in\Sigma} \mathbb{C}^n$ the sum of the ordinary squared Euclidean norms. For $1 \le d \le n$, let $A \subset E^n$ be a rational d-plane and $Y \in M_{n,d}(E)$ a basis matrix of A. At every finite place w put $|u|_w = q_w^{-v_w(u)}$ for $u \neq 0$, and $|0|_w = 0$, where q_w is the residue cardinality. Define

$$H(Y) = \prod_{\tau\in\Sigma} \det(Y_\tau^* Y_\tau) \prod_{w \text{ finite}} \max_{|I|=d} |\det Y_I|_w.$$

Here Y_I consists of the rows indexed by I. Let $L_A = A \cap O^n$ and take its covolume in $\prod_\tau A_\tau$ with the induced Euclidean measure. Write V_E for the covolume of O in E_∞.

Proposition 482.2. *For the data of Definition 482.1, L_A is a full lattice in $\prod_\tau A_\tau$, and*

$$\operatorname{covol}(L_A) = V_E^d H(Y).$$

In particular $H(A) := H(Y)$ is independent of the basis.

Proof. A change $Y \mapsto YB$ multiplies every minor by $\det B$ and every infinite Gram determinant by $|\tau(\det B)|^2$. The product of these factors is one. Indeed the principal ideal norm of a nonzero integral element is the absolute determinant of multiplication on O, by Lemma 82.2, and is also the product of its prime-power norms by Proposition 82.3. Division treats a fractional element. Only finitely many local factors differ from one for a fixed rational frame, by clearing denominators and choosing one nonzero minor. We may therefore multiply Y by a nonzero integer and assume it integral.

Let $u \in O \setminus \{0\}$ be a nonzero maximal minor. The module $Y^{-1}L_A = \{c \in E^d : Yc \in O^n\}$ lies between O^d and $u^{-1}O^d$, by the adjugate formula on the selected rows. Hence YO^d has finite index in L_A. In these same coordinates the quotient is the kernel of multiplication by Y from $(O/uO)^d$ to $(O/uO)^n$, by sending a class c to $uc \bmod u$. Factor uO and take the product of the residue rings of its prime powers. This follows by the Chinese remainder identity: for coprime I, J write $1 = i + j$, $i \in I, j \in J$; multiplication by j, i lifts the two prescribed residues and $I \cap J = IJ$. Iteration gives the assertion. The factors are the same residue rings as those for the completed valuation rings, since completion preserves residues modulo each power by its definition.

Over a valuation ring, move a nonzero entry of smallest valuation to the top left. It divides every entry, so elementary integral row and column operations clear its column and row. Repeat on the remaining rectangle. This gives a rectangular diagonal matrix with entries units times $\pi_w^{a_1}, \dots, \pi_w^{a_d}$. Integral invertible operations preserve the ideal of maximal minors by multilinearity in both directions. Its valuation is therefore $\sum_j a_j \le v_w(u)$. The local kernel just described has size $q_w^{\sum_j a_j}$, by counting each of the diagonal congruences modulo $\pi_w^{v_w(u)}$. Thus the reciprocal index is exactly the finite product in $H(Y)$.

At τ, the real volume expansion of the complex injection Y_τ is $\det(Y_\tau^* Y_\tau)$: orthogonalize its columns, then each orthogonal length expands a real two-plane in both coordinates. Lemma 82.2 gives V_E^d for the domain lattice. Multiplying these expansions gives the covolume of YO^d, and dividing by the index gives the formula. □

Theorem 482.3. *For fixed E, n, d as in Definition 482.1 there is $C > 0$ such that for $X \ge 1$*

$$\#\{A \subset E^n : \dim_E A = d,\ H(A) \le X\} \le CX^n(1 + \log X)^d.$$

Consequently $\sum_{\dim_E A = d} H(A)^{-s} < \infty$ for every real $s > n$. The sum over rational maximal isotropic subspaces in a hyperbolic space of rank $2r$ converges at $s = h/2$ whenever $h > 4r$, in particular at $h = 6r$.

Proof. Write $N = bd$. For any rank N Euclidean lattice L the proof of Lemma 107.1 gives $\lambda_j(L) \le C_N r_j$, where r_j are the successive orthogonalized lengths of an integral basis. Hence $\prod_{j=1}^N \lambda_j(L) \le C_N^N \operatorname{covol}(L)$. In L_A choose successively a shortest vector v_j outside the E-span of the preceding $j-1$ choices. This is possible through $j = d$ by Proposition 482.2 and discreteness. The real spans at all embeddings of that excluded space have total dimension $b(j-1)$. Among $b(j-1)+1$ real-linearly independent short vectors at least one lies outside it. Thus

$$\|v_j\| \le \lambda_{b(j-1)+1}(L_A), \qquad \prod_{j=1}^{d} \|v_j\|^b \le \prod_{i=1}^{N} \lambda_i(L_A) \le C' H(A),$$

where monotonicity of the successive minima gives the middle inequality.

Let $\delta > 0$ be the shortest length of a nonzero vector of O^n. Balls of radius $\delta/2$ centered at its lattice points have disjoint interiors. Volume comparison in real dimension bn bounds the number of these points of length at most R, for $R \ge \delta$, by $(1 + 2R/\delta)^{bn}$. Choose $k_j \ge 0$ with $\delta 2^{k_j} \le \|v_j\| < \delta 2^{k_j+1}$. For $H(A) \le X$ we have $2^{b\sum_j k_j} \le C'' X$, so at most $C'''(1 + \log X)^d$ tuples can occur. For a fixed tuple the number of possible ordered vectors

is at most a constant times $2^{bn\sum_j k_j} \le (C''X)^n$. Each independent list determines its span. This proves the count. In particular only finitely many heights are below one, and they are positive. Grouping the others between 2^k and 2^{k+1} gives a majorant proportional to $\sum_{k\ge 0}(1+k)^d 2^{-k(s-n)}$, which converges by the ratio test. Take $n = 2r, d = r$ and retain the isotropic subspaces to obtain the final assertion. □

483. The exterior height of a theta zero term

Definition 483.1. Use the arithmetic CM field of Lemma 57.1. Let D be a split hermitian space of dimension $2r$, $r \ge 1$, and T a hermitian target of dimension h with the finite frames of Theorem 396.1. Use a rational hyperbolic basis of D, so its Gram matrix is $J = \left(\begin{smallmatrix} 0 & I_r \\ I_r & 0 \end{smallmatrix}\right)$. Write A_0 for the first r-plane and $Y_0 = \binom{I_r}{0}$. For a basis matrix Y of any rational maximal isotropic plane A and a source adelic isometry g, define

$$H_g(A) = H(g^{-1}Y)$$

by the local products in Definition 482.1. The basis independence proof in Proposition 482.2 applies also with g^{-1} present. On a Schwartz test u of the oscillator on $\mathrm{Hom}_E(D,T)$, let $Z_A(u)$ denote the mean of its theta section over $[\mathcal{N}_A]$, where $\mathcal{N}_A = \{X : X|_A = 0\}$ as in Definition 479.1. Denote an oscillator action of the source by $\Omega(g)$. In absolute values its unitary scalar normalization is immaterial.

Lemma 483.2. *Let K be the product of the integral unitary stabilizers of O_w^{2r} at finite places (taken in conjugate pairs when split) and the isometries preserving the ordinary Euclidean metric at all infinite places. Every rational maximal isotropic plane has a basis $Y = \gamma Y_0$ for a rational unitary γ. For every adelic unitary g there is a factorization $g^{-1}\gamma = kp$, $k \in K$, $pA_0 = A_0$. If $pY_0 = Y_0 a$, then*

$$H_g(A) = |\det a|_{\mathbb{A}_E}.$$

The idele norm uses squared complex absolute values at infinity.

Proof. For any rational basis Y solve $Y^*JW = I_r$ by full row rank. Replacing W by $W - Y(W^*JW)/2$ preserves this identity and makes the Gram matrix of W zero, by expansion. The combined columns have Gram matrix J and thus define a rational isometry γ with $\gamma Y_0 = Y$. For $D = S \oplus S^-$, the diagonal copy of S is isotropic of half dimension. The same construction produces the hyperbolic basis in Definition 483.1.

Consider locally the span of $g^{-1}\gamma Y_0$. Its intersection with the integral lattice has a basis with an integral left inverse: clear denominators and use the valuation-ring row and column reduction in Proposition 482.2; saturation replaces each resulting diagonal ideal by the whole coordinate ring. In the split case perform this in each field factor. The basis so obtained is isotropic and its Gram matrix is zero. By Theorem 345.2 it is the image of Y_0 by an integral unitary matrix. The local algebras satisfy that theorem's unramified or split hypothesis by Lemma 57.1. At every archimedean place, pass by the Euclidean unitary change $(v,w) \mapsto ((v+w)/\sqrt{2}, (v-w)/\sqrt{2})$ to the form $\mathrm{diag}(I,-I)$. A maximal isotropic plane projects bijectively to the positive summand and is the graph of a unitary map onto the negative summand, by equality of the two norms. Acting independently by unitary matrices on the two summands therefore carries any one such graph to any other. This supplies k at every place. Their product belongs to K, and $p = k^{-1}g^{-1}\gamma$ preserves A_0.

Integral invertible matrices preserve the maximum finite absolute value of the maximal minors by multilinearity and the inverse inequality. Euclidean unitary matrices preserve the infinite Gram determinants by direct multiplication. Evaluating the local factors on $Y_0 a$ leaves just the absolute determinant at each place. This proves the formula. □

Theorem 483.3. *For each fixed Schwartz test u of Definition 483.1 there is a finite constant C_u such that for every g, A*

$$|Z_A(\Omega(g)u)| \le C_u H_g(A)^{-h/2}.$$

One constant works for a compact family of real Schwartz tests with finite parts in a fixed finite list. For $h > 4r$ the series of absolute zero terms converges locally uniformly in g. In particular the graph series of Theorem 481.1 at $h = 6r$ converges absolutely at every pair of source points, uniformly on compact sets of the pair.

Proof. Use rational γ and $g^{-1}\gamma = kp$ of Lemma 483.2. Transport of the additive probability by the rational isometry identifies the mean along $\mathcal{N}_A$ with that along $\mathcal{N}_{A_0}$ on the test acted on by $\gamma^{-1}g = p^{-1}k^{-1}$, in modulus. Indeed the rational theta action of Theorem 387.2 is pullback on sections, and preserves the quotient probability; any unitary normalization differs by a unit scalar, by Theorem 358.1 and Proposition 392.2.

In the rational polarization with modulation space $\mathcal{N}_{A_0}$, the configuration coordinates are the restrictions $X|_{A_0} \in \mathrm{Hom}_E(A_0, T)$, using the second hyperbolic summand to extend a restriction by zero. The zero term is evaluation at the origin: integration over the compact modulation quotient kills every nonzero rational frequency, as in Theorem 385.3. The action on $\mathrm{Hom}(D, T)$ of p^{-1} is $X \mapsto Xp$. On configuration coordinates this is right multiplication by a, of module $|\det a|_{\mathbb{A}_E}^{h}$ from its h rows. The unitary parabolic formula of Theorem 386.1 multiplies evaluation at zero by the inverse square root of this module in modulus; its quadratic phase equals one there. By the height formula it remains to bound evaluation at zero on $\Omega(k^{-1})u$ uniformly for $k \in K$.

At infinity this follows from continuity in Schwartz seminorms in Theorem 393.1, after a fixed polarization change by Proposition 392.2. At the finite places use the lattice model of Theorem 358.1. The subgroup of K_f preserving its reference lattice is open: two full adelic lattices are commensurable, and an integral congruence subgroup preserves the finitely many differing local lattices. On this subgroup the action up to phase is pullback on the lattice model. The finite test is a finite linear combination of reference-vacuum translates, by the basis argument in that theorem. An open subgroup fixes the finitely many translation classes and their phases: choose the congruence depth so that the finitely many differences lie in the reference lattice and their pairing with every listed representative lies in the kernel of the finite additive character. Thus the compact group gives only finitely many test vectors up to phases. Their evaluations in the fixed polarization are bounded. Products, then finite sums, give the required C_u. The same proof with the given compact real test family uses boundedness of the continuous seminorms on its product with the compact group.

For g in a compact set, $H_g(A) \geq cH(A)$ for a common $c > 0$. Indeed the norms of $\bigwedge^r g$ on each local exterior space are bounded and are at most one at almost every finite place uniformly: a compact subset of a restricted product is contained in a product with integral factors outside a fixed finite set, by a finite subcover. Apply g to the exterior coordinates of $g^{-1}Y$, using the squared Euclidean norm at each complex place. Theorem 482.3 now gives a summable majorant. For the last assertion take the doubled tensor test, with conjugation on the negative Hom summand, and act by the block diagonal pair. Tensoring the translation formulas gives its product theta section; this also follows by absolute convergence of the two rational sums in Theorem 385.3. Lemma 479.2 identifies each graph term and Theorem 479.3 annihilates the others. □

484. Removing one hyperbolic plane at a boundary

Definition 484.1. Write the hyperbolic basis of Definition 483.1 as $e_1, \dots, e_r, e'_1, \dots, e'_r$, with $A_0 = \sum_i Ee_i$. In the rational polarization with configurations $X \in T^r$ given by the r columns $X(e_i)$, write $s = |\Sigma|$. For $x = (x_1, \dots, x_r) \in \mathbb{Z}^r_{\geq 0}$, let d_x be identity at finite places and $\mathrm{diag}(2^{x_i})$ at each infinite place. The source element t_x has blocks d_x, d_x^{-1}. On tests set $D_x u(X) = u(Xd_x)$. For $1 \leq j \leq r$, let D' be the span of the other hyperbolic pairs, x' the tuple omitting x_j, and $u^{(j)}$ the restriction $X(e_j) = 0$ at all places. For $r = 1$ the smaller test is a scalar and its only zero term and theta value are that scalar.

Theorem 484.2. *If $h > 4r$, then for every x, A as in Definitions 483.1 and 484.1,*

$$|Z_A(D_x u)| \le C_u H(A)^{-h/2}.$$

For $e_j \notin A$ there is also a bound $|Z_A(D_x u)| \le C_{u,A} 2^{-shx_j}$. For $e_j \in A$ there is a unique maximal isotropic plane $B \subset D'$ with $A = Ee_j \oplus B$, and in that case

$$Z_A(D_x u) = Z_B(D_{x'} u^{(j)}).$$

Consequently the difference between the two zero-term series for $D_x u$ and $D_{x'} u^{(j)}$ tends to zero as $x_j \to \infty$, uniformly in the other x_i. The bounds and the limit are uniform over the compact test families in Theorem 483.3.

Proof. The action of t_x on configurations is $X \mapsto X d_x^{-1}$ and has module $2^{-2sh\sum_i x_i}$. Hence D_x is $2^{-sh\sum_i x_i}\Omega(t_x)$ up to a unit scalar, by Theorem 386.1; there is no quadratic term in this block diagonal change. Fix a rational frame Y of A. At each infinite place all coordinates of $\bigwedge^r t_x^{-1}$ in the exterior basis are scaled by positive real numbers at least $2^{-\sum_i x_i}$: in a chosen minor each negative exponent can appear only once. The squared norm of an exterior vector is the sum of the squared absolute values of its maximal minors (expand the determinant of the Gram matrix by multilinearity). At finite places $t_x = 1$. Thus $H_{t_x}(A) \ge 2^{-2s\sum_i x_i} H(A)$. The normalization cancels this factor to power $-h/2$ in Theorem 483.3.

If $e_j \notin A$, deletion of the j-th coordinate is injective on A, so one nonzero rational minor lacks that row. Its scale is at least $2^{-\sum_i x_i + x_j}$ at each place. Its absolute value is positive at every embedding. Using just this minor in the infinite factors and retaining the fixed finite factors gives the second bound. If $e_j \in A$, orthogonality to e_j forces the e_j' coordinate of every member of A to vanish. Subtract its e_j component to see that $B = A \cap D'$ has dimension $r - 1$; this gives both uniqueness and the bijection with the smaller isotropic planes.

For such a direct sum, $\mathcal{N}_A$ is the direct sum of the modulation space on the j-th hyperbolic pair and $\mathcal{N}_B$ on the other pairs. The trace pairing and its translation phases have no cross terms between these two Hom spaces. Integrate the theta section first over that modulation space. Character orthogonality kills exactly the frequencies with nonzero j-th column, leaving the smaller theta section of the restriction $X(e_j) = 0$, as in Theorem 385.3. Uniform absolute convergence on compact representatives in that theorem permits the integration for a test coupling different columns. The product of the two additive quotient probabilities is the quotient probability on their direct sum. Its remaining integral is $Z_B(D_{x'} u^{(j)})$, proving the exact identity with the explicit normalization of D_x. The difference of the two series is therefore the sum over planes missing e_j. Theorem 482.3 and the first bound control a tail of this sum uniformly. The second bound on the finitely many remaining terms tends to zero uniformly in the other exponents and over the indicated tests. □

Proposition 484.3. *For $b \in U(T)(\mathbb{A}_{E^+})$ and a test u in the polarization of Definition 484.1 put*

$$\Theta_r(u; b) = \sum_{X \in T(E)^r} u(b^{-1}X).$$

For b in a fixed compact subset and u in a compact real Schwartz family with fixed finite parts as in Theorem 483.3, for each $m > 0$ there is C_m such that

$$|\Theta_r(D_x u; b) - \Theta_{r-1}(D_{x'} u^{(j)}; b)| \le C_m 2^{-mx_j}$$

for every x. In particular this convergence holds pointwise throughout the target quotient.

Proof. By clearing denominators on the compact set of finite parts of b times the finite supports, all contributing columns belong to one fractional integer lattice L in T_∞. Within the compact set of infinite b, there is $c > 0$ with $\|b_\infty^{-1} v\| \ge c\|v\|$. Since all the diagonal factors at infinity are at least one, $\|b_\infty^{-1} X d_x\| \ge c\|X\|$; if $X(e_j) \ne 0$, the same length is at

least $c\,2^{x_j}\lambda_1(L)$. Uniform Schwartz decay of order $m+[E:\mathbb{Q}]rh+2$ bounds the sum on the nonzero j-th columns by a constant times $2^{-mx_j}\sum_{X\in L^r}(1+\|X\|)^{-[E:\mathbb{Q}]rh-2}$. The last sum converges by lattice ball counting in the proof of Theorem 482.3. On the zero column the summands are precisely those of the smaller theta value. Reindexing rational target translates leaves each sum invariant, so the pointwise limit descends to the quotient. □

485. Reduction along a shortest isotropic line

Definition 485.1. Let $F=E^+$ for the arithmetic CM field E, and let V be a hermitian E-space of rank $n>0$ admitting a lattice self-dual for its hermitian form at the finite places. Split off a rational hyperbolic pair whenever a nonzero rational isotropic vector exists. For such a vector e, choose f with $\langle e,f\rangle=1$ and subtract $e\langle f,f\rangle/2$ from f. This gives an isotropic partner. Continue on the nonsingular perpendicular. Denote the resulting pairs by e_i,f_i, $1\le i\le w$, in the order removed, and the final space by V_{an}. Thus $2w\le n$; V_{an} has no nonzero rational isotropic vector.

Choose a self-dual finite lattice in V_{an} and adjoin the standard integral lines on all pairs to obtain Λ on V. At infinity choose the Euclidean metric orthogonal on these summands, of unit length on each displayed basis vector, and coming from signed orthonormal coordinates on V_{an}. The same choices restrict to every stage. Let K be the product of the unitary stabilizers of the finite lattice and the infinite Euclidean metrics. For $g\in U(V)(\mathbb{A}_F)$ set

$$\lambda_V(g)=\min\{\max_{\tau\in\Sigma}\|g_\tau^{-1}v_\tau\| : 0\neq v\in V(E),\ g_f^{-1}v\in\Lambda\}.$$

At a positive-rank stage write n for its current rank. For $t>0$, $d(t)$ is the idele of E equal to t at every infinite place and to one at finite places, and $b=[E:\mathbb{Q}]$.

Lemma 485.2. *The lattices and minima in Definition 485.1 exist. For each fixed $\epsilon>0$ the classes with $\lambda_V(g)\ge\epsilon$ have representatives in a compact subset of the unitary group. A compact set of norm-one ideles meets every rational multiple class in the norm-one ideles of E. For any open neighborhood U of zero in $\mathbb{A}_E$ there is an integer $M\ge1$ such that, for every $\alpha\in\mathbb{A}_E$, there exist $1\le m\le M$, $c\in E$ with $m\alpha+c\in U$.*

Proof. At a finite place of F, in the original self-dual lattice choose a primitive generator e_* of its intersection with the local isotropic line of e_1. In a split pair choose a primitive generator on each branch. Such generators exist by valuation-ring elimination as in Proposition 482.2. Self-duality supplies a lattice vector y paired to one with e_*: in a lattice basis primitiveness and invertibility of the Gram matrix make its pairing row unimodular. There is an integral c_0 with $c_0+\bar{c}_0=\langle y,y\rangle$, since the arithmetic field has an integral element of relative trace one by Lemma 20.4. Subtract e_*c_0. The two vectors now span an integral hyperbolic plane with unimodular Gram matrix. Orthogonal projection splits it off integrally. Its self-dual perpendicular realizes a lattice in $e_1^{\perp}/Ee_1$, identified rationally with the prescribed next perpendicular. These operations can use the original rational pair at almost every place, since finitely many basis coordinates and pairings suffice to test integrality. Iteration supplies the announced adelic lattice on V_{an}.

For the minimum, the moved real lattice is full and has fixed covolume by the determinant argument of Theorem 397.1: a unitary determinant has absolute norm one at every real place and on each group of finite places over F. Lemma 20.2 computes the finite contribution and the real change of variables computes the infinite one. A lattice minimum is attained, and is invariant under rational left translation. For completeness, on the part with minimum at least ϵ, Lemma 107.1 gives an integral basis of the underlying real lattice of bounded lengths, since its covolume is fixed. Choose an E-basis from the corresponding rational vectors; they span over E since they span over $\mathbb{Q}$. The hermitian Gram entries are integral and bounded at all embeddings by the Euclidean bounds. Hence only finitely many Gram matrices occur by Lemma 82.2. Match to one fixed rational basis for each occurring matrix by a rational isometry. The inverse of the resulting representative takes that basis into a compact adelic

box. Thus its inverse matrix lies in a compact box; the unitary inverse formula bounds the forward matrix in another compact box. The defining matrix equations are closed in their product. This proves the compact representative assertion at every stage, including V_{an}.

For norm-one ideles use the lattice $\{va_\infty : v \in E,\ va_f \in \widehat{O}_E\}$. Its covolume is $V_E|a|_{\mathbb{A}_E} = V_E$, by the same local index calculation as in Lemma 61.2, with squared complex moduli at the infinite places. An averaging of a fixed sufficiently large ball over a lattice parallelepiped gives two points congruent modulo the lattice, hence a nonzero lattice point of bounded length. Take the corresponding v. The product formula of Proposition 482.2 bounds the integral norm of the finite ideal of va, and bounds each infinite modulus away from zero as all are bounded above. Finitely many ideals have bounded norm, by finite factorization in Proposition 82.3 and finiteness of the residue rings over any bounded rational prime. At finite places the permissible ideles therefore lie in finitely many translates of the compact integral units. This gives the asserted compact set. Finally $\mathbb{A}_E/E$ is compact by the additive lattice argument of Lemma 356.2. Choose an open set U' about zero with $U' - U' \subset U$. Finitely many translates of its image cover the quotient. Among one more multiple $0, \alpha, \ldots, M\alpha$ than there are translates two are in the same translate; their difference gives m, c. □

Theorem 485.3. *At each stage in Definition 485.1, if $\lambda_V(g) < 1/2$, the stage has a rational hyperbolic pair. Write it as $V = Ee \oplus W \oplus Ef$ using its fixed first pair. Write N for the unipotent isometries upper block triangular with identity on all three diagonal blocks, and put $m(a, z) = \mathrm{diag}(a, z, \bar{a}^{-1})$. There exist rational $\gamma \in U(V)(F)$, $k \in K$, $n_0 \in N(\mathbb{A}_F)$, and*

$$\gamma g = n_0 m(a, z) k, \qquad a = d(t)u, \qquad z \in U(W)(\mathbb{A}_F),$$

where u lies in a fixed compact set of norm-one ideles, $t \geq \lambda_V(g)^{-1}$, and $t \leq C\lambda_V(g)^{-1}$. If $W \neq 0$, then $\lambda_W(z) \geq c\lambda_V(g)$. The positive constants depend only on the stage. The unipotent factor can be taken in a fixed compact set after rational unipotent left translation.

Proof. Take a rational vector v attaining the minimum. Its norm pairing is an algebraic integer, since $g_f^{-1}v$ is integral. At every embedding its absolute value is at most $\lambda_V(g)^2 < 1$: the matrix of the form for the chosen metric is a self-adjoint involution. The norm of a nonzero algebraic integer has absolute value at least one by Lemma 82.2. Thus v is isotropic.

A rational isometry can carry v to the specified e. On the same line, scale it and its dual inversely with conjugation. For distinct nonorthogonal lines the two vectors span a hyperbolic plane: interchange them there with a scalar on the second image chosen to preserve their nonzero mutual pairing, and take identity on the perpendicular. For distinct orthogonal lines solve the two independent linear equations prescribing pairing one against both vectors. Subtract a multiple of v from the solution to make it isotropic, using half its norm as in Definition 485.1. The two pairings stay one. Two applications of the nonorthogonal case finish. After this rational move, locally K acts transitively on isotropic lines. At finite places apply Theorem 345.2 to primitive generators in the self-dual lattice and e; at infinity the positive and negative projections of a unit isotropic vector both have length $1/\sqrt{2}$ and their two unitary groups are transitive on the corresponding spheres. Thus choose k so that the remaining left factor preserves Ee. Its action on $e^\perp/Ee = W$ is unitary, and pairing with e forces the last diagonal entry to be $\bar{a}^{-1}$. Removing these diagonal blocks gives n_0. In particular the inverse image of e has finite part $k_f^{-1}a_f^{-1}e$ integral at this point and infinite component of length $|a_\tau|^{-1} \leq \lambda_V(g)$ at each τ. Hence $t = |a|_{\mathbb{A}_E}^{1/b} \geq 1/\lambda_V(g)$. Left multiply by a rational $m(c, 1)$ to put $u = a/d(t)$ in the compact set of Lemma 485.2. This leaves t unchanged by the product formula. For these new factors some fixed nonzero integer B has Ba_f^{-1} integral for all the allowed u. The rational vector Be in the new coordinates is then admissible with real lengths at most C/t. Rational invariance gives the reverse bound.

Suppose $W \neq 0$, and choose v' realizing $\lambda_W(z)$. There is an adele α with $n_0^{-1}v' = v' + \alpha e$. By compactness of u, $\bigcap_u u_f \widehat{O}_E$ contains $B'\widehat{O}_E$ for some positive integer B', and $|a_\tau|^{-1} \leq C_1/t$. Apply the last assertion of Lemma 485.2 to the neighborhood with finite part $B'\widehat{O}_E$ and

infinite moduli less than $(2C_1)^{-1}$. The rational vector $mv'+ce$ is nonzero and after the inverse group action is integral at finite places; its infinite lengths are bounded by $M\lambda_W(z)+1/(2t)$, by orthogonality or the triangle inequality. Since $1/t \le \lambda_V(g)$, the desired bound follows with $c = 1/(2M)$.

Finally the unipotent coordinates consist of a row on W and a central entry with specified relative trace, as in Lemma 20.4. Use rational rows to reduce the first coordinate modulo a compact additive set of representatives. A rational row has a rational lift, by taking minus half its hermitian norm for the required trace. Then reduce the remaining skew central coordinate by the rational skew line. Both additive quotients have compact representatives by Lemma 356.2. The lifts of the compact row set using the same quadratic formula are compact. This gives the claimed compact unipotent representatives. □

486. Ordered scales on the unitary quotient

Theorem 486.1. *Use the tower of Definition 485.1 and write W_j for the perpendicular after j pairs, $0 \le j \le w$. Write N_j for the upper triangular unipotent block group for the ordered blocks $e_1, \dots, e_j, W_j, f_j, \dots, f_1$, with diagonal blocks the identity, and write $m_j(a_1, \dots, a_j, z)$ for diagonal blocks $a_1, \dots, a_j, z, \bar{a}_j^{-1}, \dots, \bar{a}_1^{-1}$. There are compact sets in $N_j(\mathbb{A}_F)$, in the norm-one ideles of E, and in $U(W_j)(\mathbb{A}_F)$, and a fixed constant $C \ge 1$, such that every $g \in U(V)(\mathbb{A}_F)$ has a representative modulo $U(V)(F)$ of the form*

$$n_0 m_j(d(t_1)u_1, \dots, d(t_j)u_j, z)k, \qquad k \in K,$$

with n_0, u_i, z in the respective compact sets, $1 \le t_i$ and $t_{i+1} \le Ct_i$. For $j = 0$ there are no scales. When V is the split source of Definition 484.1, one compact set $\mathcal{C}$ suffices in the covering

$$U(V)(\mathbb{A}_F) = \bigcup_{\gamma \in U(V)(F),\ x \in \mathbb{Z}^r_{\ge 0}} \gamma t_x \mathcal{C}.$$

Proof. If the minimum is at least $1/2$, use the compact representatives of Lemma 485.2. Otherwise Theorem 485.3 gives t_1 and the smaller adelic group element. Apply the same procedure on successive perpendiculars until a thick stage or rank zero. It must stop at the anisotropic stage at the latest, since another short vector there would be rational isotropic by that theorem. If consecutive stages are both reduced, their minima satisfy $\lambda_{\text{next}} \ge c\lambda_{\text{prev}}$, and their scales lie between the inverse minima and a constant times these inverses. Hence $t_{i+1} \le Ct_i$ with one C, by finiteness of the tower.

Embed each rational move on a perpendicular by identity on the earlier pairs. It conjugates the earlier unipotent layers into themselves. Once the later coordinates have been chosen, reduce the earlier conjugated unipotents back into their compact sets by rational translations, from the innermost layer out. These last left translations affect only the outer layer being reduced. The scale on each earlier pair commutes with all elements of its perpendicular. The compact factors on the successive right sides combine in K by the compatible metrics and finite lattices of the tower. Thus the unipotents in their fixed compact sets give one compact product in N_j; the remaining perpendicular coordinates meet their compact set by the thick assertion. This proves the first covering.

Move $m_j(d(t_i), 1)$ to the left of n_0. In its conjugation of n_0 the entry in row block a and column block b is multiplied at infinity by the scale of b divided by that of a, for $a < b$. Along the displayed ordered block list every such ratio is bounded in terms of C, w, using $1 \le t_i,\ t_{i+1} \le Ct_i$. The inverse is bounded by the same argument on the inverse unipotents; at finite places the scale is identity. Thus the new right factor belongs to a fixed compact set. Replace t_i by $2^{\lfloor \log_2 t_i \rfloor}$, absorbing ratios in $[1, 2]$ into that compact set. For a split source set the other exponents equal to zero and take the union of finitely many compact sets. This proves the second covering. □

Lemma 486.2. *Let $G = U(V)(\mathbb{A}_F)$, $\Gamma = U(V)(F)$; use compatible Haar measures on the groups and on the left quotient. In the coordinates of Theorem 486.1, for every nonnegative measurable left Γ-invariant function Φ, its quotient integral is bounded by a constant times the sum, $0 \le j \le w$, of*

$$\int \Phi(n_0 m_j(d(t_i)u_i, z)k) \prod_{i=1}^{j} t_i^{-b(n-2i+1)} \frac{\mathrm{d}t_i}{t_i} \, \mathrm{d}n_0 \, \mathrm{d}u_1 \cdots \mathrm{d}u_j \, \mathrm{d}z \, \mathrm{d}k.$$

Here n_0, u_i, z, k range over fixed compact sets (enlarging those in the theorem is allowed), $\mathrm{d}u_i$ is Haar measure on the norm-one idele group, and the scale domain is $t_i \ge 1$, $t_{i+1} \le C t_i$.

Proof. Write $P_j = N_j M_j$ for the parabolic preserving the successive coordinate isotropic subspaces. Its Levi group M_j consists of the displayed m_j, by preservation of the hermitian form. The local transitivity argument in Theorem 485.3, repeated on each perpendicular, gives $G = P_j K$, also when applied without any condition on the minimum. The group $P_j \cap K$ is compact. For nonnegative functions on G, Haar integration equals a fixed positive constant times integration on pk for left Haar measure on P_j and probability Haar on K. Indeed average a compactly supported continuous function on the right by K. Haar measure on G is right invariant under K, since the modular character has compact image there in the positive reals. The resulting integral on G/K is left P_j-invariant. Under $G/K = P_j/(P_j \cap K)$ there is just one such nonzero measure up to scale: lifting it by averaging over right translates of the compact stabilizer gives left Haar measure on P_j. The homogeneous-space identification respects the topology, since inverse images of compact sets in G/K are compact in G (cover by projections of finitely many relatively compact open translates), and intersecting with the closed P_j preserves compactness. The formula now follows for Borel integrals by equality of measures.

For one layer of current rank $n' = n - 2(i-1)$, the row variable of Lemma 20.4 has $n' - 2$ entries over E; the central skew coordinate is one-dimensional over F. Product additive measure is Haar by the triangular translation law in Proposition 20.5. Conjugating by $m(a_i, z')$ multiplies the row by a_i and the inverse unitary transformation of the perpendicular, and multiplies the center by $a_i \bar{a}_i$. The full module on the row is $|a_i|_{\mathbb{A}_E}^{n'-2}$, since the perpendicular determinant has norm one on every group of conjugate places. On the center the module is $|a_i|_{\mathbb{A}_E}$ by the local norm identity, including the two factors at a split place. Thus the module on N_j is $\delta(m) = \prod_i |a_i|_{\mathbb{A}_E}^{n-2i+1}$, by multiplying the successive modules. Left Haar on P_j is $\delta(m)^{-1} \, \mathrm{d}n_0 \, \mathrm{d}m$: substitution by left translation cancels exactly the conjugation module. The idele norm splits by $a_i = d(t_i)u_i$ and $\mathrm{d}^\times a_i$ is a constant times $\mathrm{d}t_i/t_i \, \mathrm{d}u_i$, by the direct product law. This gives the displayed powers.

Let $\mathcal{S}_j$ be the image of the compact factors and scale domain in the first covering. It is a countable union of compact sets by bounding the scales. Rational translates of these images cover G; unfolding their indicators in a fundamental domain as in Lemma 21.4 bounds the quotient integral by $\sum_j \int_{\mathcal{S}_j} \Phi$. For every decomposition pk in one such image the possible p's differ by right multiplication by $P_j \cap K$. At infinity that compact subgroup is block diagonal: it preserves each member of the flag and its Euclidean perpendicular, and the hermitian involution interchanges the two paired flags. Its scale moduli on the lines are one. At finite places it has integral unit diagonals on the lines, and its unipotent and perpendicular projections are compact, since block projection and diagonal removal are continuous. Conjugating these finite unipotents by the allowed finite Levi coordinates stays in a compact set, since the finite $d(t_i)$ are identity. Thus all such p's satisfy the same scale inequalities and enlarged compact restrictions. Apply the Haar formula just proved to the images. This gives the stated bound. □

487. An integrable bound for rational tuples

Theorem 487.1. *Let V, n, w, b, G, Γ be as in Definition 485.1, and fix a compact subset $\mathcal{B}_f \subset V(\mathbb{A}_{E,f})^d$. For $d \geq 0$, $R \geq 1$ let $N_R(g)$ count d-tuples $X \in V(E)^d$ satisfying $g_f^{-1}X \in \mathcal{B}_f$ and $\|(g_\infty^{-1}X)_\infty\| \leq R$ in the sum Euclidean metric. If $d < n - w$, the quotient has finite positive Haar volume and*

$$\int_{\Gamma\backslash G} N_R(g)\ \mathrm{d}g \leq C(1+R)^{bnd}.$$

The constant depends on the field, space, compact set, measures and d. In particular the condition holds when $2d < n$.

Proof. The count is measurable by the countable sum of the indicators of the stated Borel conditions. Use the coordinates in Lemma 486.2. Absorb the right compact factor into the allowed compact finite set and real ball, since its action and inverse are uniformly bounded there. In $g^{-1}X$ one now first applies an upper block unipotent n_0^{-1} to X, then the inverse block diagonal element. Count the rows of X from the last block to the first. In a fixed row block the unipotent adds a shift determined by the later, already chosen rows. After multiplying the box by the diagonal block, the possible rational rows lie in a translated finite compact box and in a translated real ball. If any rational row is present, subtract one such row. The differences have finite part in one fixed fractional integral lattice: the finite parts of u_i, z lie in compact sets and $d(t_i)_f = 1$. The real differences have lengths at most CRt_i on the e_i block, CR/t_i on f_i, and CR on W_j. The use of differences makes these estimates independent of all the shifts. They also hold if a block has zero possible rows.

For a block of dimension k over E, rational differences with these finite constraints form a subset of a fixed real lattice of dimension bkd. A ball of radius L contains at most $C'(1+L)^{bkd}$ such points by packing as in Theorem 482.3. Multiplication of the successive bounds, using $t_i \geq 1$, gives

$$N_R(g) \leq C''(1+R)^{bnd} \prod_{i=1}^{j} t_i^{bd}.$$

All compact integrations have finite measure. It remains to integrate $\prod_i t_i^{-b(n-2i+1-d)}\,\mathrm{d}t_i/t_i$ in the scale domain of Lemma 486.2. Replace t_i by $s_i = C^{j-i}t_i$, paying a fixed constant, and enlarge to $s_1 \geq \cdots \geq s_j \geq 1$. With $v_i = \log s_i - \log s_{i+1}$ and $s_{j+1} = 1$, the Jacobian in the logarithms is one. The coefficient of $-bv_k$ in the exponential is

$$\sum_{i=1}^{k}(n-2i+1-d) = k(n-k-d) > 0 \qquad (1 \leq k \leq j \leq w).$$

The integrals of the resulting decaying exponentials on $[0,\infty)$ are finite. This proves the count. Using the same estimate at $d = 0$ on the singleton box gives finite quotient volume for every $n > 0$; positivity follows from an identity neighborhood disjoint from its other rational left translates as in Lemma 21.4. □

Corollary 487.2. *Let $V = T$ have rank $h > 2r$, let it have a self-dual finite lattice as above, and consider $0 \leq d \leq r$ hyperbolic pairs in the source. There is a single integrable function of $b' \in [U(T)]$ bounding*

$$\sum_{X \in T(E)^d} |(D_x u)(b'^{-1}X)|$$

for all $x \in \mathbb{Z}^d_{\geq 0}$ and all u in a compact family of real Schwartz tests with fixed finite parts as in Theorem 483.3. Define $I_d(u)$ to be the integral of $\Theta_d(u; b')$ divided by the quotient volume. For $1 \leq j \leq d$, as $x_j \to \infty$,

$$I_d(D_x u) - I_{d-1}(D_{x'}u^{(j)}) \longrightarrow 0$$

uniformly over that family and the other coordinates x'.

Proof. For a common compact finite support take the union of the finite supports of the family and sum the bounds for its elementary factors. The dilation expands all real columns and fixes the finite part. For $M > bhd + 2$, uniform Schwartz decay bounds the summands by $C(1 + \|b_\infty'^{-1}X\|)^{-M}$ times the finite indicator. Separating the unit ball and the shells $2^{k-1} < \|b_\infty'^{-1}X\| \le 2^k$, $k \ge 1$, gives the quotient majorant

$$CN_1(b') + 2^M C \sum_{k\ge1} 2^{-kM} N_{2^k}(b').$$

Its integral is finite by Theorem 487.1 and nonnegative summation. The assertion for $d = 0$ is just absolute value of the scalar test on a finite-volume quotient. The smaller restricted tests again form a compact family: restriction at zero is continuous in each real Schwartz seminorm and preserves local constancy and compact support on the finite arguments. Proposition 484.3 gives pointwise convergence along every sequence with $x_j \to \infty$, even if the other coordinates and the test vary along the sequence. The two majorants are integrable and independent of that sequence, so dominated convergence gives the integral limit. If uniform convergence failed, selecting a sequence on which the difference stayed bounded below would contradict this conclusion. □

488. The target action and a split matrix model

Proposition 488.1. *In the split source of Definition 483.1 write d for the number of hyperbolic pairs. On the configuration space $T(\mathbb{A}_E)^d$ let $V(b)u(X) = u(b^{-1}X)$ for $b \in U(T)(\mathbb{A}_F)$. For rational source isometries γ let $\mathcal{T}_\gamma$ be the operator normalized on rational theta sections by Theorem 387.2. These actions commute. If $h > 4d > 0$, put*

$$S_d(u) = \sum_A Z_A(u), \qquad \mathcal{L}_d(u) = I_d(u) - S_d(u),$$

where I_d is the normalized integral of Corollary 487.2. Set $I_0(u) = S_0(u) = u$ on scalar tests. The functionals I_d, S_d are invariant under both $V(b)$ and $\mathcal{T}_\gamma$.

Proof. The target determinant has module one on its full adelic vector space, since its product with its conjugate is one. Thus $V(b)$ is unitary. It implements the target action on translations and modulations by change of variable and preservation of the trace pairing in Definition 355.1.

We give generators over the rational field. Let $A = \gamma A_0$ and transform $A \cap A_0$ by a change of basis of A_0, extended by the conjugate inverse transpose on the opposite summand, into the first $k = \dim(A \cap A_0)$ coordinate lines. Pass to their orthogonal space modulo these lines. There A projects isomorphically to the primed summand of the remaining pairs, by disjointness from the unprimed summand and dimension. It is the graph of a skew-hermitian matrix Z, since its columns have Gram matrix $Z + Z^* = 0$. The upper unipotent shear by $-Z$ removes the graph. In the unreduced space the first k coordinates of the lifted columns can be removed using the k lines already in A. Hence elements preserving A_0 and interchanges $e_i \leftrightarrow e_i'$ generate the rational source group: the operations just described and the indicated interchanges return A to A_0.

For the stabilizer of A_0 the operator in Theorem 386.1 consists of a column change and a quadratic phase. The phase is obtained by pairing the columns with each other in T, since its bilinear expression pairs X with the source shear applied to X in the trace Hom form. Both operations commute with $V(b)$, by target isometry. For the interchange of one pair the operator is a Fourier transform on that column for the nondegenerate adelic trace pairing on T, with the sign and the identification with its dual fixed by the symplectic form. Indeed this transform interchanges translations and the prescribed modulations by direct change of variable; the normalization on rational theta sums follows from Lemma 387.1. A unit scalar in this description has no effect on commutation. The target preserves both

the pairing and additive measure, so it commutes with the partial Fourier transform. This proves commutation for all rational source isometries.

Evaluation at the origin of the theta section is the rational configuration sum. The rational action preserves it by Theorem 387.2; applying this after $V(b)$ proves rational invariance of the integrand. Right invariance of quotient Haar measure gives target invariance of its integral, using the absolute integrability in Corollary 487.2. Here the unitary group is unimodular: conjugation on its Lie algebra has absolute determinant one at real and finite places, since after splitting the quadratic algebra it is conjugation on all square matrices with determinant $(\det b)^h(\det b^{-1})^h = 1$. Equivalently use the matrix Haar calculation of Proposition 110.2. On $[\mathcal{N}_A]$ rational changes carry quotient probabilities to quotient probabilities, permuting the zero terms. The zero term of A_0 is evaluation of the test at zero and hence is target invariant. Lemma 483.2 and the commutation already proved give the same invariance for every A. Sum absolutely by Theorem 483.3. □

Lemma 488.2. *For each fixed adelic test u and $d \geq 1$ there is a split finite place of F, with completion L, integer ring O_L, uniformizer π and residue cardinality Q, where the local test in the following realization is $\phi_0 = \mathbf{1}_{M_{h,2d}(O_L)}$. The source and target groups there are respectively $\mathrm{GL}_{2d}(L)$ and $\mathrm{GL}_h(L)$. On matrix tests the local action $V(b)$ of Proposition 488.1 and a unitary choice B_c of source oscillator operators are*

$$(V(b)\phi)(Z) = |\det b|^{-d}\phi(b^{-1}Z), \qquad (B_c\phi)(Z) = |\det c|^{h/2}\phi(Zc).$$

Here the target action has the same normalization as the global one. Every other unitary source implementer differs from B_c by a unit scalar.

Proof. There are infinitely many split places available. The field contains i and $E = F(i)$, since F is real and the extension has degree two. For any finite list of rational primes, take a positive even multiple m of them. Every prime p dividing $m^2 + 1 > 1$ is odd and outside the list. The class of m is a square root of -1 modulo p with nonzero derivative $2m$; lifting one digit at a time, solving its linear congruence at each power of p, gives a root in $\mathbb{Q}_p$. Thus $X^2 + 1$ splits into distinct linear factors in every F_v above p. We can omit finitely many places, including denominators of the rational frames, trace character and their duals. At all remaining places a finite test factors with one common integral indicator: its compact support and local constancy make it a finite sum of compact open boxes, each unrestricted integral at almost all places, by a finite subcover and common refinement.

In the two field branches of a split place use the target form to identify the second branch with the dual of the first. The two configuration matrices are M, N, each with h rows and d columns, and $V(b)$ acts on functions by substitution $(b^{-1}M, b^{\mathsf{T}}N)$. Take the Fourier transform in N only, with dual variable Y, using self-dual additive measure and the pairing on the columns. Substitution $N' = b^{\mathsf{T}}N$ contributes $|\det b|^{-d}$ and substitutes $b^{-1}Y$ in the transform. The resulting Z consists of M, Y; in the dual coordinates fixed by the alternating Hom pairing these are the two blocks of columns of a map on the first source branch. This follows also by Heisenberg translation: a displacement on that branch translates its unprimed columns and, after the transform, its primed columns, while the dual branch modulates them. The source action on these displacements is $Z \mapsto Zc^{-1}$ by the definition of the Hom action. Change of configuration variables in the induced oscillator therefore gives the stated B_c, whose squared determinant factor makes it unitary. It implements the corresponding dual action on modulations by the same substitution. Uniqueness up to unit scalars is the finite translation-basis argument of Theorems 360.1 and 358.1. At the chosen good place integral coordinates are integral duals, the Fourier transform of their integral indicator is the integral dual indicator by additive character orthogonality (normalize the integral box to mass one). Thus the local matrix test is ϕ_0 as claimed, with any fixed constant absorbed into the factors at the other places. □

489. One split Hecke operator on both sides

Definition 489.1. Use the split place of Lemma 488.2 and write $N = 2d$. For every residue hyperplane D of row vectors of length N, choose $c_D \in M_N(O_L)$ with row lattice the inverse image of D and determinant π. Such a matrix is obtained by a residue basis lift and scaling a complement by π, as in Lemma 60.2. Let s_D be the adelic source element given by c_D^{-1} on the first split branch, the paired isometry on its conjugate, and identity elsewhere. Put $[k]_Q = (Q^k - 1)/(Q - 1)$, and for $h > 4d$ put

$$P_d(g) = \sum_A H_g(A)^{-h/2}, \qquad \Lambda = [d]_Q(Q^{h/2} + Q^{d-h/2}),$$

where A runs over rational maximal isotropic planes.

Lemma 489.2. *The function P_d is positive, rationally left invariant, and bounded on compact sets of the source adelic group. One has*

$$\sum_D P_d(g s_D) = \Lambda P_d(g), \qquad \Lambda > [2d]_Q.$$

Proof. There is at least A_0 in the sum. The height transforms by $H_{\gamma g}(A) = H_g(\gamma^{-1}A)$ for rational γ, by the product formula in Proposition 482.2. The local norm bound on compact sets proved in Theorem 483.3 and Theorem 482.3 give compact boundedness and convergence. Fix A, g. At the chosen split place let $B \subset L^{2d}$ be the first branch of $g^{-1}A$ and let $\bar{B}_0$ be the reduction of its saturated lattice $B \cap O_L^{2d}$. It has residue dimension d by the diagonal reduction in Proposition 482.2. The reduction of c_D has kernel the line annihilated by the row hyperplane D; these are all lines, each once. The finite exterior norm on B is multiplied by $\alpha = Q^{-1}$ if that line is contained in $\bar{B}_0$, and by $\alpha = 1$ otherwise. Indeed on a primitive basis the reduction loses exactly one or zero in rank, respectively. The elementary divisors of the image in its saturation all divide π, since πc_D^{-1} is integral. Their product gives precisely the norm multiplier.

The other branch is the annihilator of B under the perfect split pairing, and its multiplier is $\alpha/|\det c_D|$. To verify this, complete a primitive basis of B by columns to an integral invertible matrix. The final d columns of its inverse transpose span the annihilator in dual coordinates. An exterior coordinate of these columns is, up to sign, the complementary minor of the first d columns divided by the determinant. This follows by expanding the top exterior pairing between the two complementary lists, or by applying the determinant to the matrix with complementary standard columns and multiplying by the inverse. After left multiplication of the original matrix by c_D, the dual matrix is left multiplied by $c_D^{-\mathsf{T}}$. Taking maxima of the minors gives the asserted multiplier. The identification by J uses an integral invertible matrix. Ratios of norms are unchanged if the starting local bases of either branch are multiplied on the right by any invertible d-by-d matrix.

The total height multiplier is thus $Q\alpha^2$. There are $[d]_Q$ lines in $\bar{B}_0$, and $[2d]_Q - [d]_Q = Q^d[d]_Q$ outside it. Summing $(Q\alpha^2)^{-h/2}$ over them gives exactly Λ. All terms are positive, so summing over A proves the identity. Finally the difference of Λ and $[2d]_Q$ is $[d]_Q(Q^{h/2} - 1)(1 - Q^{d-h/2}) > 0$. □

Proposition 489.3. *Let ℓ be any complex linear functional on the adelic tests invariant under the target action $V(b)$ of Proposition 488.1. For fixed u, use its good split place of Lemma 488.2, and put $F_u(g) = |\ell(\Omega(g)u)|$. Then*

$$\Lambda F_u(g) \le \sum_D F_u(g s_D).$$

Proof. At the split place write ϕ_0 for its matrix indicator. For each target residue hyperplane $H \subset (O_L/\pi)^h$ of column vectors choose b_H with column lattice its preimage and determinant π. At an integral h-by-N matrix of residue rank j, exactly $[h - j]_Q$ such lattices contain every column, and exactly $[N - j]_Q$ of the source row lattices contain every row. This

follows by counting nonzero linear forms on the quotient by the corresponding span, modulo scalars. Since $[h-j]_Q = [h-N]_Q + Q^{h-N}[N-j]_Q$, the two actions in Lemma 488.2 give the identity of functions

$$Q^{-d}\sum_{H} V(b_H)\phi_0 = [h-N]_Q\phi_0 + Q^{h-N-h/2}\sum_{D} B_{c_D^{-1}}\phi_0.$$

All indicators on both sides vanish off the integral matrices. Tensor with the other factors of the fixed test, apply $\Omega(g)$, then ℓ. On the chosen branch the source action of g is B_{g_v} up to phase, hence commutes with the displayed target action, and its other local factors commute by separate variables. The sum on the left is $Q^{-d}[h]_Q\,\ell(\Omega(g)u)$ by target invariance. Solving for the source sum, the scalar on the original value is

$$Q^{N-h/2}(Q^{-d}[h]_Q - [h-N]_Q) = [d]_Q(Q^{h/2} + Q^{d-h/2})$$

by substitution of $N = 2d$. Each translated value on the right is the value for gs_D up to a unit scalar, by uniqueness of the implementers in Lemma 488.2. The triangle inequality finishes the proof. □

490. The mean of the split-source theta series

Theorem 490.1. *Let E/F be the arithmetic CM extension of Lemma 57.1, let D be rationally hyperbolic of dimension $2d$, and let T be a nonsingular hermitian space of rank $h > 4d$ admitting a self-dual finite hermitian lattice. For every Schwartz test u in the rational split polarization, the normalized theta integral equals the sum of its rational zero terms:*

$$\frac{1}{\operatorname{vol}[U(T)]}\int_{[U(T)]}\sum_{X\in T(E)^d} u(b^{-1}X)\,\mathrm{d}b = \sum_{\substack{A\subset D\\ A=A^{\perp}}} Z_A(u).$$

Here $[U(T)] = U(T)(F)\backslash U(T)(\mathbb{A}_F)$ and the additive mean defining Z_A has total mass one. Both sides are absolutely convergent, including the integral of the sum of the absolute values on the left.

Proof. Corollary 487.2 at $x = 0$ and Theorem 483.3 give the absolute convergence, together with finite positive quotient volume by Theorem 487.1. Proceed by induction in d, with fixed target. At $d = 0$ both sides equal the scalar test. Suppose $d \geq 1$ and write $\mathcal{L}_d = I_d - S_d$ as in Proposition 488.1. The induction hypothesis applies to every restricted test on $d-1$ pairs since $h > 4(d-1)$. For every compact family of tests of the kind in Theorem 483.3,

$$\sup_{u}|\mathcal{L}_d(D_x u)| \longrightarrow 0 \quad \text{as} \ \max_i x_i \to \infty.$$

Indeed for each fixed j, subtract the smaller expressions on $D_{x'}u^{(j)}$ from I_d and S_d, respectively. They cancel by induction; the two differences tend to zero uniformly in the other coordinates and tests, by Corollary 487.2 and Theorem 484.2. There are only d possible indices attaining the maximum.

Fix u, and set $F(g) = |\mathcal{L}_d(\Omega(g)u)|$ on the source adelic group. Absolute values allow any unitary normalization of the local operators, since the implementations of the same Heisenberg change differ by a unit scalar as in Theorem 483.3. By rational invariance in Proposition 488.1, F is rationally left invariant. Choose $\mathcal{C}$ from Theorem 486.1. For $g = \gamma t_x k,\ k \in \mathcal{C}$, one has

$$F(g) = 2^{sh\sum_i x_i}|\mathcal{L}_d(D_x\Omega(k)u)|, \qquad P_d(g) \geq 2^{sh\sum_i x_i}\,H_k(A_0)^{-h/2}, \qquad s = |\Sigma|.$$

The first equality is the dilation formula in the proof of Theorem 484.2. For the second remove γ by rational left invariance and retain A_0 in the positive sum. The columns of $t_x^{-1}Y_0$ are multiplied by d_x^{-1} on the right, so the exterior height of $k^{-1}t_x^{-1}Y_0$ equals $2^{-2s\sum_i x_i}H_k(A_0)$. The latter undilated height is bounded above on $\mathcal{C}$, by bounded exterior norms at the finitely varying places and at infinity. The tests $\Omega(k)u$ range, up to phases,

in a compact real Schwartz family with finitely many finite factors. For the real part use continuity in Schwartz seminorms in Theorem 393.1. At finite places the test line has open stabilizer by the lattice argument in Theorem 483.3; a compact set is covered by finitely many corresponding cosets. For finitely many small x, both functionals on these tests are bounded by their uniform summable and integrable majorants. The boundary limit just proved handles all other x. Consequently for each $\epsilon > 0$ the quantity

$$M_\epsilon = \sup_g \left(F(g) - \epsilon P_d(g)\right)$$

is a finite real number.

At a good split place for u, apply Proposition 489.3 to $\mathcal{L}_d$, which is target invariant by Proposition 488.1. Subtract ϵ times the positive identity of Lemma 489.2. For every g this gives

$$\Lambda\big(F(g) - \epsilon P_d(g)\big) \le \sum_D \big(F(gs_D) - \epsilon P_d(gs_D)\big) \le [2d]_Q M_\epsilon.$$

Taking the supremum and using $\Lambda > [2d]_Q$ gives $M_\epsilon \le 0$. In particular $F(1) \le \epsilon P_d(1)$; $P_d(1)$ is finite, so $\epsilon \downarrow 0$ proves $\mathcal{L}_d(u) = 0$. This closes the induction. □

491. The global norm pairing at the first arrow

Theorem 491.1. *Use the spaces and tests S, T, f, f' of Theorem 481.1, and suppose $h > 4r$, with the given self-dual finite frames. Put $\Theta_f(g,b) = \theta(\Omega(g,b)f)$, $H = U(T)$. For all source adelic points g, g',*

$$\frac{1}{\operatorname{vol}[H]} \int_{[H]} \overline{\Theta_{f'}(g',b)}\, \Theta_f(g,b)\, \mathrm{d}b = \sum_{\gamma \in U(S)(F)} \langle f', \Omega(g'^{-1}\gamma g, 1) f\rangle.$$

The integral and the series converge absolutely. In particular this identity applies at $h = 6r$ for all sufficiently large arithmetic scales.

Proof. In $D = S \oplus S^-$ take the rational hyperbolic summand A_0 to be the graph $\{(x,-x) : x \in S\}$. Extend it to a rational hyperbolic basis by Lemma 483.2. For two tests ξ, ξ' on $\operatorname{Hom}(S,T)$, the section $F_\xi(v_1)\overline{F_{\xi'}(v_2)}$ comes from a test u on the doubled Hom space. In the product polarization use $\xi \otimes \overline{\xi'}$; the second symplectic form and bicharacter are reversed, and the absolutely convergent configuration sum factors. Pass to the polarization with modulation space $\mathcal{N}_{A_0}$ by Theorem 387.2, which preserves the section as a function of the doubled phase variable. In particular its value at the origin is $\theta(\xi)\overline{\theta(\xi')}$.

The simultaneous actual target actions on ξ, ξ', with the second action conjugated, implement on the doubled space the same Heisenberg change as $V(b)$ in Proposition 488.1. Both operators are unitary. They can differ only by one scalar of modulus one: use the real uniqueness in Proposition 392.2 and finite uniqueness in Theorem 358.1, or their tensor product on the dense elementary tests. The scalar equals one. Indeed on the tensor of a fixed nonzero ξ with its conjugate, Lemma 479.2 gives $Z_{A_0}(u) = \|\xi\|^2$. The tensor of its simultaneous target translates gives the same value by unitarity of the actual pair. The action $V(b)$ also preserves the value, by Proposition 488.1. A nonzero value fixes the scalar for every test.

Now use $\xi = \Omega(g,1)f$, $\xi' = \Omega(g',1)f'$. The preceding identification and the pair representation law of Theorem 396.1 identify the theta value on $V(b)u$ with the product in the displayed integral. Theorem 490.1 computes its normalized mean as $\sum_A Z_A(u)$, with absolute convergence. By Theorem 479.3 only graphs contribute and, by Lemma 479.2, their values are the displayed source coefficients. The rank conditions and the marked tests at the actual scales are those of Theorem 477.2. □

Theorem 491.2. *Under the preceding hypotheses take $f' = f$, let $G = U(S)(\mathbb{A}_F)$, and put $\Phi(u) = \langle f, \Omega(u,1)f\rangle$. On compactly supported $L^2([G])$ functions define $(v_f y)(b) =$*

$\int_{[G]} y(g)\Theta_f(g,b)\,\mathrm{d}g$. *This map has a unique bounded extension* $L^2([G]) \to L^2([H])$, *and that extension satisfies*

$$\langle v_f x, v_f y\rangle_{[H]} = \mathrm{vol}[H] \int_G \Phi(u)\langle x, R(u)y\rangle_{[G]}\,\mathrm{d}u.$$

Whenever the original lift integral converges absolutely for almost every b, *its value agrees almost everywhere with this extension. This holds in particular for the rapidly decreasing functions of Proposition 398.1. For* y *at the finite level of Theorem 481.1, the right side equals*

$$\frac{\mathrm{vol}[H]\,\|v_F\|^2}{W_0(P)}\,\langle x, T_h L_\infty y\rangle_{[G]}, \qquad L_\infty y = \int_{G_\infty} \langle f_\infty, \Omega_\infty(a,1) f_\infty\rangle R(a)y\,\mathrm{d}a.$$

Proof. For g on a compact subset of the source quotient, choose representatives in a compact subset of the group. This is possible by a finite cover by projections of relatively compact open neighborhoods. The diagonal case of Theorem 491.1 bounds $\int_{[H]} |\Theta_f(g,b)|^2\,\mathrm{d}b$ uniformly there: local uniform absolute convergence of the graph series is Theorem 483.3. Cauchy–Schwarz therefore bounds the target integral of $|\Theta_f(g,b)\Theta_f(g',b)|$ uniformly for both points in prescribed compact sets. Compactly supported L^2 functions have finite L^1 norm. Thus the triple product with $|x(g')y(g)|$ is integrable. Expanding the two lift integrals and using Theorems 491.1 and 481.1 gives the stated identity on these functions. The same unfolding bounds the squared operator norm by $\mathrm{vol}[H]\|\Phi\|_1$.

The quotient has an exhaustion by compact sets since the matrix group is a countable union of compact sets, by bounding real coordinates and finite denominators for matrices and inverses. Truncating to these sets gives density by L^2 convergence. The norm bound extends the map, and the integral on the right is continuous in both arguments since $|\langle x, R(u)y\rangle| \le \|x\|_2\|y\|_2$ and $\Phi \in L^1$ by Theorem 481.1. For an absolutely convergent pointwise integral the truncations converge to that integral. They also have a subsequence converging almost everywhere to the L^2 extension: choose a subsequence with summable squared errors times 2^{2k} and bound the measures of the sets with error exceeding 2^{-k} by their squared integrals times 2^{2k}; the sums of the measures of the tails tend to zero. This proves the identification. For a rapidly decreasing input and fixed b, the theta value is bounded by a fixed power of the source height $H(g)$ on the reduced domain by Proposition 396.2, applied to the Hom action with fixed target as in the proof of Theorem 396.1; the Hom matrix acts by right multiplication by g^{-1}. Proposition 398.1 now gives absolute convergence there. Finally the finite-level identity is the second formula of Theorem 481.1, using commutation of finite and real translates. The real integral is bounded on L^2 by Theorem 480.2. □

492. The real factor after projection to the source type

Lemma 492.1. *Let* G_∞ *be the product of the real unitary groups with the measures in Theorem 99.1. At each place suppose the moment hypothesis of that theorem and take its* e_τ. *Put* $e = \prod_\tau e_\tau$ *and* $d_e = e(1) > 0$. *For a strongly continuous unitary representation* R *of* G_∞ *on a Hilbert space put*

$$P_e^R y = \int e(a)R(a)y\,\mathrm{d}a.$$

Then P_e^R *and* $P_{\bar e}^R$ *are orthogonal projections and*

$$P_e^R R(a) P_e^R = \frac{\overline{e(a)}}{d_e} P_e^R.$$

In particular for x, y *fixed by* P_e^R, *the matrix coefficient* $\langle x, R(a)y\rangle$ *is* $\overline{e(a)}\,\langle x, y\rangle/d_e$.

Proof. The kernels are integrable by Theorem 96.3 and product integration, so these integrals of vectors define bounded operators of norm at most $\|e\|_1$. Symmetry under inversion $e(z^{-1}) = \overline{e(z)}$ and unitarity give self-adjointness. Apply the second identity of Theorem 99.1

separately at each place, and multiply. In $P_e^R R(a) P_e^R$ the variable of the right translate is $v = zaw$, so its integrated coefficient is

$$\int e(z) e((za)^{-1} v) \, \mathrm{d}z = \frac{e(a^{-1}) e(v)}{d_e}.$$

The double integral converges in vector norm by $\|e\|_1^2$. This proves the formula and gives idempotence on setting $a = 1$. Conjugate the kernel identities for the same assertions about $\bar{e}$. A self-adjoint idempotent projects orthogonally onto its range. Taking the inner product with two fixed vectors gives the last identity. □

Theorem 492.2. *At the finite level and ranks of Theorem 491.2 let W_e be the subspace of $L^2([G])$ fixed by that finite level and by P_e^R for the right real action of G_∞. On the real oscillator let $Q_e = P_{\bar{e}}^{\Omega_\infty}$ for the actual source action $a \mapsto \Omega_\infty(a, 1)$. For $x, y \in W_e$,*

$$\langle v_f x, v_f y \rangle_{[H]} = \frac{\operatorname{vol}[H] \; \|v_F\|^2 \, \|Q_e f_\infty\|^2}{W_0(P) \, e(1)} \, \langle x, T_h y \rangle_{[G]}.$$

If $h = 6r$, the arithmetic scale is sufficiently large, and $W_e \neq 0$, then the lift is injective on W_e if and only if $Q_e f_\infty \neq 0$. If the integral defining v_f is multiplied by a scalar a_0, its Gram factor is multiplied by $|a_0|^2$.

Proof. The finite translates in T_h commute with every real operator, hence with P_e^R, by the norm convergence in Theorem 478.1. Thus $P_e^R T_h y = T_h y$. The integral in Theorem 491.2 consequently uses $\langle x, R(a) T_h y \rangle = \overline{e(a)} \langle x, T_h y \rangle / e(1)$ at every a, by Lemma 492.1. Its real scalar is

$$\frac{1}{e(1)} \int \overline{e(a)} \langle f_\infty, \Omega_\infty(a, 1) f_\infty \rangle \, \mathrm{d}a = \frac{\langle f_\infty, Q_e f_\infty \rangle}{e(1)} = \frac{\|Q_e f_\infty\|^2}{e(1)}.$$

The last equality is orthogonal projection. All finite coefficients outside this scalar are positive on the diagonal for nonzero y at sufficiently large scales by Theorem 478.1; the target volume and the finite test norm are positive. If the projection vanishes the squared norm is zero for every y; if the projection is nonzero every nonzero y has positive squared norm. Scalar multiplication of the lift gives the stated factor by sesquilinearity. □

493. A polar test for nonvanishing of the real projection

Proposition 493.1. *At one real place use $G, K, \Lambda, \mu, m = m_\Lambda, \psi$ of Definition 96.1, and put $c = \int_K |m(k)|^2 \, \mathrm{d}k$. Suppose the moment condition of Theorem 99.1. For a strongly continuous unitary representation R and a vector v, put*

$$v_K = c^{-1} \int_K m(k) R(k) v \, \mathrm{d}k, \qquad \Phi_v(g) = \langle v, R(g) v \rangle.$$

For $g = k_0 P$ its ordinary polar decomposition, the mean of $(\psi \Phi_v)(k_1 g k_2)$ on K^2 equals

$$c \int_K \frac{\langle v_K, R(a^{-1} P a) v_K \rangle}{m_{-\mu}(a^* P a)} \, \mathrm{d}a.$$

If $v_K \neq 0$ and $\langle v_K, R(P) v_K \rangle$ is real and nonnegative for every positive definite hermitian $P \in G$, then $P_{\bar{e}}^R v \neq 0$, where $e = (\int_G |\psi|^2)^{-1} \bar{\psi}$.

Proof. Absorb k_0 in k_1. The polar factors of $k_1 P k_2$ are $k_1 k_2$ and $k_2^{-1} P k_2$, since the latter matrix is positive hermitian. Insert the integral defining ψ and denote its compact integration variable by u. Set $a = k_2 u$, $b = k_1 a$; the three product Haar probabilities in k_1, k_2, u become those in b, a, u, by successive translations and inversion in the compact group. The numerator and denominator of the integrand, apart from the common $1/c$, are then

$$m(b) \, \overline{m(u)} \langle v, R(b a^{-1} P a u^{-1}) v \rangle, \qquad m_{-\mu}(a^* P a).$$

The principal minors of a positive hermitian matrix are positive, so the denominator is a positive continuous function on the compact set in question. The compact coefficient satisfies

$m(u^{-1}) = \overline{m(u)}$ by Definition 100.1. The u-integral replaces $R(u^{-1})v$ with its weighted integral cv_K. The operator $\int m(b)R(b)\,\mathrm{d}b$ is self-adjoint by inversion and the same symmetry of m, so the b-integral on the other side gives another cv_K. This proves the formula.

The double average is nonnegative under the displayed hypothesis and equals $c\|v_K\|^2 > 0$ at the identity. Continuity makes it positive on a neighborhood of positive Haar measure. The product $\psi\Phi_v$ is integrable by Theorem 96.3 and $|\Phi_v| \le \|v\|^2$. Its integral is unchanged by the two compact averages by bi-invariance of Haar and absolute integration. Hence $\int_G \psi(g)\Phi_v(g)\,\mathrm{d}g > 0$. By Lemma 492.1, $\|P^R_{\bar e}v\|^2 = e(1)\int_G \psi\Phi_v$, proving the assertion. □

Corollary 493.2. *For a tensor of real oscillator tests at the infinite places in Theorem 492.2, it suffices that the two hypotheses on v_K of Proposition 493.1 hold for the local test and the actual local source action at every place. Under those hypotheses the first-arrow norm on W_e is a positive real constant times its finite Gram form.*

Proof. For a tensor test the inner product of its translates factors into the local coefficients by the real place product of Theorem 396.1. The integrals against $\bar e$ factor by absolute integrability of each local e_τ and the unitary norm bound. The proof of Proposition 493.1 gives a strictly positive real result at each place. Their product is $\|Q_e f_\infty\|^2 > 0$; now apply Theorem 492.2. □

494. Positive oscillator coefficients for polynomial tests

Lemma 494.1. *Use the Darboux coordinates of Theorem 393.1 and the creation and annihilation operators C_i, A_i of Definition 408.1 on their real position space. Write $B_z = \exp(\sum_i z_i C_i)\phi_0$ for the coherent vector of Proposition 408.3. If a signed complex isometry acts in the Darboux complex coordinate $\xi = x + il$ by $\xi \mapsto U\xi + V\bar\xi$, put $B = (U^*)^{-1}$. Here U is invertible. For its unitary oscillator implementer $\mathcal{W}$, put $c_0 = \langle \phi_0, \mathcal{W}\phi_0\rangle$. Then*

$$\langle B_{\bar z}, \mathcal{W}B_y\rangle = c_0 \exp\Big(\tfrac12 z^{\mathsf T}BV^{\mathsf T}z + z^{\mathsf T}By - \tfrac12 y^{\mathsf T}\bar V^{\mathsf T}By\Big).$$

Proof. In the positive and negative signed summands write the original complex isometry in blocks $\alpha, \beta, \gamma, \delta$. In the Darboux coordinate the negative variables are conjugated, so $U = \mathrm{diag}(\alpha, \bar\delta)$ and V consists of the off-diagonal blocks $\beta, \bar\gamma$. Both diagonal blocks are invertible: the isometry equations give $\alpha^*\alpha = I + \gamma^*\gamma$ and $\delta^*\delta = I + \beta^*\beta$. The vacuum coefficient c_0 is nonzero by Theorem 393.1, also after a unit scalar twist.

Write $A(t) = \sum t_i A_i$, $C(t) = \sum t_i C_i$ with bilinear coefficient convention. Differentiating the symmetric Heisenberg translation in Lemma 392.1 at zero gives $\sqrt{\pi}(A(\bar\xi) - C(\xi))$. Intertwine this derivative by $\mathcal{W}$; using first ξ and then $i\xi$ separates the two parts, giving

$$\mathcal{W}C(t)\mathcal{W}^{-1} = C(Ut) - A(\bar V t), \qquad \mathcal{W}A(t)\mathcal{W}^{-1} = A(\bar U t) - C(Vt).$$

The difference quotients are valid on Schwartz vectors and the implementer preserves them, by Theorem 393.1. Also $A(t)B_y = t^{\mathsf T}yB_y$, since $[A_i, C_j] = \delta_{ij}$ and $A_i\phi_0 = 0$, and $\partial_{y,t}B_y = C(t)B_y$, where $\partial_{y,t}$ denotes directional differentiation. These identities also follow by differentiating the explicit Gaussian formula of Proposition 408.3. That formula gives entire dependence in Schwartz seminorms: on compact parameter sets every parameter or position derivative is bounded by a polynomial times a decaying real Gaussian after completing the square.

Let $K(z, y)$ denote the left side. Moving a creation operator from the second position to the first gives multiplication by its coefficient paired bilinearly with z, since its adjoint is annihilation with conjugate coefficients. Moving annihilation to the first position differentiates in z. Applying the second intertwining identity to B_y thus gives

$$\nabla_z K = B(y + V^{\mathsf T}z)K.$$

The first intertwining identity at $z = 0$ gives $\nabla_y K(0, y) = -\bar V^{\mathsf T}By\,K(0, y)$. Integrate each of these first-order equations along the straight line from zero to its argument, first in y at

$z = 0$, then in z at fixed y. The scalar equation on each line has solution the exponential of the integral of its coefficient, by differentiating its product with the inverse exponential. The initial value $K(0,0) = c_0$ yields the formula. □

Theorem 494.2. *At one real place index the coordinates of* $\mathrm{Hom}(S,T)$ *by a target row* a *and a source column* i*, with the hermitian signs in Theorem 393.1. Let* $P \neq 0$ *be a complex polynomial in these coordinates and suppose*

$$\partial_{ai}\partial_{aj}P = 0 \quad \textit{whenever the source signs of } i,j \textit{ differ.}$$

For the Hermite test $H_P = P(C)\phi_0$ *one then has*

$$\langle H_P, \Omega_\infty(g,1)H_P\rangle > 0 \quad \textit{for every positive hermitian source isometry } g.$$

The inequality means a strictly positive real number. The space of polynomials satisfying the derivative condition is invariant under the source maximal compact group. Hence the nonnegativity hypothesis of Proposition 493.1 holds for the compact average of such a test; it is strictly positive if the compact average is nonzero.

Proof. The Hom action is right multiplication by g^{-1}, which on each target row is represented on its column coordinates by $g^{-\mathsf{T}}$. For g positive hermitian this matrix is also positive hermitian. The full Hom isometry is its block repetition and has positive hermitian diagonal signed blocks α, δ. The U of Lemma 494.1 is thus positive hermitian, as is $B = (U^*)^{-1}$. It preserves each target row and each sign block, while V only connects opposite signs in the same target row. Both pure quadratic forms in that lemma therefore consist entirely of products of variables of opposite signs in one row. The vacuum factor is positive: its untwisted value is $1/\det\delta > 0$ by Theorem 393.1, and the determinant of a positive hermitian unitary-group element is one, since it is positive and has modulus one. The determinant twist of Theorem 396.1 is therefore one.

Apply $\overline{P}(\partial_z)P(\partial_y)$ at $z = y = 0$ to the coherent coefficient. Multiplication of any power series by z_i under evaluation by $\overline{P}(\partial_z)$ is equivalent to replacing $\overline{P}$ by $\partial_i\overline{P}$, as follows on monomials including their factorials. The same holds on the y side. Thus any term containing a factor from either pure quadratic form vanishes by the stated second derivative condition. Only finitely many Taylor coefficients enter. What remains is the same differentiation of $c_0\exp(z^{\mathsf{T}}By)$. Let $A = B^{1/2}$ be the positive hermitian square root from Lemma 79.2. The last exponential is $\langle B_{A\bar{z}}, B_{Ay}\rangle$: the coherent inner product is the exponential of the hermitian pairing by Theorem 408.2 and the factorial expansion. Its differentiated value is the squared norm of $P((\sum_j A_{ji}C_j)_i)\phi_0$. The substituted polynomial is nonzero since A is invertible; the Hermite norm formula in the same theorem makes the squared norm positive. This proves the assertion on positive elements.

On the source compact group the block V vanishes. The creation-operator formula of Lemma 494.1 then acts on a polynomial by linear changes preserving each target row and the two source-sign blocks, times the scalar on the vacuum. The chain rule preserves the derivative condition. A compact average stays in the same finite-dimensional polynomial space of bounded degree and obeys the same linear conditions. If it is zero the coefficient is zero, and otherwise the strict positivity just proved applies. □

495. A determinant polynomial with the required compact coefficient

Theorem 495.1. *At one infinite place let* S,T *have signed orthonormal bases, with source signs* $s_i \in \{\pm 1\}$ *and target signs* t_a*. Write* $h_\pm = \#\{a : t_a = \pm 1\}$*. Use the local source action of Theorem 396.1 with determinant twist* $(\det g)^\alpha$ *on the source, where* $\alpha \in \mathbb{Z}$*. Let* Λ_i *be the compact row weight of Definition 96.1, in the given parameter order, weakly decreasing separately on the two sign sets. Define*

$$b_i = \Lambda_i + \alpha + h_{-s_i}.$$

Suppose the indices with $b_i \neq 0$ admit an injection $i \mapsto a(i)$ into target rows with $t_{a(i)} = s_i \operatorname{sign}(b_i)$. Then there is an explicit nonzero Hermite test H_P with

$$c^{-1} \int_K m_\Lambda(k) \Omega_\infty(k,1) H_P \, \mathrm{d}k = H_P, \qquad \partial_{ai} \partial_{aj} P = 0 \quad (s_i \neq s_j).$$

Under the moment condition of Theorem 99.1 this test has $P^{\Omega_\infty}_{\bar{e}} H_P \neq 0$.

Proof. For each sign s and each integer $j \geq 1$ form the initial and final segments

$$I^+_{s,j} = \{i : s_i = s,\ b_i \geq j\}, \qquad I^-_{s,j} = \{i : s_i = s,\ b_i \leq -j\}.$$

They have the stated segment property by the compact weight ordering. For every nonempty segment I take the determinant $\det(Z_{a(i),l})_{i,l \in I}$ in the source order. Let P be the product of all these finitely many determinants, including repetitions. Put $Z_{a(i),i} = 1$ and all other entries zero to see $P \neq 0$, also for the empty product. A target row used in the product occurs only against source columns of one sign by the injection and the definitions of the segments. This proves the derivative assertion.

We compute the compact projection before multiplying the minors. For $k = (k_+, k_-) \in K$, its Hom action on the column vector of a fixed target row of entries is $k_s^{-\mathsf{T}} = \bar{k}_s$ in a source-sign block s. In signed Darboux coordinates the creation labels of that row therefore transform by $\bar{k}_s$ when $s = t_a$, and by k_s when $s = -t_a$, by Lemma 494.1. The vacuum transforms by $\prod_s (\det k_s)^{\alpha + h_{-s}}$. Indeed the negative Hom block contains h_{-s} copies of $\bar{k}_s$ and its inverse determinant gives the untwisted vacuum factor by Theorem 393.1. Since k is compact the negative block is unitary; the vacuum coefficient of absolute value one gives the actual vacuum character.

For a segment of length l, use the l-th exterior power of the indicated label representation, and take its unit coordinate wedge on the segment I. The alternating map from l label vectors to the determinant of their entries in the assigned target rows is equivariant in the source: apply the same k_s or $\bar{k}_s$ to each label and expand multilinearly. Equivalently its value on a coordinate wedge $i_1 \wedge \cdots \wedge i_l$ is the minor with those columns in the chosen rows. Tensor all the wedges with the one-dimensional vacuum character and multiply the resulting polynomials acting on ϕ_0 by creation operators. This gives a linear compact intertwiner sending their tensor of unit wedges v to H_P.

The coefficient of v is the vacuum character times the product of the conjugate leading minors on the positive segments and the ordinary trailing minors on the negative segments. For a unitary matrix a trailing principal minor equals its determinant times the conjugate of the complementary leading minor. This follows from the complementary minor and inverse-transpose identity proved in Lemma 489.2. Thus on each block the coefficient is a determinant character times conjugate leading minors. On a diagonal element its exponent at i is $\alpha + h_{-s_i} - b_i = -\Lambda_i$. Such exponents determine the product on the whole block: the adjacent differences of exponents give the multiplicities of the proper leading minors and the last exponent fixes the determinant character. Hence the coefficient is precisely $\overline{m_\Lambda(k)}$, by the compact formula of Theorem 77.2 on both sign blocks.

We verify that this coefficient suffices for the projection. For its irreducible compact representation with unit vector v_0 and coefficient m_Λ, average the rank-one orthogonal projection onto v_0 over conjugation by the compact group. It commutes with the representation, so is scalar: diagonalize the averaged self-adjoint matrix, whose eigenspaces are invariant, and use irreducibility. Its v_0 entry is $c = \int_K |m_\Lambda|^2$. Taking the coefficient at an arbitrary group element gives $\int_K m_\Lambda(k) m_\Lambda(k^{-1}u) \, \mathrm{d}k = c \, m_\Lambda(u)$. The same computation on the product compact group follows by multiplying the two block identities. Thus $c^{-1} \int_K m_\Lambda(k) R(k) \, \mathrm{d}k$ is a self-adjoint idempotent in any unitary action R, by convolution and inversion as in Lemma 492.1. In the tensor of wedges its inner product with the unit vector v on both sides is one by the coefficient just computed; the projection therefore fixes v. Passing through the

intertwiner proves it fixes H_P. Theorem 494.2 and Proposition 493.1 now give the claimed nonzero real projection. □

496. Insertion of a positive block of parameters

Theorem 496.1. *Let the source have rank n, signs s_i, and decreasing parameters λ_i, $0 \le i < n$. Let the target have rank $H = n + D$, where $D \ge 1$. Fix $0 \le M \le n$. List the target indices in order as the first M old indices, D new indices, and the remaining old indices. The signs are s_i in the prefix, positive on the new block, and $-s_i$ in the suffix. Write $\alpha, \beta \in \mathbb{Z}$ for the exponents of the extra source and target determinant characters in the Hom oscillator. Set $\Delta = \alpha + \beta + D/2$. On the target take the parameters*

$$\lambda'_{\mathrm{old(i)}} = \lambda_i + \Delta, \qquad \lambda'_{\mathrm{new(j)}} = \beta - (H+1)/2 + D - j, \quad 0 \le j < D.$$

Suppose they are strictly decreasing in the indicated order, that the compact weights Λ, Λ' of Definition 96.1 are integral and dominant within their sign blocks, and that $b_i = \Lambda_i + \alpha + h_{-s_i}$ is positive for $i < M$ and negative for $i \ge M$, with h_t the target sign counts. Use the injection in Theorem 495.1 sending an index to its old copy in the target. The resulting nonzero test H_P is fixed by both compact coefficient averages, using m_Λ for the source and $\overline{m_{\Lambda'}}$ for the target, each divided by its own squared coefficient integral. Its polynomial satisfies

$$\partial_{ai}\partial_{aj} P = 0 \ \ (s_i \ne s_j), \qquad \partial_{ai}\partial_{bi} P = 0 \ \ (t_a \ne t_b).$$

Proof. Write n_s for the number of source indices of sign s, P_s for their count in the prefix, and $U_s = n_s - P_s$ for their count in the suffix. Then $h_t = P_t + U_{-t} + D\mathbf{1}_{t=+}$, where $\mathbf{1}$ denotes the stated indicator. For an index i let B_i count the preceding indices of the same source sign. The weight formula of Definition 96.1, now indexed from zero, reads

$$\Lambda_i = \lambda_i + (n+1)/2 - i - n_{s_i} + 2B_i.$$

Use the identical formula on the target. Its compact vacuum exponent for a sign t is $a'_t = \beta - n_{-t}$, since the target acts by left multiplication on the Hom entries and the negative block for k_t contains n_{-t} copies of k_t. Its inverse determinant gives the untwisted vacuum character in Theorem 393.1.

For a prefix position i the position, sign and count of preceding equal signs stay the same. Hence $\Lambda'_{\mathrm{old(i)}} - \Lambda_i = \Delta + D/2 - h_{s_i} + n_{s_i} = \alpha + \beta + h_{-s_i} - n_{-s_i}$. For a suffix position of old sign s, new sign $t = -s$, the new position is $i + D$ and the preceding equal signs number $P_t + D\mathbf{1}_{t=+} + B_i - P_s$. Using $h_t = n_s + P_t - P_s + D\mathbf{1}_{t=+}$ gives

$$\Lambda'_{\mathrm{old(i)}} - \Lambda_i = \Delta - D/2 + h_t - n_s = \alpha + \beta + h_{-s} - n_{-t}.$$

Thus on every old position $\Lambda'_{\mathrm{old(i)}} = a'_t + b_i$. The j-th new index has position $M + j$ and preceding positive count $P_+ + j$. Since $M + h_+ - 2P_+ = n_- + D$, substitution of its parameter gives $\Lambda'_{\mathrm{new(j)}} = a'_+$.

The injection satisfies the sign condition in Theorem 495.1, which supplies the source assertion and the first derivative condition. For the second, a fixed source column can occur only in segments of its own source sign and its own sign of b_i. All target rows of those segments therefore have the same target sign. This proves the other vanishing.

For the target compact action, the creation label transforms by k_t on rows of the same sign as the source column and by $\bar{k}_t$ on rows of opposite sign, by the signed coordinate calculation in Lemma 494.1. The rows of the segments with $b_i \ge j > 0$ are initial in their target-sign set: every row of that sign in the prefix has that same source sign, and the b_i decrease within it. The rows with $b_i \le -j$ are final in their target-sign set by the identical order in the suffix, whose source sign is the opposite one. In the exterior tensor construction of Theorem 495.1, now alternating the row labels for fixed columns, the compact coefficient is the vacuum character times ordinary leading minors from the prefix and conjugate trailing minors from the suffix. Its torus exponents are $a'_t + b_i$ on old and a'_+ on new indices. By

the weight identities proved above and the same complementary-minor calculation, the coefficient equals $m_{\Lambda'}$. The weighted compact projection argument of that theorem, with complex conjugation in the averaging kernel, therefore fixes the polynomial test. □

497. Real first arrows at the arithmetic parameters

Definition 497.1. Fix a sufficiently large arithmetic scale, $k, r, m, h_2 = r^8,\ t = 5r$ and $w_j(\tau)$ as in Definition 12.3. For an offset $0 \le o < m$ and $z \in \{0, 1\}$ put $n = r + z,\ D = t - z,\ H = n + D = 6r$, $M = k + o$. At an infinite place write $W = W_\tau$ for the rounded weight of Definition 56.1 and $w = w_{o+1}(\tau)$. For $0 \le i < n$ set

$$a_i = \max(0, i + 1 - k), \qquad \lambda_i = \frac{z}{2} + \begin{cases} k/2 - 1 - i, & i < k - 1, \\ -W - (i + 2 - k)2h_2, & i \ge k - 1. \end{cases}$$

For the offset grid choose $s_i = (-1)^{o+a_i}$. For the auxiliary group at $z = 0$ choose $s_i = 1$ for $i < M$ and $s_i = -1$ for $i \ge M$. Use the target signs of Theorem 496.1. The target old parameters are $\lambda_i - (1 - z)/2$. At its inserted block they are $(t - 2)/2 - w - j$, $0 \le j < D$. The two determinant exponents in the actual Hom action are

$$\alpha = w - H/2, \qquad \beta = (n + z - 1)/2 - w.$$

In particular the auxiliary target is definite. In the grid the target sign of an old index is $(-1)^{o+a_i+\min(\max(a_i-o,0),1)}$; every inserted direction is positive.

Theorem 497.2. *Uniformly in the parameters of Definition 497.1, at all sufficiently large scales the construction in Theorem 496.1 applies at every infinite place. For the resulting tensor of Hermite tests f_∞ and the source kernel e of those parameters,*

$$\|Q_e f_\infty\|^2 > 0.$$

The test can be chosen the same for all finite twists at the fixed signatures, weights and offset. For the initial row arrow $Y_0 \to Y_1$ of Definition 348.2 these are the data at $o = z = 0$, in the ordered signed frames.

Proof. The exponents are integers since r is odd. For the ordinary row and auxiliary arrows the character of weight $-w$ is rank-twisted by $\lfloor H/2 \rfloor$ on the source determinant in the denominator and by $\lfloor n/2 \rfloor$ on the target in the numerator, as in Definition 376.1 and Theorem 376.2. For the second grid the upper rounding of these half ranks has the same value as the lower rounding, since $n = r + 1$ and $H = 6r$ are even. This gives exactly the exponents above. The source signs of the offset-zero row at $z = 0$ are positive on the initial block, then alternate starting with a negative sign at $i = k$, as in Proposition 122.1. Its target removes that negative sign and alternates thereafter, with the t new directions positive; the signature counts agree with Theorem 348.3. More generally $a_i \le o$ holds exactly on $i < M$; hence the stated target formula preserves the prefix signs and reverses the suffix signs at every offset.

Writing $\Delta = \alpha + \beta + D/2$ gives $\Delta = -(1 - z)/2$, and $\beta - (H + 1)/2 + D = (t - 2)/2 - w$. Thus the target parameters have exactly the insertion formula. For $i \ge k - 1$,

$$\lambda_i + w = z/2 + 2h_2(o - a_i) + h_2.$$

Its values are at least h_2 in the prefix and at most $-h_2 + 1/2$ in the suffix. For $i < k - 1$ one has $w = W + 2(o + 1)h_2 + h_2$, $W \ge 0$, and $\lambda_i \ge -r$; hence the prefix bound also holds there. In particular the old prefix lies strictly above all D inserted target parameters and the suffix strictly below them: relative to $-w$ the inserted parameters are at most t in absolute value, and the old shift from λ_i is at most $1/2$ in absolute value. Within the old and inserted lists the parameters decrease by at least one by their formulas. Both formal parameter lists have the required half-integral parity for their ranks by the parity of r, t.

For a list of rank n_0 its sign correction in Definition 96.1 satisfies $|\Lambda_i - \lambda_i| \le 3n_0/2$, by bounding the full half sum and the same-sign difference of counts separately. For successive

indices $i < j$ within one sign block their weight difference equals $\lambda_i - \lambda_j + j - i - 2$ by the formula in Theorem 496.1. It is nonnegative since $\lambda_i - \lambda_j \geq j - i$. Thus the compact weights are integral and dominant on both groups. For the source,

$$|b_i - (\lambda_i + w)| \leq 3n/2 + H/2,$$

since $-H/2 \leq h_{-s_i} - H/2 \leq H/2$. This is strictly smaller than $h_2 - 1/2$ at large scales. Hence precisely the prefix has positive b_i. Mapping each index to its old copy in the target gives the required injection.

Finally, adjacent opposite source signs occur only outside the consecutive first block or at its end; their parameter gaps are at least $2h_2 - O(r)$. In the target the only additional possible changes of sign occur at the two ends of the inserted block, with gaps at least $h_2 - O(r)$ by the preceding separation. The constants here are absolute: the first consecutive block has magnitude at most r, the inserted block has magnitude at most t after adding w, and the old shifts have magnitude at most one. Thus Lemma 96.2 and Theorem 96.3 apply with $B = 0$, since $h_2 - O(r) > 1 + 4H^2$ at large scales. If all signs at a place agree, integrability follows directly by compactness. Apply Theorems 496.1 and 494.2, then Corollary 493.2 or its local proof at each place. The determinant data used here involve only real parameters and signs, so the same choice works for every allowed finite twist. □

498. The first transferred rows

Proposition 498.1. *At a nonempty finite set M of split pairs, in the first-branch coordinates of Theorem 476.1 use a nonzero matrix test depending on residues modulo $\mathfrak{p}_w^{l_w}$, $l_w \geq 1$, supported on integral H-by-n matrices. Assume that on its support $L_w X_w \equiv I_n \bmod \mathfrak{p}_w^{l_w}$ for a fixed integral n-by-H matrix L_w. Use the vacuum elsewhere, a unitary determinant character trivial on the principal subgroups at these depths, and unramified on determinant units elsewhere. Put*

$$I_l = \prod_{w \in M} |\mathrm{GL}_n(O_w/\mathfrak{p}_w^{l_w})|, \qquad D_l = \prod_{w \in M} q_w^{l_w}.$$

The squared norm of the finite test F is

$$N_F = D_l^{-Hn} \sum_X |F(X)|^2.$$

For $H/2 - 2n \geq 2$, on the indicated principal source level and source real type e one has

$$\langle v_f x, v_f y \rangle_{[U(T)]} = \frac{\mathrm{vol}[U(T)]\, N_F\, \|Q_e f_\infty\|^2}{I_l e(1)} \langle x, T_H y \rangle.$$

Here T_H and its tail estimate use rank n on the source.

Proof. Each integral residue class in the split matrix model has volume D_l^{-Hn} by its number of additive cosets, proving the norm statement. If the support at X meets that at Xg, then $L_w X_w$ and $L_w X_w g_w$ are both integral and congruent to I_n at the indicated depth. The former has integral inverse and the quotient gives $g_w \equiv I_n \bmod \mathfrak{p}_w^{l_w}$. Conversely this congruence preserves every residue of integral matrices and their integrality in both directions. The determinant character and other scalar factors are one there by the unitary integral frame calculation of Theorem 476.1. Thus the finite coefficient is supported there and equals $N_F \rho_H(g_*)$. Reduction of the integral group onto the invertible residue matrices is surjective: lift the entries and use the unit determinant to invert the lift by its adjugate. Consequently its principal subgroup has mass I_l^{-1}.

On this support X_w is injective over the field since $L_w X_w$ is invertible. Hence the nongraph vanishing in Theorem 479.3 applies. Theorem 490.1 and the proof of Theorem 491.1 give the same graph formula, since $H/2 - 2n \geq 2$ implies $H > 4n$. The real coefficient is integrable by Theorem 480.2; the absolute tail of the unchanged finite complement is bounded by

Theorem 477.2. The absolute unfolding of Theorem 481.1 therefore applies with the just-computed marked mass N_F/I_l. Use the compact truncation argument of Theorem 491.2 and compress on the real type by Lemma 492.1 to obtain the asserted formula. □

Theorem 498.2. *For an offset and signature in Definition 497.1 use the real test of Theorem 497.2. At all sufficiently large scales the lift of Proposition 498.1 is bounded below on its prescribed real and finite source subspace. More precisely its squared norm at y is at least*

$$\frac{\operatorname{vol}[U(T)]\, N_F\, \|Q_e f_\infty\|^2}{I_l e(1)}(1-\eta_H)\|y\|^2, \qquad \eta_H = \exp(24|\Sigma|\, 2^{-H/2+2n}) - 1.$$

In particular choose this real test in the first arrow $\mathcal{F}_0$ of Definition 422.1 for $Y_0 \to Y_1$. For every allowed twist the images $\mathcal{F}_0 y_i$ of the independent source rows $y_i = cZ_{s_i}\phi_i$ in Corollary 470.2 are linearly independent. Their number is at least $(1 - 4/b_{\mathrm{sc}}^2)D$ for the numerical rank D there.

Proof. Here $H/2 - 2n = r - 2z$, which is at least two eventually. The field degree factor $|\Sigma|$ has polynomial growth in the arithmetic scale and r grows at least as a positive power by Proposition 10.5. Hence $\eta_H = o(1)$, uniformly in all finite twists and levels. By Theorem 478.1, $\langle y, T_H y\rangle \geq (1-\eta_H)\|y\|^2$. All the displayed prefactors are strictly positive by Theorem 497.2 and Proposition 498.1. This proves the norm bound.

At $o = z = 0$ every selected row is in the actual real source type. Indeed Theorem 204.4 reproduces the Poincaré functions by the product of the actual e_J, which are transports of e by Theorem 122.2. The coordinate combinations, central frame change and marked-group operators forming the rows in Proposition 470.1 are linear combinations and translations at finite places, so commute with that real projection. These rows have principal level at the original marked places and integral invariance at the twist places, thus also the required principal invariance on their union. The first finite test is the inclusion test of Corollary 476.2. The scalar $[K_0 : J_0]$ multiplying the theta kernel in the definition of $\mathcal{F}_0$ is finite and positive. The bounded source rows have absolutely convergent lift integrals by Theorem 422.2. They agree with the L^2 extension by Theorem 491.2. Applying the lower bound to every linear combination of them and using their source independence proves the conclusion and the stated count. □

499. Complementary frames over a finite ring

Definition 499.1. Let R be a finite commutative ring with unit, of order Q. Write $M_{i,j}$ for the i-by-j matrices over R and $G_i = \mathrm{GL}_i(R)$. Fix a coordinate decomposition $J = R^a \oplus R^d$ of a free module, with inclusion $y_0 : R^a \to J$ and projection $x_0 : J \to R^d$. The two summands may occupy interleaved positions in an ordered basis of J. For each $k \in \mathrm{GL}(J)$ put $Y_k = ky_0$, $X_k = x_0 k^{-1}$. Fix a character $\chi : R^\times \to \mathbb{C}^\times$. A pair of functions Φ on $\mathrm{Hom}(J, R^d)$ and Ψ on $\mathrm{Hom}(R^a, J)$ is matched if they are supported on the X_k and Y_k, respectively, and

$$\Phi(X_k) = \chi(\det k)\Psi(Y_k) \qquad (k \in \mathrm{GL}(J)).$$

Given $c \in \mathbb{C}$ and functions L, L' on the same respective sets as Φ, Ψ, the exchange identity on this finite ring means

$$\sum_Y \Psi(Y)L'(Y) = c\sum_X \Phi(X)L(X) \qquad \text{for every matched pair.}$$

Proposition 499.2. *In Definition 499.1 assume $L(hX) = \chi(\det h)L(X)$ and $L'(Yg) = \chi(\det g)L'(Y)$ for all $h \in G_d$, $g \in G_a$. The exchange identity is equivalent to the pointwise equations*

$$|G_a|L'(Y_k) = c\,|G_d|\,\chi(\det k)L(X_k) \qquad (k \in \mathrm{GL}(J)).$$

For the exchange identity it suffices to test one pair per submodule $S = ky_0(R^a)$.

Proof. Fix k_* and put $Y_* = Y_{k_*}, X_* = X_{k_*}$. An invertible k gives the same submodule precisely when $k_*^{-1}k$ stabilizes the first summand. In the decomposition of Definition 499.1 it then has blocks $\left(\begin{smallmatrix} g & v \\ 0 & u \end{smallmatrix}\right)$ with $g \in G_a,\ u \in G_d$. Indeed both diagonal blocks are invertible since their determinants have invertible product, and the converse follows by block inversion. Multiplication gives $Y_k = Y_* g,\ X_k = u^{-1}X_*$. The two orbits determine the same submodule since $\operatorname{im} Y_k = \ker X_k$. Their respective parameters $g, h = u^{-1}$ are unique since Y_* is split injective and X_* split surjective.

Take $\Psi(Y_* g) = \chi(\det g)^{-1}$ and $\Phi(hX_*) = \chi(\det k_*)\chi(\det h)^{-1}$, and give both functions value zero off these orbits. The block determinant verifies matching, including its support condition. Every matched pair on these orbits is a multiple of this one. Indeed one can choose g, u independently in the block formula and then set first $u = 1$ and next $g = 1$. Running through the finitely many S therefore gives a basis of matched pairs. Equivariance cancels each character in the two sums. The orbit for Y_* contributes $|G_a|L'(Y_*)$ and that for X_* contributes $|G_d|\chi(\det k_*)L(X_*)$. This gives both implications, including $c = 0$. □

Theorem 499.3. *Let a, n, t be nonnegative integers, $d = n + t$, and write $J = R^n \oplus R^a \oplus R^t$ in the displayed order. In Definition 499.1 take the middle summand first and its ordered complement $R^n \oplus R^t$ second. Let L, L', c satisfy the equivariances and exchange identity of Proposition 499.2. For $y \in M_{n,a}$, $Z \in M_{d,a+t}$, put*

$$Y(y) = \begin{pmatrix} y \\ I_a \\ 0 \end{pmatrix}, \qquad X(Z) = \left(\begin{pmatrix} I_n \\ 0 \end{pmatrix} \quad Z \right).$$

For $p : M_{n,a} \to \mathbb{C}$, define p^+ on $M_{d,a+t}$ by block rows n, t and columns a, t:

$$p^+ \begin{pmatrix} -y & u \\ 0 & I_t \end{pmatrix} = Q^{-nt} p(y) \quad (u \in M_{n,t}),$$

with value zero on all other matrices. Then

$$|G_a| \sum_y p(y) L'(Y(y)) = c\,|G_d| \sum_Z p^+(Z) L(X(Z)).$$

Only the pointwise equations of Proposition 499.2 for submodules $Y(y)R^a$ are needed.

Proof. The substitution $k(v, b, w) = (v + yb, b, w)$ has determinant one, first columns $Y(y)$ in the specified decomposition, and quotient map $X_k = \left(\begin{smallmatrix} I_n & -y & 0 \\ 0 & 0 & I_t \end{smallmatrix}\right)$. The pointwise equation gives $|G_a|L'(Y(y)) = c|G_d|L(X_k)$. Multiplication of X_k on the left by $\left(\begin{smallmatrix} I_n & u \\ 0 & I_t \end{smallmatrix}\right)$ inserts u into its last columns. Its determinant is one, so this leaves the value of L unchanged. Average over the Q^{nt} choices of u, multiply by $p(y)$, and sum over y. The sign in the definition of p^+ is the one in the inverse of k. □

500. The Fourier scalar for a complementary pair

Definition 500.1. Use the finite ring of Definition 499.1. An additive character $\psi : R \to \mathbb{C}^\times$ is primitive if for each $v \neq 0$ there is $z \in R$ with $\psi(vz) \neq 1$. A character $\chi : R^\times \to \mathbb{C}^\times$ is primitive here if for each $v \neq 0$ some z gives $1 + zv \in R^\times$ and $\chi(1 + zv) \neq 1$. Extend χ and χ^{-1} by zero on nonunits. All nonzero character values have absolute value one since the underlying groups are finite. If $f : M_{a,b} \to \mathbb{C}$, put

$$\widehat{f}(Y) = \sum_{V \in M_{a,b}} f(V)\psi(\operatorname{tr}(YV)) \qquad (Y \in M_{b,a}).$$

All finite Fourier transforms in this section use this normalization.

Lemma 500.2. *Let ψ, χ be primitive in Definition 500.1. For $t \geq 0$ set $D(V) = \chi(\det V)$ on $M_{t,t}$ and $G = \widehat{D}(I_t)$. If $S(gV) = \chi(\det g)S(V)$ for all $g \in G_t$, then*

$$S(V) = S(I_t)D(V), \qquad \widehat{D}(Y) = \chi^{-1}(\det Y)G, \qquad |G|^2 = Q^{t^2}.$$

Proof. When V is invertible the first equation is the given transformation applied to the identity. Otherwise right multiplication by V on the finite set of row vectors cannot be injective. If it were injective it would be onto, giving a left inverse to V by taking preimages of the standard rows. Taking determinants and using the adjugate would make V invertible. Choose a nonzero row u with $uV = 0$, and choose j with $u_j \neq 0$. For the j-th coordinate column e_j, the matrix $g = I + ze_j u$ has determinant $1 + zu_j$ by expansion in row j. Primitivity gives a choice of z making g invertible and $\chi(\det g) \neq 1$. Since $gV = V$, the transformation forces $S(V) = 0$.

Changing variables in the Fourier sum gives $\widehat{D}(Yg) = \chi(\det g)^{-1}\widehat{D}(Y)$. Apply the first assertion to $S(V) = \widehat{D}(V^{\mathsf{T}})$ with character χ^{-1}, which has the same primitivity. This gives the middle formula. For any rectangle, summing $\psi(\mathrm{tr}(YZ))$ over all Y gives the size of that matrix set when $Z = 0$, and zero otherwise. In the second case, translate Y in one entry paired with a nonzero entry of Z to multiply the sum by a scalar different from one. Expanding the square of the transform and using this orthogonality gives $\sum_Y |\widehat{f}(Y)|^2 = Q^{ab} \sum_V |f(V)|^2$. For D the latter sum before the factor is $|G_t|$; the middle formula makes the former $|G|^2|G_t|$. This proves the norm, also when $t = 0$, in which case there is one matrix and its determinant is one. □

Theorem 500.3. *Let $n, t \geq 0$, $a = n + t$, and use $X(Z), Y(y), L, L', c$ of Theorem 499.3 with $d = a$. Let χ, ψ satisfy Lemma 500.2. In the notation of that lemma put $\gamma = (Q^{nt}G)^{-1}$. For every $p : M_{a,n} \to \mathbb{C}$, define its extension p^+ on $M_{a+t,n+t}$ by the rule of Theorem 499.3 with a, n interchanged. Then*

$$\sum_{y \in M_{n,a}} \widehat{\widehat{p}}(y) L'(Y(y)) = \gamma c \sum_{Z \in M_{a,a+t}} \widehat{\widehat{p^+}}(Z) L(X(Z)),$$

$$|\gamma|^2 Q^{\,t(t+2n)} = 1.$$

Here too it suffices to assume the pointwise incidence equations on the graph submodules listed in Theorem 499.3.

Proof. Write $q(y) = \widehat{\widehat{p}}(y)$. At $Z = \left(\begin{smallmatrix} A & B \\ C & D_0 \end{smallmatrix}\right)$ use block rows n, t and columns a, t. The trace after multiplication by $\left(\begin{smallmatrix} -v & w \\ 0 & I_t \end{smallmatrix}\right)$ is $-\mathrm{tr}(Av) + \mathrm{tr}(Cw) + \mathrm{tr}(D_0)$. Averaging over $w \in M_{a,t}$ cancels the factor Q^{-at} on the support of p^+ when $C = 0$ and gives zero otherwise, by the entrywise orthogonality in Lemma 500.2. Thus the transform is $\mathbf{1}_{C=0} q(-A) \psi(\mathrm{tr} D_0)$.

For fixed A, B, the function

$$S(D_0) = L \begin{pmatrix} I_n & A & B \\ 0 & 0 & D_0 \end{pmatrix}$$

satisfies $S(gD_0) = \chi(\det g) S(D_0)$. This follows by acting on the bottom rows with g. Lemma 500.2 evaluates its sum against $\psi(\mathrm{tr} D_0)$ as $GS(I_t)$. At $D_0 = I_t$ the block B can be removed by a left shear of determinant one. The sum over B therefore gives Q^{nt} copies of the same value. Changing A to $-y$, the full sum over Z on the right before multiplication by γc equals

$$GQ^{nt} \sum_y q(y) L \begin{pmatrix} I_n & -y & 0 \\ 0 & 0 & I_t \end{pmatrix}.$$

The pointwise equation in the proof of Theorem 499.3 now gives the assertion, since $|G_a| = |G_d|$. The displayed norm of γ follows from $|G|^2 = Q^{t^2}$. □

501. The marked Fourier normalization and its conductor

Definition 501.1. Let S be a nonempty finite set of chosen branches of split places, with valuation rings O_w, uniformizers π_w, residue field orders q_w and integers $l_w \geq 1$. Write $O_S = \prod_{w \in S} O_w$, $\delta = (\pi_w^{l_w})_w$, $R = O_S/\delta O_S$, and $Q_R = |R| = \prod_w q_w^{l_w}$. The cardinality follows by the successive quotients of the powers of a uniformizer, each identified with the

residue field by division by that power. Let ψ_w^0 be the normalized additive character ψ_f^0 of Proposition 18.5 restricted to the component at w. Fix arbitrary $\epsilon_w \in O_w^\times$, using $\epsilon_w = 1$ unless specified, and put

$$\psi_R(v) = \prod_{w \in S} \psi_w^0(\epsilon_w \pi_w^{-l_w} \widetilde{v}_w), \qquad v \in R,$$

where $\widetilde{v}_w$ is any integral lift. For $p : M_{a,b}(R) \to \mathbb{C}$ let $\widetilde{p}(X)$ take the value $p(X \bmod \delta)$ on $M_{a,b}(O_S)$ and zero outside it. Define

$$\mathcal{U}_{a,b} p = Q_R^{-ab/2} \widehat{\widetilde{p}} \quad \text{on } M_{b,a}(R)$$

using ψ_R in Definition 500.1.

Proposition 501.2. *The character ψ_R in Definition 501.1 is well defined and primitive. With the integral matrix box of mass one, the additive Fourier integral of $\widetilde{\overline{p}}$ at a dual matrix Z is zero outside $\delta^{-1} M_{b,a}(O_S)$. At $Z = \delta^{-1} Y$ for integral Y it equals $Q_R^{-ab} \widehat{\widetilde{p}}(Y \bmod \delta)$. After the norm-preserving change from Z to δZ the transformed function is the integral lift of $\mathcal{U}_{a,b} p$. In the finite counting norm and in the finite probability norm this last transform is an antilinear isometry onto the reversed matrix space.*

Proof. The annihilator of O_w under multiplication followed by ψ_w^0 is O_w. To see this, vary only the w-entry of the integral argument in Proposition 18.5; conjugating the other variable permutes the branches and their integral lattices. Hence changes of integral lift do not alter ψ_R. For $v_w \notin \pi_w^{l_w} O_w$, the annihilator assertion gives $z_w \in O_w$ for which $\psi_w^0(\pi_w^{-l_w} \widetilde{v}_w z_w) \neq 1$. Use zero at all other entries to obtain primitivity.

The Fourier integral uses $\prod_w \psi_w^0(\epsilon_w \operatorname{tr}(Z_w X_w))$. Multiplication by a unit preserves each of the annihilator lattices just computed. Translating X by δB, $B \in M_{a,b}(O_S)$, fixes the test. If δZ has a nonintegral entry the annihilator assertion in that entry makes the integral zero. For integral $Y = \delta Z$ split X into its Q_R^{ab} equal-volume residue classes. The phase on each class is constant, giving the stated sum. Replacing Z by $\delta^{-1} Y$ multiplies the measure by Q_R^{ab}, by the index of the new box. Consequently the norm-preserving substitution multiplies the function value at $\delta^{-1} Y$ by $Q_R^{ab/2}$. This proves the normalization. In the finite spaces the norm identity follows directly from the orthogonality computation in Lemma 500.2, which uses only additive primitivity. Injectivity gives surjectivity by equal finite dimensions. Dividing both counting measures by Q_R^{ab} preserves the isometry. □

Proposition 501.3. *In Definition 376.1, let M_g be the marked places and let the twist at each $w \in P$ satisfy $\vartheta_w^j \neq 1$ for $1 \le j \le m$. This is the m-fold twist condition. Use the squarefree ring on $S = M_g \cup P$ in Definition 501.1. For $1 \le j \le m$ and any integer u with $0 < |u| \le 8m$, the character with components ω_w^u on M_g and ϑ_w^j on P is primitive for all sufficiently large scales. Its inverse is primitive as well.*

Proof. Proposition 373.2 gives ω_w order $M_\omega = (Md)^{s+6}$. Proposition 10.5 gives $2^s = m^{22}$; since $Md \ge 2$, eventually $M_\omega > 8m$. Thus every component is nontrivial, including those in P by hypothesis. The sets are disjoint by Definition 357.1. For $v \neq 0$ in the product of residue fields choose w with $v_w \neq 0$ and choose a unit b there on which its character is different from one. Put $z_w = (b-1)/v_w$ and put $z_{w'} = 0$ elsewhere. Then $1 + zv$ is a unit and its character value equals that at b. This is exactly the required primitivity, also after inversion. □

Theorem 501.4. *With $a = n + t$ in Theorem 500.3, use the normalized transform of Definition 501.1 on both data. Then*

$$\sum_y (\mathcal{U}_{a,n} p)(y) L'(Y(y)) = c\,\Gamma \sum_Z (\mathcal{U}_{a+t,n+t} p^+)(Z) L(X(Z)), \qquad |\Gamma|^2 = Q_R^{t^2}.$$

In particular the squared modulus of the product of these Fourier multipliers over the pairs $i,j\ge 0,\ i+j+2\le m$ is $Q_R^{t^2\binom{m}{2}}$, independently of the intermediate ranks. On the squarefree arithmetic level the ratio of this squared modulus to its value at the trivial twist is $Q_P^{t^2\binom{m}{2}}$, where $Q_P=\prod_{w\in P}q_w$.

Proof. Multiply the Fourier identity of Theorem 500.3 by $Q_R^{-an/2}$. Proposition 501.2 gives the new multiplier

$$\Gamma=\gamma Q_R^{((a+t)(n+t)-an)/2}=\gamma Q_R^{t(n+t)}.$$

Since $|\gamma|^2=Q_R^{-t(t+2n)}$, the exponent in $|\Gamma|^2$ is $2t(n+t)-t(t+2n)=t^2$. There are $\sum_{i=0}^{m-2}(m-i-1)=\binom{m}{2}$ pairs, including zero pairs when $m=1$. Proposition 501.3 checks multiplicative primitivity for the indicated squarefree characters, and Proposition 501.2 checks the additive character. Removing P divides Q_R by Q_P, giving the claimed ratio. The proportionality constant c in the exchange identity remains in the formula. □

502. Adjunction on rational and real theta data

Definition 502.1. Let S,T be nonsingular hermitian spaces over the CM field E, with matrices J_S,J_T. Use $V=\mathrm{Hom}_E(S,T)$, $V'=\mathrm{Hom}_E(T,S)$, with trace hermitian forms H,H' as in Definition 355.1. Put

$$X^\dagger=J_S^{-1}X^*J_T,\qquad \sigma X=-X^\dagger.$$

On V,V' denote by b,b' the rational bilinear forms of Definition 385.1 and by a,a' their alternating parts. Write $G_{g,h}(X)=hXg^{-1}$ for $g\in U(S),\ h\in U(T)$; on V' use $G'_{h,g}(Z)=gZh^{-1}$.

Proposition 502.2. *The map σ in Definition 502.1 is a rational linear isomorphism with*

$$H'(\sigma X,\sigma Y)=\overline{H(X,Y)},\qquad a'(\sigma X,\sigma Y)=-a(X,Y),\qquad \sigma G_{g,h}=G'_{h,g}\sigma.$$

It induces an antiunitary bijection C_σ on rational Schrödinger test spaces, characterized in the symmetric coordinates of Theorem 387.2 by

$$F^{a'}_{C_\sigma f}(\sigma v)=\overline{F^a_f(v)}.$$

In particular the theta value at zero is conjugated. The map intertwines the symmetric translations at v and σv.

Proof. Taking the adjoint twice gives the original map. Moreover $\mathrm{tr}(XY^\dagger)=\mathrm{tr}(Y^\dagger X)$, by expanding both traces entry by entry. This proves the first identity, and $\bar J=-J$ gives the second by the trace formula in Lemma 385.2. The adjoints of g,h are their inverses, proving the third identity.

Choose complementary rational isotropic spaces P,L for a, with dual bases as in Lemma 385.2. Use $\sigma P,\sigma L$ on V' with the transported basis on the first and the negative of the transported basis on the second. These are dual by the alternating-form identity. On positions define $(C_\sigma f)(\sigma x)=\overline{f(x)}$. In these coordinates it preserves the L^2 norm and takes tests bijectively onto tests. The position coordinate of $\sigma(x+l)$ is x and its momentum coordinate is $-l$. Consequently the symmetric translation formula in Theorem 387.2 intertwines under conjugation. The bijection of rational positions gives $\theta_{\sigma P}(C_\sigma f)=\overline{\theta_P(f)}$ by absolute convergence from Theorem 385.3. Apply this identity after translation to get the section formula. The injectivity and change of polarization in Theorem 387.2 prove characterization and independence of the rational coordinates. □

Theorem 502.3. *At one real place choose signed orthonormal coordinates on S,T, with signs s_j,t_i, respectively. In the reversed coordinates define the real orthogonal signed permutation*

B by $(Bz)_{ji} = -s_j t_i z_{ij}$. If H_P is the Hermite test of Definition 408.1, conjugate the coefficients of P to form $\bar{P}$ and put

$$P^{\#}(Z') = \bar{P}(B^{\mathsf{T}} Z').$$

There is an antiunitary real-place intertwiner C_∞ implementing σ on the symmetric translations, sending H_P to $H_{P^\#}$. It intertwines the actual untwisted real splittings for $G_{g,h}, G'_{h,g}$. For unitary determinant characters a_0, b_0, let $\lambda = a_0(\det h)/b_0(\det g)$. The same operator intertwines the first splitting multiplied by λ with the second multiplied by $\bar{\lambda}$.

Proof. The adjoint formula of Definition 502.1 in these bases is the signed conjugation $z \mapsto B\bar{z}$. A position ij and its reversed position ji have identical Hom sign $s_j t_i$. Thus even after the negative coordinates are conjugated in the Darboux convention of Theorem 393.1, the action is $\xi = x + il \mapsto B\bar{\xi}$. On real position functions set $(C_\infty f)(x') = \overline{f(B^{\mathsf{T}} x')}$. It is antiunitary, fixes the standard Gaussian and interchanges the translated coordinates (x, l) with $(Bx, -Bl)$, by Lemma 392.1. Each creation operator is a real linear combination of its position and the corresponding derivative. Changing signs and permuting variables in that formula gives the substitution $B^{\mathsf{T}} Z'$ and conjugates each coefficient. This proves the Hermite formula.

By Proposition 502.2, conjugation by this operator transports an implementer of $G_{g,h}$ to one of $G'_{h,g}$. In the original signed complex coordinates the negative-to-negative block of the latter is $B_- \bar{\delta} B_-^{-1}$, where δ is that block for $G_{g,h}$ and B_- is the restriction of B to the negative positions. Its inverse determinant is the conjugate of the vacuum coefficient for $G_{g,h}$. Since C_∞ fixes the vacuum, uniqueness with this coefficient in Theorem 393.1 gives equality of the transported and reversed real splittings. The determinant ratios are complex conjugates because the characters are unitary, which proves the last assertion. □

503. The finite adjoint and the marked shift

Proposition 503.1. *Use the self-dual finite stage frames of Theorem 348.3 on S, T, the scale α of Lemma 348.1, and σ of Definition 502.1. Set $\eta = (\alpha/\bar{\alpha})\sigma$ at the finite places. On the finite oscillator spaces there is an antiunitary map C_η taking the reference vacuum to the reversed reference vacuum and intertwining symmetric translations through η. For every finite pair g, h one has*

$$C_\eta \, \mathcal{V}_{G_{g,h}} = \mathcal{V}'_{G'_{h,g}} \, C_\eta,$$

where the splittings are those of Theorem 390.2.

Proof. In the finite isometric frames the adjoint is ordinary conjugate transpose. Hence $\eta(\alpha X) = -\alpha X^*$, and η takes the reference lattice L bijectively onto L'. It still conjugates the hermitian forms and negates the alternating form, since $\alpha/\bar{\alpha}$ has norm one. In particular $b'(\eta v, \eta v) = b(v, v)$. The symmetric bilinear form $b(v, z) + b'(\eta v, \eta z)$ has diagonal $2b(v, v)$; its symmetry follows because the alternating parts cancel.

Write $\mathbf{e}_f$ for the finite part of the additive character of Definition 385.1. Use the full finite lattice realization of Theorem 358.1 and set

$$(C_\eta F)(\eta v) = \mathbf{e}_f(-b(v, v)) \, \overline{F(v)}.$$

With $D^a_z = \mathbf{e}_f(b(z, z)/2) D_z$, expand $b(v + z, v + z) - b(v, v) - b(z, z)$ using the symmetric sum above. Its value is $b(v, z) + b'(\eta v, \eta z)$. Substitution into the translation formula of Lemma 356.2 gives $C_\eta D^a_z = D^{a'}_{\eta z} C_\eta$. The phase is one on L, since B_f is one on $L \times L$ by Theorem 358.1. Thus the initial vacuum goes to the new vacuum. Additive measure in each space is normalized on its vacuum lattice; η preserves this normalization. The translates span both test spaces by Definition 357.1, so the displayed map is an antiunitary bijection.

The transported $W_{G_{g,h}}$ and the reversed $W'_{G'_{h,g}}$ implement the same Heisenberg change by Proposition 502.2; the extra scalar in η commutes with the Hom action. Both have

positive real vacuum coefficient, since only the identity coset in the vacuum formula of Theorem 358.1 has support meeting the initial box. Its uniqueness up to a unit scalar identifies the operators. To include the splitting phase, the definition of J_G in Definition 375.1 counts the valuation of the domain lattice on which G is integral. On coordinates divided by α, the map $X \mapsto -X^*$ exchanges the two conjugate places and permutes integral entries. It therefore sends the domain lattice for G to that for G', exchanging w and $\bar{w}$ in its index. Consequently $J_{G'} = \overline{J_G}$. The ideal phase of Theorem 374.2 takes conjugate values on these two ideals. This proves the asserted identity. □

Theorem 503.2. *In Proposition 503.1 let $a = \dim_E T$, $b = \dim_E S$. At a marked set M of split branches disjoint from $\bar{M}$ use the depths of Definition 501.1. Extend its δ to a finite idele by one off M, and let $s_\delta = \bar{\delta}/\delta$. Write v_D for the finite test with first-branch matrix function $\widetilde{D}$ in coordinates divided by α, and vacuum elsewhere. Use analogous notation v'_D in the reversed pair. For $D : M_{a,b}(R) \to \mathbb{C}$ one has*

$$C_\eta v_D = W'_{s_\delta} v'_{\mathcal{U}_{a,b} D}.$$

Here W'_{s_δ} implements multiplication by s_δ on the reversed Hom space with positive vacuum coefficient, and $\mathcal{U}_{a,b}$ uses $\epsilon_w = (-\varpi \jmath \alpha \bar{\alpha})_w$ in Definition 501.1.

Proof. The specified ϵ_w is a unit since $\jmath\alpha\bar{\alpha}\,\widehat{O}_E = \varpi^{-1}\widehat{O}_E$ by Lemma 355.2. Thus the local character in the trace pairing in these coordinates is $\zeta_w(t) = \psi^0_w(\epsilon_w t)$. Indeed $\psi^0_w(u) = \psi_{f,w}(-u/\varpi)$ by Proposition 18.5. In the split model of Theorem 360.1 a first-branch displacement x translates the argument by x. A displacement y on the conjugate branch multiplies the value at X by $\prod_w \zeta_w(\mathrm{tr}(y^* X))$. On either pure branch the quadratic correction from symmetric translations is one. The normalized adjoint map on these displacements is $x \mapsto -x^*$, $y \mapsto -y^*$.

Consider the integral transform

$$\mathfrak{F} f(Z) = \int \overline{f(X)} \prod_{w \in M} \zeta_w(\mathrm{tr}(Z_w X_w))\, \mathrm{d}X.$$

Translating X by x in the input multiplies the result by the character at $-\mathrm{tr}(Zx)$; modulation by the character at $\mathrm{tr}(y^* X)$ translates the output by $-y^*$. The transform takes the integral indicator to the integral indicator, by the entrywise annihilator in Proposition 501.2. The two intertwining properties and the value on the vacuum determine its values on the span of the translates. This span is the whole split test space by Theorem 360.1. It agrees there with C_η; at all other places the vacuum is taken to the vacuum by Proposition 503.1. These assertions on products follow by using the spanning vacuum translates separately on each factor.

On the first branches the scalar s_δ is δ^{-1}. The normalized substitution for W'_{s_δ} in Theorem 360.1 sends $f(Z)$ to $Q_R^{-ab/2} f(\delta Z)$; it acts as the identity away from the pairs. Apply Proposition 501.2 to $\mathfrak{F}(\widetilde{D})$. It equals that substitution on the integral lift of $\mathcal{U}_{a,b} D$, proving the claim. □

504. Reversal of the global theta kernel

Definition 504.1. Let S, T and their frames satisfy Proposition 503.1. Write $G = U(S)(\mathbb{A}_F)$, $H = U(T)(\mathbb{A}_F)$ with $F = E^+$, and write $[G], [H]$ for their rational left quotients. Choose unitary characters a_0, b_0 of the adelic norm-one torus, trivial on its rational points. On $V = \mathrm{Hom}_E(S, T)$ set

$$\Omega(g, h) = \frac{a_0(\det h)}{b_0(\det g)} \Omega_V(G_{g,h}).$$

On V' use $\Omega'(h, g) = b_0(\det g)\Omega_{V'}(G'_{h,g})/a_0(\det h)$. Use the splittings of Theorem 395.2. Let P denote a polynomial in the real Hermite variables at all infinite places and D the residue

matrix function of Theorem 503.2. Put

$$K_{P,D}(g,h) = \theta\big(\Omega(g,h)(H_P \otimes v_D)\big).$$

Here the Hermite polynomial may couple variables at different places, by expansion into products of monomials. Write $K'_{P',D'}(h,g)$ for the corresponding kernel in the reversed order. The rational theta invariance and the triviality of a_0, b_0 on rational points make both kernels functions on the indicated quotients.

Theorem 504.2. *With the finite ideles δ, α of Theorem 503.2, put*

$$u_\delta = \frac{\overline{\alpha\delta}}{\alpha\delta},$$

and give it real component one. Regard it as a central scalar in G. For fixed spaces, realizations, characters and modulus there is c_δ of modulus one such that for every P, D, g, h,

$$\overline{K_{P,D}(g,h)} = c_\delta\, K'_{P^\#, \mathcal{U}_{a,b}D}(h, gu_\delta), \qquad a = \dim_E T, \quad b = \dim_E S.$$

The substitution $P^\#$ of Theorem 502.3 is applied in all places, conjugating the coefficients once. For the unnormalized transform $\widehat{\widehat{D}}$ the corresponding multiplier has squared modulus Q_R^{-ab}.

Proof. Transport both rational models to the local models used for their splittings as in Theorem 395.2. On elementary test tensors combine C_∞ at the infinite places with C_η at the finite places to give an antiunitary C_0. This is well defined on the complex tensor product, since moving a scalar between tensor factors conjugates it on either side. It is bijective on tests, and multiplication of the local inner products proves antiunitarity. In the real variables the product substitution also acts on arbitrary Schwartz functions, by conjugation and one signed permutation in all real variables. Theorem 502.3 and Proposition 503.1 give $C_0\Omega(g,h) = \Omega'(h,g)C_0$ because the determinant multipliers are conjugate.

Let $s_\alpha = \bar\alpha/\alpha$, extended by one at infinity. For symmetric translations the operator $\Omega'(1, s_\alpha)C_0$ implements the rational map σ at every place. Indeed at finite places $s_\alpha \eta = \sigma$, and the archimedean implementation is the one in Theorem 502.3. Its quotient with the antiunitary C_σ of Proposition 502.2 is a linear unitary commuting with all symmetric translations. These differ from ordinary translations only by scalars, so Lemma 395.1 gives $C_\sigma = c_0\Omega'(1, s_\alpha)C_0$ for a fixed $|c_0| = 1$.

By Theorem 503.2, the finite part of C_0 on v_D equals W'_{s_δ} on $v'_{\mathcal{U}_{a,b}D}$, where $s_\delta = \bar\delta/\delta$. The operator $\Omega'(1, s_\delta)$ differs from this finite operator by the unitary ideal phase of Theorem 390.2 and a unitary determinant value. Both are independent of P, D, g, h. Apply the equality for C_σ to $\Omega(g,h)(H_P \otimes v_D)$. Commute the central scalars, use the representation law, and evaluate the resulting rational theta value at zero. Proposition 502.2 conjugates that evaluation. The two central elements multiply to u_δ, and the two fixed phases give c_δ. Finally the normalization of Definition 501.1 replaces the unitary transform by $Q_R^{-ab/2}\widehat{\widehat{D}}$. □

Corollary 504.3. *For the bounded first lift A of Theorem 491.2 take $K = K_{P,D}$ of Definition 504.1. For compactly supported $z \in L^2([H])$ put*

$$Bz(g) = \int_{[H]} K'_{P^\#, \mathcal{U}_{a,b}D}(h,g)z(h)\, \mathrm{d}h.$$

With $R(u)v(g) = v(gu)$, its Hilbert adjoint satisfies

$$A^*z = c_\delta R(u_\delta)Bz.$$

In particular the right side is square integrable. It extends from such z to the adjoint on all of $L^2([H])$.

Proof. The rational theta sum converges locally uniformly in g, h. Indeed the group action is continuous on real Schwartz tests and locally constant on the finite test, as in Theorem 396.1, and the rational lattice tail on a bounded family of Schwartz tests is summable by the proof of Theorem 385.3. Quotient compact sets have lifts in compact sets by the finite open cover argument in Theorem 491.2. Thus the reverse integral is locally bounded. For compactly supported $y \in L^2([G])$, both y, z are integrable by Cauchy–Schwarz on their finite-volume supports. The kernel is bounded on their product, so integration in either order is allowed. Theorem 504.2 gives $\langle z, Ay\rangle = \langle v, y\rangle$ with $v = c_\delta R(u_\delta)Bz$.

For a compact exhaustion K_j of $[G]$, test with $y = \mathbf{1}_{K_j} v$. Boundedness of A gives $\|\mathbf{1}_{K_j} v\|_2^2 \le \|z\|_2 \|A\| \|\mathbf{1}_{K_j} v\|_2$. Increasing the compact sets and applying Lemma A.2 gives $\|v\|_2 \le \|A\| \|z\|_2$. The same exhaustion argument as in Theorem 491.2 gives density of compactly supported functions on both sides. Thus the assignment on z extends boundedly and its pairing identity extends to all y, z, which characterizes the Hilbert adjoint. □

505. Descent to the opposite source

Lemma 505.1. *Let S have an actual stage frame as in Theorem 348.3. There is a finite integral idele $\nu \in \widehat{O}_E^{\times}$ with $\nu\bar{\nu} = -1$. On S^-, obtained by negating J_S, take the same ordered real frame with all signs reversed and multiply the old finite frame by ν. This finite frame again has identity Gram matrix. The two rational groups, their adelic quotients, and the Haar measures with the stage normalizations coincide under the identity on rational matrix coordinates. For a fixed real parameter sequence the polar coefficient of Definition 96.1 is also identical.*

Proof. At a split pair choose local components $1, -1$. At every nonsplit finite place use the residue norm lift in the proof of Theorem 390.2: there the extension is unramified of degree two, the characteristic of the residue field is odd, and the successive unit corrections lift a residue norm of -1 to an exact norm of -1. All these components are units, so their product is a finite idele. The norm of the scalar cancels the sign of the new Gram matrix. The defining equation $g^* J_S g = J_S$ is unchanged by negation. The scalar change of finite frame commutes with all matrices and preserves the integral levels. At infinity $g^*(-J_S)\,\mathrm{d}g = -g^* J_S\,\mathrm{d}g$ gives the same density in Definition 95.1. Thus the identical rational and adelic groups have the same quotient measure. For the polar coefficient, equality of two signs at specified indices is unaffected by a simultaneous sign change. Consequently Λ, μ, the compact subgroup, its probability measure, and the ordinary polar decomposition used in Definition 96.1 are the same. □

Theorem 505.2. *Use $a = \dim_E T$, $b = \dim_E S$ and the kernel of Definition 504.1. Write υ_T for the character of Theorem 374.2 restricted to the norm-one torus. On $S^- \to T$ use the characters*

$$a_- = a_0^{-1} \upsilon_T^{b}, \qquad b_- = b_0^{-1} \upsilon_T^{a}$$

in the same numerator and denominator convention. Define $D^-(X) = \overline{D(\nu^{-1}X)}$ on the residue matrices, reducing ν in the first branches. For a unit scalar c_- independent of P, D, g, h,

$$\overline{K_{P,D}(g,h)} = c_- K^-_{\bar{P},D^-}(g,h).$$

The variables g, h on the right use the identification of Lemma 505.1. In particular a point mass at the standard inclusion gives a point mass at ν times that inclusion. On compactly supported target functions the first Hilbert adjoint is integration against this opposite-source kernel with the scalar c_-.

Proof. In rational coordinates the identity from $\mathrm{Hom}(S, T)$ to $\mathrm{Hom}(S^-, T)$ sends H to $-H$. It therefore negates both the bilinear form and its alternating part in Definition 385.1. Use the same position and modulation subspaces on both sides, negating the modulation basis.

The proof of Proposition 502.2 then gives an antiunitary map conjugating theta at zero and intertwining symmetric translations through this rational identity.

At infinity each positive Hom coordinate becomes negative and each negative coordinate becomes positive. The Darboux coordinates of Theorem 393.1 are therefore conjugated, with a relabeling if the signs were grouped. Complex conjugation on real position functions, with that relabeling, sends H_P to $H_{\bar{P}}$, by Definition 408.1. It intertwines the translations. If A_+, A_- are the original positive and negative diagonal blocks of the Hom isometry $G_0 = G_{g,h}$, the conjugated old vacuum coefficient divided by the new one is $\det A_+ / \overline{\det A_-} = \det_{\mathbb{C}} G_0$, by the determinant calculation in Theorem 393.1. Uniqueness with a specified vacuum coefficient shows that this ratio is exactly the multiplier on the new real operator.

At finite places the rational identity sends normalized old matrix coordinates X to νX in the new source frame. The conjugate pullback on the full lattice model takes the old vacuum to the new vacuum and intertwines the Heisenberg translations. This follows directly from negation of b in rational coordinates, so in this case no quadratic gauge factor is needed. By the positive vacuum coefficient argument of Proposition 503.1, the bare operators W_{G_0} agree under this transport. The finite ideal J_{G_0} is unchanged since multiplication by ν preserves the lattice. The conjugate ideal phase divided by the new one is consequently $\beta(\bar{J}_{G_0}/J_{G_0})$. The valuation calculation in the proof of Theorem 400.1 gives $\bar{J}_{G_0}/J_{G_0} = (\det_E G_0)_f$ as fractional ideals, where the right side is the ideal of the finite idele. Together with the real determinant just found this gives the factor $\upsilon_T(\det_E G_0)$. In the split matrix realization the same conjugate pullback is $f(X) \mapsto \overline{f(\nu^{-1}X)}$. Indeed this formula intertwines pure translations under $x \mapsto \nu x$, and the modulation character is conjugated since $\bar{\nu}\nu = -1$. It fixes the integral indicator, so Theorem 360.1 determines it on the test. It gives precisely D^-.

The determinant of $X \mapsto hXg^{-1}$ is $(\det h)^b (\det g)^{-a}$, by multiplying the column and row determinants. Thus multiplication of the conjugated old character ratio by $\upsilon_T(\det_E G_0)$ gives the stated new ratio. The product of the local antiunitaries and the rational antiunitary implement the same physical change. Their quotient is a unit scalar by Lemma 395.1. Evaluating at the origin proves the formula. Substitution gives the assertion about the point mass. For the adjoint repeat the compact-support integration and exhaustion argument of Corollary 504.3 with this expression for $\bar{K}$. □

506. The return operator on the prescribed source type

Lemma 506.1. *Use the real product group G_∞, kernel e, positive constant $d_e = e(1)$, and projection P_e^R of Lemma 492.1 in a strongly continuous unitary representation R. Write $K = \prod_\tau K_\tau$, $m(k) = \prod_\tau m_{\Lambda_\tau}(k_\tau)$ and $s = \prod_\tau \dim T_{\Lambda_\tau}$, with the conventions of Definition 100.1. Let Ω be any strongly continuous unitary representation of G_∞. Suppose f satisfies*

$$f = s \int_K m(k) \Omega(k) f \, \mathrm{d}k, \qquad \phi(g) = \langle f, \Omega(g) f \rangle \in L^1(G_\infty).$$

On every $y = P_e^R y$,

$$\int_{G_\infty} \phi(g) R(g) y \, \mathrm{d}g = \frac{\| P_{\bar{e}}^{\Omega} f \|^2}{d_e} \, y.$$

Proof. It suffices to consider $\|y\| = 1$. Its matrix coefficient is $\bar{e}(g)/d_e = \prod_\tau \psi_\tau(g_\tau)$ by Lemma 492.1. In the product of the cyclic row representations of Lemma 100.2, the tensor v of their unit generators has exactly this coefficient. The map $R(g)y \mapsto \pi(g)v$ therefore preserves all Gram entries by taking the coefficient at $g^{-1}g'$. Extending first linearly and then by norm limits identifies the two closed cyclic spans isometrically, intertwining translations. The latter span is the completed tensor product of the single-place cyclic spans, since products of independent translates give all their elementary spanning tensors.

On either cyclic span put $P_K = s \int_K \overline{m(k)} R(k)\,\mathrm{d}k$, using π in the tensor realization. It has rank one by Theorem 100.3: on elementary tensors it is the tensor product of the one-place projections, and the formula extends by boundedness. Its image contains v. Indeed the compact coefficient of each generator is m_{Λ_τ} by Definition 96.1 at positive polar part I and the compact convolution computation in Theorem 100.3. Its squared integral on the compact group is the reciprocal compact dimension by that computation. Hence the expectation of the orthogonal projection at the unit vector v is one. Transporting back, the image of P_K on the original cyclic span is $\mathbb{C}y$.

Write $v_f = \int \phi(g) R(g) y \,\mathrm{d}g$; absolute norm convergence places it in the cyclic span. After applying P_K, the coefficient of $R(g)y$ in the integral becomes

$$s \int_K \overline{m(k)} \langle \Omega(k) f, \Omega(g) f \rangle \,\mathrm{d}k = \phi(g).$$

This follows by left translation in g and the assumed compact identity for f. Absolute integrability on the product follows from $\phi \in L^1$, compactness and translation invariance. Thus $v_f \in \mathbb{C}y$. Its coefficient on y is

$$\langle y, v_f \rangle = d_e^{-1} \int_{G_\infty} \bar{e}(g) \langle f, \Omega(g) f \rangle \,\mathrm{d}g = d_e^{-1} \| P_{\bar{e}}^{\Omega} f \|^2$$

by the orthogonal projection in Lemma 492.1. Scaling proves the formula for arbitrary y. □

Theorem 506.2. *At the arithmetic first arrows of Theorem 498.2, use the Hermite test of Theorem 497.2 and the marked finite test of Proposition 498.1. Write A for the first lift and d_0 for the bounded opposite-source descent of Theorem 505.2, including c_- in its test. Let j be identification of opposite-source functions by Lemma 505.1, from original to opposite. On every vector y of the finite principal source level and real type e,*

$$d_0 A y = j\, C T_H y, \qquad C = \frac{\mathrm{vol}[U(T)]\, N_F\, \| Q_e f_\infty \|^2}{I_l e(1)} > 0.$$

This is an identity of vectors in the full opposite-source L^2 space. The return preserves the initial finite and real conditions under j. It preserves any additional integral invariance or finite group-algebra projection supported at the marked pairs that is imposed on this source subspace. It is injective there for all sufficiently large scales.

Proof. The integral for d_0 on compactly supported target vectors is jA^* there by Theorem 505.2. Its unique bounded extension is therefore jA^*. For arbitrary $x \in L^2([G])$ and any y at the indicated finite level, Theorem 491.2 gives

$$\langle Ax, Ay \rangle = \mathrm{vol}[U(T)]\, (N_F / I_l) \langle x, T_H L_\infty y \rangle.$$

At the marked prime-power depths the same formula follows before real compression in Proposition 498.1: the finite coefficient is zero off the principal level at the marked places, constant there with mass N_F / I_l, and the remaining factor is ρ_H. Only y must be fixed under the compact groups to replace each right-coset integral by its translation on y. At the real places $\phi(g) = \langle f_\infty, \Omega(g, 1) f_\infty \rangle$ is integrable by Theorem 480.2. The compact fixed-vector identity for f_∞ holds factor by factor by Theorems 495.1 and 497.2. Lemma 506.1 evaluates $L_\infty y$ as an actual multiple of y, with the displayed positive numerator by the latter arithmetic theorem. Testing against every x identifies $A^* A y = C T_H y$.

The finite sum defining T_H in Theorem 478.1 converges in operator norm on the fixed level by its summable coset bounds. It preserves the complementary unmarked level by left invariance of its coefficient. Each translate in the sum commutes with the marked group actions and with all real translations because it is supported away from those factors. Thus the finite and real conditions are preserved, including the additional marked projections. The same theorem and $H/2 - 2(r + z) = r - 2z$ give a strictly positive lower bound on

the real quadratic form of T_H at sufficiently large scales, as in Theorem 498.2. This proves injectivity. □

507. Convergence of the two complementary periods

Definition 507.1. Use the field, isometric frames and finite oscillator of Definitions 357.1 and 501.1. Let U_1, U_2, A, A' be positive-rank hermitian spaces with those frames, including opposites as in Lemma 505.1. Put $W = U_1 \perp U_2$, with the sum frames, and assume $\dim W = \dim A + \dim A'$. Retain a marked branch $w \neq \bar{w}$ above the rational prime p. Write k_w for its residue field and denote reduction there modulo its maximal ideal by a subscript 0. For any of these spaces M, put $h_M(s) = \max(1, \lambda_M(s)^{-1})$, where the length and minimum are those of Theorem 397.1, and use $[M]$ for the rational quotient of its adelic unitary group. For a residue matrix B write v_B for the oscillator vector named by the point mass at B, with the normalizing index of the finite Fourier sum in Proposition 357.2. For Schwartz real data f, f', use the kernels of Definition 504.1 on the pairs (W, A) and (A', W), respectively, permitting independent unitary determinant characters in the two kernels. Write

$$I_B(g) = \int_{[A]} K_{f,B}^{W,A}(g,h)\,\mathrm{d}h,$$
$$I'_C(g) = \int_{[A']} K_{f',C}^{A',W}(h,g)\,\mathrm{d}h.$$

Here $B : W_R \to A_R$ and $C : A'_R \to W_R$ are matrices in the finite reference frames over the residue ring R; a framed space of dimension j has coordinate module R^j. The kernels use v_B, v_C and the indicated Schwartz vectors in place of the Hermite vectors there.

Lemma 507.2. *In Definition 507.1 consider one pair (S,T), group arguments (s,t), and a named point D. When D_0 is injective there is a rational rank-one $z : S \to T$ such that $x = t^{-1}zs$ satisfies*

$$D_{x_f} v_D = \zeta v_D, \qquad \|x_\infty\| \le C\lambda_S(s)H(t).$$

When D_0 is surjective the same assertion holds with bound $CH(s)\lambda_T(t)$. In each case ζ is a nontrivial p-th root of unity. The constant is uniform in s, t.

Proof. Choose the finite scalar convention and the lattice $L_0 = \alpha \operatorname{Hom}(\Lambda_S, \Lambda_T)$ of Definition 357.1. Choose $\beta \in E^\times$ in $\delta\alpha\widehat{O}_E$ at every place except $\bar{w}$, with valuation $v_{\bar{w}}(\alpha) - 1$ there. The difference of the two fractional ideals obtained by increasing this single valuation bound by one is nonempty by Proposition 82.3. Here δ has components $\pi_v^{l_v}$ at the marked branches and one elsewhere.

For vectors $u \in L(s)$, $v \in L(t)$ take $z(y) = \beta v \langle u, y\rangle_S$ with the hermitian form linear in y. In finite coordinates let a be the reduction of $s_w^{-1}u$ and b^* the row given by the conjugate of the reduction of $t_{\bar{w}}^{-1}v$. The frames are isometries with identity finite Gram matrices, so both residues can be specified independently. Apart from its $\bar{w}$-component, x_f lies in δL_0. Translation by that part fixes v_D by the Fourier sum in Proposition 357.2: it fixes the vacuum and commutes with the translations indexed by $\bar{\delta}^{-1}L_-/L_-$, since their pairings are integral. This argument works at depth l_v by the same trace duality as in Proposition 501.2. The product-rule phase between the two parts of x_f is one, as the w-part has valuation at least $l_w \ge 1$ relative to L_0. The $\bar{w}$-part acts on the Fourier sum by reindexing. Its character at D is the additive character of the residue field evaluated on a fixed nonzero scalar times $b^* D_0 a$. Indeed the outer product at $\bar{w}$ pairs by matrix trace with D at w, and has at most one denominator relative to L_0. The character on this first residue layer is nontrivial by Lemma 355.2. The additive group of k_w has exponent p. Choose $t_0 \in k_w$ at which this additive character, with the fixed nonzero scalar in its argument, is nontrivial. Its value there is a nontrivial p-th root.

In the injective case take u short and primitive at w by Lemma 420.1. Then $D_0 a \neq 0$. Choose b^* taking value t_0 on that vector. This is possible by assigning its value in one

nonzero coordinate. Lift this residue to v of length at most $CH(t)$ by the other part of the same lemma. In the surjective case take v short and primitive at $\bar{w}$. Then $b^*D_0 \neq 0$. This row maps onto k_w, so choose a with $b^*D_0a = t_0$ and lift it to u of length at most $CH(s)$. The proof of the short-vector lemma applies to each space here: it uses integral lattices and the fixed real frames, both preserved under orthogonal sum and change of sign. The two vectors are nonzero, giving rank one, and the norm of their framed real outer product is bounded by a constant times the product of their lengths. This proves both estimates. □

Theorem 507.3. *If B_0 is onto and C_0 is one-to-one, both periods in Definition 507.1 converge absolutely and are continuous in g. For each integer $N \geq 0$ there are $C_1 > 0$ and integers $b, k \geq 0$ such that*

$$|K_{f,B}^{W,A}(g,h)| \leq C_1 p_k(f)\, h_A(h)^{-N} h_W(g)^b,$$
$$|K_{f',C}^{A',W}(h',g)| \leq C_1 p_k(f')\, h_{A'}(h')^{-N} h_W(g)^b.$$

Here p_k is the real Schwartz seminorm of Lemma 421.1, and constants can be chosen for all named points satisfying the stated ranks at fixed spaces and level. If y_i are rapidly decreasing on $[U_i]$ as in Proposition 398.1, then both kernels at $g = g_1 \oplus g_2$, multiplied by $y_1(g_1)y_2(g_2)$, are absolutely integrable on the respective triple quotients. Finite matched sums and all three integrations can be interchanged.

Proof. Use rational representatives with $H \ll h_M$ by Theorem 397.1. Its proof for an integral hermitian lattice applies also to the orthogonal sums here. Write $u = h_M(h)$ for the height of the variable of integration, where $M = A$ or A', and $v = h_W(g)$. Lemma 421.1 gives a coarse bound $Cp_j(f)u^a v^a$ for either kernel: its proof uses only the finite datum, the height of the Hom action and the real Schwartz seminorms. Use f' on the second pair. For $u > 1$, Lemma 507.2 gives a translation of real norm at most $C_2 v/u$; use its surjective case on the first pair and its injective case on the second.

Write $T(f)$ for the value of the fixed kernel. The rational translation preserves theta evaluation, and its finite part acts by ζ. Thus $T((D_{x_\infty}^{\infty} - 1)^j f) = (\zeta^{-1} - 1)^j T(f)$ for every integer $j \geq 0$, by the intertwining in Theorem 395.2. If $u > \max(1, C_2)v$, apply the coarse bound to the difference of order $N + a$. The real difference estimate of Lemma 421.1 and $|1-\zeta| \geq 2\sin(\pi/p)$ give a bound by $C_N p_k(f) u^{-N} v^{N+2a}$. For the other u, the coarse estimate itself gives this bound with an increased constant. There are only finitely many named points at the fixed level, so maxima of the constants and indices suffice.

Proposition 398.1 bounds the measure of $h_M \leq R$ by $C(1 + R)^d$, using comparability with the reduced height. With $N > d + 1$ the integral of h_M^{-N} converges by summing over $2^j \leq h_M < 2^{j+1}$. This proves absolute convergence. The endpoint height is bounded on compact sets since $h_W \ll H$; the kernels are continuous by Theorem 395.2 and its Schwartz convergence estimate. The same majorant therefore gives continuity by Lemma A.2. For the sum frames the finite lattice is a direct sum and the minimum on the sum is the smaller of its two minima. In fact the norm of a vector dominates the norm of each component and vectors with one component zero realize the two minima. Thus $h_W(g_1 \oplus g_2) = \max_i h_{U_i}(g_i)$. Bound its b-th power by $\prod_i h_{U_i}(g_i)^b$. The absolute integral is now bounded by a constant times the integral just estimated and the product of the two finite integrals $\int_{[U_i]} |y_i| h_{U_i}^b$ from Proposition 398.1. Each matched sum is finite and every matrix in it has the required rank by Definition 499.1. Lemmas A.2 and 123.1 now permit every interchange. □

508. A fixed pair of complementary frames

Definition 508.1. In Definition 507.1 write $a = \dim A'$, $d = \dim A$ and $N = a + d$. Choose a character ν of the adelic norm-one torus, trivial on rational points, whose restriction to its finite integral units is given by a residue character $\chi : R^\times \to \mathbb{C}^\times$. Explicitly, on a norm-one finite unit its value is χ of the tuple of the reductions at the chosen branches. Put

$\gamma = \nu^{-1} \upsilon_T^{\lfloor N/2 \rfloor}$, using Definition 376.1. In I_B take the two characters $(\gamma, 1)$, and in I'_C take (γ, γ), in the order of Definition 504.1. Identify W_R with $J = R^a \oplus R^d$, with the chosen ordering of the coordinates, and use x_0, y_0, X_k, Y_k, G_i from Definition 499.1. For residue functions supported on the indicated frames set $I_\Phi = \sum_B \Phi(B) I_B$ and $I'_\Psi = \sum_C \Psi(C) I'_C$.

Lemma 508.2. *For $k \in G_N$, let $\widetilde{k}$ be any integral finite unitary lift, supported over the marked split pairs with real component one. Such lifts exist. For all g and for all quotient and subspace frames X, Y one has*

$$I_{Xk^{-1}}(g) = I_X(g\widetilde{k}), \qquad I'_{kY}(g) = \chi(\det k) I'_Y(g\widetilde{k}),$$
$$I_{hX}(g) = \chi(\det h) I_X(g), \quad I'_{Yj}(g) = \chi(\det j) I'_Y(g)$$

where $h \in G_d$, $j \in G_a$.

Proof. At each chosen branch lift the entries of k to the local integers. Its determinant is a unit since its image modulo the maximal ideal is a unit. At the conjugate branch use the inverse conjugate transpose in the finite isometric frame. This preserves the hermitian form, and using the identity elsewhere gives $\widetilde{k}$. The same construction applies on A, A'. The splitting ideal phase in Theorem 390.2 is one for these integral isometries, since they preserve the reference lattice. Likewise υ_T is trivial on integral units by Theorem 374.2, so $\gamma(\det \widetilde{k}) = \chi(\det k)^{-1}$.

Acting on a point mass in the finite oscillator by the change of variables for source s and target t sends its name D to tDs^{-1}, before multiplication by the determinant character. This follows by reindexing the finite Fourier sum as in Corollary 357.3; at a higher depth the same reindexing is on the two dual boxes of Proposition 501.2. Consequently right multiplication of the ambient argument by $\widetilde{k}$ produces Xk^{-1} in the large kernel, and $\chi(\det k)^{-1}$ times the point mass kY in the small kernel. This proves the first two equations using the integral in Theorem 507.3.

In the large integral right translate its integration variable by $\widetilde{h}$. In the small integral right translate that variable by $\widetilde{j}^{-1}$. These compact translations preserve the rational quotient measure of Lemma 21.4: their modular function is one, since a compact subgroup of the positive real multiplicative group is trivial. The resulting factors are $\chi(\det h)^{-1}$ on the name hX and $\chi(\det j)^{-1}$ on the name Yj, respectively. This gives the second line. □

Theorem 508.3. *For any $c \in \mathbb{C}$ in Definition 508.1, the identity $I_\Phi(g) = c I'_\Psi(g)$ for every matched pair and every g is equivalent to the single identity of functions*

$$|G_d| I_{x_0} = c\, |G_a| I'_{y_0}.$$

For the graph sums of Theorem 499.3 on any subset B of ambient arguments it suffices that this last identity hold on the right translates $B\widetilde{k}_y$, where $k_y(v, b, w) = (v + yb, b, w)$ in the coordinates of that theorem. For $c \neq 0$ the multiplier in those finite frame sums in the orientation from I' to I is c^{-1}.

Proof. The basis of matched pairs in the proof of Proposition 499.2, applied to $k = 1$, consists on its supporting orbits of the functions $hx_0 \mapsto \chi(\det h)^{-1}$ and $y_0 j \mapsto \chi(\det j)^{-1}$. The second line of Lemma 508.2 evaluates their periods as $|G_d| I_{x_0}$ and $|G_a| I'_{y_0}$, which proves necessity. For a general orbit representative k the same basis calculation gives contributions

$$|G_d| \chi(\det k) I_{X_k}(g), \qquad |G_a| I'_{Y_k}(g).$$

By the first line of that lemma these are $\chi(\det k)$ times the respective expressions for $k = 1$ evaluated at $g\widetilde{k}$. Thus the single identity implies the one for every orbit, and addition proves sufficiency without division by c. In the graph calculation only the displayed shears occur, each with determinant one. Using these translates and the proof of Theorem 499.3 gives the restricted assertion and its reciprocal when $c \neq 0$. □

509. Inversion of the standard determinant tests

Definition 509.1. Use $J = R^a \oplus R^d$, $N = a + d$, $Q = |R|$ and the ordered coordinate identification in Definition 508.1. Write $\mathcal{G} = \{ky_0(R^a) : k \in G_N\}$. For an a-row matrix $S : J \to R^a$, define the pair

$$D_S(Y) = \chi^{-1}(\det(SY)), \qquad E_S(X) = \chi^{-1}\left(\det\begin{pmatrix} S \\ X \end{pmatrix}\right).$$

Determinants are relative to the ordered decomposition, and the character has value zero on nonunits. Denote projection onto the first summand by s_0.

Lemma 509.2. *Every (E_S, D_S) in Definition 509.1 is matched. The pair vanishes unless $S = s_0k^{-1}$ for some $k \in G_N$. For such k,*

$$E_{s_0k^{-1}}(X) = \chi(\det k)E_{s_0}(Xk), \quad D_{s_0k^{-1}}(Y) = D_{s_0}(k^{-1}Y).$$

Proof. If SY is invertible then the reduction of Y at every residue field has independent columns. Extend these to a basis at each field by successively adjoining coordinate vectors outside the previous span. Lift the new columns to the factors $O_w/\pi_w^{l_w}$ of R and combine the entries into columns over R. The completed determinant is a unit in every factor, so $Y = Y_k$. Applying this argument to the transpose of S shows $S = s_0k^{-1}$ whenever SY is invertible. If $\binom{S}{X}$ is invertible, its inverse itself completes X as a quotient frame and its first rows give S as a quotient frame. For Y_k, X_k, $\binom{S}{X_k}k$ has diagonal blocks SY_k, I_d and zero lower-left block. Its determinant is $\det(SY_k)$. This proves matching even when the common determinant factor is a nonunit. Finally $\binom{s_0k^{-1}}{X} = \binom{s_0}{Xk}k^{-1}$; this proves both transformation formulas. □

Theorem 509.3. *Suppose the characters χ, ψ on the multiplicative units and additive group of R, respectively, are primitive in Definition 500.1. Choose one frame Y_V for each $V \in \mathcal{G}$. For any matched pair (Φ, Ψ), put*

$$b_S = \sum_{V \in \mathcal{G}} \Psi(Y_V)\overline{D_S(Y_V)}.$$

Then b_S is independent of the choice of these frames, and

$$(\Phi, \Psi) = \frac{1}{Q^{ad}|G_a|} \sum_{S : J \to R^a} b_S(E_S, D_S).$$

Proof. For $Y = ky_0$ consider $F_Y(S) = D_S(Y)$ as a function of S. For its Fourier transform use the trace pairing with $Z : R^a \to J$. Write $Sk = (S_1, S_2)$. In the exponent $\operatorname{tr}(SZ) = \operatorname{tr}((Sk)(k^{-1}Z))$ the second component of $k^{-1}Z$ pairs with the free matrix S_2, while $F_Y(S) = \chi^{-1}(\det S_1)$. The character sum in S_2 vanishes unless this second component is zero, in which case it equals Q^{ad}, by the entrywise orthogonality in Lemma 500.2. When $Z = YB$ the remaining factor is $G_-\,\chi(\det B)$, where G_- is the transform of $\chi^{-1} \circ \det$ at I_a in that lemma. Thus $\widehat{F_Y}$ is supported precisely on YG_a, since $|G_-|^2 = Q^{a^2}$. Transforms attached to different submodules have disjoint supports. The same character orthogonality, expanding the product of one transform and the conjugate of the other, gives orthogonality of the original functions. At a single frame exactly $|G_a|Q^{ad}$ matrices S satisfy $SY \in G_a$, by the coordinates S_1, S_2. Consequently

$$\sum_S \overline{D_S(Y_V)}D_S(Y_U) = Q^{ad}|G_a|\,\mathbf{1}_{U=V}.$$

Every matched Ψ satisfies $\Psi(Y_Vj) = \chi(\det j)^{-1}\Psi(Y_V)$ for $j \in G_a$, as proved by the block calculation in Proposition 499.2. The corresponding law for D_S follows from its determinant. Their product in b_S is independent of j, as claimed. The orthogonality formula recovers Ψ on each representative, and equivariance recovers it on all frames. Both sides vanish off

the frames by Lemma 509.2 and the support condition. The first component of a matched pair is determined by its second via Definition 499.1. This proves the formula for both components. □

Corollary 509.4. *Under the hypotheses of Theorem 509.3 the identity of functions $I_{E_{s_0}} = cI'_{D_{s_0}}$ is equivalent to $I_\Phi = cI'_\Psi$ for every matched pair at this level. This equivalence applies to the squarefree levels and characters of Proposition 501.3.*

Proof. In Lemma 508.2 reindex the finite sum in X by Xk^{-1} and the one in Y by kY. The formulas of Lemma 509.2 then give

$$I_{E_{s_0k^{-1}}}(g) = \chi(\det k) I_{E_{s_0}}(g\widetilde{k}),$$
$$I'_{D_{s_0k^{-1}}}(g) = \chi(\det k) I'_{D_{s_0}}(g\widetilde{k}).$$

All other standard pairs vanish. The one equality therefore gives the identity on every standard pair; the identical coefficients in the reconstruction formula give it on all matched pairs. The converse follows by choosing the matched pair at s_0. On the stated levels the multiplicative primitivity is precisely Proposition 501.3 and the additive primitivity is Proposition 501.2. □

Proposition 509.5. *If $R = \mathbb{Z}/p^2\mathbb{Z}$, p prime, $a = d = 1$, and χ is inflated from $\mathbb{F}_p^\times$, the standard pairs need not span all matched pairs.*

Proof. The columns $Y = (1,0)^{\mathsf{T}}$ and $Y' = (1,p)^{\mathsf{T}}$ give distinct summands since their first entries are one. For every $S = (u,v)$ the numbers u and $u+pv$ have the same reduction modulo p. They are either both nonunits, or units with the same character value. Thus $D_S(Y) = D_S(Y')$ for every S. The single-orbit matched pair constructed in Proposition 499.2 for the summand of Y takes respective values $1, 0$ on these two frames, which is impossible for a combination of standard pairs. □

510. Fourier coefficients indexed by hermitian forms

Definition 510.1. Let the ambient space have a rational splitting $W = L \oplus W_0 \oplus L'$, with L, L' of dimension $b > 0$, both totally isotropic, mutually in duality, and orthogonal to the nonsingular W_0. Choose dual ordered bases of L, L'. Write J_0, J_T for the hermitian matrices of W_0 and a nonsingular space T, and assume both have unimodular finite lattices as in Lemma 403.1. For $X \in \mathrm{Hom}_E(W,T)$ use coordinates (x,m,u) on those three summands, so x, u are matrices with b columns. In $\mathrm{Hom}(W_0,T)$ choose a rational symplectic polarization with position space P_0 by the construction in Lemma 385.2. In the full oscillator take x and P_0 as positions, and u and the complementary space of the residual polarization as momenta. Use the trace pairing and the additive character $\mathbf{e}$ of Definition 385.1. For a test f in the product of these rational coordinates put $f_{g,h} = \Omega(g,h)f$, with any unitary pair characters as in Definition 504.1. For rational x set

$$t_x(g,h;f) = \sum_{\xi \in P_0(\mathbb{Q})} f_{g,h}(x,\xi).$$

Write $\mathcal{Z}$ for the additive rational space of skew-hermitian b-by-b matrices over E and $[\mathcal{Z}] = \mathcal{Z}(\mathbb{A}_\mathbb{Q})/\mathcal{Z}(\mathbb{Q})$. For $z \in \mathcal{Z}(\mathbb{A}_\mathbb{Q})$ set $n(z)(l,w,l') = (l + zl', w, l')$ in W. For each hermitian rational b-by-b matrix B, set

$$\xi_B(z) = \mathbf{e}\left(-\tfrac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(\jmath\,\mathrm{tr}(zB))\right).$$

For a continuous automorphic function I on $[W]$ define its B-coefficient at g by

$$\int_{[\mathcal{Z}]} \overline{\xi_B(z)} I(n(z)g)\,\mathrm{d}z$$

with additive probability measure. The space $[\mathcal{Z}]$ is compact since any rational basis identifies it with a finite product of $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$, as in Lemma 356.2. Both factors in the integrand are invariant under rational translations of z.

Lemma 510.2. *In Definition 510.1,*

$$K_f(n(z)g,h)=\sum_{x\in \mathrm{Hom}_E(L,T)} \xi_{x^*J_Tx}(z)t_x(g,h;f).$$

The sum converges absolutely, uniformly in z for fixed g,h. Moreover $t_{\delta x}(g,\delta h;f)=t_x(g,h;f)$ for $\delta\in U(T)(E)$.

Proof. In the dual bases the off-diagonal Gram block of W is I_b. Thus $n(z)$ preserves the form since $z+z^*=0$, and on the Hom space its action is $(x,m,u)\mapsto(x,m,u-xz)$. The factors on $\mathrm{Hom}(L\oplus L',T)$ and $\mathrm{Hom}(W_0,T)$ split in the oscillator: choose orthogonal real frames and a direct sum of the hyperbolic dual lattices and the residual lattice, as in the proof of Theorem 403.2. There the vacuum coefficient, the splitting character and the translation action give precisely the product operators, and Lemma 403.1 allows the original reference frame. The shear is the identity on the residual factor. On the hyperbolic part the position block is the identity and the lower block C sends x to $-xz$. By Theorems 394.2 and 386.1 its action on a position value is multiplication by $\mathbf{e}(-a(x,Cx)/2)$. All determinant factors are one. Indeed $a(x,Cx)=\mathrm{Tr}_{E/\mathbb{Q}}(\jmath H(Cx,x))$; the trace in H is $\mathrm{tr}((-xz)^*J_Tx)=\mathrm{tr}(zx^*J_Tx)$ by Definition 355.1. This gives the specified sign and character. The rational theta sum in Theorem 385.3 now gives the asserted expansion. Its sum of absolute values before grouping by x is finite by Proposition 388.2. The phase has modulus one, giving uniformity in z.

A rational target isometry δ acts on the hyperbolic part by $x\mapsto\delta x$ and the dual action on u. The same parabolic formula gives the substitution $x\mapsto\delta^{-1}x$ on functions, with factor one: the adelic determinant modulus of a rational invertible matrix is one and υ is trivial on rational determinants by Theorem 374.2. The residual operator preserves its rational theta evaluation by Theorem 395.2. The pair character is trivial on the rational point as well. Evaluating at δx proves the covariance. □

Theorem 510.3. *In Definition 510.1 take f to be a finite sum of real Schwartz tests times point data mapping onto T at the chosen branch of Definition 507.1. Set $I(g)=\int_{[T]}K_f(g,h)\,\mathrm{d}h$. For any rational hermitian matrix B its coefficient is*

$$\int_{[T]}\sum_{\substack{x\in \mathrm{Hom}_E(L,T)\\ x^*J_Tx=B}} t_x(g,h;f)\ \mathrm{d}h.$$

The grouped integrand is absolutely integrable. Truncating the outer integral to any increasing compact exhaustion of the quotient gives the same value in the limit. In particular the coefficient vanishes if B has no representation over E by T. For nonsingular B with a representation x_*, all the indices of the sum lie in the single orbit $U(T)(E)x_*$; its rational stabilizer acts identically on $x_*(L)$ and is the full unitary group of its orthogonal complement there.

Proof. The proof of Theorem 507.3 for a surjective matrix uses only W,T, its ranks at the chosen prime and its two heights. Thus it applies to this pair. A compact set of z represents $[\mathcal{Z}]$; its elements $n(z)g$ have bounded height for fixed g by Theorem 397.1. The bound for the full theta kernel by $C_Nh_T(h)^{-N}$ on this set is integrable for large N. Lemma 123.1 therefore interchanges the two integrations. For fixed h, Lemma 510.2 permits termwise integration on the compact group. Characters with different hermitian indices are distinct. Indeed for a nonzero difference D, at a complex place take $z=r\jmath^{-1}D$ there and zero at the other places, with $r\in\mathbb{R}$ arbitrary. This component is skew-hermitian, and $\mathrm{tr}(D^2)=\sum_{ij}|D_{ij}|^2>0$ at that place. By varying r the value of the character differs from one. Translation of additive probability by such a z shows its integral is zero; the trivial character has integral

one. This proves the formula. Pointwise the absolute value of the retained sum is bounded by the integral of the absolute value of the full theta kernel on $[\mathcal{Z}]$, which has the same integrable majorant. Covariance in Lemma 510.2 makes the grouped sum a function on $[T]$. Lemma A.2 gives the asserted exhaustion and the empty index set gives the vanishing.

For nonsingular B, every representing matrix is injective since its kernel is in the kernel of B. The rule $x_*v \mapsto xv$ defines an isometry between nonsingular rational subspaces. Lemma 381.1 extends this map to a rational isometry of T. A unitary fixing all columns of x_* preserves their perpendicular; conversely the direct sum of the identity on their span and any unitary on that perpendicular fixes the matrix. This proves the orbit and stabilizer assertions. □

511. Absolute unfolding of a nonsingular Gram coefficient

Definition 511.1. Use Definition 510.1 with the marked branch w of Definition 507.1. In the marked finite frame of W put

$$M_{g,w} = g_w^{-1}(L^\perp \otimes_E E_w) \cap \Lambda_{W,w}.$$

Here $L^\perp = L \oplus W_0$. Write $\overline{M}_{g,w}$ for the image of $M_{g,w}$ in the residue vector space of the lattice. For a named point $D : W_R \to T_R$ the condition at g is that D_0 maps $\overline{M}_{g,w}$ onto the full target residue space. For a finite linear combination of point data impose the condition separately on each point with nonzero coefficient.

Theorem 511.2. *Fix g and a named point D satisfying Definition 511.1. Write $f = v \otimes v_D$, with v a real Schwartz function. For each integer $M \geq 0$ there are $C_{g,M} > 0$ and an integer $j \geq 0$, independent of h, v, such that*

$$\sum_{x \in \mathrm{Hom}_E(L,T)} |t_x(g,h;f)| \leq C_{g,M}\, p_j(v)\, h_T(h)^{-M}.$$

In particular the sum of the absolute values is integrable on $[T]$. These conclusions apply by addition to finite sums of the allowed elementary data.

Proof. Fix g. Each residue $a \in \overline{M}_{g,w}$ lifts to a rational $u \in L^\perp$ with $g_f^{-1}u$ integral at all places and residue a at w. Indeed pull the compact open lattice $g_f\Lambda_W \cap L^\perp(\mathbb{A}_{E,f})$ back to a rational basis of $L^\perp$. It contains an integer multiple of the standard integral lattice and lies in an inverse-integer multiple. Increasing the first integer by a factor divisible by p if needed gives a contained lattice mapping to zero in the required residue. The quotient of these two bounding lattices is finite and every class lifts to the rational space, by the integer Chinese remainder construction in Lemma 13.4. Lift a local representative in the specified intersection, setting the other components to zero. Fix one rational lift for every residue. Their g-lengths are bounded by a constant depending on g.

Choose $v' \in L(h)$ short and primitive at $\bar{w}$ by Lemma 420.1; here $L(h)$ is the target lattice of Theorem 397.1. Its conjugate residue row b^* is nonzero. By the surjectivity hypothesis the row b^*D_0 on $\overline{M}_{g,w}$ maps onto k_w. Take an a on which its value is the t_0 selected in Lemma 507.2. Use its fixed lift u to form $z = \beta v'\langle u, \cdot\rangle_W$ with the scalar of Lemma 507.2. That lemma's finite calculation gives $D_{e_f} v_D = \zeta v_D$ for $e = h^{-1}zg$, with $|1-\zeta| \geq 2\sin(\pi/p)$; the outer product bound gives $\|e_\infty\| \leq C_g \lambda_T(h)$. Crucially $z|_L = 0$ since $u \in L^\perp$.

A rational translation by this z preserves *each* functional $F \mapsto \sum_{\eta \in P_0(\mathbb{Q})} F(x,\eta)$. In the formula of Theorem 385.3 its position increment in x is zero, its residual position increment is rational, and all its phase arguments at the rational positions (x,η) are rational. The additive character there is one. Moving the translation through $\Omega(g,h)$ therefore gives, simultaneously for all x,

$$t_x(g,h;(D_{e_\infty}^{\infty} - 1)^j v \otimes v_D) = (\zeta^{-1} - 1)^j\, t_x(g,h;v \otimes v_D) \qquad (j \geq 0).$$

This follows for $j = 1$ from the intertwining in Theorem 395.2, and iteration is valid for every Schwartz vector.

The sum on the left of the proposed estimate is invariant under rational left translation of h, by Lemma 510.2. Take a representative with $H(h) \ll h_T(h)$. Before using the translation, the sum is at most $\sum_{x,\eta} |f_{g,h}(x,\eta)|$; this in turn is at most $C_g p_i(v) h_T(h)^k$ for fixed integers i, k. This is Proposition 396.2 in the present rational polarization, with the seminorm dependence computed in Lemma 421.1. When $h_T(h) > \max(1, C_g)$, increase C_g if needed so $\|e_\infty\| \le 1$. Apply this coarse sum bound to the difference of order $j = M + k$, and the preceding identity and the small difference estimate of the same lemma to get $C_{g,M} p_{i+2j}(v) h_T(h)^{k-j}$. At smaller $h_T(h)$ the coarse estimate suffices with a larger constant. Finally Proposition 398.1 and its dyadic sublevel estimate give integrability on the quotient. □

Corollary 511.3. *Fix a nonsingular represented rational hermitian $B = x_*^* J_T x_*$, and let $\Gamma_* = \{\delta \in U(T)(E) : \delta x_* = x_*\}$. For the data of Theorem 511.2, the B-coefficient of I at g is the absolutely convergent integral*

$$\int_{\Gamma_* \backslash U(T)(\mathbb{A}_F)} t_{x_*}(g, h; f)\, \mathrm{d}h, \qquad F = E^+.$$

The measure is induced by the same Haar measure used for $[T]$. When T is the orthogonal sum of the framed space $x_* L$ and T_1, the group Γ_* is $U(T_1)(E)$ embedded as the identity on $x_* L$; when $\dim T = b$ it is trivial.

Proof. The group and orbit are precisely those of Theorem 510.3. Use the covariance of Lemma 510.2 and apply Lemma 384.1 to the singleton x_*. Its integrability requirement follows from Theorem 511.2 since the sum on the orbit is bounded in absolute sum by the sum over all rational matrices. The lemma integrates over the disjoint translates of a domain for the rational group, so the Haar normalizations agree. The description of the rational stabilizer gives both special cases. □

Proposition 511.4. *In Definition 509.1 take $a = b$, $d = \dim W - b = \dim T$. There is a quotient frame S for which the standard test E_S satisfies Definition 511.1 at every adelic g preserving L. The criterion of Corollary 509.4 may use this S in place of s_0.*

Proof. At w the intersection $L^\perp(E_w) \cap \Lambda_{W,w}$ is a direct summand of rank d of the valuation-ring lattice. To see this, clear denominators of a basis of $L^\perp(E_w)$ and diagonalize its column matrix by integral row and column elimination, successively using an entry of least valuation as pivot. In the resulting ambient basis the field subspace is a coordinate subspace, so its full intersection is the corresponding integral coordinate summand. Choose S_w to project away from this summand, then reduce modulo $\pi_w^{l_w}$. At other marked branches use any coordinate projection of rank b. Their product is a quotient frame by the completion argument of Lemma 509.2. An isometry preserving L preserves $L^\perp$, so $\overline{M}_{g,w} = \ker(S_w \bmod \pi_w)$. Whenever $E_S(D) \neq 0$, the matrix $\binom{S}{D}$ is invertible over the residue ring. Its reduction shows that D_0 maps the indicated kernel isomorphically to the d-dimensional target residue space. This proves the rank condition. Finally $S = s_0 k^{-1}$ for some k; the last two period transformation formulas in the proof of Corollary 509.4 are invertible and have the same character factor. □

512. The small period in the same polarization

Lemma 512.1. *In Proposition 503.1 use the chosen set M for naming the original datum and $\bar{M}$ for naming the reversed datum. At conjugate places take conjugate uniformizer powers, so entrywise field conjugation identifies the two residue rings. Write Z^* for conjugation between these rings followed by transposition. The finite antiunitary C_η takes v_D to the reversed test named at $\bar{M}$ by $Z \mapsto \overline{D(-Z^*)}$.*

Proof. In the split model at M, a displacement supported on M is a translation in the position variable X, normalized by α, and a displacement on $\bar{M}$ is a modulation, by Theorem 360.1. In the reversed model take the position variable Z on $\bar{M}$. Under η a position displacement x goes to $-x^*$, now again a position displacement, and the same formula sends the modulation variable to a modulation variable. The pairing character of the two transformed displacements is the conjugate of the original one, since η negates the alternating pairing by Proposition 503.1. On each pure branch the quadratic change to symmetric translations is one. It follows that $\phi(X) \mapsto \overline{\phi(-Z^*)}$ implements both intertwining relations. It fixes the integral indicator since conjugation and transposition take one integral box onto the other. The vacuum translates span the split test space by Theorem 360.1, so the formula agrees with C_η. Off the marked pairs the vacuum maps to the vacuum. Reduction at the prescribed conjugate depths gives the stated residue function. □

Theorem 512.2. *Use Definition 508.1, with $a = \dim A'$, $N = \dim W$. On $T = (A')^-$ use the finite frame $\epsilon C_{A'}$ of Lemma 505.1, where $\epsilon\bar{\epsilon} = -1$. Given small-side residue data D read at M, read the following data at $\bar{M}$ on the pair (W, T):*

$$D^\flat(Z) = D(-\epsilon^{-1}Z^*), \qquad a^\flat = \gamma^{-1}\upsilon_T^N, \qquad b^\flat = \gamma^{-1}\upsilon_T^a.$$

Here the symbol υ_T denotes the splitting character on the norm-one torus from Definition 376.1. For a Schwartz real test v on the small pair there is a Schwartz test $v^\flat$, obtained by a fixed linear coordinate identification of real oscillator spaces. For some $|c_0| = 1$ independent of the tests and group arguments,

$$K_{v,D}^{A',W}(h, g) = c_0 K_{v^\flat,D^\flat}^{W,T}(g, hs_\alpha), \qquad s_\alpha = \bar{\alpha}/\alpha \quad \text{at finite places}, \quad s_{\alpha,\infty} = 1.$$

The new kernel uses the characters $a^\flat, b^\flat$. If D is supported on subspace frames, its period is c_0 times the period of the new kernel with arguments (g, h).

Proof. Conjugation to the opposite source in Theorem 505.2 first changes the name to $D^-(X) = \overline{D(\epsilon^{-1}X)}$ and changes the two characters to $\gamma^{-1}\upsilon_T^a, \gamma^{-1}\upsilon_T^N$. Now apply the rational adjoint conjugation of Proposition 502.2 on this opposite pair. As in the proof of Theorem 504.2, its real implementation is C_∞, and at finite places the normalized adjoint η differs from the rational adjoint by $\alpha/\bar{\alpha}$. Multiplication by s_α in the reversed target group corrects this difference. The quotient of these implementations is a unit scalar by Lemma 395.1. The adjoint interchanges the two characters. Lemma 512.1 gives directly $\overline{D^-(-Z^*)} = D(-\epsilon^{-1}Z^*)$. Both uses of conjugation are antiunitary, and their product is linear. At infinity they act on real position functions by conjugation and fixed signed coordinate changes, as shown in Theorems 502.3 and 505.2. These formulas preserve Schwartz tests and give $v^\flat$. The two scalar comparison factors give c_0.

The groups of A' and T and their quotient measures agree by Lemma 505.1. The old period converges absolutely by Theorem 507.3. On the new side translation by s_α preserves quotient Haar measure: this element is central, so its right translation is also left translation upstairs. Consequently the shift disappears in the period integral. □

Corollary 512.3. *In Proposition 511.4 choose the standard test at a frame S whose integral lift at w has kernel $L^\perp(E_w) \cap \Lambda_{W,w}$. For every g preserving L, every named point in the conjugate-branch test for D_S satisfies Definition 511.1 at $\bar{w}$. If a nonsingular B is represented by x_* in $T = (A')^-$, then the small-side B-coefficient equals*

$$c_0 \int_{U(T)(\mathbb{A}_F)} t_{x_*}(g, h; v^\flat \otimes v_{D_S^\flat})\, \mathrm{d}h.$$

Here a finite combination of point vectors has the linear meaning of Definition 508.1. For a nonsingular B unrepresented by T, this coefficient is zero.

Proof. At the split pair the finite frames identify the lattice on $\bar{w}$ with the conjugate dual of the one on w. The columns of the conjugate transpose of the chosen integral lift of S_w form

a basis of the annihilator of its kernel: a split surjection onto the free module identifies that annihilator with the dual of the quotient, by extending a basis of the kernel to a basis of the lattice. The field span of the annihilator is $L(E_{\bar{w}})$, by the perfect hermitian pairing and the identity $(L^{\perp})^{\perp} = L$. It is the full intersection with the lattice, by the same dual description. In particular its reduced columns lie in $\overline{M}_{g,\bar{w}}$ for every g preserving L, since $L \subset L^{\perp}$.

For a point Z of the support the determinant of $S(-\epsilon^{-1}Z^{*})$ is a unit by Definition 509.1. Conjugation, transposition and multiplication by the unit scalar show that ZS^{*} is invertible at the conjugate branch. Thus Z_0 maps the reduced columns just described onto the target residue space. Corollary 511.3 applies to every term of this finite combination. The target has dimension b, so the stabilizer of a frame of nonsingular Gram matrix is trivial. Using Theorem 512.2 gives the displayed formula, and Theorem 510.3 gives the vanishing in the absence of a frame. □

513. Support of the individual frame terms

Lemma 513.1. *Use the mixed polarization of Definition 510.1, with $gL = L$ and $g_L = g|_L$. At each place the map induced on the frame coordinate by the Hom action is $R(x) = hxg_L^{-1}$. At a real place suppose the original Schwartz test vanishes for frame coordinates outside a closed set C. The moved test vanishes for frame coordinates outside $R(C)$.*

At a chosen branch v of a split pair write $\mathfrak{l} = L(E_v) \cap \Lambda_{W,v}$. For a finite point test v_D choose any integral lift $\widetilde{D}$ of the residue matrix there in the full finite frames. If $t_x(g,h;f) \neq 0$, then in these frames

$$(h^{-1}xg_L)_v\big|_{\mathfrak{l}} \;\in\; \alpha_v\widetilde{D}\big|_{\mathfrak{l}} + \alpha_v\pi_v^{l_v}\mathrm{Hom}_{O_v}(\mathfrak{l},\Lambda_{T,v}).$$

A linear combination of point tests uses the union of their cosets.

Proof. Restriction of Hom matrices to L intertwines the two linear maps since $(hXg^{-1})|_L = h(X|_L)g_L^{-1}$. For its dual variables use displacements in the momentum summand $\mathrm{Hom}(L',T)$, extended by zero on $L^{\perp}$. The alternating form pairs them perfectly with the frame position and pairs them trivially with the residual variables, as seen from the off-diagonal Gram blocks in Lemma 510.2. Such displacements remain in that summand under the Hom action since $gL^{\perp} = L^{\perp}$. On functions they multiply by the additive character of a linear functional on the frame coordinate, by Theorem 385.3. Their quadratic gauge factor is one on the isotropic momentum summand. The intertwining law in Theorem 395.2 therefore transports the multiplier of a functional ℓ to the multiplier of $\ell \circ R^{-1}$.

At infinity let b be a smooth function of compact support on the frame coordinates. Fourier inversion expresses its multiplication operator as the integral of the character multipliers against $\widehat{b}$ in real dual coordinates. Indeed integration by parts in each coordinate gives an integrable Fourier transform, and the inversion formula follows by the Gaussian insertion in Lemma 392.1, one coordinate at a time. The identity can be integrated in L^2 because the multipliers are unitary and $\int |\widehat{b}| < \infty$. Thus conjugation carries multiplication by b to multiplication by $b \circ R^{-1}$. Take b supported in the complement of C and nonzero at the preimage of a proposed position outside $R(C)$. Such a bump on a small ball follows by scaling $\exp(-1/(1-|y|^2))$ for $|y| < 1$, extended by zero: each derivative approaches zero at the boundary since an exponential dominates any fixed power. The old product is zero, so the moved product is zero in L^2 and pointwise by continuity. This proves the real assertion.

At the split place the branch opposite v contains the dual of the whole position matrix space on v, by Theorem 360.1. Restrict its modulations to the dual of the frame coordinate just described. The original support on the chosen branch is $\alpha_v(\widetilde{D} + \pi_v^{l_v}\mathrm{Hom}_{O_v}(\Lambda_{W,v},\Lambda_{T,v}))$. Its restriction to $\mathfrak{l}$ is exactly the coset in the assertion: $\mathfrak{l}$ is an integral direct summand by the elimination argument in Proposition 511.4 applied to L, so every integral map on it extends. Every character trivial on the difference lattice of that coset acts on the named vector by its value on the coset. After transport it acts on the moved test by its

value on $R^{-1}x$, independently of the residual position. If $R^{-1}x$ lies outside the coset some such character distinguishes it. Indeed choose bases for the two lattices, divide by $\alpha_v \pi_v^{l_v}$, and take a nonintegral entry of the difference. By the additive annihilator calculation of Proposition 501.2 an integral multiple of this entry has nontrivial normalized character. The corresponding dual functional is trivial on all integral entries. Hence the moved slice at that x is zero pointwise. Summation over residual positions proves the claim for t_x, and linearity proves the last assertion. □

Theorem 513.2. *For the standard test S of Corollary 512.3 all singular Gram coefficients of the small period vanish at g preserving L. Every nonzero coefficient has index rationally isometric to $(A')^-$. For the corresponding large-side test E_S, every contributing rational matrix has rank b. If the large real test, in its mixed position variables, is supported at each infinite place in a closed set of frames whose Gram matrices are nonsingular of the signature of $(A')^-$ there, every large-side coefficient of a different signature or of singular index vanishes at these g.*

Proof. On the small side use the conjugate marked branch of Theorem 512.2. The proof of Corollary 512.3 shows that each support matrix restricts invertibly to the reduction of ſ: the columns of the conjugate transpose of the lifted S are an integral basis there. Lemma 513.1 now shows that $\alpha_v^{-1}(h^{-1}xg_L)_v$ is an integral isomorphism of lattices whenever $t_x \neq 0$. Its determinant is a unit by reduction. Since h, g_L and α_v are invertible in the field, x is invertible at this place and hence over E. Thus $x^* J_T x$ is a nonsingular rational form isometric to $T = (A')^-$. The coefficient formula of Theorem 510.3 gives the two assertions for the small period.

On the large side, every named point in E_S is invertible on the reduced lattice of $L^\perp$ by Proposition 511.4. The lattice of L has independent reductions inside that of $L^\perp$, since it is a direct summand there by the same basis elimination. Thus the restriction to ſ has rank b modulo the maximal ideal. The coset condition of Lemma 513.1 implies full column rank at w, hence rational rank b. At infinity that lemma puts $h^{-1}xg_L$ in the prescribed closed set whenever the slice is nonzero. Its Gram matrix is $g_L^*(x^* J_T x) g_L$, since h is unitary. Congruence by an invertible matrix preserves nonsingularity and signature: it bijects the subspaces on which the form is positive definite and also those on which it is negative definite. The maximal dimensions of these two kinds of subspaces are the respective sign counts by diagonalization as in Lemma 381.1. Thus the retained rational indices have the claimed signatures, and the same coefficient formula finishes the proof. □

514. Real data on a neighborhood of a nondegenerate plane

Lemma 514.1. *Let J be an invertible hermitian complex m-by-m matrix, X_0 an m-by-b frame with invertible $J_b = X_0^* J X_0$, and $P_0 = X_0 J_b^{-1} X_0^* J$. For X with $X^* J X$ invertible set $P_X = X(X^* J X)^{-1} X^* J$. On the neighborhood $\|P_X - P_0\| < 1$ in these frames there are smooth functions $k(P_X) \in U(J)$ and $c(X) \in \mathrm{GL}_b(\mathbb{C})$ with*

$$k(P_0) = I, \qquad X = k(P_X) X_0 c(X), \qquad c(X)^* J_b c(X) = X^* J X.$$

The norm is the Euclidean operator norm. For every invertible D one has $P_{XD} = P_X$ and $c(XD) = c(X)D$.

Proof. Write $M^\dagger = J^{-1} M^* J$. The formula for $P = P_X$ gives $P^2 = P = P^\dagger$ and $\operatorname{im} P = \operatorname{im} X$; the corresponding identities hold for P_0. Put

$$A = PP_0 + (1-P)(1-P_0), \quad Z = (P - P_0)^2, \qquad k(P) = A \sum_{n \geq 0} 4^{-n} \binom{2n}{n} Z^n.$$

Then $PA = AP_0$, $Z^\dagger = Z$, $ZP_0 = P_0 Z$ and $A^\dagger A = 1 - Z$, by multiplication and idempotence. Since $\|Z\| < 1$, the series is convergent and smooth there. Indeed its coefficient is at most one, and on $\|Z\| \leq r < 1$ any derivative of order s has summable bounds by $n^s r^{n-s}$ times the

norms of the s directions. For $u(z) = \sum 4^{-n}\binom{2n}{n} z^n$, the ratio of successive coefficients gives $2(1-z)u' = u$, $u(0) = 1$; hence $(1-z)u^2 = 1$. Multiplication of the absolutely convergent matrix series thus gives $k(P)^\dagger k(P) = I$. This identity implies invertibility by determinants. Since $Pk(P) = k(P)P_0$, $k(P)$ carries the original plane onto $\operatorname{im} X$, and at $P = P_0$ it is the identity. Take $c(X) = J_b^{-1} X_0^* J k(P)^{-1} X$. This is the unique coordinate matrix of the frame in the transported basis and is invertible. Taking Gram matrices gives the asserted equality. The projection formula is invariant under $X \mapsto XD$; the coordinate formula then gives the final transformation. □

Lemma 514.2. *For the real oscillator on a hermitian Hom space used in Definition 504.1, with determinant characters of integer weights, the function $k \mapsto \pi(k)v$ near the identity in either real unitary factor is smooth in every Schwartz seminorm when v is Schwartz.*

Proof. In real symplectic position and dual coordinates write the matrix of the Hom action as $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Near the identity it factors as the lower shear CA^{-1}, the diagonal with blocks $A, A^{-\mathsf{T}}$, and the upper shear $A^{-1}B$, in that order. Indeed preservation of the alternating pairing makes both shear matrices symmetric and gives $D = CA^{-1}B + A^{-\mathsf{T}}$. By Theorem 386.1 and Lemma 392.1, unitary implementers of these factors are respectively multiplication by a quadratic phase, normalized linear substitution, and a quadratic phase conjugated by a fixed Fourier transform. The corresponding product $V(k)$ acts smoothly in Schwartz seminorms. To see this, each parameter derivative of the phase inserts a polynomial in the position variables. Differentiation of a linear substitution inserts a first derivative multiplied by a linear function. On a compact neighborhood the substitution and its inverse have bounded norms. These observations bound each differentiated expression by finitely many original seminorms; the integral mean-value formula in the parameter verifies convergence of the difference quotients in each seminorm. Fourier transform is continuous in these seminorms by differentiation and integration by parts in Lemma 392.1.

The true implementer differs by a scalar. Its vacuum coefficient is the inverse determinant of the negative diagonal block by Theorem 393.1, before multiplication by the pair character. The vacuum coefficient of $V(k)$ is smooth by the preceding seminorm bounds and equals one at identity. In a smaller neighborhood the ratio of the prescribed coefficient to the computed one is therefore smooth. The determinant characters there are integer powers of smooth unit complex determinants by Definition 372.1. Multiplying by these scalars proves the claim. □

Theorem 514.3. *At the infinite places take $J = J_b \oplus J_n$, with signed diagonal forms of ranks $b, n > 0$, and $X_0 = \binom{I_b}{0}$ in Lemma 514.1. Let $n = 2j$ and μ_τ be nonnegative integer weights. Use the mixed positions of Definition 510.1 for targets of rank $b+n$ and b. In the residual oscillator for the first target identify the two orthogonal factors as in Theorem 403.2. Let v, v_1 be arbitrary real Schwartz vectors on the two respective factors. Denote by π the residual representation of the real target group with any fixed unitary determinant characters of integer weights. There exist real oscillator data F, F' of Schwartz class and a nonnegative smooth function ρ on $\prod_\tau U(J)$, positive near identity, with these properties. The opposite slice of F' at zero equals v in self-dual Fourier coordinates. The frame support of F is compact and has nonsingular Gram matrices of signature J_b at each place. For every $h \in \prod_\tau U(J)$ some $p \in \prod_\tau (U(J_b) \times U(J_n))$ satisfies, for every $B \in \prod_\tau \mathrm{GL}_b(\mathbb{C})$,*

$$\pi(ph)F((ph)^{-1}X_0 B) = \rho(h) \prod_\tau \left[\left(\frac{\det B_\tau}{\overline{\det B_\tau}} \right)^{-\mu_\tau} |\det B_\tau|^{-2j} \right] (F'(B) \otimes v_1).$$

Here evaluation in a frame variable denotes the remaining Schwartz vector. The last product, including both determinant factors at each place, is the inverse of the archimedean base-change character of weight μ times $\prod_\tau |\det B_\tau|^{-n}$.

Proof. Choose smooth nonnegative bumps $\beta(P)$ and $\eta(C)$ on the finite products of matrix spaces, positive near P_0 and I, respectively. For β take compact support so small that

$\|P - P_0\| < 1$ placewise and the $k(P)$'s on the support are in the neighborhoods of Lemma 514.2. For η take compact support inside the invertible matrices. Scaled translates of the ball bump in Lemma 513.1 give these choices. Use self-dual real position measure $\mathrm{d}C$ for the mixed coordinates on the small space and put

$$m(C) = \eta(C) \prod_\tau |\det C_\tau|^{2(j+\mu_\tau)}, \qquad A = \left(\int m(C)\, \mathrm{d}C \right)^{-1}.$$

The integral is finite and positive by compact support and positivity near the identity. Set $F'(C) = Am(C)v$. Exchanging the two isotropic spaces is partial Fourier transformation in C; evaluation at zero is integration in C, by Lemma 392.1 in dual coordinates. This gives the stated opposite slice.

For $P = P_X$ and $c = c(X)$ in the plane chart put

$$F(X) = A\, \beta(P) \eta(c) \prod_\tau \overline{\det c_\tau}^{\,2\mu_\tau}\, \pi(k(P))(v \otimes v_1),$$

and extend by zero outside the chart. Its frame support lies in the image under $(P, c) \mapsto k(P) X_0 c$ of a compact set: use the intersection of the support of β with the closed set of self-adjoint idempotents and the support of η. On the component used here the rank equals b, since $k(P)$ carries P_0 onto P. These image frames have invertible Gram matrices of the desired signature and lie inside the chart. The extended function is smooth by Lemmas 514.1 and 514.2; all derivatives in the frame variables take bounded values in every residual Schwartz seminorm on this compact support. Thus it is Schwartz in the joint position variables and defines real oscillator data.

Put $P = P_{h^{-1} X_0}$ and $\rho(h) = \beta(P)$. This plane is always nondegenerate. If $\rho(h) = 0$, take $p = 1$; right multiplication of the frame by B leaves its plane unchanged, so both sides vanish. Otherwise $h^{-1} X_0 = k(P) X_0 c_0$, with $c_0 \in U(J_b)$ by its Gram matrix. Hence $hk(P)$ preserves the original plane and its perpendicular, and acts there as c_0^{-1} and some $q \in U(J_n)$. Take $p = \mathrm{diag}(c_0, q^{-1})$ at each place. Then $ph = k(P)^{-1}$. The argument of F has plane P and coordinate $c = B$. The two group operators cancel by the representation law. On substituting $F'(B)$ the remaining quotient is the product of $\bar{z}^{2\mu} / |z|^{2(j+\mu)} = (z/\bar{z})^{-\mu} |z|^{-2j}$, where $z = \det B_\tau$. A torus character of weight μ_τ evaluates on $z/\bar{z}$ as its μ_τ-th power by Definition 372.1. The support and normalization already obtained prove all assertions. □

515. The Hecke values at the positive integers

Definition 515.1. Let E be the CM field, $D = [E : \mathbb{Q}] = 2|\Phi|$, and let ν be a character of the norm-one torus with weight $\mu = (\mu_\tau)_{\tau \in \Phi}$ as in Definition 372.1. On the ideles of E put $\lambda(x) = \nu(x/\bar{x})$; write λ_f for its finite part. Let S be a finite set of places, disjoint from its conjugate, such that λ_f is trivial on the local unit groups outside $S \cup \bar{S}$. For each such unmarked place v take any uniformizer π_v, inserted as an idele with other components one. Define in $\Re s > 1$

$$L^S(s, \nu) = \prod_{v \notin S \cup \bar{S}} (1 - \lambda_f(\pi_v) q_v^{-s})^{-1}, \qquad Q_S = \prod_{w \in S} q_w.$$

The values of $\lambda_f(\pi_v)$ are independent of the uniformizers by triviality on units. The character on principal ideles is one by rational triviality of ν, and its infinite value on x is $\prod_\tau (x_\tau / |x_\tau|)^{2\mu_\tau}$.

Lemma 515.2. *For a nonzero tuple $k \in \mathbb{Z}^\Phi$ and $x_\tau \neq 0$ put $u_k(x) = \prod_\tau (x_\tau/|x_\tau|)^{k_\tau}$. Use the sector $\mathcal{R}$ of Lemma 83.2 in the totally imaginary field E. For each full real lattice $\Lambda \subset E_\infty$, each translate $z + \Lambda$, and $Y \geq 1$, one has*

$$\left| \sum_{x \in (z+\Lambda) \cap Y^{1/D} \mathcal{R}} u_k(x) \right| \ll_{k, \Lambda, \mathcal{R}} 1 + Y^{1 - 1/D}.$$

Proof. The integral of u_k over the dilated sector is zero. A rotation in a complex coordinate with $k_\tau \neq 0$ preserves the sector and measure and may be chosen to multiply the integrand by a scalar different from one. Put $T = Y^{1/D}$. Tile by the fixed lattice cells based at the points of $z + \Lambda$ as in Lemma 83.2. The boundary meets $O(1 + T^{D-1})$ cells by its proof, each contributing a bounded error when the lattice sum is compared with the integral divided by the cell volume. The finitely many cells in a fixed ball also give a bounded error, uniformly in the translate.

On all other interior cells, for $\|x\|$ between R and $2R$ at a corner, every $|x_\tau| \gg R$ on that cell. Indeed the logarithmic coordinates of the corner lie in the bounded parallelepiped of the sector and each cell has bounded diameter. The real gradient of u_k on the cell has size $O_k(R^{-1})$, by differentiation of each unit complex phase. Its variation is therefore $O_k(R^{-1})$ by integration along segments. There are $O_\Lambda(R^D)$ such cells for $R \geq 1$, by comparing the volumes of their disjoint interiors with a ball of radius $2R + O_\Lambda(1)$. The sector lies in a ball of radius $O(T)$. Summing the resulting $O(R^{D-1})$ over successive dyadic radii gives $O(1 + T^{D-1})$ since $D \geq 2$. This proves the estimate. □

Theorem 515.3. *If $\mu \neq 0$ and the finite unit character of ν factors through residues at the indicated split places, the product in Definition 515.1 converges absolutely, with nonzero value, in $\Re s > 1$. Its continuation is holomorphic on $\Re s > 1 - 1/D$ and satisfies $L^S(1, \nu) \neq 0$. In particular all its positive integer values are well defined and nonzero.*

Proof. Temporarily omit all primes over a positive rational integer m divisible by the rational primes below $S \cup \bar{S}$; take m divisible by their required conductor powers. At every divisor of m, the character of finite units congruent to one modulo m is then one by the residue hypothesis. For an ideal $\mathfrak{a}$ prime to m write $u(\mathfrak{a})$ for the multiplicative product of the specified uniformizer values, with powers the ideal valuations. By Proposition 82.3, expanding the Euler factors gives the ideal Dirichlet series with coefficients $u(\mathfrak{a})$ in $\Re s > 1$. In fact the counting estimate of Theorem 83.3 gives $\sum_{\mathfrak{a}} (N\mathfrak{a})^{-\sigma} < \infty$ for $\sigma > 1$, by integration. The series of prime-power logarithms is absolutely convergent there, by comparison with $\sum_v q_v^{-\sigma}/(1 - 2^{-\sigma})$, so its exponential is nonzero.

For a nonzero integer r, apply the ray class enumeration in Proposition 219.2. In a fixed class choose an integral ideal $\mathfrak{b}$ in the inverse class prime to m. The integral ideals $\mathfrak{a}$ in the class correspond to $\mathfrak{a}\mathfrak{b} = (x)$, with $0 \neq x \in \mathfrak{b}$, $x \equiv 1 \bmod mO_E$, modulo the units U congruent to one there. Since E is totally imaginary there is no sign condition. On such generators

$$u(\mathfrak{a})^r = u(\mathfrak{b})^{-r} \prod_{\tau \in \Phi} \left(\frac{\tau(x)}{|\tau(x)|} \right)^{-2r\mu_\tau} .$$

This follows by evaluating λ on the principal idele of x: its components above m contribute one, and every other component gives the uniformizer value to its valuation. The right side is constant on U-orbits by the same argument on units. The congruence and membership in $\mathfrak{b}$ specify a translate of $m\mathfrak{b}$, by the Chinese remainder argument in Proposition 219.2. In the sector of a logarithmic basis subgroup $U_0 \subset U$ every ideal occurs exactly $[U : U_0]$ times by Lemma 83.1 and the orbit argument of Theorem 83.3. Its norm bound Y becomes $|N_{E/\mathbb{Q}} x| \leq Y N\mathfrak{b}$. Apply Lemma 515.2 with $k = -2r\mu$, which is nonzero. Summing the bounds over finitely many classes gives

$$\sum_{\substack{N\mathfrak{a} \leq Y \\ (\mathfrak{a}, m) = 1}} u(\mathfrak{a})^r = O_{E,m,\nu,r}(1 + Y^{1-1/D}).$$

Partial summation, first for $\Re s > 1$, writes the corresponding series $L_m(s, u^r)$ as s times the integral of this counting function against $Y^{-s-1}\,\mathrm{d}Y$ from one to infinity. Lemma A.3 gives holomorphy on $\Re s > 1 - 1/D$.

Write $\zeta_m(s)$ for the series of the trivial ideal character with the same deletions. It has a simple pole at one by Proposition 219.2. For each real $\sigma > 1$ the prime-power logarithms give

$$\zeta_m(\sigma)^3 |L_m(\sigma, u)|^4 |L_m(\sigma, u^2)| \geq 1.$$

Indeed the coefficient in the logarithm at the l-th power of a prime is $(3 + 4\Re t + \Re(t^2))/(l q_v^{l\sigma})$, where $t = u(\mathfrak{p}_v)^l$ has modulus one. Its numerator is $2(1 + \Re t)^2$. If $L_m(1, u) = 0$, holomorphy bounds the fourth power by $O((\sigma - 1)^4)$, and holomorphy for $r = 2$ bounds the last factor near one. The left side would tend to zero. This proves nonvanishing at one. Restoring the finitely many unramified factors over m uses factors holomorphic and nonzero on $\Re s > 0$, proving every assertion for L^S. □

Corollary 515.4. *Suppose the ambient rank N, the number q of its negative directions, and the integer weights satisfy $2q + 2 \leq N \leq 2\mu_\tau + 2q$ for every τ. For $e \in \mathbb{Z}_{\geq 0}$ put*

$$\Pi_S(e, \nu) = Q_S^{e^2} \prod_{j=1}^{e} |L^S(j, \nu)|^2.$$

Then $\Pi_S(0, \nu) = 1$ and $\Pi_S(e, \nu) > 0$ for every e, under the finite hypotheses of Theorem 515.3. If a comparison satisfies $|c_\nu|^2 = C \Pi_S(e, \nu)$ with $C > 0$, division by c_ν in Theorem 508.3 is valid.

Proof. The two rank inequalities give $\mu_\tau \geq 1$. Each factor is positive by Theorem 515.3, and the empty product and Q_S^0 equal one. The indicated equality then gives a positive squared modulus. □

516. A finite Mellin transform with one displaced unit

Lemma 516.1. *Let O be a discrete valuation ring in a nonarchimedean local field K, π a uniformizer, and $k = O/\pi O$ of order q. Give O additive mass one. Let ψ be an additive character whose annihilator of O under multiplication is O. Let χ be a nontrivial character of $k^\times$, and put $d(x) = \chi(x \bmod \pi)^{-1}$ on $O^\times$, zero elsewhere. For $t \in K$ define $g_\chi = \int_O d(z) \psi(\pi^{-1} z) \, \mathrm{d}z$. Then $|g_\chi|^2 = q^{-1}$ and*

$$\int_O d(z) \psi(tz) \, \mathrm{d}z = \begin{cases} g_\chi \chi(\pi t \bmod \pi), & \pi t \in O^\times, \\ 0, & \text{otherwise.} \end{cases}$$

For a fixed square matrix Y, $a \in K$, take the test on a block upper triangular matrix

$$H = \begin{pmatrix} a & u & v \\ 0 & Y & w \\ 0 & 0 & z \end{pmatrix}, \qquad \mathbf{1}_{H \text{ integral}} \, d(\det H).$$

Its integral in the three off-diagonal blocks and in z, against $\psi(tz)$ and additive measures giving each integral box mass one, is $\mathbf{1}_{Y \text{ integral}} d(\det Y) d(a)$ times the displayed scalar. The same conclusion holds after interchanging the two scalar diagonal entries.

Proof. On a residue coset $r + \pi O$ the multiplier $d(z)$ is constant. If $t\pi \notin O$, translation by an element of πO multiplies the character integral there by a number different from one. If $t \in O$, the character is trivial and the sum of the nontrivial multiplicative character over $k^\times$ vanishes by translation in that group. In the remaining case $t = \pi^{-1} u_0$ with $u_0 \in O^\times$, use the substitution $y = u_0 z$; it preserves additive measure and gives the factor $\chi(u_0 \bmod \pi)$. The character $r \mapsto \psi(\pi^{-1} \widetilde{r})$ on k is nontrivial by the annihilator assumption. Thus $q g_\chi$ is the one-by-one determinant Gauss sum of Lemma 500.2 with multiplicative character χ^{-1}. Its primitivity on a field follows by taking $1 + zr$ to be any prescribed unit for $r \neq 0$. This gives the claimed norm. The full matrix test factors on the integral box because $\det H = a(\det Y) z$, and an integral product is a unit exactly when each factor is a unit. The off-diagonal integrals have mass one. The determinant equality holds in the other order as well. □

Definition 516.2. In Definition 515.1 assume the finite unit part of ν is $\prod_{w\in S}\chi_w$, with every χ_w a nontrivial residue character at a split branch and with trivial remaining unit characters. Use conjugate uniformizers on conjugate branches, and local additive characters as in Proposition 501.2. Put $G_S=\prod_{w\in S} g_{\chi_w}$ and take a finite idele c with $c_{\bar w}=\bar\pi_w^{-1}$ for $w\in S$ and one elsewhere. For a finite idele x let $D_S(x)$ be zero unless $x_v\in O_v$ off $S\cup\bar S$, and $x_w,\pi_w\overline{x_{\bar w}}\in O_w^\times$ for all $w\in S$. On this support put

$$D_S(x)=\prod_{w\in S}\chi_w(x_w \bmod \pi_w)^{-1}\chi_w(\pi_w\overline{x_{\bar w}} \bmod \pi_w).$$

Give the finite idele group the invariant multiplicative measure for which $\prod_v O_v^\times$ has mass one. This means summation over the valuation vectors of finite support, with the translated unit probability measure on each coset. Write $|x|_f=\prod_v q_v^{-v(x_v)}$. The finite Mellin integral for real $\sigma>1$ is

$$M_S(\sigma)=G_S\int D_S(x)\lambda_f(x)|x|_f^\sigma\,\mathrm{d}^\times x.$$

The D_S in this definition is an idele function; the determinant tests in Definition 509.1 are functions on matrices.

Theorem 516.3. *In Definition 516.2, for $\sigma>1$ the integral converges absolutely and*

$$M_S(\sigma)=G_S\,\lambda_f(c)\,Q_S^\sigma L^S(\sigma,\nu).$$

If $\mu\neq 0$, define $M_S(1)=\lim_{\sigma\downarrow 1}M_S(\sigma)$. This limit exists and is nonzero. At $\sigma=1$ the integral of the absolute value of the original finite integrand is infinite. For any integer e, put $\Pi^{\mathrm{int}}(e)=\Pi_S(e,\nu)$ when $e\geq 0$ and $\Pi^{\mathrm{int}}(e)=\Pi_S(-e,\nu)^{-1}$ otherwise. With $s_e=\max(e+1,-e)$ and $\mu\neq 0$, one has

$$|M_S(s_e)|^2=\frac{\Pi^{\mathrm{int}}(e+1)}{\Pi^{\mathrm{int}}(e)}.$$

Proof. Lemma 516.1 at each chosen branch with $a=x_w$ and $t=\overline{x_{\bar w}}$ gives the factor $G_SD_S(x)$ there with $|G_S|^2=Q_S^{-1}$. Write $x=cy$. Now y has unit components on all the marked pairs and integral components elsewhere. Write $y=up$ with a unit idele u and an idele p given by nonnegative uniformizer powers off the marked pairs and by one there. On units, $\lambda_f(u)=\nu(u/\bar u)$ takes the value $\prod_w\chi_w(u_w)\chi_w(\overline{u_{\bar w}})^{-1}$, with reductions understood. Thus $D_S(x)\lambda_f(u)=1$. Since $|c|_f=Q_S$, every allowed valuation coset has constant integrand after the unit phases cancel. Summing over these cosets gives the ideal series of Theorem 515.3, with the factor $G_S\lambda_f(c)Q_S^\sigma$. For $\sigma>1$ the absolute sum is bounded by the unrestricted positive ideal series in Corollary 83.4, so this calculation is valid. For $\mu\neq 0$, continuation at one and nonvanishing are given by Theorem 515.3.

For absolute values at one the same coset calculation gives $|G_S|Q_S\sum_{\mathfrak a}(N\mathfrak a)^{-1}$ with the finitely many marked divisors excluded. As $\sigma\downarrow 1$ the unrestricted positive series has a simple pole by Corollary 83.4; deleting finitely many Euler factors multiplies by a factor with positive limit. Monotonicity of the terms proves divergence at one. Finally if $e\geq 0$, the ratio in the assertion is $Q_S^{2e+1}|L^S(e+1,\nu)|^2$, by Corollary 515.4 and its definition. For $e=-k-1<0$ it is instead $\Pi_S(k+1,\nu)/\Pi_S(k,\nu)$, giving $Q_S^{2k+1}|L^S(k+1,\nu)|^2$. In both cases $s_e\geq 1$ and the evaluated Mellin transform has precisely that squared modulus because $|\lambda_f(c)|=1$ and $|G_S|^2=Q_S^{-1}$. □

517. Finite slices on an integral Witt decomposition

Definition 517.1. Use the finite places of the arithmetic field E. Suppose that the splitting $W=L\oplus W_0\oplus L'$ of Definition 510.1 splits the finite lattice as the sum of lattices on the three spaces. The first and last lattices are in perfect hermitian duality, and the middle lattice is unimodular. Let $b=\dim L>0$, $n=\dim W_0$, $T=T_b\perp W_0$, where T_b has dimension b. Use unimodular finite lattices on both summands and their direct sum on T. At each

marked split pair choose ordered integral frames in the order L, W_0, L' and conjugate dual frames with the pairing between L, L' equal to the identity. Do the same in the order T_b, W_0 on the target. At the branch $w \in S$ take S_w in Definition 509.1 to be projection on the last b coordinates. Its conjugate adjoint inserts the first b columns at $\bar{w}$. Here and below coordinates on Hom lattices are divided by the finite scale α of Lemma 355.2.

In the mixed model use both branches of the restriction to L as frame position Y. At the pair $\{w, \bar{w}\}$ the remaining position is the restriction Z to W_0 at the branch used to name the test. Normalize the finite models to take each reference vacuum to its lattice indicator, with position lattice of mass one. Write $\mathbf{1}_{\mathrm{int}}$ for the indicator that all specified entries are integral. Extend χ_w^{-1} by zero from residue units, and denote this extension by d_w, also applied to integral lifts. The depth is the one in the determinant datum. Write F^+ for the finite large datum E_S, and F^- for the transformed small datum $D_S^\flat$ of Theorem 512.2.

Lemma 517.2. *In Definition 517.1, an integral-box test at a split pair named at w, whose value on the box has the form $H(Y_w, Z_w)$, becomes*

$$\mathbf{1}_{\mathrm{int}}(Y_w, Y_{\bar{w}}, Z_w)\, H(Y_w, Z_w)$$

in the mixed model. The same formula with the branches interchanged holds for a test named at $\bar{w}$. Off the marked pairs the vacuum slice is the integral-frame indicator times the residual vacuum.

Proof. At w the three source blocks of a position matrix are (Y_w, Z_w, U_w) by Theorem 360.1. In the conjugate branch the block paired with U_w is $Y_{\bar{w}}$, because L' pairs with L and is orthogonal to W_0 and to itself. The change of polarization therefore takes the Fourier transform in U_w alone. It implements the exchange of translations and multipliers on these coordinates by substitution in the Fourier integral, as in Lemma 387.1. The kernel is the trace character from Lemma 355.2. The annihilator of the integral U_w-box is exactly the integral $Y_{\bar{w}}$-box by the trace-dual identity there and perfect duality of the two frames. Fourier transformation of the first box is consequently the indicator of the second box. Indeed it integrates to one for a trivial character and translation by an element on which the character is nontrivial forces the integral to vanish otherwise. This also fixes the normalization of the intertwiner, since it takes the vacuum to the vacuum. The other variables retain their factors. Interchanging the places gives the second assertion. At an unmarked place the hyperbolic position and momentum lattices split and are dual, so their vacuum is the normalized frame indicator by Proposition 391.1. Translations in the residual Hom space act independently, giving the tensor product with its vacuum by the direct sum argument of Theorem 403.2. □

Theorem 517.3. *In Definition 517.1 the large and small finite slices at a marked pair are respectively*

$$F^+(Y; Z) = \mathbf{1}_{\mathrm{int}}(Y_w, Y_{\bar{w}}, Z_w)\, d_w\Big((-1)^{b(b+n)} \det[\, Y_w \; Z_w \,]\Big),$$

$$F^-(X; Z') = \mathbf{1}_{\mathrm{int}}(X_w, X_{\bar{w}}, Z'_{\bar{w}})\, d_w\Big((-\epsilon_w^{-1})^b \overline{\det X_{\bar{w}}}\Big).$$

Here Y has target T, X has target T_b, and ϵ is the integral frame unit of Lemma 505.1. The bar takes scalars from $E_{\bar{w}}$ to E_w.

Let $X = (X_v)_v$ be a finite frame in T_b which is an integral isomorphism of the frame lattice onto the lattice of T_b at every finite place. Set $Y = \binom{X}{0}$ and $\lambda_f(x) = \nu_f(x/\bar{x})$, with unit character as in Definition 516.2. In the compatible frames, the finite residual vector satisfies

$$F^+\Big(\tbinom{X}{0}\Big) = u_S\, \lambda_f(\det X)^{-1}\, F^-(X) \otimes F_{\mathrm{eq}},$$

where F_{eq} is the integral determinant test $d_w(\det Z_1)$ on $\mathrm{Hom}(W_0, W_0)$ at $w \in S$ and the vacuum elsewhere. The fixed scalar u_S has modulus one and is the product, over w, of

$$d_w((-1)^{b(b+n)}) \,/\, d_w((-\epsilon_w^{-1})^b).$$

The determinant of X here is the finite unit idele in the chosen ordered frames.

Proof. On the large box the determinant is that of the matrix with first block row $(0, 0, I_b)$ and remaining block rows (Y_w, Z_w, U_w). Moving its last b columns ahead of its first $b+n$ contributes the displayed sign, after which the matrix is block triangular. On the small box the determinant of $S(-\epsilon^{-1}Z^*)$ in Theorem 512.2 is the determinant of the conjugate transpose of the restriction to L, times $(-\epsilon_w^{-1})^b$. In each case the box value is independent of the L'-block, so Lemma 517.2 proves both slice formulas.

At $Y = \binom{X}{0}$, write $Z_w = \binom{Z_b}{Z_1}$. Then $\det[Y_w\ Z_w] = \det X_w \det Z_1$. The free integral Z_b-block is the reference vacuum on $\mathrm{Hom}(W_0, T_b)$. The small test has its vacuum on the same space in the conjugate-branch model. These give the same abstract vector since both identifications take the reference vacuum to the appropriate box, by Theorem 360.1. On comparing the two scalar unit factors at the marked pair one obtains, besides the displayed constant, $\chi_w(\overline{\det X_{\bar{w}}})/\chi_w(\det X_w)$. This is the inverse of the w-factor of $\nu_f(\det X/\overline{\det X})$ on units. All frame indicators are one and at the unmarked places Lemma 517.2 leaves only the two residual vacua. Their tensor product gives the original residual vacuum by the same splitting argument. Multiplication over places gives the asserted equality. All the factors of u_S are unit character values. □

518. The finite orbit in the nonsingular coefficient

Lemma 518.1. *Let $T = T_b \perp T_n$ have the orthogonal direct sum of two unimodular finite hermitian lattices over the arithmetic CM field E. Let $F = E^+$, $G_f = U(T)(\mathbb{A}_{F,f})$, $U_i = U(T_i)(\mathbb{A}_{F,f})$ for $i = b, n$, and $K_T \subset G_f$ be the stabilizer of the sum lattice. Write $\iota : T_b \to T$ for the first inclusion. In integral frames let $Y = uA$, where $u \in U_b$ and $A \in \mathrm{GL}_b(\widehat{O}_E)$. Then*

$$\{h \in G_f : h^{-1}\iota Y \text{ is integral}\} = \mathrm{diag}(u, U_n)K_T.$$

For $y \in U_b$, if $y^{-1}Y$ is integral then $y^{-1}Y \in \mathrm{GL}_b(\widehat{O}_E)$.

Proof. Multiplication on the right by A and its integral inverse shows that the first integrality condition is equivalent to integrality of $B = h^{-1}\iota u$. Its Gram matrix in the chosen source frame is J_b, the unimodular Gram matrix of the source lattice. If B is integral then $J_b^{-1}B^*J_T$ is an integral left inverse, where J_T is the Gram matrix of the sum lattice. The inclusion ι also has this Gram matrix and an integral left inverse. Apply Theorem 345.2 at each finite place of F. Its two possibilities for the local integer algebra exhaust all places, including those above two, by Lemma 57.1. The product of the resulting isometries is $k \in K_T$ with $kB = \iota$, as in Corollary 345.3. Consequently $kh^{-1}\mathrm{diag}(u, 1)$ fixes ι pointwise. An isometry with this property restricts to an isometry on the orthogonal complement, so it has the form $\mathrm{diag}(1, l)$. Hence $h = \mathrm{diag}(u, l^{-1})k$. Conversely each element of the indicated set takes ιY by its inverse to $k^{-1}\iota A$, which is integral. For the last assertion, $y^{-1}u$ is integral if $y^{-1}Y$ is integral. Its inverse is $J_b^{-1}(y^{-1}u)^*J_b$, also integral by unimodularity. Multiplication by A gives the assertion. □

Lemma 518.2. *With characters $(\gamma, 1)$ as in Definition 508.1, the entire standard large datum F^+ of Definition 517.1 is fixed by $(1, k)$ for every $k \in K_T$, acting at finite places and as the identity at infinity.*

Proof. The finite lattice action of k takes a named point D to kD on the chosen branches, as shown in Lemma 508.2. This holds for the full integral group because the complementary places carry the vacuum, which an integral isometry fixes by Theorem 390.2. This theorem also gives phase one on an integral lattice stabilizer. Thus the action on a function of the residue names replaces D by $k^{-1}D$ in the function. In $E_S(k^{-1}D)$ the lower block rows are multiplied by the reduction of k^{-1}, so this replacement contributes $\chi(\det k)$. It preserves integrality and the support on unit determinants. Finally the target twist in Definition 504.1 is $\gamma(\det k) = \chi(\det k)^{-1}$ by Theorem 374.2 and Definition 508.1. The factors multiply to one. □

Proposition 518.3. *Use the adapted integral coordinates of Definition 517.1. Consider a term $t_{x_*}(g,h;f)$ of the large standard test in Corollary 511.3, with $gL = L$. Suppose the normalized finite coordinates of $x_*g|_L$ are ιY, where $Y = uA$ as in Lemma 518.1. The finite coordinate h_f of every nonzero term lies in $\mathrm{diag}(u, U_n)K_T$. Changing h_f on the right by any $k \in K_T$ leaves the term unchanged. In particular at a block element $h_f = \mathrm{diag}(y, l)$ the only contributing small frames $y^{-1}Y$ belong to $\mathrm{GL}_b(\widehat{O}_E)$, so that Theorem 517.3 applies there.*

Proof. The initial large test is supported on integral normalized frame coordinates at every finite place by Lemma 517.2. We check that this support condition is transported on the whole finite frame and remains valid for the residual theta sum. All modulations in the annihilator of the frame lattice fix the initial test, since they multiply it by a character equal to one on this lattice. The inverse image of this annihilator law on the transformed test is obtained by composing its frame variable with $x \mapsto h^{-1}xg|_L$, by the intertwining and pure momentum calculation in Lemma 513.1. A finite frame outside the lattice is detected by one such modulation supported at a single place. Indeed the dual lattice has that original lattice as annihilator by the character argument in Proposition 391.1. The resulting nontrivial scalar multiplies every residual value of the slice, so they all vanish. Thus $h_f^{-1}\iota Y$ is integral whenever the term is nonzero. The coset description follows from Lemma 518.1. Right invariance holds before taking the frame slice by the representation law and Lemma 518.2. At a block element the same integrality condition reads $y^{-1}Y$ integral. The last assertion of Lemma 518.1 now proves the hypothesis on $X = y^{-1}Y$ in Theorem 517.3. □

519. Integrating the coefficient in two stages

Definition 519.1. Continue with $T = T_b \perp T_n$ and the finite groups of Lemma 518.1. Use the rational block decomposition, set $G = U(T)(\mathbb{A}_F)$, and let $H = U(T_b)(\mathbb{A}_F) \times U(T_n)(\mathbb{A}_F)$ act block diagonally. Write $H_K = H_f \cap K_T$ and $\Gamma_n = \mathrm{diag}(1, U(T_n)(F))$, with rational points here meaning isometries over E defined over F. Give the groups product Haar measures, using the densities of Definition 95.1 at infinity and any fixed positive Haar normalizations at the finite places. On a discrete subgroup use counting measure.

In the real coordinates of Lemma 514.1 let P_0 be projection on the first block. For a tuple Z of complex n-by-b matrices let P_Z be orthogonal projection for J_T onto the graph of Z, when that graph is nondegenerate. Choose an open neighborhood $\mathcal{U}$ of zero on which these graphs are nondegenerate and $\|P_Z - P_0\| < 1$ placewise. Take the support of the real plane bump β of Theorem 514.3 inside this chart, with its graph coordinates compactly contained in $\mathcal{U}$. Write $k_Z = k(P_Z)$ and put

$$\mathcal{V}_\infty = \{s_\infty : P_{s_\infty^{-1}\iota} = P_Z \text{ for } Z \in \mathcal{U}\}, \qquad \mathcal{V} = \mathcal{V}_\infty \times H_f K_T.$$

Use ordinary coordinate Lebesgue measure $\mathrm{d}Z$ on the product of matrix spaces.

Lemma 519.2. *There is a positive smooth density factor $J(Z)$, determined by the real Haar measures in Definition 519.1, such that for each nonnegative measurable A supported on $\mathcal{V}$,*

$$\int_G A(s)\,\mathrm{d}s = c_f \int_{\mathcal{U}} J(Z) \int_{K_T}\int_H A\big(h(k_Z^{-1}, k)\big)\,\mathrm{d}h\ \mathrm{d}k\ \mathrm{d}Z, \qquad c_f = \frac{\mathrm{vol}_{G_f}(K_T)}{\mathrm{vol}_{H_f}(H_K)}.$$

Here $\mathrm{d}k$ is probability on K_T and (k_Z^{-1}, k) has the indicated infinite and finite parts. If A is left Γ_n-invariant, the same formula holds with the left group G and the inner group H both replaced by their left quotients by Γ_n. Each version holds for absolutely integrable complex functions.

Proof. The map $(h_\infty, Z) \mapsto h_\infty k_Z^{-1}$ is a smooth bijection from $H_\infty \times \mathcal{U}$ onto $\mathcal{V}_\infty$ with smooth inverse. Indeed if $s_\infty^{-1}\iota = \binom{V_1}{V_2}$, its plane is a graph exactly when V_1 is invertible. Then $Z = V_2V_1^{-1}$, and $s_\infty k_Z$ preserves the first plane by Lemma 514.1. It preserves the perpendicular as well and belongs to H_∞. This gives the inverse and proves smoothness in

both directions. Pull back the real group density in this chart and divide by the product of the density on H_∞ and $\mathrm{d}Z$. The result is smooth and positive since the coordinate differentials are inverse isomorphisms. Left invariance under H_∞ on both sides, by Lemma 95.2, makes the ratio independent of the H_∞ coordinate. This is $J(Z)$.

The finite-adelic region $H_f K_T$ is the union of the disjoint translates tK_T, where t ranges over representatives for H_f/H_K. This union can be countably infinite. The translates are open and the local fields have countable bases, as does their restricted product since only finitely many integrality conditions may be changed at a time. Each translate has measure $\mathrm{vol}_{G_f}(K_T)$; in the integral on H_f the translate tH_K has measure $\mathrm{vol}_{H_f}(H_K)$. Inserting the normalized average on K_T after the H_f coordinate gives exactly the claimed ratio, since translating that average on the left by any element of H_K preserves it. Applying Lemma 123.1 gives the first formula.

For the quotient, take Borel fundamental domains as in Lemma 21.4. Its construction applies on both adelic groups: a compact set of translating rational matrices has common finite denominators and bounded real entries, hence is finite by Lemma 13.4. Cover by a countable family of neighborhoods disjoint from their nonidentity rational translates and keep in each only the orbits which did not meet an earlier neighborhood. The resulting domains meet each orbit once. Multiply A upstairs by the indicator of the domain on G. In the formula already proved, decompose the H-integral over translates of the domain on H. At fixed Z, k, h, the sum of the domain indicators at $\delta h(k_Z^{-1}, k)$, $\delta \in \Gamma_n$, is one. The value of A is invariant under these translations. Nonnegative summation gives the quotient formula, and applying it first to the modulus gives the complex case. □

Theorem 519.3. *In Proposition 518.3, let $T_n = W_0$, $x_* = \iota$ in rational coordinates, and take the real test F of Theorem 514.3 with the plane bump of Definition 519.1. Take g at which the specified finite frame condition holds and $gL = L$. Let $A_g(s) = t_{x_*}(g, s; F \otimes F^+)$. Write $[U(T_n)] = U(T_n)(F)\backslash U(T_n)(\mathbb{A}_F)$, and let $B_* = x_*^* J_T x_*$. The B_*-coefficient of the large period at g equals*

$$c_f \int_{\mathcal{U}} J(Z) \int_{U(T_b)(\mathbb{A}_F)} \int_{[U(T_n)]} A_g\big(\mathrm{diag}(y, l)(k_Z^{-1}, 1)\big)\, \mathrm{d}l\ \mathrm{d}y\ \mathrm{d}Z.$$

The integral of the modulus in this expression is finite. The transverse constant

$$R = c_f \int_{\mathcal{U}} J(Z) \beta(P_Z)\, \mathrm{d}Z$$

is strictly positive, finite, and independent of S, χ, ν when the real bump, spaces, lattices and measures are fixed.

Proof. By Corollary 511.3 the coefficient is the integral of A_g over $\Gamma_n\backslash G$ and is absolutely integrable. Its finite support lies in $H_f K_T$ by Proposition 518.3. At infinity the transported frame coordinate for a nonzero term is $s_\infty^{-1} \iota g|_{L,\infty}$, in the real frames and scales. Its column plane is the plane of $s_\infty^{-1}\iota$, since the last factor is invertible on L and the real scale is scalar. The mixed-test support and its transport in Lemma 513.1 therefore put s_∞ in $\mathcal{V}_\infty$. Apply the quotient formula in Lemma 519.2. Right K_T-invariance in Proposition 518.3 removes the last finite compact average. The quotient of H is the direct product displayed since Γ_n acts only in the second factor. This proves the formula and the absolute integrability assertion. The continuity and positivity of J, the compact support of $Z \mapsto \beta(P_Z)$, its nonnegativity and its positivity in a neighborhood of zero give $0 < R < \infty$. Its defining factors involve only the data listed in the statement. □

520. Levi action and the characters of the two periods

Definition 520.1. In Definition 517.1 let $n = 2j$, $N = 2b + n$ and $d = b + n$, and identify T_b with the opposite target of Theorem 512.2. Write $\upsilon_T = \upsilon|_{U(1)}$ with υ from Theorem 374.2,

and put

$$\gamma_0 = \nu^{-1}\upsilon_T^j, \qquad \eta_0 = \nu\upsilon_T^j, \qquad \gamma = \gamma_0\upsilon_T^b, \qquad \lambda(t) = \nu(t/\bar{t}).$$

Write $\delta(s) = \det_E s$ for an isometry, and $|t|_{\mathbb{A}}$ for the idele modulus of E, using $\prod_\tau |t_\tau|^2$ at infinity. For an adelic $q \in \mathrm{GL}(L)$, $g_0 \in U(W_0)(\mathbb{A}_F)$, let $m(q, g_0)$ act on L, W_0, L' by $q, g_0, (q^*)^{-1}$, respectively. For the large and the flipped small period multiply both functions at g by $\gamma(\delta(g))^{-1}$. Denote the resulting kernels by the corresponding normalized kernels, using for the small one its directly transformed finite datum F^-.

Lemma 520.2. *In the mixed position model for* $\mathrm{Hom}(W, T')$ *write* $f[x]$ *for evaluation in the full adelic frame variable at* x*, taking values in the residual oscillator on* $\mathrm{Hom}(W_0, T')$*. Put* $t = \dim T'$*,* $D = \det q$ *and let* $h \in U(T')(\mathbb{A}_F)$*. For unitary characters* a, B *of the norm-one torus use* $a\upsilon_T^b, B$ *on the full pair. Then*

$$\big(\Omega_{a\upsilon_T^b, B}(m(q, g_0), h)f\big)[x] = p_{B,t}(D)\,\Omega_{a,B}(g_0, h)\big(f[h^{-1}xq]\big),$$

$$p_{B,t}(D) = \upsilon(D)^t\, B(D/\bar{D})^{-1}|D|_{\mathbb{A}}^{t/2}.$$

The symbols $x, h^{-1}xq$ *in this identity are physical frame coordinates. Expressing them in the same normalized frames on the two sides gives the identical formula.*

Proof. The map on the hyperbolic position is $X \mapsto hXq^{-1}$. Its E-determinant is $\delta(h)^b D^{-t}$, since left multiplication acts separately on the columns and right multiplication separately on the rows. The hyperbolic and residual actions are independent factors by the direct sum calculation of Theorem 403.2. The whole hyperbolic operator, at any position, is substitution by the inverse map times the inverse splitting character of its determinant and the inverse square root of its modulus. This follows from Theorems 394.2 and 386.1, since an operator preserving both isotropic summands has zero shear. As $\delta(h)\overline{\delta(h)} = 1$ and conjugation preserves the idele modulus, $|\delta(h)|_{\mathbb{A}} = 1$. Thus the two factors are $\upsilon_T(\delta(h))^{-b}\upsilon(D)^t$ and $|D|_{\mathbb{A}}^{t/2}$. The target determinant twist cancels the first power of $\upsilon_T(\delta(h))$, leaving $a(\delta(h))$. The source determinant is $(D/\bar{D})\delta(g_0)$, giving the other displayed scalar and $B(\delta(g_0))^{-1}$. Together the remaining action and characters give $\Omega_{a,B}(g_0, h)$ by Definition 504.1. This calculation applies in a dense family of pure tests and also pointwise for arbitrary Schwartz tests: in a product polarization the block substitutions act on separate variables and the residual Fourier integrals commute with evaluation in the frame variables, as shown in Theorem 403.2. □

Proposition 520.3. *After the common normalization of Definition 520.1, the full characters on the large pair are* (γ, γ)*, and on the small pair they are* $(\eta_0\upsilon_T^b, \upsilon_T^b)$*. The ratio of their Levi scalars in Lemma 520.2 is*

$$p_{\gamma,d}(D)/p_{\upsilon_T^b,b}(D) = \lambda(D)|D|_{\mathbb{A}}^{n/2}.$$

On the residual pair at a block element $\mathrm{diag}(y, l)$ *of the target, the normalized large action on a tensor* $u \otimes v$ *is*

$$\nu(\delta(y))^{-2}\,\Omega_{\eta_0,\upsilon_T^b}(g_0, y)u \;\otimes\; \Omega_{\gamma_0,\gamma_0}(g_0, l)v.$$

Proof. The original large characters are $(\gamma, 1)$ by Definition 508.1, and the flipped small ones are $(\gamma^{-1}\upsilon_T^N, \gamma^{-1}\upsilon_T^b)$ by Theorem 512.2. Multiplication by the common scalar on the ambient group multiplies the denominator character by γ. Since $N = 2(b + j)$, $\gamma^{-1}\upsilon_T^N = \eta_0\upsilon_T^b$. For the ratio of Levi scalars the unit character before simplification is $\upsilon(D)^n\gamma_0(D/\bar{D})^{-1}$. The identity $\upsilon_T(D/\bar{D}) = \upsilon(D)^2$ follows from the conjugation rule of Theorem 374.2. Use $n = 2j$ and the definition of γ_0 to get $\lambda(D)$. The ratio of moduli is $|D|_{\mathbb{A}}^{(d-b)/2}$.

The residual target character is γ_0 on both blocks. The original residual source character is $\gamma = \upsilon_T^b\gamma_0$ and the untwisted action on the orthogonal Hom sum is the tensor product, by the product calculation in Theorem 403.2. Putting υ_T^b on the first source denominator and

γ_0 on the second accounts for this character at g_0. Changing the first target numerator to η_0 multiplies it by $\eta_0(\delta(y))/\gamma_0(\delta(y)) = \nu(\delta(y))^2$, giving the displayed correcting factor. □

521. The two period coefficients on the Levi subgroup

Definition 521.1. With the hypotheses of Theorem 519.3 and Definition 520.1, assume $S \neq \varnothing$ and that its marked branches are distinct from their conjugates, as in Definition 516.2. Let v_1 be the real residual vector on $\mathrm{Hom}(W_0, W_0)$ occurring in Theorem 514.3. Define

$$J_\nu(g_0) = \int_{[U(W_0)]} \theta\big(\Omega_{\gamma_0,\gamma_0}(g_0, l)(v_1 \otimes F_{\mathrm{eq}})\big)\, \mathrm{d}l,$$

where F_{eq} is the finite determinant vector of Theorem 517.3. This integral is absolutely convergent. Indeed the vector is a finite linear combination of named residue points, all surjective at the chosen branch. The large-side estimate and the height integral in Theorem 507.3 apply with source and target W_0. The unitary twists leave its estimate unchanged.

Write $A^+_\nu(m)$ and $A^-_\nu(m)$ for the B_*-coefficients of the two normalized periods in Definition 520.1, using $F \otimes F^+$ and $F' \otimes F^-$ respectively, with the real data of Theorem 514.3. The small period uses the directly transformed model, so the phase c_0 of Theorem 512.2 does not enter these symbols. Identify L with T_b by the rational frame x_*. Write $Y(q)$ for $x_* q$ in the chosen normalized frame coordinates and $a = (\det Y(1))^{-1}$. Then $\det q = a \det Y(q)$: changing from rational to normalized frames multiplies the frame matrix on the left and right by fixed invertible adelic matrices.

Lemma 521.2. *Use rational product polarizations on the hyperbolic and residual Hom summands in Lemma 520.2, and on the two residual summands for $T = T_b \perp W_0$. In any fixed physical position measures the normalized finite slice formulas of Theorem 517.3 acquire one nonzero constant c_t for a target of rank t. It is fixed by the space, frames, measures and choice of vacuum phase, independently of the set of branches and of the residue function. With positive frame vacuum and unit residual vacuum one has $c_t = \mathrm{vol}(\Lambda_t)^{-1/2}$, where Λ_t is the physical hyperbolic frame lattice.*

Proof. The full finite vacuum in the orthogonal sum splits as the hyperbolic vacuum times the residual vacuum, since the factors are independently fixed by their lattice translations. Both fixed spaces are lines by Proposition 391.1. In the hyperbolic frame model this line is spanned by the lattice indicator. Its squared norm is the volume of the lattice. These assertions hold before specifying any marked branches. Normalize the tensor factors to be unit vectors and take the positive frame indicator. This gives the stated c_t. With any other fixed phase multiply it by that phase. A named residue function acts by a finite linear combination of the physical translations in Proposition 357.2; each acts in both normalizations by the same substitutions and characters. Thus their common scalar persists for the full linear combination. In particular the transformation of the unused L'-block in Lemma 517.2 introduces no further scalar varying with the marked set, because its intertwiner takes the unit vacuum to the unit vacuum. The residual vacuum on the direct sum is the product of the two residual vacua with the compatible product convention. □

Theorem 521.3. *Suppose $Y(q)_f \in U_b\, \mathrm{GL}_b(\widehat{O}_E)$ as in Lemma 518.1. For every g_0, with $m = m(q, g_0)$ of Definition 520.1,*

$$A^+_\nu(m) = K_\nu\, J_\nu(g_0)\, A^-_\nu(m), \qquad K_\nu = R\,(c_d/c_b)\, u_S\, \lambda(a) |a|^{n/2}_{\mathbb{A}}.$$

Here $R > 0$ is as in Theorem 519.3. In particular $|K_\nu|$ is positive and independent of the chosen twist and its marked set for fixed spaces, real bump, lattices and measure choices.

Proof. Use the integral over Z, y, l of Theorem 519.3 and put $B = y^{-1} Y(q)$ in target coordinates. At the finite places a nonintegral B_f annihilates both frame slices by Lemma 517.2. Otherwise B_f is unimodular by Lemma 518.1. At an integral B_f, the finite slice ratio of the

large to the tensor of the small and equal-rank vectors means the scalar in the equality of vectors in Theorem 517.3. By Lemma 521.2 this scalar is $(c_d/c_b)u_S\lambda_f(\det B_f)^{-1}$ in physical coordinates. Also $|\det B_f|_f = 1$.

At infinity $(k_Z^{-1})^{-1}\iota B_\infty = k_Z\iota B_\infty$. In the construction of Theorem 514.3 the plane of this frame is P_Z and its coordinate c is B_∞. Apply the real target action of k_Z^{-1} to the explicit F in its proof, taking π to be the residual target action of γ_0. The factor $\pi(k_Z)$ in F cancels. This gives the scalar $\beta(P_Z)\lambda_\infty(\det B_\infty)^{-1}|\det B_\infty|_\infty^{-n/2}$ times $F'(B_\infty)\otimes v_1$. Here $|\ |_\infty$ is the product of the squared complex absolute values. The infinite character of γ_0 depends only on the fixed weight by Definition 372.1, so this choice of the real data works for all these twists.

Apply Lemma 520.2 and Proposition 520.3 to the remaining block $\mathrm{diag}(y,l)$. In addition to $(c_d/c_b)u_S\beta(P_Z)$ the total scalar on comparing the large slice to the small slice and the equal-rank vector is

$$\lambda(\det q)|\det q|_{\mathbb{A}}^{n/2}\,\lambda(\det B)^{-1}|\det B|_{\mathbb{A}}^{-n/2}\nu(\delta(y))^{-2}.$$

Since $\delta(y)\overline{\delta(y)} = 1$, $\lambda(\delta(y)) = \nu(\delta(y))^2$ and $|\delta(y)|_{\mathbb{A}} = 1$. As $\det B = \delta(y)^{-1}\det Y(q)$, the expression is $\lambda(a)|a|_{\mathbb{A}}^{n/2}$, even at zeros of the remaining theta functions. These computations establish a vector equality before taking theta values.

In the product rational position polarization the theta sum of a tensor is the product of the theta sums. Indeed its index set is the direct product of the two rational position spaces and the sums converge absolutely by Theorem 385.3. The two sums now depend on y and l respectively. The integral of the first over $U(T_b)(\mathbb{A}_F)$ is the small coefficient with trivial frame stabilizer by Corollary 512.3, in its directly transformed model. The second integrates to $J_\nu(g_0)$ by Definition 521.1. Both are absolutely integrable by those results. The common normalization commutes with taking the coefficient because the central shears in Definition 510.1 have determinant one. The last transverse integration gives R, proving the formula. The constants c_b, c_d and the idele a are fixed, $R > 0$, and both $\lambda(a)$ and u_S have modulus one. □

522. The unipotent radical in the coefficient comparison

Definition 522.1. In the ordered dual frames L, W_0, L' use the Gram matrix with identity off-diagonal blocks between L, L', and block J_0 on W_0. For an adelic b-by-n matrix C and an adelic b-by-b matrix S_0 with $S_0^* = -S_0$, put

$$u(C,S_0) = \begin{pmatrix} I & C & S_0 - \frac{1}{2}CJ_0^{-1}C^* \\ 0 & I & -J_0^{-1}C^* \\ 0 & 0 & I \end{pmatrix}, \qquad U = \{u(C,S_0)\}.$$

These are precisely the isometries preserving L, acting as identity on L, $L^\perp/L$ and $W/L^\perp$. Indeed such a matrix is upper triangular with identity diagonal blocks. In the Gram equation the middle-right block must equal $-J_0^{-1}C^*$, and the upper-right block plus its adjoint must equal $-CJ_0^{-1}C^*$. Conversely substitution gives the Gram equation. Every adelic isometry preserving L is a unique product $u\,m(q,g_0)$ with $u \in U$, by taking its diagonal blocks and using the same equation.

Lemma 522.2. *For a target T' let the pair action of $u \in U$ on $\mathrm{Hom}(W,T')$ be the one induced by precomposition with u^{-1}. In the frame polarization of Lemma 520.2 it induces, at each frame x, an operator $D_{T'}(u,x)$ on residual Schwartz vectors satisfying*

$$(\Omega(u,1)f)[x] = D_{T'}(u,x)f[x], \qquad D_{T'}(u,0) = 1.$$

For compatible product polarizations on $T' = T_b \perp T_n$,

$$D_{T'}(u,(x_b,x_n)) = D_{T_b}(u,x_b)\otimes D_{T_n}(u,x_n).$$

These identities hold for all the pair characters in Definition 520.1.

Proof. Write a Hom matrix in three column blocks (x, M, V) according to L, W_0, L'. Under precomposition by u^{-1}, its first block remains x, its middle block changes by $-xC$, and the third changes by a linear function of x, M. Thus if $x = 0$ and M lies in the chosen residual momentum, all position coordinates remain zero. The whole momentum space is preserved. This also holds for $d = m(2I, 1)$. On the group generated by U, d use the operators $\mathcal{W}$ of Theorem 386.1. Both these operators and the physical Ω represent this group and intertwine the same Heisenberg translations by Theorem 395.2. Their ratio is scalar by Lemma 395.1 and is therefore a character.

Conjugation by d carries (C, S_0) to $(2C, 4S_0)$. On the central subgroup $C = 0$, invariance of the character under this conjugation makes it trivial on every $3S_0$, hence on all S_0. Modulo that central subgroup the radical is additive in C, by block multiplication in Definition 522.1. The descended character has equal values at $2C$ and C, so is one. This proves $\Omega(u, 1) = \mathcal{W}_u$. Pair characters do not change the conclusion since $\delta(u) = 1$.

In the explicit formula of Theorem 386.1 the quotient substitution fixes the frame coordinate, so the answer at a fixed x depends only on $f[x]$. At $x = 0$ the residual position is unchanged. Its change in momentum lies entirely in the third column block and therefore pairs trivially with the residual position, by the Gram blocks. The quadratic phase is consequently one there. The quotient substitution has determinant one since it fixes x and adds a linear function of x to the residual position. This proves the zero-frame assertion. For an orthogonal sum of targets, the linear maps are direct sums and the two summands pair trivially. Thus the quotient substitution acts on the two factors separately and the exponent of the quadratic phase is the sum of the two exponents. The displayed formula for $\mathcal{W}_u$ gives the tensor identity. □

Theorem 522.3. *In Theorem 521.3, for every $u \in U$ and $m = m(q, g_0)$ in its range,*

$$A_\nu^+(um) = K_\nu J_\nu(g_0) A_\nu^-(um).$$

Here each side uses the coefficient of the same index B_* with the same left central integration as in Definition 510.1.

Proof. In the two-stage formula of Theorem 519.3 the finite support condition is unchanged because $(um)|_L = q$. The same statement holds for the real plane chart. In the proof of Theorem 521.3 the equality at m holds on residual vectors at the frames $(x_*, 0)$ and x_* in the two targets, writing x_* here for the identification with the first summand. It holds at each of the indicated Z, y, l before taking any residual theta sum. Apply $\Omega(u, 1)$ on the left of each transformed kernel and evaluate at the same frames. By Lemma 522.2 the large residual action is $D_{T_b}(u, x_*) \otimes 1$, and the small residual action is $D_{T_b}(u, x_*)$. Hence the first vector is still the same scalar times the tensor of the new small vector and the previous equal-rank vector. Taking rational theta sums as in Theorem 521.3 and integrating gives the asserted identity. Its absolute convergence at um follows from Corollary 511.3 on the large side, Corollary 512.3 on the small side, and Definition 521.1 on the residual factor. □

523. The equal-rank period at a marked split place

Lemma 523.1. *Let K be a nonarchimedean local field with valuation ring O and normalized absolute value. Put $G = \mathrm{GL}_n(K)$ for $n > 0$. On locally constant compactly supported functions on G write $L_h f(X) = f(h^{-1}X)$, $R_s f(X) = f(Xs)$. Give G the measure $\mathrm{d}^\times X$ of mass one on $\mathrm{GL}_n(O)$ obtained by normalizing $|\det X|^{-n}\,\mathrm{d}X$, where $\mathrm{d}X$ is additive measure on the matrix entries. For a locally constant character $\omega : G \to \mathbb{C}^\times$, any linear functional ℓ with $\ell(\omega(h) L_h f) = \ell(f)$ satisfies*

$$\ell(f) = c \int_G f(X)\omega(X)^{-1}\,\mathrm{d}^\times X, \qquad \ell\big(\omega(s)^{-1} R_s f\big) = \ell(f).$$

Here $c = \ell(\omega\, \mathbf{1}_{\mathrm{GL}_n(O)})$. Continuity of ℓ is unnecessary.

Proof. Left and right multiplication of a matrix by h each change additive measure by $|\det h|^n$, using the columns and the rows, respectively, and the determinant change of variables in Theorem 360.1. Thus $\mathrm{d}^{\times}X$ is invariant on both sides. The functional $T(f) = \ell(\omega f)$ is left invariant because $\omega L_h f = \omega(h) L_h(\omega f)$. Let $K_m = 1 + \pi^m M_n(O)$ for $m \geq 1$ and a uniformizer π. It is a subgroup: the inverse series for $1 + \pi^m A$ converges entrywise by completeness. Its finite index in $K_0 = \mathrm{GL}_n(O)$ follows by reduction modulo π^m. Partitioning K_0 into its K_m cosets gives $T(\mathbf{1}_{K_m}) = T(\mathbf{1}_{K_0})/[K_0 : K_m]$. Every test is constant on the cosets of some K_m: cover its compact open support by finitely many constancy cosets and refine to the largest depth. Outside their union it vanishes. Translation invariance proves $T(f) = c \int f \, \mathrm{d}^{\times}X$ on all such tests. Substituting $\omega^{-1} f$ gives the formula for ℓ, and right invariance of the measure gives the last assertion upon substituting $Y = Xs$. □

Theorem 523.2. *Use Definition 521.1 and fix a marked branch w to which its rank estimate applies. In split integral frames identify $G_w = \mathrm{GL}_n(E_w)$ with the entire unitary factor over the place below w by taking the conjugate dual inverse on $\bar{w}$. Write $\widetilde{h}$ for the corresponding group element supported on this pair. Replacing the determinant test at w by $f \in C_c^{\infty}(G_w)$, extended by zero on singular matrices, defines an absolutely convergent equal-rank period $\ell_{g_0}(f)$. If $e_w = \beta(\mathfrak{p}_w)$ is the ideal character of Theorem 375.2, set*

$$\omega(h) = \gamma_0(\delta(\widetilde{h}))\, e_w^{-n v_w(\det h)} |\det h|_w^{-n/2}.$$

With the measure of Lemma 523.1,

$$\ell_{g_0}(f) = J_\nu(g_0) \int_{G_w} f(X) \omega(X)^{-1} \, \mathrm{d}^{\times}X, \qquad J_\nu(g_0 \widetilde{s}) = J_\nu(g_0) \quad (s \in G_w).$$

At every unmarked finite place J_ν is right invariant under the integral unitary group there.

Proof. In the split Hom coordinates X the target and source act by $X \mapsto hXs^{-1}$. On the chosen branch the determinant of the Hom action is $(\det h)^n (\det s)^{-n}$. The normalized substitution of Theorem 360.1, the splitting phase of Theorem 375.2, and the two identical determinant twists of Definition 521.1 therefore give on functions the action $\omega(h)\omega(s)^{-1} L_h R_s$. The character ω is locally constant because valuation is locally constant and on integral units the torus character factors through the finite residue group by Definition 508.1.

Any such f is a finite sum of indicators of translates of K_m in the proof of Lemma 523.1. Extended by zero on the full matrix space these indicators are locally constant and compactly supported, since each coset is both compact and open there. The test $\mathbf{1}_{K_m}$ is a residue test at depth m with invertible reduction. Hence the target decay proof of Theorem 507.3 applies: the detecting displacement in Lemma 507.2 uses the first residue layer and the test is a finite sum of named points at this depth. For a left translate $\mathbf{1}_{hK_m}$, the target action already computed amounts to right translation of the integration variable by $\widetilde{h}$ and multiplication by a fixed nonzero scalar. This translation preserves the quotient measure, since it preserves Haar measure upstairs by Lemma 523.1 and transports a rational fundamental domain to another one. We thus obtain absolute convergence for every f, even when this is the only marked pair. The same change of variables gives $\ell_{g_0}(\omega(h) L_h f) = \ell_{g_0}(f)$.

On K_0 the scalar $\omega(h)$ equals $\chi_w(\det h)^{-1}$ by integral triviality of v_T in Theorem 374.2 and the unit restriction of ν. The defining determinant test of J_ν at w is therefore $\omega \mathbf{1}_{K_0}$. Lemma 523.1 gives the first equality with exactly the stated normalization. Translating g_0 on the right by $\widetilde{s}$ acts on this test by $\omega(s)^{-1} R_s$. The second equality follows from the same lemma. At an unmarked place the integral group fixes the vacuum by Theorem 390.2, and all the finite determinant twists there are trivial on integral units. This gives the last invariance. □

524. Removing one place in additive approximation

Lemma 524.1. *Let F be a number field and v a finite place. The diagonal copy of F is dense in its additive adeles with the component at v omitted. The same holds for any finite-dimensional F-vector space in rational coordinates.*

Proof. Write $\mathfrak{p}$ for the prime at v. Some positive power $\mathfrak{p}^h$ is principal by Lemma 82.5; choose its generator a. Its absolute norm is $(N\mathfrak{p})^h > 1$ by Proposition 82.3. There is an element ϖ generating a positive power of $\mathfrak{p}$ with $|\sigma\varpi| > 1$ at every infinite place. Indeed put $d = [F : \mathbb{Q}]$, $A = d^{-1} \log |N_{F/\mathbb{Q}} a| > 0$. The vector $(A - \log |\sigma a|)_\sigma$ has weighted sum zero. By Lemma 83.1 the multiples of this vector can be approximated by unit logarithms with bounded error in every coordinate, by subtracting a vector in the bounded fundamental parallelepiped. For a sufficiently large integer m, choose the corresponding unit u and set $\varpi = a^m u$. Then $\log |\sigma\varpi| = mA + O(1) > 0$ and ϖ is a local unit at every finite place other than v.

A basic approximation problem prescribes balls at finitely many places outside v, arbitrary open neighborhoods at infinity and integrality at the remaining finite places other than v. Choose $x_0 \in F$ satisfying the finite conditions. For detail, first multiply the finite targets by a positive integer D so that all prescribed centers and ball ideals become integral. Include every other prime dividing D among the conditions, requiring integrality after division by D, and impose any integral condition at v if needed. Using residues of the local centers, prescribe the resulting congruences for Dx_0 in O_F. Local residues modulo powers of a prime ideal lift to O_F: denominators in its localization are invertible modulo that power, and completion has the same finite quotients. Powers of distinct prime ideals are coprime by Proposition 82.3, so their congruences can be solved together by taking a combination of elements equal to one in a single quotient and zero in the others, using the unit ideal generated by coprime ideals. This produces x_0.

There is a nonzero integral ideal $I \subset O_F$ whose addition to x_0 preserves all the finite conditions. Take the product of sufficiently high prime powers at the specified places. Since ϖ is a unit away from v, every lattice $x_0 + \varpi^{-k} I$, $k \geq 0$, still satisfies them. The ideal I is a full real lattice under the archimedean embeddings by Lemma 82.2. A bounded fundamental parallelepiped for it shrinks in diameter to zero after multiplication by ϖ^{-k}, since each $|\sigma\varpi| > 1$. Thus for large k a point of that translate meets the prescribed real and complex conditions. Applying the scalar assertion to the coordinates of a basis gives the vector-space case. □

Theorem 524.2. *Let W_0 of Definition 521.1 admit a rational Witt splitting and let U_0, U_0^- be its two opposite radicals as in Definition 522.1. For all $g \in U(W_0)(\mathbb{A}_F)$ and $u \in U_0(\mathbb{A}_F)$ or $U_0^-(\mathbb{A}_F)$,*

$$J_\nu(ug) = J_\nu(g).$$

In particular the integral of $J_\nu(ug)$ over a rational quotient of either radical with probability measure is $J_\nu(g)$.

Proof. The period is continuous by Theorem 507.3 and is left rationally invariant by Theorem 395.2 and rational triviality of both twists. Choose the finite place v of F below the branch of Theorem 523.2. Invariance on the right under the entire local factor at v also permits changing that coordinate independently at any point, since two points agreeing at all other places differ on the right by an element of that local factor.

The coordinates C, S_0 in Definition 522.1 range over finite-dimensional F-vector spaces. Skew-hermitian matrices form such a rational space, with the diagonal entries in the anti-invariant line and one free E-entry in each off-diagonal pair. The formulas from those coordinates to the group and back are polynomial and preserve integrality outside finitely many places. Lemma 524.1 therefore gives a sequence $u_i \in U_0(F)$ tending to u in every adelic coordinate away from v, in the restricted product topology. The topology is first countable by

the countable basis argument of Lemma 519.2. In $u_i g$ replace the v-component by $(ug)_v$. The modified sequence tends adelically to ug and each modified point has value $J_\nu(u_i g) = J_\nu(g)$. Continuity gives the identity. The identical argument with L, L' interchanged applies to U_0^-. The last assertion is the integral of a constant against a measure of mass one. □

525. Generation by the two radicals

Definition 525.1. Let k be a field of characteristic zero and K/k a quadratic field extension with conjugation, or $K = k \times k$ with its exchange involution. Let $W = Ke \oplus V \oplus Kf$ have the hyperbolic pair e, f with $\langle e, f\rangle = 1$, perpendicular to V of invertible hermitian matrix J. For $v \in V,\ z \in K$ with $z + \bar{z} = -v^* J v$, write

$$N(v,z) = \begin{pmatrix} 1 & -v^*J & z \\ 0 & I & v \\ 0 & 0 & 1 \end{pmatrix}, \qquad N^-(v,z) = \begin{pmatrix} 1 & 0 & 0 \\ v & I & 0 \\ z & -v^*J & 1 \end{pmatrix}.$$

Let D be the subgroup generated by all these matrices and $M(t,r) = \operatorname{diag}(t, r, \bar{t}^{-1})$ for $t \in K^\times$, $r \in U(V)(k)$. The two displayed families are the radicals of the opposite isotropic lines by Definition 522.1.

Lemma 525.2. *In Definition 525.1, if $\theta \in K^\times$ has $\bar{\theta} = -\theta$, then*

$$w_\theta = \begin{pmatrix} 0 & 0 & \theta \\ 0 & I & 0 \\ -\theta^{-1} & 0 & 0 \end{pmatrix} \in D.$$

If $v \in V$, $\sigma \in K^\times$ and $\sigma + \bar{\sigma} = v^* J v$, then $r = I - v\sigma^{-1} v^* J$ is an isometry of V and $M(-\theta\sigma, r) \in D$.

Proof. Multiply $N(0,\theta)N^-(0,-\theta^{-1})N(0,\theta)$ to get w_θ. Put $z = -\sigma$ and $w = -v/z$, so that $1/\bar{z} + 1/z = -w^* J w$. In the product

$$g = w_\theta N(w, 1/\bar{z}) N^-(v,z)$$

the rightmost factor takes e to $(1, v, z)$. The middle factor sends it to $(0, 0, z)$: its middle coordinate is $v + wz = 0$, and its first is $1 + (v^* J v)/\bar{z} + z/\bar{z} = 0$. Thus $ge = \theta z e$. On a vector $(0, x, 0)$, the middle coordinate after the last two factors is $x - w v^* J x = rx$; w_θ fixes this coordinate. This product is an isometry preserving Ke, so its induced map on $e^\perp/Ke$ is an isometry and equals r. Factoring the parabolic matrix as in Definition 522.1 gives $g = uM(\theta z, r)$ with u in the upper radical. Hence $M(-\theta\sigma, r) \in D$. □

Theorem 525.3. *In Definition 525.1 one has $D = \{g \in U(W)(k) : \det_K g = 1\}$. Every $g \in U(W)(k)$ has the form $dM(t,1)$, $d \in D,\ t \in K^\times$.*

Proof. First suppose K is a field. Every nonsingular hermitian space over K is spanned by an orthogonal basis of nonzero norms. Indeed if all norms vanish, expanding the norm of $x + cy$ for $c = 1$ and for a nonzero anti-invariant c shows that all pairings vanish. Split off a nonzero norm vector by orthogonal projection and repeat. The same reasoning supplies an anti-invariant $\theta \neq 0$ by subtracting the conjugate of an element outside k.

Every isometry of V is a product of the maps r of Lemma 525.2. To see this, let x, y have the same nonzero norm and set $v = x - y$, $\sigma = v^* J x$. Then $\sigma + \bar{\sigma} = v^* J v$, and when $\sigma \neq 0$ the indicated map sends x to y. If $\sigma = 0$, replace y by $-y$, for which the new scalar is $2x^* J x \neq 0$. A second such map with vector y and scalar $y^* J y/2$ sends $-y$ to y. Use this procedure to align the image of the first member of an orthogonal basis with that member, then repeat in its perpendicular. Reflections chosen in the perpendicular fix the aligned member. Induction on the dimension proves generation, also in dimension zero. By the lemma and multiplication, for each such isometry r some t_r satisfies $M(t_r, r) \in D$.

Now take $ge = (a, v, c)$. If $c \neq 0$, multiply on the left by $N(-v/c, \bar{a}/\bar{c})$. This is a permissible parameter since $v^* J v = -\bar{a}c - \bar{c}a$, and sends the vector to $(0, 0, c)$ by direct

substitution. Applying w_θ moves it onto Ke. If $c = 0$ and $a \neq 0$, a first w_θ makes the last entry nonzero. If $c = a = 0$, then $v \neq 0$. Choose w with $w^*Jv \neq 0$ by nonsingularity. Applying $N(w, -w^*Jw/2)$ makes the first entry nonzero. We have thus left multiplied by elements of D to reach the parabolic of Ke. Its upper radical lies in D, and its Levi part $M(t, r)$ lies in $DM(K^\times, 1)$ by the preceding paragraph. This proves the asserted representation.

Both radicals have determinant one. If $dM(t, 1)$ has determinant one, then $t = \bar{t}$, so $t \in k^\times$. In this case $M(t, 1) = w_{t\theta} w_\theta^{-1} \in D$, proving equality. For the split algebra use either branch to identify the unitary group with $\mathrm{GL}_{\dim V+2}(k)$, the other branch being forced by the pairing. In the upper and lower radicals the rectangular blocks adjacent to the end indices can be varied independently on this branch; their conjugate entries belong to the other branch. Thus $I + xE_{ij}$ with $i \neq j$ belongs to D if at least one index is an end index. For two interior indices use the commutator of $I + xE_{i1}$ with $I + E_{1j}$, which equals $I + xE_{ij}$. These row additions reduce an invertible matrix to diagonal form, exchanging two pivots up to sign by the two-by-two product giving w_θ with entries $t, -t^{-1}$ in those two positions. Every two-by-two diagonal block $\mathrm{diag}(t, t^{-1})$ is a product of two such pivot exchanges. They reduce any determinant-one diagonal to identity by successively clearing diagonal entries. Consequently D contains the special linear group, and is contained in it by determinants. The determinant of $M((t, 1), 1)$ on the chosen branch is t, giving the representation for every determinant in this case as well. □

526. Equal-rank invariance at unmarked split places

Lemma 526.1. *Let K, O, π be as in Lemma 523.1, and let $Q = |O/\pi O|$. Use the topology induced by the matrix entries. Let $\omega : \mathrm{GL}_n(K) \to \mathbb{C}^\times$ be a locally constant character of the determinant, and put $a = \omega(\mathrm{diag}(\pi, 1, \dots, 1))$. Suppose*

$$a \neq Q^r \qquad (0 \leq r < n).$$

Any linear functional Δ on $C_c^\infty(M_n(K))$ which satisfies $\Delta(\omega(h)L_h f) = \Delta(f)$ for all h and vanishes on tests compactly supported on $\mathrm{GL}_n(K)$ is zero.

Proof. For a closed subset Y of a locally closed subset X of matrix space, restriction gives an exact sequence of test spaces

$$0 \longrightarrow C_c^\infty(X \setminus Y) \longrightarrow C_c^\infty(X) \longrightarrow C_c^\infty(Y) \longrightarrow 0,$$

where the first arrow is extension by zero. All the spaces have a basis of compact open neighborhoods: intersect small matrix balls with a closed set in an open neighborhood where the locally closed set is closed. To extend a compact open indicator on Y, cover its compact support by finitely many such neighborhoods in X whose intersections with Y lie inside the indicator set. Their union works. A test is a finite sum of these indicators. A test in the kernel has compact open support disjoint from Y, giving exactness. For subsets preserved by left multiplication the maps commute with that action.

Let X_r be the closed locus of rank at most r, defined by vanishing of its larger minors. Descend from X_n. The hypothesized vanishing on the invertible matrices makes Δ factor through X_{n-1}. At each step it suffices by exactness to prove that the induced functional is zero on compactly supported tests of the rank r stratum. This stratum is covered by the finitely many left invariant open charts where a prescribed set of r columns is independent. After placing those columns first, the matrix on the chart is $[V\ VC]$, with V an n-by-r frame and C arbitrary of the indicated size. These are homeomorphic coordinates since on a nonzero minor the entries of C are obtained by inverting that minor. Left multiplication acts only on V. Compactly supported tests can be partitioned among the charts by a finite compact open refinement. On each chart they are sums of indicators of products of compact open sets in V, C, by the product topology and compactness. Fixing a test in C, it thus suffices to treat a linear functional on frames with the same transformation law.

Let $V_0 = \binom{I_r}{0}$, and set $B_m = V_0 + \pi^m M_{n,r}(O)$ for $m \geq 1$. Each element is kV_0 with $k \in 1 + \pi^m M_n(O)$, by adjoining the unchanged final columns of the identity. Take h which scales just row $r+1$ by π. Then B_m is the union of Q^r disjoint translates of hB_m by unipotent matrices. Indeed the row $r+1$ in hB_m lies in $\pi^{m+1}O^r$. Add to it $\pi^m t$ times the first r rows, for t running over row vectors modulo π. The leading square block is I_r mod π; these additions give exactly the Q^r possible residues of the row divided by π^m. Subtracting the same row multiple proves surjectivity onto each piece. Applying the functional to the partition gives $\Delta(\mathbf{1}_{B_m}) = Q^r a^{-1}\Delta(\mathbf{1}_{B_m})$, since the row additions have determinant one. Thus its value is zero. For $r = 0$ the frame space is a point, the empty row partition has one piece, and the same conclusion holds.

Every full frame is gV_0 by extension to a basis. The sets gB_m give compact open neighborhoods at every frame. Inside a fixed gB_m, smaller boxes of the form $gkB_{m'}$, $k \in 1 + \pi^m M_n(O)$, for $m' \geq m$, partition it into the ordinary residue boxes after applying g^{-1}. Their indicators have zero functional by covariance. Cover a compact open test support with finitely many such chart boxes, subtract earlier boxes to get disjoint compact open portions, and subdivide each portion in its box to a common sufficient depth by compactness. The functional is zero on every portion. This proves the required vanishing on rank r and completes the descent to the empty locus. □

Theorem 526.2. *Let J_ν satisfy Definition 521.1. Let $w \notin S \cup \bar{S}$ lie above a split finite place of F. If $u_w = \lambda_f(\pi_w) \neq 1$ for a uniformizer idele on the chosen branch, then*

$$J_\nu(g_0 \widetilde{s}) = J_\nu(g_0) \qquad (s \in \mathrm{GL}_n(E_w)).$$

At every finite place, including the marked pairs, J_ν is right invariant under its full integral unitary group.

Proof. First the integral assertion holds off the marked set by Theorem 523.2. On a marked pair an integral change of variables on the source multiplies $d_w(\det X)$ by $\chi_w(\det s)^{-1}$. The inverse source twist $\gamma_0(\delta(\widetilde{s}))^{-1}$ is $\chi_w(\det s)$ and the splitting phase is one by Theorem 375.2. Hence the entire vector is fixed. Products over all integral factors act in the same way since off the finite set the vector is a tensor of vacua.

At the unmarked w vary the finite test over all $C_c^\infty(M_n(E_w))$, and denote the period by $\ell(f)$. Absolute convergence for every such test follows using the fixed marked place of Definition 521.1. In detail the test at w is invariant under finite Heisenberg translations in a sufficiently small lattice on the two branches: for positions use compact support and local constancy, and for momenta use the annihilator of its compact support in Theorem 360.1. In the rational displacement proof of Lemma 507.2, multiply the displacement coefficient by an integer of E that is one modulo the marked prime and its conjugate and divisible by a sufficiently high power of both primes here. Such an integer exists by the coprime ideal argument in Lemma 524.1. It preserves the detecting character at the marked place and increases the finite depths here to those fixing f. Its fixed infinite norms change the constant in the difference estimate only. The polynomial theta estimate in the proof of Theorem 507.3 holds for a fixed compactly supported locally constant finite test by clearing its finite support denominators as in Proposition 388.2. The same iterated real difference and height integration now give absolute convergence.

The source and target action remains $\omega(h)\omega(s)^{-1} L_h R_s$ by the split operator calculation of Theorem 523.2. Here

$$\omega(h) = u_w^{-v_w(\det h)} |\det h|_w^{-n/2}.$$

Indeed $\delta(\widetilde{h}) = t/\bar{t}$ for the idele t supported on w with value $\det h$. In γ_0 its ν^{-1}-value is $\lambda_f(t)^{-1}$, while its v_T^j-value is $v(t)^{2j}$ by Theorem 374.2. This last factor cancels the split ideal phase since $n = 2j$ and v is unramified here. The extension of the period satisfies $\ell(\omega(h) L_h f) = \ell(f)$ by the quotient change of variables of Theorem 523.2.

For fixed s subtract the two functionals by setting $\Delta(f) = \ell(\omega(s)^{-1}R_s f) - \ell(f)$. It has the same left transformation since the two translations commute. On compactly supported tests of the invertible locus it is zero by Lemma 523.1. The number a of Lemma 526.1 is $u_w^{-1}Q^{n/2}$. If $a = Q^r$, absolute values force $r = n/2$ and then $u_w = 1$, contrary to the hypothesis. The lemma makes Δ vanish on the whole matrix space. Apply it to the unmarked full integral matrix indicator. This is the finite vacuum in the split model of Theorem 360.1. Right translating the source point acts by $\omega(s)^{-1}R_s$, which proves the desired equality. □

527. Split primes generating the ideal class group

Lemma 527.1. *Let E, F, ν, λ, S be as in Definition 515.1, with nonzero weight μ and finite angular conductor as in Theorem 515.3. For an unmarked prime ideal $\mathfrak{p}$ write $u_\mathfrak{p} = \lambda_f(\pi_\mathfrak{p})$ and $Q_\mathfrak{p} = N\mathfrak{p}$. As real $\sigma \downarrow 1$,*

$$\sum_{\mathfrak{p}\ \text{unmarked}} Q_\mathfrak{p}^{-\sigma} = \log\frac{1}{\sigma-1} + O(1), \qquad \sum_{\substack{\mathfrak{p}\ \text{unmarked}\\ u_\mathfrak{p}=1}} Q_\mathfrak{p}^{-\sigma} = o\Big(\log\frac{1}{\sigma-1}\Big).$$

Exclude any additional fixed finite set in either sum if desired.

Proof. In the logarithm of an absolutely convergent Euler product of unitary factors the terms with prime power at least two have bounded total absolute value uniformly for $\sigma > 1$. Indeed their sum is bounded by $2\sum_\mathfrak{p} Q_\mathfrak{p}^{-2}$, finite by Corollary 83.4. The simple pole of its trivial ideal series then gives the first assertion, since a finite set contributes a bounded amount. For every fixed positive integer l, apply Theorem 515.3 to ν^l. Its weight $l\mu$ is nonzero, and one may omit the same finite set even if the actual conductor decreases. Its continued value at one is finite and nonzero, so taking the real logarithm of its absolute value in the Euler region gives

$$\sum_{\mathfrak{p}\ \text{unmarked}} \Re(u_\mathfrak{p}^l)\, Q_\mathfrak{p}^{-\sigma} = O_l(1).$$

For $m \geq 1$ and $|t| = 1$ one has

$$\mathbf{1}_{t=1} \ \leq\ m^{-2}|1 + t + \cdots + t^{m-1}|^2 = \frac{1}{m} + \frac{2}{m^2}\sum_{l=1}^{m-1}(m-l)\Re(t^l).$$

The inequality holds since the right side equals one at $t = 1$ and is everywhere nonnegative. Sum with prime weights, divide by $\log(1/(\sigma-1))$, and let $\sigma \downarrow 1$ with m fixed. The upper limit is at most $1/m$. Letting m grow proves the second assertion. □

Theorem 527.2. *Under the hypotheses of Lemma 527.1 the ordinary ideal class group of E is generated by the unmarked primes $\mathfrak{p}$ lying over split places of F with $u_\mathfrak{p} \neq 1$. The same holds after omitting any fixed finite set of primes.*

Proof. The sum of $Q_\mathfrak{p}^{-\sigma}$ over primes lying over inert finite places of F is bounded for $\sigma > 1$: their absolute residue degrees are at least two. Above a rational prime p there are at most $[E:\mathbb{Q}]$ primes: their norms are at least p and their product divides $N(pO_E) = p^{[E:\mathbb{Q}]}$ by Proposition 82.3. The bound $[E:\mathbb{Q}]\sum_p p^{-2}$ suffices. The arithmetic CM field has only split and unramified quadratic finite local algebras by Lemma 57.1. Consequently the primes excluded from the generating set have total weight $o(\log(1/(\sigma-1)))$ by Lemma 527.1.

If the generated subgroup is proper, the finite abelian class group in Lemma 82.5 has a nontrivial character ξ trivial on this subgroup. Indeed its nontrivial quotient has a maximal proper subgroup. The resulting simple abelian quotient is cyclic of prime order, by taking any nonzero element and its cyclic subgroup, and admits a nontrivial root-of-unity character. Write $L(s,\xi)$ for the ideal series of this character. It is holomorphic near one by the ordinary ideal count in Theorem 83.3: all classes have the same linear main term and their character sum vanishes. Summing its error $O(1 + Y^{1-1/[E:\mathbb{Q}]})$ by partial summation gives the asserted

continuation as in Corollary 83.4. In the Euler region the convergent prime-power logarithm therefore gives

$$\sum_{\mathfrak{p}} \Re\xi([\mathfrak{p}])Q_{\mathfrak{p}}^{-\sigma} = \log|L(\sigma,\xi)| + O(1) \le O(1).$$

On the generating primes the real part in this sum is one. Everywhere else its difference from one is at most two, so the sum equals $\log(1/(\sigma-1)) + o(\log(1/(\sigma-1)))$ by the preceding weight estimate. This contradiction proves generation. □

528. Constancy of the equal-rank period

Lemma 528.1. *The period J_ν of Definition 521.1 is invariant on the right under central isometries zI, where $z\bar{z} = 1$ adelically.*

Proof. The action of (zI, zI) on $\mathrm{Hom}(W_0, W_0)$ is the identity, and the determinant twists of the two equal-rank groups agree by Definition 521.1. Consequently $\Omega(gzI, l) = \Omega(g, lz^{-1}I)$. Right translation by the central point $z^{-1}I$ is also left translation on the full target group and preserves Haar measure there. It takes a rational fundamental domain to another one, so also preserves its quotient integral by the argument of Lemma 21.4. The defining integral is absolutely convergent, giving the identity. □

Theorem 528.2. *In Definition 521.1 let the weight of ν be nonzero. Suppose W_0 contains a rational hyperbolic plane whose splitting over its perpendicular splits the reference lattice at every finite place, with dual line lattices. For every Schwartz v_1 used in that definition,*

$$J_\nu(g) = J_\nu(1) \qquad \bigl(g \in U(W_0)(\mathbb{A}_F)\bigr).$$

Proof. Use $M(t,1)$ of Definition 525.1 on this particular plane, defined over F and on all its adeles. At each place of F regard local points as supported there. By Theorem 524.2, J_ν is invariant on the left under the two radicals supported at that place. It is therefore invariant on the left under the entire local special unitary group, by Theorem 525.3. At an infinite place the determinant of a local isometry lies on the complex unit circle. It is the n-th power of some point z on this circle, since $n = \dim W_0 > 0$, by division of its real argument by n. Removing zI leaves determinant one. The preceding left invariance and the central invariance of Lemma 528.1 thus account for the full archimedean unitary factors.

Define a function φ on the ideles of E by $\varphi(t) = J_\nu(M(t,1))$. It is invariant under multiplying t by $E^\times$, since a rational M acts on the left and J_ν is rationally invariant by Theorem 524.2 and its proof. It is invariant under every infinite component by the preceding paragraph. It is invariant under $\widehat{O}_E^\times$: the corresponding M preserves the split finite lattice because both its line factors are units, and all integral unitary points fix the function on the right by Theorem 526.2. Consequently φ factors through the ordinary ideal class group of E. Explicitly $t \mapsto \prod_w \mathfrak{p}_w^{v_w(t_w)}$ is onto the fractional ideals by choosing uniformizer powers at finitely many places. Its kernel consists of finite units and arbitrary infinite components. On rational ideles it gives the principal ideals by Proposition 82.3.

At every prime $\mathfrak{p}$ of Theorem 527.2 the matrix $M(\pi_{\mathfrak{p}}, 1)$ belongs to the full local factor fixing J_ν on the right by Theorem 526.2. Thus the function on classes is invariant under multiplication by $[\mathfrak{p}]$, hence also by its inverse. These classes generate the class group, so $\varphi(t) = J_\nu(1)$ for every idele t.

Finally consider g supported at infinity and finitely many finite places. Apply Theorem 525.3 at each of these places to write $g = dM(t,1)$, where d is a finite product of supported special unitary elements. Left invariance gives $J_\nu(g) = \varphi(t)$. Such elements g are dense in the full restricted product: for any point and any basic neighborhood one may keep its components where conditions are prescribed and replace the others by identity in the prescribed integral groups. Continuity of the period in Definition 521.1, proved by Theorem 507.3, extends the equality to every g. □

Corollary 528.3. *Under the splitting and weight assumptions of Theorem 528.2, put $j_\nu = J_\nu(1)$. Throughout the unimodular parabolic range of Theorem 522.3,*

$$A_\nu^+(um) = K_\nu j_\nu \, A_\nu^-(um).$$

The same scalar applies at every such u, q, g_0 and depends linearly on v_1 for fixed remaining data and measures.

Proof. Substitute Theorem 528.2 in Theorem 522.3. By Definition 521.1 the dependence of j_ν on v_1 is linear, as absolute convergence permits addition under the integral. The factors defining K_ν in Theorem 521.3 are independent of v_1. □

529. Fiber integrals of complementary frame tests

Definition 529.1. Let $K = \prod_{w\in S} E_w$, $O = \prod_{w\in S} O_w$, $G_i = \mathrm{GL}_i(K)$, $K_i = \mathrm{GL}_i(O)$, and $N = b + d$ be even with $b, d > 0$. Use the original large and small orientations of Definition 508.1. For $t \in K^\times$ let $\lambda_S(t)$ be λ_f on t supported at the chosen branches, equal to one as an idele outside them. Put

$$\omega(t) = \lambda_S(t)^{-1} \prod_{w\in S} |t_w|_w^{-N/2}, \qquad \omega(k) = \omega(\det k) \quad (k \in G_i).$$

Give G_i the product measures of Lemma 523.1. A row test Φ is locally constant with compact support on the surjections $K^N \to K^d$; a column test Ψ has the same properties on the injections $K^b \to K^N$, in each case of full rank at every factor. Both extend by zero. Write $x_0 = [0\ I_d]$ and $y_0 = \binom{I_b}{0}$. For $k \in G_N$ put

$$B_\Phi^+(k) = \int_{G_d} \Phi(rx_0k^{-1})\omega(r)^{-1} \, \mathrm{d}^\times r, \qquad B_\Psi^-(k) = \int_{G_b} \Psi(ky_0t)\omega(t)^{-1} \, \mathrm{d}^\times t.$$

The preimages of the compact supports on each fiber are compact. Indeed the row and column coordinates describe the frames of a fixed quotient or subspace, with continuous inverses obtained by any full minor. This fiber is closed in the respective full-rank locus, being defined by vanishing on a fixed plane or taking values in it.

Lemma 529.2. *With fixed real tests and all unmarked factors as in Definition 508.1, the original periods $I_\Phi(g), I'_\Psi(g)$ converge absolutely for all the tests of Definition 529.1. For fixed g, $B_\Phi^+ = 0$ implies $I_\Phi(g) = 0$, and $B_\Psi^- = 0$ implies $I'_\Psi(g) = 0$.*

Proof. For a surjection X the image XO^N is a full lattice in K^d. Multiply X on the left by an invertible matrix taking that lattice onto O^d. This makes X integral and its reduction surjective at each branch. Such matrices form an open set, since integrality and a unit minor at each branch are open conditions. For a column test apply the same procedure to its transpose. By compactness, partition the support into finitely many compact open pieces on each of which a single such row or column operation works. Each transformed test then depends on a finite residue depth inside the integral box. The surjective or injective estimate of Theorem 507.3 applies on these pieces. Undoing the transformation translates the integration variable at the split places and multiplies by a constant. It preserves finiteness because their Haar measures are right invariant by Lemma 523.1 and the change of quotient domains in Theorem 523.2.

More precisely an integrated large target element $r \in G_d$ acts on Φ by $\omega(r)\Phi(r^{-1}\,\cdot)$, and an integrated small source element $t \in G_b$ acts on Ψ by $\omega(t)^{-1}\Psi(\cdot\, t)$. Indeed each acts on N columns or rows, and has target character γ or inverse source character γ^{-1}. The split substitution and ideal phase of Theorems 360.1 and 375.2 give the modulus powers $-N/2, N/2$ and the ideal phases $e_w^{-Nv_w(\det r)}, e_w^{Nv_w(\det t)}$, respectively, where $e_w = \beta(\mathfrak{p}_w)$. The $v_T^{N/2}$ part of γ cancels these phases by Theorem 374.2. Thus both claimed actions follow from the definition of ω. The corresponding changes of integration variable leave the periods invariant.

For surjections cover the Grassmannian of their kernels by the charts on which a prescribed complementary set of d columns is invertible. After ordering the columns the chart on frames has coordinates $X = [rC\ r]$, $r \in G_d$, with arbitrary C. Order columns separately at different branches if needed. For fixed C this is exactly a fiber $rx_0k(C)^{-1}$, by taking an invertible shear and the indicated permutations. If the row fiber integral vanishes, multiply the test by a locally constant cutoff in C; its fiber integral still vanishes. A finite compact open partition of the compact projected support allows these cutoffs inside individual charts. Write a test there as a sum of $\phi_i(r)\mathbf{1}_{U_i}(C)$ with the U_i disjoint compact open, by finite rectangle refinement as in Lemma 526.1. For each nonempty U_i the integral of $\phi_i\omega^{-1}$ vanishes. Holding the cutoff fixed, the large-period functional on ϕ_i satisfies the left transformation law of Lemma 523.1. That proof also works on the finite product K, by taking products of its compact open cosets. The functional therefore vanishes on each piece. For columns use coordinates $Y = \binom{I_b}{C}t$, with the invertible chosen rows first. Translation on t is now on the right. The same coset proof with right translates, or the substitution $t \mapsto t^{-1}$, gives the functional as a scalar times $\int \phi(t)\omega(t)^{-1}\,\mathrm{d}^{\times}t$. Thus the column assertion follows as well. □

Proposition 529.3. *Let $\mathcal{M}$ be the column tests for which a row test exists satisfying $B^-_\Psi(k) = \omega(k)B^+_\Phi(k)$ for all $k \in G_N$. For a fixed scalar c, the automorphic function $D\Psi = I_\Phi - cI'_\Psi$ depends only on Ψ and is linear on $\mathcal{M}$. Every original matched pair at the squarefree residue level belongs to this relation after integral lifting. The kernel of D is stable under $\Psi \mapsto \Psi(q\,\cdot)$ for every $q \in G_N$.*

Proof. Two choices for the row test have difference with zero fiber integral, so Lemma 529.2 proves the first assertion. On K_N, for an original residue pair the row integrand can be nonzero only for $r \in K_d$, since the induced matrix of a primitive quotient frame generates the entire target lattice. The column integrand can be nonzero only for $t \in K_b$ by the transposed assertion. The determinant equivariance on ordered frames in Definition 499.1 cancels the weight ω^{-1} there, and the groups have mass one. Thus on K_N the relation is exactly $\Psi(ky_0) = \chi(\det k)^{-1}\Phi(x_0k^{-1})$ of that definition. To extend it, for $m = \left(\begin{smallmatrix} a & u \\ 0 & d_0 \end{smallmatrix}\right)$ the functions $B^+_\Phi(km)$, $B^-_\Psi(km)$ acquire the factors $\omega(d_0)^{-1}, \omega(a)$, respectively, by changing their integration variables. These give the same law after inclusion of $\omega(k)$. Moreover K_N is transitive on b-planes over K: intersect the plane with O^N and extend a basis of this primitive summand as in Proposition 511.4. Hence every k is a product of an integral element with such a block m.

The pair transformed by q is $(\omega(q)\Phi(\cdot\, q^{-1}), \Psi(q\,\cdot))$. Its two fiber functions use the original functions at qk, with the extra $\omega(q)$ on the row side. This preserves the relation. On both periods this transformation is right translation by the ambient element $\widetilde{q}^{-1}$ times the same nonzero scalar. Indeed, before division by the small-side multiplier, a right translation by $\widetilde{k}$ multiplies $\Phi(\cdot\, k)$ by $\prod_w e_w^{dv_w(\det k)}|\det k_w|_w^{d/2}$, and $\Psi(k^{-1}\,\cdot)$ by $\gamma(\delta(\widetilde{k}))\prod_w e_w^{-bv_w(\det k)}|\det k_w|_w^{-b/2}$. These follow from the same split substitutions, now acting on the ambient variable in the two original orientations. Their ratio at $k = q^{-1}$ is $\omega(q)$ since $N = b + d$ and the splitting characters cancel as above. This proves kernel stability for arbitrary q. □

530. The determinant sum of shears

Definition 530.1. In Definition 529.1 let $N = 2b+n$ and write the column frame Y in three successive blocks P, H, Q of respective row sizes b, n, b. Let $R = \prod_{w\in S} O_w/\pi_w O_w$, $\pi = (\pi_w)$, and $d(M) = \chi^{-1}(\det M)$ on invertible residue matrices and zero elsewhere. Use the same d on integral matrices by reduction. Assume multiplicative primitivity as in Lemma 500.2. Write $D_Q(Y) = \mathbf{1}_{\mathrm{int}}(Y)d(Q)$ and $D_P(Y) = \mathbf{1}_{\mathrm{int}}(Y)d(P)$, putting either expression zero off

the integral box. The upper-right shear $v(t)$ adds tQ to P, leaving H, Q fixed. Set

$$S_t(Y) = D_Q(v(t)Y), \qquad G(Y) = \sum_{t \in M_b(R)} d(t)\, S_{\pi^{-1}\widetilde{t}}(Y),$$

where $\widetilde{t}$ is any integral lift of t. The functions belong to $\mathcal{M}$ of Proposition 529.3. Indeed D_Q, D_P are standard partners of Lemma 509.2, and the action of G_N there preserves $\mathcal{M}$.

Theorem 530.2. *In Definition 530.1, the function G is zero unless H is integral and $Q, \pi P \in \mathrm{GL}_b(O)$; on that set it equals $d(-\pi P)$. For $a = \mathrm{diag}(-\pi^{-1} I_b, I_n, I_b)$, let $w(t)$ add tP to Q, preserving P, H. Then*

$$\sum_{t \in M_b(R)} G(aw(\widetilde{t})Y) = |\mathrm{GL}_b(R)|\, D_P(Y).$$

In particular, if the global difference DG in Proposition 529.3 is identically zero, the original comparison holds for every matched pair at this residue level.

Proof. If a summand is nonzero, Q is an integral unit matrix and H is integral. The remaining integrality condition is $P + \pi^{-1}\widetilde{t}Q$ integral, equivalent to πP integral and $tQ \equiv -\pi P \bmod \pi$. This gives exactly one allowable residue matrix t; changes of its lift do not affect the condition. The weight is zero unless t is invertible. Otherwise its product with the initial datum is $d(t)d(Q) = d(-\pi P)$. This gives the first formula. Upon substituting $aw(\widetilde{t})Y$ it says that the summand is $d(P)$ exactly when $P \in \mathrm{GL}_b(O)$, H is integral, and $Q + \widetilde{t}P \in \mathrm{GL}_b(O)$. In particular Q must be integral. Conversely for integral Y with P invertible, $t \mapsto Q + tP \bmod \pi$ is a bijection of $M_b(R)$. Exactly $|\mathrm{GL}_b(R)|$ terms therefore contribute, proving the formula, including all zero cases outside that support.

The kernel in Proposition 529.3 is a linear space stable under all the changes just used. Thus $D(G) = 0$ implies $D(D_P) = 0$. The first-block projection is an integral quotient frame. By Corollary 509.4 the comparison for this one standard pair implies it for every matched pair at the level. □

Lemma 530.3. *Suppose $L : \mathcal{M} \to \mathbb{C}$ is linear and there are matrices $T_w \in M_b(E_w)$ and additive characters ψ_w with annihilator O_w such that, for every $z \in M_b(K)$,*

$$L(S_z) = \prod_{w \in S} \psi_w(\mathrm{tr}(T_w z_w))\, L(D_Q).$$

If $L(G) \neq 0$, then $L(D_Q) \neq 0$ and every T_w belongs to $\mathrm{GL}_b(O_w)$.

Proof. If $L(D_Q) = 0$, the stated law annihilates every summand of G. Otherwise $S_z = D_Q$ for integral z, since on the support where Q, H are integral, $P + zQ$ is integral exactly when P is. The product character is then one on all integral matrices. Varying a single entry at a single place and using the annihilator shows that T_w has integral entries there. On integral z divided by π the product character factors through the finite matrix pairing with $\overline{T} = (T_w \bmod \pi_w)_w$. The residue additive character is primitive because at each field the original conductor is the integer ring. The ratio $L(G)/L(D_Q)$ is the finite matrix Fourier sum of $d(t)$ at $\overline{T}$, with its sign adjusted if the opposite Fourier convention is used. Lemma 500.2 applied to χ^{-1} makes this sum zero on noninvertible matrices. This proves the necessity of integral unit determinant at every marked branch. □

531. A global criterion at the marked branches

Lemma 531.1. *For the compact additive quotient $[\mathcal{Z}]$ in Definition 510.1, a continuous function all of whose coefficients against the ξ_B vanish is zero. Here B ranges over the rational hermitian b-by-b matrices.*

Proof. The rational bilinear pairing

$$(B, z) \longmapsto -\tfrac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(\jmath\,\mathrm{tr}(zB))$$

identifies the hermitian matrix space with the rational dual of the skew-hermitian one. Their dimensions agree by multiplication by $\jmath$, and a nonzero B pairs nontrivially with $\jmath^{-1}B$: the trace of B^2 in each complex embedding is a sum of squared moduli of its entries. Thus in a rational basis of dimension h the ξ_B include the characters for every rational frequency in $\mathbb{Q}^h$.

For a positive integer m average the function over the image of $m\widehat{\mathbb{Z}}^h$ at the finite adeles, with probability measure. The average is continuous and all of its same coefficients vanish, by translation and compact integration as in Lemma 123.1. The quotient by this compact subgroup is $\mathbb{R}^h/m\mathbb{Z}^h$: subtract a rational vector to put the finite part into $m\widehat{\mathbb{Z}}^h$ by the finite congruence argument of Lemma 524.1, and the rational choices then differ by $m\mathbb{Z}^h$. The induced map from the real torus is a continuous bijection from a compact space to a Hausdorff group, hence a homeomorphism. Pushing forward the invariant probability measure gives uniform measure there: translated equal boxes of side m/k partition the torus up to their faces, parallel faces having measure zero since infinitely many disjoint translates exist. Approximation by such boxes gives the asserted measure on continuous tests. All ordinary Fourier coefficients of the average vanish since their frequencies belong to $m^{-1}\mathbb{Z}^h$. The product Fejér-kernel argument of Lemma 14.2 shows that the average is zero. Along $m = j!$, the averaging groups shrink into every neighborhood of zero at the finite adeles. Uniform continuity on the compact quotient makes the averages converge uniformly to the function, proving the claim. □

Definition 531.2. At the chosen branches use the integral frame of Definition 517.1. For $t \in M_b(K)$ let $z(t)$ be the adelic skew-hermitian matrix giving the shear $v(t)$ there, its forced conjugate on the opposite branches and zero elsewhere. Thus $\widetilde{v(t)} = n(z(t))$ in Definition 510.1. For a rational hermitian B define its marked index $T(B) = (T_w(B))$ by

$$\xi_B(-z(t)) = \prod_{w\in S} \psi_w(\mathrm{tr}(T_w(B)t_w)).$$

Use the conductor-O_w characters from the trace-dual identity in Lemma 355.2. This specifies a unique matrix at each branch. Indeed the two terms of the trace in ξ_B at a split pair are conjugate field transports with the same local trace: the two minus signs from $\jmath$ and the skew matrix cancel. In the adapted coordinates the resulting character is the local additive character of a trace linear form in the entries of t_w. Divide its coefficients by the nonzero scalar defining ψ_w in the trace character to obtain $T_w(B)$, with indices transposed for the matrix trace. Uniqueness follows by testing a single matrix entry at arbitrary multiples, since ψ_w has annihilator O_w.

Theorem 531.3. *Use D of Proposition 529.3 and the data of Definition 530.1. The exchange equality at this twist and scalar c, for all matched pairs at the prescribed residue level and every ambient point, is equivalent to the following condition. For every g whose components at the marked pairs are identity and each rational hermitian B with $T_w(B) \in \mathrm{GL}_b(O_w)$ for every $w \in S$, the coefficient at B, g of $D(D_Q)$ is zero.*

Proof. At fixed B, g integration against $\overline{\xi_B}$ on the compact center defines a linear functional L on $\mathcal{M}$ by taking this coefficient of $D(\cdot)$. The defining integrals exist since the periods are continuous: in the compact partitions in Lemma 529.2, each piece is a fixed translate of primitive tests, for which Theorem 507.3 gives continuity. By the transformation in Proposition 529.3 and $\det v(t) = 1$, the function $D(S_t)$ equals the right translate of $D(D_Q)$ by $n(-z(t))$. If g is identity at the marked pairs, it commutes with this supported element. Moving the translate through g and changing variables in the central integral gives $L(S_t) = \xi_B(-z(t))L(D_Q)$. Thus if $L(G) \neq 0$, Lemma 530.3 gives $T_w(B) \in \mathrm{GL}_b(O_w)$ at all marked branches and $L(D_Q) \neq 0$. Under the stated condition every coefficient of $D(G)$ at this g therefore vanishes. Lemma 531.1 applied to $z \mapsto D(G)(n(z)g)$ gives $D(G)(g) = 0$.

We verify that such points and their left rational translates are dense. At the finitely many marked places the images of rational isometries are dense in the product of the local unitary groups. To see this, the Cayley map $Z \mapsto (1+Z)(1-Z)^{-1}$ and its inverse $g \mapsto (g-1)(g+1)^{-1}$ give inverse rational maps between the open skew-adjoint and unitary charts, by the Gram equation as in Lemma 95.2. At a split pair the chart where $g+1$ is invertible is dense: on one branch multiply by a scalar arbitrarily close to one, avoiding the finitely many roots of $\det(tg+1)$; the inverse conjugate branch then also avoids eigenvalue -1. The skew-adjoint matrices form an F-vector space, whose rational points approximate any prescribed tuple in these charts by Lemma 524.1, omitting a place outside this finite set. Such a place exists since each rational prime has a prime of F above it by Proposition 82.3. Inverting the rational Cayley transform is legitimate for sufficiently close approximants because their denominators are invertible at these places, hence nonzero over the field. This proves density.

For arbitrary g choose rational δ_i tending to its marked components, and take g_i equal to identity there and to $\delta_i^{-1} g$ elsewhere. Then $\delta_i g_i \to g$. The function $D(G)$ is continuous and left rationally invariant by Theorem 395.2 and rational triviality of the twists, so it is identically zero. Theorem 530.2 gives exchange on all the required pairs. Conversely that exchange makes $D(D_Q)$ identically zero by using its standard partner, and consequently gives every coefficient in the condition. □

532. Transport of unimodular Fourier indices

Definition 532.1. Use the data of Corollary 528.3. Identify L rationally with T_b by the reference frame of Definition 521.1. Its Gram matrix on L is B_*. On the finite lattice of L choose an ordered adelic basis with columns C_L in rational coordinates. Let C_T be such a basis for the finite lattice of T_b, and $J_f = C_T^* B_* C_T$, in the identified rational coordinates. For $g = u\, m(q, g_0)$ as in Definition 522.1 and any rational hermitian matrix B on L, put

$$H_f(B, q) = (\alpha\bar{\alpha})^{-1} C_L^* q_f^* B q_f C_L.$$

Here α is the finite scale in Definition 517.1. Call $H_f(B,q)$ unimodular if it and its inverse have integral entries at every finite place. The definition is unchanged by replacing C_L by $C_L k$ with $k \in \mathrm{GL}_b(\widehat{O}_E)$, since this conjugates the form by $k^*(\cdot)k$. For either normalized period $P^{\pm}$ of Definition 520.1 write $P_B^{\pm}(g)$ for its coefficient in Definition 510.1.

Lemma 532.2. *For any continuous left rationally invariant function P on the ambient group and any $c \in \mathrm{GL}_b(E)$, one has*

$$P_{B_*}(m(c,1)g) = P_{c^* B_* c}(g).$$

This applies to both $P^{\pm}$.

Proof. Conjugation by $m(c,1)$ sends $n(z)$ to $n(czc^*)$ by the three blocks of Definition 520.1. The linear map $z \mapsto czc^*$ on the skew-hermitian space is invertible and rational. It induces an automorphism of its compact additive adelic quotient preserving probability measure: the pushed-forward measure is again translation invariant with mass one. Uniqueness here follows directly by integrating $f(x+y)$ against two invariant probabilities in the two orders, as in Lemma 123.1. Now change the integration variable to czc^* in the coefficient at $m(c,1)g$. The rational left factor $m(c,1)$ drops out. The trace formula of Definition 510.1 gives $\xi_{B_*}(czc^*) = \xi_{c^* B_* c}(z)$ by cyclicity of matrix trace, which follows by summing products of entries. The two periods are rationally invariant by Theorem 395.2, since all determinant characters in the normalization are trivial on rational points. □

Theorem 532.3. *Assume the hypotheses of Definition 532.1, including the real tests of Theorem 514.3. For every $g = u\, m(q, g_0)$ preserving L and every rational hermitian B for which $H_f(B,q)$ is unimodular,*

$$P_B^{+}(g) = K_\nu j_\nu P_B^{-}(g).$$

The same scalar works for all these B, g.

Proof. Unimodularity forces B to be nonsingular. If at any real place its signature differs from that of B_*, both coefficients vanish by Theorem 513.2. Indeed the real test on the large side has compact frame support inside the specified nonsingular signature locus by Theorem 514.3. The character used to normalize either period is constant on $n(z)g$, since $\det n(z) = 1$.

Suppose the signatures agree at every real place. At each finite place the two integral unimodular matrices $H_f(B, q)$ and J_f are integrally isometric. The argument in Lemma 403.1 applies to these two matrices directly: at a split prime use a basis and its dual on the two branches. At an inert prime reduce the two matrices modulo the uniformizer, apply the finite-field diagonalization in Lemma 345.1, and lift the isometry one digit at a time by the trace correction in Lemma 403.1. This gives $A \in \mathrm{GL}_b(\widehat{O}_E)$ with $A^* J_f A = H_f(B, q)$: each local choice and its inverse are integral, hence they form adelic matrices. It follows by undoing both changes of basis, including the scalar α^{-1}, that B, B_* are isometric over every finite completion. Our field $E = F(i)$ has $\sqrt{3} \in F$ by the twelfth root of unity in Lemma 20.1. Theorem 90.2 therefore gives $c \in \mathrm{GL}_b(E)$ with $c^* B_* c = B$.

In the identified coordinates the normalized finite matrix of the moved reference frame is

$$Y(cq)_f = \alpha^{-1} C_T^{-1} c q_f C_L, \qquad Y(cq)_f^* J_f Y(cq)_f = H_f(B, q) = A^* J_f A.$$

Thus $Y(cq)_f A^{-1}$ is unitary. The element $m(c, 1)g$ still preserves L, with Levi frame map cq, since conjugating u by $m(c, 1)$ preserves the radical in Definition 522.1. Corollary 528.3 now compares the two B_*-coefficients there. Lemma 532.2 gives the stated equality at B, g. The factors K_ν, j_ν in that corollary are fixed before the rational change of frame. □

Corollary 532.4. *Under the same hypotheses, if $H_f(B, q)$ has a nonintegral entry then $P_B^+(g) = P_B^-(g) = 0$. This also holds for every singular B and every B with an incorrect real signature.*

Proof. For the latter two conclusions use Theorem 513.2 as in the preceding theorem. For the first, any contributing rational frame x has $x^* J_{T'} x = B$ for its target T', by Theorem 510.3. The transported finite frame $h_f^{-1} x q_f C_L / \alpha$ has columns in the target lattice: at every marked place both slices in Theorem 517.3 carry the frame integrality indicator, and elsewhere the vacuum does so by Lemma 517.2. Transport of this support for a parabolic element follows from the dual modulation argument in Proposition 518.3: a frame outside the lattice is detected by a dual character trivial on the lattice; that argument precedes any use of the Gram determinant. Since h_f is unitary, the Gram matrix of these columns is precisely $H_f(B, q)$. Entries of the form on the finite target lattice are integral by Definition 517.1. Thus no frame contributes at a nonintegral entry. The truncated formula and its limiting assertion give the vanishing for both periods. □

533. A rank bound from one split place

Definition 533.1. Fix a branch w in the nonempty marked set of Definition 529.1. Write $K = E_w$, $O = O_w$, take a uniformizer π, and choose the additive character ψ of conductor O whose annihilator on O is O, as in Proposition 501.2. Fix a rational isotropic splitting as in Definition 510.1, now with isotropic dimension k, $1 \le k \le N/2$, allowing k to differ from the column count b. Choose dual rational bases on the two isotropic spaces. At w write the columns in their three ambient blocks as (P, H, Q), with P, Q of size k-by-b. The component at $\bar{w}$ of a central shear is determined by the component at w. On these columns the latter acts as $v(z)(P, H, Q) = (P + zQ, H, Q)$ for $z \in M_k(K)$. These coordinates may be obtained from the original coordinates on the w-component by an invertible change of basis. For a locally constant compactly supported column test f, extended by zero off its

full-rank support, put

$$\widehat{f}(C,H,Q)=\int_{M_{k,b}(K)} f(P,H,Q)\psi(-\operatorname{tr}(CP))\,\mathrm{d}P, \qquad C\in M_{b,k}(K).$$

Other marked variables are kept as parameters. Use additive measure giving the integral matrices mass one in each matrix space.

Lemma 533.2. *Let f belong to a linear space of such tests stable under $f\mapsto f(v(z)\,\cdot)$ for every $z\in M_k(K)$. Suppose a linear functional Λ on this space satisfies*

$$\Lambda(f(v(z)\,\cdot))=\psi(\operatorname{tr}(Tz))\Lambda(f)$$

for every test and every z, with $T\in M_k(K)$. If $\widehat{f}(C,H,Q)=0$ whenever $QC=T$, for all H and the other parameters, then $\Lambda(f)=0$. In particular $\Lambda=0$ when $\operatorname{rank}_K T>b$.

Proof. A compactly supported locally constant function on a product of matrix spaces is a finite linear combination of indicators of products of compact open boxes. Indeed cover its compact support by boxes on which it is constant, and subdivide at a common precision in each variable. Its zero extension from the open full-rank locus is locally constant also on the complement because the support is compact and closed in the ambient space. The transform of a P-box $P_0+\pi^a M_{k,b}(O)$ is its volume times $\psi(-\operatorname{tr}(CP_0))$ on $\pi^{-a}M_{b,k}(O)$ and zero elsewhere. Integration of a nontrivial character over a compact group vanishes by translation, and the trace pairs the entries in opposite positions. The annihilator assertion in Definition 533.1 gives exactly this dual box. Transforming back with the positive character recovers the indicator: the two box volumes are reciprocals by counting the cosets at successive precisions. Thus the partial transform is injective and its result is again locally constant and compactly supported in all the variables.

For $m\geq 0$ form the probability average of $\psi(-\operatorname{tr}(Tz))f(v(z)\,\cdot)$ over $z\in\pi^{-m}M_k(O)$. This is a finite linear combination in the original space. A sufficiently small open subgroup fixes the test by substitution, since Q is bounded on its support, translations of P by a sufficiently small box preserve it, and the inverse shear has the same property. Shrink the subgroup to kill the scalar character there too. The functional of the average equals $\Lambda(f)$. Substituting $P+zQ$ in the partial transform gives its original value multiplied by $\psi(\operatorname{tr}(QCz))$. Hence the transform of the average is

$$\mathbf{1}_{QC-T\in\pi^m M_k(O)}\,\widehat{f}(C,H,Q).$$

The support of $\widehat{f}$ is compact and equals its nonzero set, since it is locally constant. The continuous image under $(C,H,Q)\mapsto QC$ is compact and disjoint from T by hypothesis. A sufficiently small box about T misses this image, so the transform and hence the average are zero for all large m. This proves the first assertion. The product QC factors through K^b, proving the rank assertion. □

Theorem 533.3. *Let D and $\mathcal{M}$ be as in Proposition 529.3, for any scalar of comparison. For every rational isotropic splitting of dimension k as in Definition 533.1, every $\Psi\in\mathcal{M}$, every adelic argument g and every rational hermitian k-by-k index R with $\operatorname{rank}_E R>b$, the R-coefficient of $D\Psi$ at g is zero.*

Proof. First let g have identity component at the split pair of w. Taking the R-coefficient at g gives a linear functional Λ on $\mathcal{M}$. A shear supported at this split pair has determinant one. By the transformation formula in the proof of Proposition 529.3, $D(\Psi(v(z)\,\cdot))$ is the right translate of $D\Psi$ by the inverse shear. This supported shear commutes with g. In the compact central integral of Definition 510.1 its effect is therefore multiplication by $\xi_R(-z_{\mathbb{A}})$, where $z_{\mathbb{A}}$ denotes z at w, its negative adjoint at $\bar{w}$, and zero elsewhere. In dual rational bases, $\xi_R(-z_{\mathbb{A}})=\psi(\operatorname{tr}(Tz))$ with T a nonzero local scalar times R_w. Indeed the two branch traces are equal: both $\jmath$ and the skew-hermitian $z_{\mathbb{A}}$ change sign under conjugation, whereas $R^*=R$. The two copies cancel the factor $1/2$ in the defining trace. Expressing the resulting

field trace character in the fixed normalization ψ multiplies its argument by a nonzero scalar, as in Proposition 501.2. This also shows directly that $\operatorname{rank}_K T = \operatorname{rank}_E R$, since a rational minor is nonzero exactly when its embedding in K is nonzero. The linear space of tests is stable under every such shear by Proposition 529.3. Lemma 533.2 applies.

For arbitrary g, write $g = g^{(w)} h_w$, with $g^{(w)}$ identity at the split pair and h_w supported there. Right translation of $D\Psi$ by h_w is a nonzero scalar times D applied to a transformed column test, by the same proposition, which applies to the entire split general linear group. This translation passes through the defining left central integral. Applying the first case at $g^{(w)}$ proves the result. □

534. Invariance at a bordered index of maximal rank

Definition 534.1. Use a rational isotropic space $L_+ = Ee_0 \oplus L$ and its dual isotropic space with bases e_0' and a basis dual to L, where $\dim_E L = b$. Denote the orthogonal complement of the two dual spaces by W_2, and put $M = L \oplus W_2 \oplus L'$, where L' is the chosen dual of L. Thus $W = Ee_0 \oplus M \oplus Ee_0'$ is also a rational isotropic splitting. Write U_e for the unipotent radical of the line Ee_0, given explicitly by Definition 522.1 with frame dimension one. It acts as the identity on the three successive quotients for $Ee_0 \subset e_0^{\perp} \subset W$. For any nonsingular rational hermitian B on L use the index

$$R_0 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

on L_+. For $\Psi \in \mathcal{M}$ and fixed comparison scalar let $F_\Psi(g) = (D\Psi)_{R_0}(g)$ denote the coefficient along L_+ as in Theorem 533.3.

Lemma 534.2. *For any u supported at the split pair of one marked branch and belonging there to U_e, one has $F_\Psi(ug) = F_\Psi(g)$ at every g.*

Proof. First take g with identity component at this pair. As in Theorem 533.3, the coefficient there is a functional Λ with local frequency $T = \operatorname{diag}(0, T_1)$, T_1 invertible, in the variables of Definition 533.1. Its value at ug equals its value at g on the test $\Psi(u^{-1}\cdot)$, because g commutes with the supported element and $\det u = 1$. Work at the selected branch, leaving all other marked variables fixed. In the enlarged splitting write $P = (P_0, P_1)^{\mathsf{t}}$, $Q = (Q_0, Q_1)^{\mathsf{t}}$ in row blocks and $C = (C_0\ C_1)$ in column blocks, with index zero referring to the first row or column. On $QC = T$, both $Q : K^b \to K^{b+1}$ and $C : K^{b+1} \to K^b$ have rank b. The zero first row of T gives $Q_0 C = 0$, hence $Q_0 = 0$ by surjectivity. The zero first column gives $QC_0 = 0$, hence $C_0 = 0$ by injectivity.

The row ordering for the line splitting is $(P_0; (P_1, H, Q_1); Q_0)$, with H the positions in W_2. In these coordinates u^{-1} is upper triangular in three blocks with identity diagonal by Definition 534.1. At $Q_0 = 0$ it therefore fixes P_1, H, Q_1 and adds to P_0 a row depending only on P_1, H, Q_1. In the partial transform integral, $\operatorname{tr}(CP)$ is independent of P_0 when $C_0 = 0$. For fixed P_1, H, Q_1, the indicated translation of P_0 preserves its measure. The integrals for the moved and original test consequently agree on the whole fiber $QC = T$, also when the original test couples the various variables. The difference of the two tests belongs to $\mathcal{M}$ by Proposition 529.3. Lemma 533.2 gives the asserted equality at g. For general $g = g^{(w)} h_w$ absorb the rightmost h_w into the datum in both coefficients as in Theorem 533.3. □

Theorem 534.3. *For every $u \in U_e(\mathbb{A}_F)$, every adelic g, and every test in Definition 534.1,*

$$F_\Psi(ug) = F_\Psi(g).$$

The assertion applies to any rational rank-b hermitian index on an isotropic $(b+1)$-plane, with Ee_0 its kernel.

Proof. A rational $v \in U_e(F)$ preserves L_+. Its action on that plane has matrix $a = \left(\begin{smallmatrix} 1 & \ell \\ 0 & I_b \end{smallmatrix}\right)$ since it fixes e_0 and acts trivially on $e_0^{\perp}/Ee_0$. Therefore $a^* R_0 a = R_0$. Conjugation by v on

the center of L_+ sends $n(z)$ to $n(aza^*)$. Indeed $n(z)-1$ maps $W/L_+^{\perp}$ to L_+, and on the former quotient v acts by $(a^*)^{-1}$ by the perfect pairing. The rational change of variables and probability measure argument of Lemma 532.2 therefore shows $F_\Psi(vg)=F_\Psi(g)$.

Let v_0 be the finite place of F below the chosen marked branch. By the rational coordinates of Definition 522.1 and Lemma 524.1, there are $v_i\in U_e(F)$ tending to u on the adeles away from v_0. Replace the v_0-component of v_i by that of u, obtaining u_i. This changes v_ig by left multiplication by an element of the supported local radical, so $F_\Psi(u_ig)=F_\Psi(v_ig)=F_\Psi(g)$ by Lemma 534.2. Since $u_ig\to ug$, take limits. The coefficient is continuous: the period is continuous as in Theorem 531.3; translation is continuous uniformly on the compact center quotient over a compact neighborhood of g. Integrating on that quotient preserves this convergence.

For the final assertion choose a rational basis of the kernel line and extend it to a basis of the given plane. The restriction of the form to the chosen complement is nonsingular: a vector of the complement orthogonal to it pairs also to zero with the kernel, hence belongs to the kernel. The index is R_0 in this basis. Change the opposite isotropic basis by the inverse adjoint, giving a rational splitting of the required kind, and use the result just proved. □

535. Two means on a hyperbolic plane

Definition 535.1. In Definition 534.1 use the labels $e=e_0,\ e'=e_0'$, with $\langle e,e'\rangle=1$. In the rational frame of L identify $V=E^b$ with L and its dual coordinates with L'. For $x,y\in V_{\mathbb{A}}=V\otimes_{\mathbb{Q}}\mathbb{A}_{\mathbb{Q}}$ let $X(x)$ add x to e and send $l'\in L'$ to $l'-e'x^*l'$; let $Y(y)$ add y to e' and send l' to $l'-e\,y^*l'$. Each fixes all other source summands. Thus, for example, $X(x)e=e+x$ and $X(x)e'=e'$. Let $Z(z)=n(z)$ be the center for the isotropic space L. For fixed $\Psi\in\mathcal{M}$ let $F(g)=(D\Psi)_B(g)$. With probability on $[V]=V_{\mathbb{A}}/V$, put

$$M_+(g)=\int_{[V]}F(Y(y)g)\,\mathrm{d}y,\quad \phi_+(x;g)=M_+(X(x)g),$$

$$M_-(g)=\int_{[V]}F(X(x)g)\,\mathrm{d}x,\quad \phi_-(y;g)=M_-(Y(y)g).$$

Lemma 535.2. *The preceding formulas define continuous functions. Write* $\eta_B(x,y)=\mathbf{e}(-\mathrm{Tr}_{E/\mathbb{Q}}(\jmath x^*By))$ *with the adelic trace and character of Definition 510.1. Then*

$$\phi_+(x;Y(y)g)=\eta_B(x,y)\phi_+(x;g),\qquad \phi_-(y;X(x)g)=\eta_B(x,y)^{-1}\phi_-(y;g).$$

Moreover M_+ *is the bordered coefficient at* $\mathrm{diag}(0,B)$ *for* $Ee\oplus L$, *and* M_- *is the corresponding one for* $Ee'\oplus L$. *In particular* $M_\pm$ *are left invariant under the respective adelic line radicals.*

Proof. The maps $X(x),Y(y)$ are isometries: the added pairing of x with a vector of L' cancels the pairing of e with $-e'x^*l'$, and likewise on interchanging e,e'. All pairings of two added vectors in either map vanish. Both maps are additive in their parameters under multiplication. They fix L pointwise and commute with $Z(z)$. The latter acts on the left on F by $\xi_B(z)$, by a translation in the defining central integral. A rational X or Y fixes F on the left by rational invariance of $D\Psi$ and commutation with Z. This proves periodicity of each integrand. Continuity follows from the compact integration argument of Theorem 534.3.

The definitions on L' give

$$X(x)Y(y)=Z(yx^*-xy^*)Y(y)X(x).$$

Indeed the mixed products of the two added maps on L' are $-xy^*$ and $-yx^*$, respectively, and all longer products vanish. The central character on their difference is $\eta_B(x,y)$, by moving the row to the front in each trace and using $\bar\jmath=-\jmath$, $B^*=B$. Substitute this commutation into each mean and translate its integration variable on $[V]$. This gives both formulas.

For the first bordering, the skew matrices on $Ee\oplus L$ have independent coordinates consisting of the skew scalar on Ee, the off-diagonal column y, and the skew block on L.

Their central elements are the products of the shear $Z_e(s)$ sending e' to $e' + se$, the element $Y(y)$, and $Z(z)$, by the displayed actions. The compact quotient and its probability measure are the products in these rational coordinates, by uniqueness of invariant probability as in Lemma 532.2. Therefore the coefficient at $\mathrm{diag}(t, B)$, $t \in F_{\mathrm{re}}$, is the t-th Fourier coefficient of $s \mapsto M_+(Z_e(s)g)$; here $F_{\mathrm{re}} = E^{\mathfrak{c}}$ denotes the base field to distinguish it from the function F. For every $t \neq 0$ this index has rank $b + 1$, so the coefficient vanishes by Theorem 533.3. Subtract the constant coefficient and use Lemma 531.1 for the one dimensional hermitian index space. The function of s is constant. Its value at zero is $M_+(g)$, so M_+ equals the coefficient for $t = 0$. The computation with e, e' exchanged gives the assertion for M_-. Theorem 534.3 gives both radical invariances. □

Theorem 535.3. *At any two points g, h and for any complex number c, each of the conditions $\phi_+(\cdot; g) = c\phi_+(\cdot; h)$ and $\phi_-(\cdot; g) = c\phi_-(\cdot; h)$ implies the other. Either one implies $F(g) = cF(h)$.*

Proof. Suppose the plus functions agree with this factor. For rational $r \in V$, move $X(r)$ to the far left using the commutator in Lemma 535.2. Left rational invariance of F then gives

$$\phi_+(x + r; g) = \int_{[V]} \eta_B(r, y)^{-1} F(Y(y)X(x)g)\,\mathrm{d}y.$$

The characters $y \mapsto \eta_B(r, y)$, $r \in V$, supply all rational frequencies on the compact vector quotient. Indeed $r \mapsto (y \mapsto -\mathrm{Tr}_{E/\mathbb{Q}}(\jmath r^* By))$ is an injective rational linear map to its rational dual. For $r \neq 0$ the E-linear row $\jmath r^* B$ is onto E, so it takes the value one whose trace is $[E : \mathbb{Q}]$. The two rational dimensions coincide, giving bijectivity. The Fourier uniqueness proof of Lemma 531.1 in rational coordinates therefore applies on $[V]$. Subtract c times the same formula at h and use uniqueness for every x. It follows that $F(Y(y)X(x)g) = cF(Y(y)X(x)h)$ for every y, x. Taking both zero gives the assertion for F. Commuting $X(x)$ and $Y(y)$ by the same common unit scalar and integrating in x gives the equality of the minus functions. Starting with the minus equality gives the reverse implication: the commutator pairing is the inverse under reversing the group order, and $x \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(\jmath x^* By)$ is a nonzero rational linear functional for every $y \neq 0$ by the same row argument and conjugation. Thus Fourier uniqueness applies with the two parameters exchanged. □

536. The projected lattice at the unmarked places

Definition 536.1. Use Definition 535.1. Suppose the finite unimodular lattice Λ of W is the direct sum in the ordered splitting Ee, L, W_2, L', Ee'. Its paired isotropic summands are perfect integral duals, and the lattice on W_2 is unimodular. Use the single standard pair E_S, D_Q in Proposition 529.3, and write $f = D(D_Q)$. Let $\mathcal{U}$ consist of all finite places of $E^{\mathfrak{c}}$ away from the marked pairs. Call g admissible here if it fixes L pointwise at every place outside $\mathcal{U}$, including infinity. For a sign $\epsilon = +$ or $-$ write $e_+ = e$, $e_- = e'$ and $L_\epsilon = Ee_\epsilon \oplus L$. Define, at $v \in \mathcal{U}$,

$$\Gamma_\epsilon(g)_v = \mathrm{pr}_L(g_v \Lambda_v \cap L_{\epsilon,v}),$$

where projection has kernel the first line. Call an admissible p a point preserving the decomposition at the unmarked places if, at every $v \in \mathcal{U}$, it preserves each of Ee, Ee', M and also $L \subset M$. Here M is as in Definition 534.1. At such a point the two projected lattices coincide and equal $\Gamma(p)_v = p_v \Lambda_{L,v}$, where $\Lambda_{L,v} = \Lambda_v \cap L_v$. Write $\tau_v O_{E,v} e$ for $p_v \Lambda_{e,v}$. Its dual $p_v \Lambda_{e',v}$ is $\bar{\tau}_v^{-1} O_{E,v} e'$, by the perfect pairing. Here $O_{E,v}$ is the full local integer algebra, including both factors at split places. These assertions follow by applying p_v to the indicated original intersections, since p_v preserves both larger planes and their common subspace L_v.

Lemma 536.2. *For every admissible g and either sign there is an admissible p preserving the decomposition at the unmarked places, with*

$$M_\epsilon(p) = M_\epsilon(g), \qquad \Gamma(p)_v = \Gamma_\epsilon(g)_v \quad (v \in \mathcal{U}).$$

It can be chosen equal to g at every place outside $\mathcal{U}$.

Proof. At every unmarked place the original periods are fixed on the right by the stabilizer $K_v = U(\Lambda_v)$ in their ambient variable. Indeed the local test there is the vacuum of Lemma 517.2; the integral pair action fixes its unit vector by the lattice coefficient in Theorem 390.2, for which the intersection lattice is the original one and the ideal phase is one. The determinant twists are trivial there on integral units by Definition 508.1 and Theorem 374.2. This proves invariance under the product of the K_v: it also follows from invariance under finite subproducts and continuity on the restricted product. The coefficient and both its means retain right invariance by their definitions.

For a place $v \in \mathcal{U}$, the integral isometries act transitively on isotropic flags of a line in a $(b+1)$-plane. Indeed the intersection of a local lattice with a field subspace is a saturated submodule: if a nonzero scalar multiple of a lattice vector lies in the subspace, the vector does also. Over a discrete valuation ring a saturated submodule is spanned by part of a lattice basis. For detail, move an entry of least valuation in any finite generating matrix to the first position and clear its row and column by integral elementary operations. Iterating puts the submodule in diagonal form, and saturation forces each nonzero diagonal entry to be a unit. Such a matrix of generators exists since multiplying a subspace basis by suitable powers of a uniformizer gives a full sublattice, and its intersection with a bounded box is contained in a fixed denominator multiple of that lattice. The intervening quotient is finite. Apply this first inside the plane for the line, then inside the ambient lattice for the plane. At split places perform this argument on each field factor. The resulting frame has an integral left inverse and zero Gram matrix. Theorem 345.2 carries any such ordered frame to another, including its distinguished first vector. This proves transitivity.

Choose $k_v \in K_v$ taking the reference flag to the inverse image of the flag by g_v. Then $g_v k_v$ preserves the flag. In the line parabolic it uniquely decomposes as $u_v m_v$ by Definition 522.1, where m_v preserves both hyperbolic lines and their common orthogonal complement M_v. The radical acts trivially on $e_\epsilon^\perp / E e_\epsilon$. Flag preservation therefore implies that m_v preserves L_v. If g_v is integral one may choose $k_v = g_v^{-1}$. Thus we may replace g_v by $m_v = u_v^{-1} g_v k_v$ at the unmarked places to form an adelic p. Right compact invariance and left radical invariance from Lemma 535.2 preserve the mean. The compact factor preserves Λ_v, and u_v preserves $L_{\epsilon,v}$ and acts trivially on its quotient by the line. These facts preserve the projected lattice. □

Lemma 536.3. *On L_v let $h_v(u,w) = (\alpha\bar{\alpha})_v^{-1} u^* B w$. For a product of full lattices $\Gamma = (\Gamma_v)_{v\in\mathcal{U}}$ equal to $\Lambda_{L,v}$ at almost all places, write $\Gamma_v^\# = \{u : h_v(u, \Gamma_v) \subset O_{E,v}\}$. If each Γ_v is integral for h_v, set*

$$\Delta(\Gamma) = \prod_{v\in\mathcal{U}} [\Gamma_v^\# : \Gamma_v].$$

This is a positive integer. If integral Γ' contains Γ at every place, then

$$\Delta(\Gamma) = \Delta(\Gamma') \prod_{v\in\mathcal{U}} [\Gamma'_v : \Gamma_v]^2.$$

In particular a strict enlargement at one place strictly lowers this integer. For a point p of Definition 536.1 preserving the decomposition, $M_\epsilon(p) \neq 0$ implies that $\Gamma(p)$ is integral.

Proof. In an integral basis with nonsingular Gram matrix H the dual lattice has coordinates $H^{-1} O_{E,v}^b$, since $H^* = H$. The index is the cardinality of $O_{E,v}^b / H O_{E,v}^b$. By the diagonal elimination in Lemma 536.2, on a field factor of residue size q_w this is $q_w^{v_w(\det H)}$. At almost all places the rational B and its inverse are integral in a fixed rational basis, α is a unit, and the lattice has that basis, so the index is one. For a change from Γ' to its sublattice Γ by an integral basis matrix A, the new Gram matrix is $A^* H' A$. Its determinant gains the factor $\overline{\det A} \det A$. In the product over all branches at v, conjugation permutes branches

of equal residue size. The additional index is consequently exactly $[\Gamma'_v : \Gamma_v]^2$, by the same elimination for A. Every strict local inclusion has index at least two.

For the last assertion nonvanishing of the mean supplies a shift $Y(y)$, or $X(x)$ for the opposite sign, at which the original B-coefficient F is nonzero. The shift fixes L pointwise. The shifted point is therefore in the parabolic at every place and induces on L the same frame action as p. Corollary 532.4 excludes every nonintegral entry of the transported Gram matrix for either period. It applies also to their difference before normalization: the determinants are constant along the central integration and the small-side change of model is the constant multiple of Theorem 512.2. At the unmarked places the transported matrix is the matrix of h_v on $\Gamma(p)_v$. This proves integrality. □

537. A least discriminant argument

Lemma 537.1. *Keep the data of Definition 536.1 and fix a nonsingular rational hermitian B. If $F(g) \neq 0$ at some admissible g, there exist a choice of order of e, e' and an admissible p preserving the decomposition at the unmarked places, such that $M_+(p) \neq 0$ and*

$$\phi_+(x;p) \neq 0 \implies \tau_v x_v \in \Gamma(p)_v \quad (v \in \mathcal{U}),$$
$$\phi_-(y;p) \neq 0 \implies \bar{\tau}_v^{-1} y_v \in \Gamma(p)_v \quad (v \in \mathcal{U}).$$

Proof. By Theorem 535.3 with $c = 0$, some $\phi_+(x;g)$ is nonzero. The point $X(x)g$ is admissible because the shift fixes L pointwise. Lemma 536.2 gives a point preserving the decomposition with a nonzero mean at the origin. Among all such points for either sign choose one p minimizing the positive integer $\Delta(\Gamma(p))$, defined by Lemma 536.3. Exchange the names of the two lines if necessary, so that $M_+(p) \neq 0$.

If $\phi_+(x;p) \neq 0$, straighten $X(x)p$ for the plus line. Its projected lattice at $v \in \mathcal{U}$ is

$$\Gamma_+(X(x)p)_v = \Gamma(p)_v + O_{E,v}\tau_v x_v.$$

Indeed $p_v\Lambda_v \cap L_{+,v}$ is the direct sum of $\tau_v O_{E,v} e$ and $\Gamma(p)_v$. The shift preserves $L_{+,v}$, takes e to $e + x_v$ and fixes L_v. Intersecting its moved lattice with this plane and projecting gives the equality. The new point has nonzero mean and the same projected lattice by Lemma 536.2; hence this lattice is integral by Lemma 536.3. A strict enlargement anywhere would lower Δ, contrary to its choice. This gives the first implication. For the second use $Y(y)p$ and the minus line. Its projected lattice is $\Gamma(p)_v + O_{E,v}\bar{\tau}_v^{-1} y_v$ by the same projection, using the actual dual line lattice in Definition 536.1. Minimality was taken over both signs, so again there is equality. □

Theorem 537.2. *At the point p of Lemma 537.1, one has $\Gamma(p)_v^{\#} = \Gamma(p)_v$ for every $v \in \mathcal{U}$.*

Proof. Fix $v \in \mathcal{U}$, write $\Gamma = \Gamma(p)_v$, and let $u \in \Gamma^{\#}$. Let $a \in V_{\mathbb{A}}$ be supported at v, with $a_v = \tau_v^{-1} u$, using both field factors when v is split. If $\phi_-(y;p) \neq 0$, then $w = \bar{\tau}_v^{-1} y_v \in \Gamma$ by Lemma 537.1. Thus

$$a_v^* B y_v = u^* B w \in (\alpha\bar{\alpha})_v O_{E,v}.$$

The trace dual identity $\jmath\alpha\bar{\alpha}\widehat{O}_E = \widehat{O}_E^{\vee}$ in Lemma 355.2 gives $\eta_B(a, y) = 1$, since the pairing in the exponent is supported at v. Where the minus function is zero the same multiplier leaves its value unchanged. The commutator formula of Lemma 535.2 therefore gives $\phi_-(\cdot; X(a)p) = \phi_-(\cdot; p)$. By Theorem 535.3 the plus functions at these two points agree. At zero their equality reads $\phi_+(a;p) = \phi_+(0;p) = M_+(p) \neq 0$. The plus support implication of Lemma 537.1 now gives $u = \tau_v a_v \in \Gamma$. The reverse containment follows from integrality in Lemma 536.3. □

Corollary 537.3. *Take the real tests and the scalar $K_\nu j_\nu$ of Theorem 532.3, with the nested integral splitting of Definition 536.1. In the original orientations put $c = K_\nu j_\nu / c_0$, where c_0 is the unit scalar of Theorem 512.2. If $H_f(B, 1)$ is unimodular at all marked pairs, then the B-coefficient of $f = D(D_Q)$ vanishes at every admissible point.*

Proof. There is a marked pair by Definition 521.1, so unimodularity there implies B is nonsingular over E. If the coefficient is nonzero, choose p by Lemma 537.1. Since $M_+(p) \neq 0$, some $F(Y(y)p) \neq 0$, where Y denotes the opposite-line shift for the chosen order. This point preserves L everywhere and has the same action q as p on L. At a marked pair this action is identity, by admissibility. At every unmarked place, its transported Gram matrix is the matrix of h_v on $\Gamma(p)_v$ in Definition 536.1 and Lemma 536.3. This is an invertible integral matrix with integral inverse by Theorem 537.2 and the basis formula for the dual lattice. Hence $H_f(B, q)$ is unimodular at all finite places. Theorem 532.3 compares the two normalized coefficients at this point with factor $K_\nu j_\nu$. Restoring the common determinant normalization multiplies both sides by the same character, since the central shear has determinant one. By Theorem 512.2, the original small period is c_0 times its directly transformed one. Thus $c = K_\nu j_\nu / c_0$ makes $F(Y(y)p) = 0$, a contradiction. □

538. The complementary identity on the enlarged spaces

Lemma 538.1. *For the standard integral shear of Definition 531.2 and the finite Gram matrix of Definition 532.1, a rational B satisfies*

$$T_w(B) \in \mathrm{GL}_b(O_w) \quad (w \in S) \quad \Longleftrightarrow \quad H_f(B, 1) \text{ is unimodular at every marked pair.}$$

Proof. In rational coordinates on the skew center, the shear specified by t in integral dual bases is $C_L z C_L^*$, where the coordinate $z_w = t_w$, $z_{\bar{w}} = -t_w^*$ at a split pair. This follows by applying it to the dual basis, whose basis matrix is the inverse adjoint of C_L. Cyclicity in the trace of Definition 510.1 replaces B by $C_L^* B C_L$. As in Theorem 533.3, the two branch traces are equal, canceling the $1/2$; the remaining character at w is the ordinary local trace character applied to $\jmath_w \mathrm{tr}(t_w (C_L^* B C_L)_w)$ with the sign prescribed by $\xi_B(-z)$. The trace dual identity in Lemma 355.2 says that the character with coefficient $\jmath_w(\alpha\bar{\alpha})_w$ has annihilator O_w on O_w. The trace character chosen in Definition 531.2 has the same annihilator. Its coefficient therefore differs from this one by a unit, since the annihilator for a scalar multiple is the inverse scalar multiple of the original annihilator. It follows by the entry tests of that definition that $T_w(B)$ is $H_f(B, 1)_w$ times a unit. A sign in the choice of additive character also contributes a unit. Since $H_f(B, 1)$ is hermitian its component at the opposite branch is the conjugate transpose, preserving integrality of the matrix and its inverse. This gives the equivalence. □

Lemma 538.2. *In Theorem 531.3 it suffices to check the stated primitive indices only at points whose infinite components and marked components are all identity.*

Proof. At every such point the shear calculation in Theorem 531.3 and Lemma 531.1 give $D(G)(g) = 0$, where G is the finite shear sum of Definition 530.1. We prove density of the left rational translates of these points. It suffices to approximate a prescribed element at all infinite and marked places by one rational isometry. On each local group the locus $\det(h + 1) \neq 0$ is dense. For a split pair multiply the matrix on one branch by a scalar close to one and avoid the finitely many roots of $\det(th + 1)$; the other branch is the inverse adjoint. At a real place multiply by a unit complex scalar tending to one, avoiding those same finitely many roots. This preserves the hermitian form for every signature. On the locus in question the skew-adjoint Cayley coordinates of Lemma 95.2 and their inverse are rational and continuous. The skew-adjoint matrices form a rational vector space over the totally real field. Apply Lemma 89.1 to the coordinates in a basis of this space at all the chosen places together. Sufficiently close rational coordinates have nonzero denominator determinant at those places, hence over the field, and yield a rational isometry. This proves the required approximation.

For arbitrary g, choose rational points r_i approaching its components at the specified places and let g_i be identity there and $r_i^{-1} g$ at the other finite places. Then $r_i g_i \to g$ since the other finite components are already exactly those of g. Continuity and rational invariance

of $D(G)$, established in Theorem 531.3, give its vanishing everywhere. Theorem 530.2 then gives the original exchange identity. □

Theorem 538.3. *Assume the nonzero weight and isotropic hypotheses of Corollary 528.3 and the nested integral hyperbolic splitting in Definition 536.1. Take the real tests of Theorem 514.3, using the inverse of the real identification of Theorem 512.2 for the original small orientation. For every allowed twist ν of the fixed weight with squarefree marked conductor as in Corollary 509.4, for all original matched residue tests (Φ, Ψ) at that level and at every adelic ambient argument g, one has*

$$I_\Phi(g) = c_\nu I'_\Psi(g), \qquad c_\nu = K_\nu j_\nu / c_0.$$

Here $j_\nu = J_\nu(1)$ for the chosen residual real vector, K_ν is the constant of Theorem 521.3, and $|c_0| = 1$. In particular $|c_\nu| = C_*|j_\nu|$ with $C_* > 0$ independent of the twist and its marked set.

Proof. An argument g supported at the unmarked finite places is admissible in Definition 536.1. For every marked primitive index, Lemma 538.1 gives unimodularity of the Gram matrix at the marked pairs. Therefore Corollary 537.3 gives the required vanishing of the B-coefficient for the single fixed test D_Q and the scalar displayed here. Lemma 538.2 proves the identity on all tests and at every g. The real data are chosen using only the fixed weight, real frames and the two residual real vectors by Theorem 514.3 and the real action in Theorem 521.3. Taking absolute values in the formula for c_ν gives $C_* = |K_\nu| > 0$, independent of the marked set and twist by that last theorem. □

539. Averaging a theta period along one line

Definition 539.1. Write $W' = Ee \oplus W \oplus Ee'$ as a rational hermitian direct sum of a hyperbolic plane and its perpendicular, with $\langle e, e' \rangle = 1$; the order displayed is used for matrix blocks. Use a finite lattice adapted to this splitting as in Definition 517.1. For any rational nonsingular target T with unimodular finite lattice, identify a frame $x : Ee \to T$ with its value $x(e) \in T(E)$. Write J_T for its hermitian matrix. Let N_e consist of $u(C, S)$ in Definition 522.1, with C a row on W and $S + \bar{S} = 0$ scalar. For a function P on the ambient quotient put

$$\mathcal{C}_e P(g) = \int_{N_e(F_0)\backslash N_e(\mathbb{A}_{F_0})} P(ng)\,\mathrm{d}n, \qquad F_0 = E^{\mathfrak{c}},$$

using right invariant probability. For a theta test f let $t_x(g, h; f)$ have the meaning of Definition 510.1, and put

$$d_x(g, h; f) = \int_{[N_e]} t_x(ng, h; f)\,\mathrm{d}n, \quad [N_e] = N_e(F_0)\backslash N_e(\mathbb{A}_{F_0}).$$

Lemma 539.2. *The quotient $[N_e]$ has compact representatives. Integration against its right invariant probability is obtained by taking the mean first in S modulo rational skew scalars, then in C modulo rational rows, using $u(C, S)$. For a fixed $x \in T(E)$, the integrand defining d_x is a left rationally invariant function of the radical variable.*

Proof. By multiplying the three blocks in Definition 522.1, the product law adds the two rows C, D and adds their two skew scalars with the additional skew scalar $(DJ_W^{-1}C^* - CJ_W^{-1}D^*)/2$. Either left or right translation is thus a translation on the row variable and a translation depending on that row on the scalar variable. Both rational additive quotients have compact representatives: a rational basis reduces this to $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ as in Definition 510.1. Concretely the finite part can be made integral by a rational translation, and then an integer translation puts the real coordinate in $[0, 1]$. First put C in its representatives by a rational radical element and then put S there by a rational central element. The two means are independent of the choice of lifts of the row because the additional translation in S preserves its mean. The same product law proves invariance on the right. This is the normalized quotient

measure: in these coordinates ordinary additive Haar product is both left and right invariant by successive translation, and the indicated representatives give a fundamental domain up to null boundaries, using half-open intervals in the rational coordinates.

A rational radical element preserves the momentum polarization used in Lemma 522.2. Its Schrödinger action is the rational substitution and quadratic phase in Theorem 386.1, with trivial scalar by that lemma. It fixes the frame position x. On rational positions its phase is one because the global additive character is trivial on rational arguments, and its invertible rational substitution permutes the residual rational positions at this x. Summing as in Definition 510.1 proves the claimed rational invariance. □

Proposition 539.3. *For a period I_f satisfying the estimates of Theorem 507.3 in the orientation of Definition 539.1,*

$$\mathcal{C}_e I_f(g) = \int_{[U(T)]} \sum_{\substack{x \in T(E) \\ x^* J_T x = 0}} d_x(g, h; f) \, \mathrm{d}h.$$

For fixed h, g the sum is absolutely convergent. Its grouped value is absolutely integrable in h. Moreover $d_0(g, h; f) = t_0(g, h; f)$.

Proof. For fixed g, h, the full rational position sum after moving the test by n is absolutely convergent uniformly on the compact representatives in Lemma 539.2. Indeed in a rational polarization the finite position supports are contained in one bounded box after clearing denominators. The moved real tests range in a set with uniformly bounded Schwartz seminorms: n acts by the polynomial phase and triangular substitution of Lemma 522.2, whose coefficients and inverse coefficients range over a compact set. The finite substitutions and phases give a common finite supremum bound on this compact set as well. In a rational basis the resulting sum of absolute values is bounded by a constant times the sum of $(1 + \|a\|)^{-m}$ on one integer-denominator lattice, with m greater than its real dimension plus one; this converges by counting integer points in concentric boxes as in Proposition 396.2. Consequently the frame sum may be integrated term by term for fixed h. The inner central translation acts at x by $\xi_{x^* J_T x}$ by Lemma 510.2. Its mean vanishes unless $x^* J_T x = 0$, by character orthogonality as in Theorem 510.3. At $x = 0$ the entire residual radical operator is identity by Lemma 522.2, giving $d_0 = t_0$.

For the integral in h, ng ranges in a compact set of representatives at fixed g. The full kernel is bounded there by an integrable function of h, by Theorem 507.3 and Proposition 398.1. Thus the radical and target integrations may be interchanged. At each h the grouped inner sum is the mean of that full kernel. Bounding its modulus by the mean of the full kernel modulus gives the asserted integrability and proves the formula. □

Lemma 539.4. *Let $\dim W = N$, let $b \geq 0$, let V' have rank $N + 2 - b$, and at a marked branch choose an integral matrix $s : W \to E_w^b$ in the reference frames. On the enlarged space use the standard large determinant test with upper rows $\widetilde{s} = s \, \mathrm{pr}_W$, as in Lemma 509.2. When read at several marked branches impose the same form at each. For this test f and every g preserving Ee, one has $t_0(g, h; f) = 0$ for every h.*

Proof. At the chosen branch let e_v generate the integral line $\Lambda_{e,v}$. A matrix D contributing to this test has $\begin{pmatrix} \widetilde{s} \\ D \end{pmatrix}$ invertible on reduction, and $\widetilde{s}(e_v) = 0$. Since e_v has nonzero reduction by the adapted direct sum, $D(e_v)$ has nonzero reduction. This holds for every contributing residue matrix. By Lemma 513.1 the transported value of the zero frame would have to reduce to $D(e_v)$ in scaled integral coordinates. It is zero for every g, h in the stated range. Thus the frame slice is zero already at this branch, and its residual theta term is zero. □

540. Descent of the small period

Definition 540.1. Write $W' = Ee \oplus W \oplus Ee'$ as in Definition 539.1, $N = \dim W$ even, and let $T_b = (A')^-$ be the directly transformed small target of rank b. Use compatible finite

frames for the summands at both ranks and the same opposite-target unit ϵ of Theorem 512.2. For integral s of size b-by-N at the marked branches, take the small standard determinant of $s\,\mathrm{pr}_W$ at rank $N+2$, and that of s at rank N. Let $P^-_{W'}(v', s\,\mathrm{pr}_W)$ and $P^-_W(v, s)$ be their directly transformed normalized periods as in Definition 520.1, with real vectors in this transformed orientation. Their pair characters are respectively

$$(\nu \upsilon_T^{(N+2)/2}, \upsilon_T^b), \qquad (\nu \upsilon_T^{N/2}, \upsilon_T^b).$$

Indeed this is Proposition 520.3 at each ambient rank, or follows from the two characters in Theorem 512.2 by the common ambient normalization. For v' written in the line frame polarization, write $v'[0]$ for its residual real vector at frame zero. Use the same rational residual polarization at rank N.

Lemma 540.2. *For the small test of Definition 540.1 and every $x \neq 0$ rational in T_b,*

$$d_x(m(1, g), h; f) = 0$$

for all $g \in U(W)(\mathbb{A}_{F_0})$, $h \in U(T_b)(\mathbb{A}_{F_0})$.

Proof. Select one marked split pair and work in the position model on the conjugate branch, where the directly transformed test is named. In that model write the matrix as (L_1, Z, L_2) in source column blocks e, W, e', where Z has b rows. The finite test before making the frame polarization is the integral-box indicator times the determinant character from the residual block Z. This is because the original $s\,\mathrm{pr}_W$ kills both line summands and the split adjoint in Theorem 512.2 identifies the residual block with its own adjoint. On its support Z has row rank b over the local field: the b-by-b determinant of its adjoint composed with s, up to the fixed units, is nonzero on reduction. Applying $m(1, g)$ and h here only multiplies Z on its two sides by invertible matrices, by Theorem 360.1. It also acts separately on the two outside columns. The transformed test has compact support, and its residual block still has row rank b throughout that support.

For each local column d of length N choose the source line radical element whose matrix at this branch adds the W-column d to e' and fixes e, W. At the other branch use the inverse adjoint, elsewhere the identity. The line radical in Definition 522.1 allows this arbitrary block in a split pair. These elements form an additive subgroup; on positions before Fourier change they act by $L_2 \mapsto L_2 + Zd$, up to the common sign from inversion, which can be absorbed by replacing d by $-d$. The split character and modulus for this matrix are one since its determinant is one. In the line frame polarization one Fourier transforms in L_2, by the duality argument of Lemma 517.2. With rational frame x fixed, its Fourier variable is the opposite-branch component of x in the dual target coordinates, including the invertible trace scale. It gives a nonzero local row r_x: the rational nonzero vector has nonzero image at every embedding, and the target pairing with the opposite branch is perfect. Thus the supported radical acts on the slice by the character $d \mapsto \psi(r_x Z d)$ with nontrivial additive ψ.

The set of Z's in the projection of the original transformed support is compact and consists of full-row-rank matrices. The rows $r_x Z$ on it form a compact set avoiding zero. Consequently the average of these characters over a sufficiently large compact additive box in d vanishes for every such Z, by the shrinking-annihilator computation of Lemma 533.2. Fourier transformation in L_2 preserves the projection bound for Z. The averaged test therefore has identically zero residual slice at x. To insert this local average in the defining integral for d_x, translate the integrating radical on the right by each of the supported elements and use Lemma 539.2. The operations commute with the integral since the average on a compact box of a locally constant moved finite test is a finite sum. For the remaining left radical element the residual action at x depends only on the slice there, by Lemma 522.2. Thus the entire mean vanishes. If $b = 0$ there is no nonzero x. □

Theorem 540.3. *With compatible physical measures in Definition 540.1 there is a fixed nonzero constant c_{sm} such that for every real Schwartz vector v', every allowed twist of the*

fixed weight, every integral standard s at the squarefree level, and every g,

$$\mathcal{C}_e P^-_{W'}(v', s\,\mathrm{pr}_W)(m(1,g)) = c_{\mathrm{sm}}\, P^-_W(v'[0], s)(g).$$

With positive unit finite frame vacuum, c_{sm} is the inverse square root of the volume of the finite line frame lattice for this target. It is independent of the twist and marked set.

Proof. When the standard test is zero both sides vanish. Otherwise the original small data at each rank are supported on full column frames since their determinant is a unit. Convergence and the compact uniform estimate in Proposition 539.3 follow by Theorem 507.3 and the fixed central change of variables in Theorem 512.2. That proposition and Lemma 540.2 leave only t_0 under the target integral.

At the finite places the slice at zero is c_{sm} times the smaller finite test. At a marked pair the original function of (L_1, Z, L_2) is the integral indicator in the two outside columns times the smaller function of Z. Fourier transforming the last column takes its box to its dual box, leaving the central block unchanged, by Lemma 517.2. Both frame boxes are one at zero. Away from the marked pairs the frame vacuum and residual vacuum factor as in the same lemma. In physical coordinates the positive unit frame vacuum is the box indicator divided by the square root of its volume, by Lemma 521.2. The same fixed physical normalization therefore works at every marked set. At infinity the zero slice is $v'[0]$. Evaluation at zero is continuous on the real Schwartz space by its defining seminorms, so the residual vector is Schwartz. Apply Lemma 520.2 with one source line and $q = 1$. Its frame scalar is one, and the target determinant character loses one power of υ_T; these are precisely the two pairs of characters in Definition 540.1. The residual theta sum and its integral over the same target quotient are consequently the claimed smaller period. □

541. The two scalar sections on a target line

Definition 541.1. Let $W' = Ee \oplus W \oplus Ee'$ and $V' = Ea \oplus V \oplus Ea'$ be rational hyperbolic splittings with the indicated dual line pairings one. Write $N = \dim W$, $r = \dim V$, and denote the Gram matrices on these summands by J_W, J_V. In these bases an element of $\mathrm{Hom}_E(W', V')$ has block form

$$\begin{pmatrix} A & B_1 & C_1 \\ D & Z & H_1 \\ I & J & K \end{pmatrix}.$$

Use as positions the whole first column (A, D, I), the row J on W, and rational positions ζ for Z as in Lemma 385.2. The paired momenta belong to the last column, the row B_1, and the remaining coordinates of Z. This is a polarization: in the hermitian trace form of Definition 355.1 the first and last columns pair, B_1 pairs with J through J_W^{-1}, and the Z block is orthogonal to all these blocks. For a real-times-finite Schwartz test f written in this model and $t \in \mathbb{A}_E$, set

$$\mathcal{L}_+(t) f(\zeta) = f(t, 0, 0, 0, \zeta),$$
$$\mathcal{L}_-(t) f(\zeta) = \int_{\mathbb{A}_E} \mathbf{e}\big(\mathrm{Tr}_{E/\mathbb{Q}}(\jmath\bar{t}u)\big) \mathcal{L}_+(u) f(\zeta)\, \mathrm{d}u.$$

Use the measure on the scalar adeles given in a rational basis by Lemma 387.1; in particular the rational scalar lattice has covolume one. Write $\theta_{W,V}$ for the residual rational position sum on ζ.

Proposition 541.2. *For every $\xi \in E^\times$, the mean of the frame term at $x = \xi a$ of Definition 539.1 is*

$$d_{\xi a}(g, h; f) = \theta_{W,V}\big(\mathcal{L}_+(\xi) \Omega(g, h) f\big).$$

For any $t \in \mathbb{A}_E$, taking this plus section of a test is unchanged by first applying the radical of the target line Ea.

Proof. Write $f_1 = \Omega(g,h)f$ and consider the mean for f_1 at the identity. For $x = \xi a$ the central source phase is one by Lemma 510.2. We compute the action of the remaining source radical $u(C,0)$, where C is a row on W. On columns (x, M, U) its underlying linear map by precomposition with the inverse sends

$$(x, M, U) \longmapsto (x,\ M - xC,\ U + MJ_W^{-1}C^* - \tfrac{1}{2}xCJ_W^{-1}C^*).$$

At $x = \xi a$ the correction to M is purely in row B_1, which is momentum. Hence the map fixes every position with this x. On such a position put the original momenta equal to zero. Let Δ be the change under the displayed map and let p denote the original position. Put $t = x^*J_{V'}MJ_W^{-1}C^* = \bar{\xi}JJ_W^{-1}C^*$. In the hermitian trace $H(\Delta, p)$ the last-column correction contributes $\bar{t}$ by the pairing with x, its quadratic term contributing zero since $x^*J_{V'}x = 0$. The middle-column correction contributes $-t$. Thus the total is $\bar{t} - t$. The phase of Theorem 386.1 is $\mathbf{e}(-\mathrm{Tr}_{E/\mathbb{Q}}(\jmath(\bar{t} - t))/2) = \mathbf{e}(\mathrm{Tr}_{E/\mathbb{Q}}(\jmath t))$; the radical scalar is trivial by Lemma 522.2. For a rational residual row $J \neq 0$, as C varies this is a nontrivial character of the compact row quotient: J_W is invertible, $\xi \neq 0$, and at one infinite place a suitable component of C realizes every real trace value. For rational C the trace is rational and its global character is one. The mean kills the terms with $J \neq 0$ and retains every term with $J = 0$. The interchange with the rational position sum is justified by the fixed-g, h compact majorant of Proposition 539.3. The remaining variables are precisely $A = \xi,\ D = I = J = 0, \zeta$, proving the formula.

For the target radical, its linear action preserves separately the first and last source columns. On those hyperbolic variables it acts by ordinary substitution with determinant one, by Theorem 394.2, and fixes ta. In the residual middle column its upper triangular target blocks leave J fixed and change Z only by a multiple of J. They preserve the momentum space. At $J = 0$ the residual position is fixed and its change in momentum is entirely in B_1, pairing only with $J = 0$. The quadratic phase is therefore one. To see that the Schrödinger parabolic formula carries no further scalar on this radical, conjugate by the target element scaling a by two and a' by one half; this scales its central parameters by four and its quotient row by two. The character-ratio argument in Lemma 522.2 applies verbatim, since here also the group preserves the chosen residual momentum and the determinant twist is trivial on the radical. This proves invariance of the residual section. □

Lemma 541.3. *The two sections have the same rational theta sum:*

$$\sum_{\xi \in E} \theta_{W,V}(\mathcal{L}_+(\xi)f) = \sum_{\xi \in E} \theta_{W,V}(\mathcal{L}_-(\xi)f),$$

both absolutely convergent for fixed f. Let $\ell(c,h)$ act on Ea, V, Ea' by $c^{-1}, h, \bar{c}$, where c is an idele of E, $h \in U(V)(\mathbb{A}_{F_0})$, and take $g \in U(W)(\mathbb{A}_{F_0})$. For even N put $\gamma = \nu^{-1}\upsilon_T^{N/2}$, $\lambda(c) = \nu(c/\bar{c})$. Use the characters $(\gamma\upsilon_T, \gamma\upsilon_T)$ on the large pair at the upper rank, (γ, γ) on the residual pair. Writing $|c| = |c|_{\mathbb{A}_E}$, one has

$$\mathcal{L}_+(t)\Omega(m(1,g), \ell(c,h))f = \lambda(c)|c|^{-N/2}\Omega(g,h)\mathcal{L}_+(ct)f,$$
$$\mathcal{L}_-(t)\Omega(m(1,g), \ell(c,h))f = \lambda(c)|c|^{-N/2-1}\Omega(g,h)\mathcal{L}_-(t/\bar{c})f.$$

Proof. The rational linear pairing $(t,u) \mapsto \mathrm{Tr}(\jmath\bar{t}u)$ is perfect over $\mathbb{Q}$: a nonzero coefficient may be multiplied by its inverse to obtain the nonzero trace of one. Thus in rational scalar coordinates its matrix gives a bijection of the rational dual lattice of frequencies with E. Lemma 387.1 gives the theta identity at each residual rational position. On taking the sum there, absolute convergence on both sides follows from the bounded finite support and all real Schwartz seminorms, preserved by the partial transform as proved in that lemma.

For the first transformation formula use Lemma 520.2 on the first source column with line multiplier one. The target inverse takes ta to cta, and the residual pair acting on the middle column has characters $(\gamma, \gamma\upsilon_T)$. Within that column the position row J is taken by the underlying linear map to $\bar{c}Jg^{-1}$. Write $\delta(g) = \det_E g$. The determinant on this row space is

$\bar{c}^N/\delta(g)$. By Theorem 403.2 at $J=0$, the untwisted scalar is $\upsilon(\bar{c})^{-N}\upsilon_T(\delta(g))|c|^{-N/2}$; here $|\delta(g)|=1$. Its factor at g cancels the extra power in the source determinant character. The line contribution to the target character is $\gamma(c/\bar{c})^{-1}$. Since $\upsilon(\bar{c})=\upsilon(c)^{-1}$ and $\upsilon_T(c/\bar{c})=\upsilon(c)^2$ by Theorem 374.2, the remaining unit scalar is $\lambda(c)$, with residual characters (γ,γ). This proves the first formula. Fourier transformation in t multiplies the expression by $|c|^{-1}$ by the additive change of variables $u\mapsto cu$, and transports the dual argument to $t/\bar{c}$ by its trace pairing. The residual pair operator acts in independent variables and commutes with this integral as in Theorem 403.2. This gives the second formula. □

542. Finite and infinite factors of the line sections

Definition 542.1. Use the two splittings of Definition 541.1. Endow each added hyperbolic plane with the direct sum of the standard finite lines $\widehat{O}_E e, \widehat{O}_E e'$, respectively $\widehat{O}_E a, \widehat{O}_E a'$, with dual pairings one. Keep the given unimodular lattices on the two perpendiculars. Thus all four scalar Hom entries between the added planes have the same physical scale α of Lemma 355.2. At split places use bases putting the line pairings equal to one and use the same complementary bases at the upper and lower ranks. Let $b=N-r\geq 0$. For the integral b-by-N matrix s of Lemma 539.4 let F_s^{up} be the large test whose top rows are $s\,\mathrm{pr}_W$, and let F_s^{low} be the smaller large test with top rows s. At a marked branch both use the same ordering of the fixed rows and operator rows in the determinant. Use positive normalized finite vacua in the two Schrödinger models. Write $\mathcal{L}_{\beta,f}$ and $\mathcal{L}_{\beta,\infty}$ for the finite and infinite factors of Definition 541.1, with compatible product measures. Thus on elementary tensors the full section is the tensor of its two factors, by successive integration.

Proposition 542.2. *For each $\beta=+,-$ there is a fixed nonzero b_β, depending only on the framed spaces, the measures and phases. At every allowable squarefree marked set S there is a number $G_{\nu,\beta}$, $|G_{\nu,\beta}|^2=Q_S^{-1}$, such that for finite ideles t*

$$\mathcal{L}_{\beta,f}(t)F_s^{\mathrm{up}}=b_\beta G_{\nu,\beta}\,D_S(t/\alpha)F_s^{\mathrm{low}}.$$

Here D_S, Q_S are as in Definition 516.2. Both sections vanish at the scalar adele zero when $S\neq\varnothing$.

Proof. First use at each finite place the lattice models in which the vacua are their integral-box indicators. At an unmarked place the orthogonal paired blocks in Definition 541.1 separate the vacuum into the frame box and the residual vacuum. A partial transform of a box is its dual indicator in these normalized models, by Proposition 391.1 and Lemma 355.2. Evaluating at the indicated zeros leaves the smaller vacuum times the condition t/α integral, for either scalar coordinate.

At a marked pair use the full split matrix position at w. Making the plus polarization of Definition 541.1 retains A, D, I, J, Z at this branch and transforms K, H_1, C_1, B_1 into the entries at the opposite branch paired to them. This follows by pairing the blocks in the hermitian trace as in Lemma 517.2; the residual Z is kept in its original local model. In units of α set $A=u_w$, $D=I=J=0$, where $u=t/\alpha$. In the transformed direction integrate the compact boxes of H_1, C_1, B_1 at zero frequency. The top rows of the stacked matrix are $(0,s,0)$, and its remaining rows are those of the Hom matrix with the displayed entries. Expansion in its first column and last row gives determinant

$$(-1)^b u_w K \det\begin{pmatrix} s \\ Z \end{pmatrix},$$

where K is also in normalized coordinates. The ordinary integral in K has the additive multiplier at $\overline{u_{\bar{w}}}$, up to a fixed unit and sign: the paired scalar coordinates have line pairing one, and their trace coefficient is $\jmath\alpha\bar{\alpha}$, a generator of the trace dual. Indeed a scalar supported in the K_w block pairs only with the $A_{\bar{w}}$ block. The hermitian product is supported on a single branch. Moving its trace to w conjugates $\jmath$ and introduces only a sign, as in the

normalized split transform of Proposition 501.2. Lemma 516.1 gives g_{χ_w} times the local factor of $D_S(u)$ and the residual determinant test. The sign, and any unit in the chosen conductor normalization, change the Gauss factor by a character of a unit, hence preserve its absolute value. The three other box integrals have mass one.

For the minus section one additional Fourier transform is made in the full scalar adele A. At this split pair it transforms the old A_w into the variable paired to it at $\bar{w}$, and inverts the preceding transform of K_w, up to a sign. Indeed the Fourier transform of the Fourier transform with opposite pairing recovers a function by the box computation in Lemma 533.2; using the same sign just negates its argument in the symmetric convention. Thus one sets K_w equal to u_w up to a unit and integrates A_w with the same unit-normalized frequency $\overline{u_{\bar{w}}}$. The other three frequencies are again zero. The product of the two scalar diagonal entries in the stacked determinant has the same character law, and the interchanged-entry calculation in Lemma 516.1 applies. For both sections multiplying the local Gauss moduli gives $\prod_{w\in S} q_w^{-1} = Q_S^{-1}$ after squaring. The calculation also gives zero at $t = 0$.

Passing back to fixed physical measures and unit vacua multiplies each full finite section by the fixed factor b_β. For detail, evaluate the same polarizations on the unmarked vacuum at frame zero. Each normalized box is there replaced by its nonzero unit-vector value; each transform has a fixed measure conversion. The chosen local normalized transforms map the box to its dual box, so these factors are exactly the same for every test processed in that polarization. They are one at almost all places, where the scale is a unit and the physical self-dual lattice measure is one. This is the product normalization argument of Lemma 521.2, now also applied to the scalar Fourier variable. The two choices of polarization may have different fixed factors. □

Theorem 542.3. *Let K_∞ be the product of the compact $U(p)\times U(q)$ in the real target group for V', using signed orthonormal bases. Give it invariant probability and give $(\mathbb{C}^\times)^\Phi$ product multiplicative Haar measure $d^\times t = \prod_\tau (dr_\tau/r_\tau)d\theta_\tau/(2\pi)$, $t_\tau = r_\tau e^{i\theta_\tau}$. Write $|t|_\infty = \prod_\tau r_\tau^2$ and let $\lambda_\infty(t) = \nu_\infty(t/\bar{t})$ for the prescribed weight. For either β, a real Schwartz vector v' on the upper pair and real $\sigma > 0$, define*

$$V_\beta(\sigma; v') = \int_{K_\infty} \int_{(\mathbb{C}^\times)^\Phi} \lambda_\infty(t)|t|_\infty^\sigma \, \mathcal{L}_{\beta,\infty}(t)\Omega_\infty(1,k)v' \; \mathrm{d}^\times t \; \mathrm{d}k.$$

Use the upper character pair of Lemma 541.3. This integral converges in every real Schwartz seminorm, gives a Schwartz vector on the smaller pair, and depends continuously on σ in those seminorms. The construction depends only on the archimedean character.

Proof. For either section, v' is written after a fixed linear change of polarization and, for the minus section, a partial Fourier transform. These are continuous Schwartz operators by Lemma 387.1 and the substitution and phase formulas of Lemma 514.2. Restricting a Schwartz function to the coordinates $(t, 0, 0, 0, \zeta)$ bounds any residual Schwartz seminorm by $C_m \prod_\tau (1 + r_\tau)^{-m}$, for every m. Indeed multiply its residual derivatives by the required residual coordinate powers and also by sufficiently many powers of the scalar coordinates before taking the supremum. The constants are uniform for $\Omega_\infty(1, k)v'$, $k \in K_\infty$: near each fixed k use the representation law and the continuity in every seminorm of Lemma 514.2, and cover the compact group by finitely many such neighborhoods. Fix $0 < \sigma_0 \le \sigma \le \sigma_1$. For $r_\tau \le 1$ the radial majorant is $r_\tau^{2\sigma_0-1}\,\mathrm{d}r_\tau$; for $r_\tau \ge 1$ it is $r_\tau^{2\sigma_1-m-1}\,\mathrm{d}r_\tau$. Both are integrable for $m > 2\sigma_1$. Their product dominates each prescribed seminorm. Differentiating in ζ under the integral is justified by the same majorants for the next derivatives. Tail truncation gives convergence in each seminorm, and dominated convergence gives continuity in σ by applying the bounds to the difference of the scalar factors. The unitary phase and the compact action at infinity use only the fixed archimedean weight. □

543. Integration over rational isotropic lines

Lemma 543.1. *Use the splittings of Definition 541.1 and put $G = U(V')(\mathbb{A}_{F_0})$, $\Gamma = U(V')(F_0)$. Let P be the adelic stabilizer of Ea, N_a its unipotent radical, $\Gamma_P = \Gamma \cap P$, and $K = K_\infty K_f$, where K_f preserves the given unimodular lattice and K_∞ is as in Theorem 542.3. Use probability on K and on $[N_a] = N_a(F_0)\backslash N_a(\mathbb{A}_{F_0})$. Let $C_E = E^\times\backslash\mathbb{A}_E^\times$, and normalize its quotient measure by counting on $E^\times$ and the product of the multiplicative measures of Definition 516.2 and Theorem 542.3. Write $[U(V)] = U(V)(F_0)\backslash U(V)(\mathbb{A}_{F_0})$. There is a positive C_P, depending only on the spaces, compact groups and measures, such that for any nonnegative measurable A on $\Gamma_P\backslash G$,*

$$\int_{\Gamma_P\backslash G} A(y)\,\mathrm{d}y = C_P \int_{C_E\times[U(V)]\times K} |c|^{r+1} \int_{[N_a]} A(n\ell(c,h)k)\,\mathrm{d}n\ \mathrm{d}^\times c\ \mathrm{d}h\ \mathrm{d}k.$$

The inner mean is independent of the choices of quotient representatives for c, h. The same equality holds for absolutely integrable complex functions.

Proof. At a finite place the intersection of an isotropic field line with the integral lattice has a primitive generator. This follows by clearing denominators and then dividing a generator by the largest common uniformizer power in its coordinates, separately in both factors at a split place. The local algebra is split or unramified by Lemma 57.1. Theorem 345.2 sends such a generator to the reference generator. At a real place an isotropic vector in the orthogonal sum of the positive and negative spaces has components of equal nonzero length. After rescaling, the unitary groups of these two definite spaces can carry the two components to those of a fixed generator of Ea. To see this, extend each unit vector to an orthonormal basis by successively subtracting projections from a spanning list and normalizing its nonzero remainders. Thus at all places K is transitive on the respective isotropic lines. For $y \in G$, choose compact elements carrying the reference line to its inverse image by y. At almost all places take y_v^{-1}. This proves $G = PK$. Every member of P is uniquely $n\ell(c,h)$: its actions on the line and on $a^\perp/Ea$ determine c, h, and the remaining triangular matrix belongs to the radical by Definition 522.1. The same assertion with rational entries describes Γ_P.

Conjugation by $\ell(c,h)$ on the radical row is $C \mapsto c^{-1}Ch^{-1}$. Its additive modulus is $|c|^{-r}$, since the determinant of h has conjugate product one. On the skew scalar the action is multiplication by $(c\bar{c})^{-1}$, of additive modulus $|c|^{-1}$: at split places the two ordinary local absolute values multiply, at inert places the residue field degree is two, and at a real place $c\bar{c}$ is the square of the complex modulus. The triangular product law in Lemma 539.2 therefore gives the factor $|c|^{-(r+1)}$ on radical Haar measure. The measure

$$dp = |c|^{r+1}\,\mathrm{d}n\ \mathrm{d}^\times c\ \mathrm{d}h$$

is left invariant on P, by substitution in these coordinates. We take radical Haar with the compact quotient of mass one. Right translation by a radical element preserves dp. For right translation by $\ell(c_0,h_0)$ the integral of a translated test acquires the factor $|c_0|^{-(r+1)}$, since the two Levi measures are right invariant. Right invariance for the unitary group follows at infinity from Lemma 21.4, at finite places from Lemma 161.1, and on finite restricted products by normalization on the integral compact.

We justify proportionality of measures in $G = PK$. The natural bijection $P/(P\cap K) \to G/K$ is a homeomorphism: a closed saturated set in P is closed in G since P is closed, and its product with the compact K is closed. Indeed a convergent sequence of products has a subsequence whose compact factors converge, and the remaining factors converge in the closed set. These second countable groups are metrizable in their local coordinates. Push the measure on G to G/K. The result is locally finite because the quotient by a compact subgroup has compact preimages of compact sets: translates of the image of a relatively compact open identity neighborhood give a finite cover of any such set, and their closed lifts multiplied by K are compact. Lift this measure from $P/(P\cap K)$ to P by integrating over right $P\cap K$ against its invariant probability. This defines a nonzero left invariant locally

finite measure Λ. For completeness, for nonnegative continuous compactly supported f, b integrate $f(x)b(x^{-1}y)$ against $\mathrm{d}\Lambda(x)\,dp(y)$. Integrating in y first gives $\Lambda(f)p(b)$. In the opposite order write $x = yz$ by left invariance of Λ. The result is $p(f)\int b(z^{-1})R(z)\,\mathrm{d}\Lambda(z)$, where $R(z) > 0$ is the right translation factor of dp computed above. Choose b positive near one with $p(b) > 0$. Thus Λ is a fixed positive multiple of dp. Equality on positive compact tests gives equality on open sets by exhaustion with compactly supported continuous cutoffs, then on Borel sets by countable monotone approximation of the measures. Integrating over right K, which preserves G-measure, now gives the formula on G with this constant C_P.

For its quotient form first take $A(y) = \sum_{\delta\in\Gamma_P} A_0(\delta y)$, $A_0 \geq 0$. Use the rational semidirect product just obtained to sum first over the rational radical and then over the rational Levi. Rational Levi conjugation preserves the rational radical and its Haar measure: the adelic modulus of a rational scalar is one, by the rational determinant calculation in the proof of Theorem 383.1, applied to multiplication on E in a rational basis. The rational h has determinant of norm one as above. The inner mean is consequently unchanged on replacing c, h by rational translates. Unfolding the three sums against their fundamental domains gives the formula on G just proved. Conversely every nonnegative quotient function is a periodization: restrict its pullback to a measurable fundamental domain and put it zero outside. For these countable rational groups such a domain is obtained by the disjoint chart construction of Lemma 21.4. Indeed a compact set in any of the adelic matrix groups contains only finitely many rational matrices, by real bounds and fixed finite denominators, so a sufficiently small neighborhood has disjoint translates. Taking a countable chart cover and assigning each orbit to its first chart gives a measurable selector. For complex functions apply the nonnegative identity to the positive and negative parts of their real and imaginary parts. □

Proposition 543.2. *Use the standard upper large test $f = v' \otimes F_s^{\mathrm{up}}$ of Proposition 542.2, its upper normalized character pair of Lemma 541.3, and a nonempty marked set. For $g \in U(W)(\mathbb{A}_{F_0})$ write $g' = m(1, g)$. On $C_E \times [U(V)] \times K_\infty$ define the function*

$$B(c, h, k) = |c|^{r+1} \sum_{\xi\in E^\times} \theta_{W,V}(\mathcal{L}_+(\xi)\Omega(g', \ell(c, h)k)f).$$

If $\int |B(c, h, k)|\,\mathrm{d}^\times c\,\mathrm{d}h\,\mathrm{d}k < \infty$, then

$$\mathcal{C}_e P^+_{W',V'}(v', s\,\mathrm{pr}_W)(g') = C_P \int B(c, h, k)\,\mathrm{d}^\times c\,\mathrm{d}h\,\mathrm{d}k.$$

Here P^+ denotes the normalized large period, obtained from Definition 507.1 by the common ambient determinant character in Definition 520.1.

Proof. Each rational isotropic line in V' is the image of Ea by a member of Γ. Indeed any map sending the reference generator to a nonzero vector on that line is an isometry on the line, so Lemma 381.1 extends it to the whole space. For $y \in G$ set $A(y) = \sum_{\xi\in E^\times} d_{\xi a}(g', y; f)$, using Definition 539.1. These are absolutely convergent sums for fixed y, by Proposition 539.3. For rational $\delta \in \Gamma$, rational substitution on the frame gives $d_{\delta x}(g', \delta y; f) = d_x(g', y; f)$ by Lemma 510.2 followed by the source radical mean. Thus A is left Γ_P-invariant and the sum of $A(\delta y)$ over $\Gamma_P\backslash\Gamma$ is the sum over all nonzero rational isotropic frames. The zero frame contributes zero by Lemma 539.4.

By Proposition 541.2, A is left invariant by the adelic target radical and its value at $\ell(c, h)k$ is the indicated plus sum. The finite part of k fixes f: in the proof of Lemma 518.2 left multiplication of the operator rows contributes the residue determinant character, whose inverse is exactly the finite target twist. This proof uses no condition on the fixed upper rows. The additional power of the unramified character in the enlarged rank is trivial on integral units. The unmarked vacuum is fixed by Theorem 390.2. Consequently $|c|^{r+1}A(n\ell(c, h)k) = B(c, h, k_\infty)$. Lemma 543.1 proves that A is absolutely integrable under the stated condition. The sum over lines therefore unfolds in the target quotient, by tiling

a Γ_P-domain with translates of a Γ-domain as in Lemma 384.1. Absolute integrability of A justifies this last exchange of sum and integral. Proposition 539.3 identifies the original grouped integral with the constant term. The quotient formula yields the asserted equality. □

544. Removing a plane from the large period

Lemma 544.1. *Use Proposition 543.2 with even N, $0 \le r \le N$, and put $d = r - N/2$. For real t write $B_t(c,h,k) = |c|^t B(c,h,k)$. For $\sigma > 1$ set $t_+(\sigma) = \sigma - d - 1$, $t_-(\sigma) = -\sigma - d$. For either sign β, $B_{t_\beta(\sigma)}$ is absolutely integrable on the product quotient in that proposition. Its integral equals*

$$b_\beta \frac{G_{\nu,\beta}}{G_S} \lambda_f(\alpha)|\alpha|_f^\sigma M_S(\sigma) \cdot P^+_{W,V}(V_\beta(\sigma; v'), s)(g).$$

Here the real vector and all finite factors use Theorem 542.3 and Proposition 542.2, and the normalized lower period uses (γ,γ) of Lemma 541.3. For a zero-dimensional target its group has just one element and its Hom space is zero. Its theta evaluation is evaluation on the unique position, with the indicated group characters.

Proof. By Lemma 541.3, the sum in B may be computed with either section. Indeed first add the summand at $\xi = 0$, apply the rational theta identity there, then remove the summand at zero on the other side. For either sign the zero summand vanishes by transporting the section past $g', \ell(c,h)$ using that lemma and evaluating the finite section of Proposition 542.2 at zero. Here K_f fixes the finite datum by Proposition 543.2. Write $j_+(c) = c$, $j_-(c) = 1/\bar{c}$, and put

$$T_\beta(u,h,k) = \theta_{W,V}(\Omega(g,h)\mathcal{L}_\beta(u)\Omega(1,k)f).$$

The same Levi formulas give the two identities

$$B_{t_\beta(\sigma)} = \sum_{\xi \in E^\times} \lambda(c)|j_\beta(c)|^\sigma T_\beta(\xi j_\beta(c), h, k).$$

For plus the total exponent of $|c|$ is $t + d + 1$, and for minus it is $t + d$. Taking the indicated values of t gives respectively $\sigma, -\sigma$.

Both j_β preserve the multiplicative measure on $\mathbb{A}_E^\times$ and carry $E^\times$ onto itself. Conjugation permutes equally normalized finite factors and preserves the real radial and angular measures. Inversion negates the finite valuation vectors and preserves unit probabilities, and negates the real log radii and angles. It follows from the defining quotient integral that j_β preserves the quotient measure. Moreover $\lambda(j_\beta(c)) = \lambda(c)$, since $j_-(c)/\overline{j_-(c)} = c/\bar{c}$. The character $\lambda(u) = \nu(u/\bar{u})$ and the adelic modulus are trivial on rational scalars, by Definition 515.1 and the rational determinant computation of Theorem 383.1. Consequently the fundamental domain for C_E unfolds the sum of the absolute values to the integral over full ideles of $|u|^\sigma |T_\beta(u,h,k)|$ for fixed h,k. The domain calculation holds for nonnegative functions without any finiteness assumption, by Lemma A.2.

To bound this last integral, put $w(u_\infty, k) = \mathcal{L}_{\beta,\infty}(u_\infty)\Omega_\infty(1,k)v'$. The finite section formula separates T_β as $b_\beta G_{\nu,\beta} D_S(u_f/\alpha)$ times the residual theta kernel with real test w and finite test F_s^{low}. Each residue matrix contributing to F_s^{low} has independent operator rows, since the stacked determinant is a unit. For fixed g its theta kernel is therefore bounded by $C_g p_m(w) h_V(h)^{-L}$, with L chosen so that the last factor is integrable, by Theorem 507.3 and Proposition 398.1. More explicitly one applies the single-point estimate there to each of the finitely many named matrices in this standard test and adds their bounds. The estimate in the integration variable requires only that the named map onto V be surjective. Its proof uses the height of g only as a multiplicative constant for fixed g, so it also applies when the entire source is presented as one space. For $r = 0$ take $h_V = 1$ and use the bound on evaluation on the unique position, with the one-point quotient measure. The

proof of Theorem 542.3 gives $\int_{K_\infty} \int |u_\infty|^\sigma p_m(w(u_\infty,k))\,\mathrm{d}^\times u_\infty\,\mathrm{d}k < \infty$. The finite factor has absolute integral

$$\int |u_f|_f^\sigma |D_S(u_f/\alpha)|\,\mathrm{d}^\times u_f = |\alpha|_f^\sigma Q_S^\sigma \sum_{\substack{\mathfrak{a}\ \text{integral}\\ (\mathfrak{a},S\cup\tilde{S})=1}} (N\mathfrak{a})^{-\sigma} < \infty$$

by the valuation-coset computation of Theorem 516.3. The sum is over nonzero ideals coprime to the primes in the indicated set. These three bounds prove absolute integrability of the unfolded terms, hence also of their grouped sum on the quotient.

We may now unfold the complex integrand by the same translations. Lemma 123.1 separates the real and finite variables. Substitution of $u_f = \alpha x$ gives

$$b_\beta G_{\nu,\beta} \lambda_f(\alpha) |\alpha|_f^\sigma M_S(\sigma)/G_S$$

for the finite integral. The residual theta functional at fixed g, h is continuous in the real test by the seminorm estimate just proved. Its integral in h is likewise continuous, by integrating that estimate. Thus the real and compact integrals may be taken in the real Schwartz vector first: on compact truncations approximate the continuous Schwartz-valued integrand by finite sums in the controlling seminorm, then use the preceding integrable tail bound. This gives precisely $V_\beta(\sigma; v')$ and proves the equality. □

Theorem 544.2. *In Lemma 544.1 assume that the prescribed weight of ν is nonzero. Put $s_d = \max(d+1, -d)$, and take $\beta = +$ if $d \geq 0$, $\beta = -$ if $d < 0$. There is a fixed $C_{\mathrm{lg}} > 0$ such that each allowed twist has a scalar $\Gamma_\nu \neq 0$ for which, for every real Schwartz v', integral standard s and adelic g,*

$$\mathcal{C}_e P^+_{W',V'}(v', s\,\mathrm{pr}_W)(m(1,g)) = \Gamma_\nu P^+_{W,V}(V_\beta(s_d; v'), s)(g),$$
$$|\Gamma_\nu|^2 = C_{\mathrm{lg}} \frac{\Pi^{\mathrm{int}}(d+1)}{\Pi^{\mathrm{int}}(d)}.$$

The sign, exponent, and operation on real tests depend only on the fixed spaces, polarizations and archimedean weight.

Proof. Choose two real numbers $a < 0 < b$ of the form $a = t_-(\sigma_-)$, $b = t_+(\sigma_+)$ with $\sigma_-, \sigma_+ > 1$, possible by taking both large. For $a \leq t \leq b$,

$$|B_t| \leq |B_a| + |B_b|.$$

Indeed use $|c|^t \leq |c|^a$ for $|c| \leq 1$ and $|c|^t \leq |c|^b$ for $|c| \geq 1$, multiplying by the modulus of the same summed function B. Both bounding functions are integrable by Lemma 544.1. At $t = 0$ the absolute integrability hypothesis of Proposition 543.2 follows. The integral of B_t is continuous for $a < t < b$ by dominated convergence, Lemma A.2.

For the stated sign, $t_\beta(s_d) = 0$. If $s_d > 1$ use Lemma 544.1 directly. If $s_d = 1$, let $\sigma \downarrow 1$ in that lemma. The left integrals converge by the domination above and the real datum on the right converges in each seminorm by Theorem 542.3. The residual period is continuous in the controlling seminorm as in the preceding proof. Finally $M_S(\sigma) \to M_S(1) \neq 0$ by Theorem 516.3. This uses a one-sided limit of the finite integrals. In both cases the scalar is

$$\Gamma_\nu = C_P b_\beta (G_{\nu,\beta}/G_S)\, \lambda_f(\alpha) |\alpha|_f^{s_d} M_S(s_d).$$

The two Gauss factors have the same nonzero absolute value by Proposition 542.2 and Theorem 516.3. Since λ_f is unitary, this last theorem gives the required squared modulus with $C_{\mathrm{lg}} = C_P^2 |b_\beta|^2 |\alpha|_f^{2s_d} > 0$. The nonzero weight gives $M_S(s_d) \neq 0$, so division by Γ_ν at any allowed twist is legitimate. □

545. Successive removal of hyperbolic planes

Lemma 545.1. *Take $p \geq 0$ mutually perpendicular hyperbolic planes in a source and a fixed nonsingular small target T. In each plane choose one of the two isotropic generators as the frame position, and use the dual generator for the momentum. In compatible real polarizations the p successive zero slices on $\mathrm{Hom}(\cdot, T)$ equal the joint zero slice in the chosen p frame variables, in any removal order. Every real Schwartz vector on the perpendicular occurs as such a joint slice. These assertions include $T = 0$.*

Proof. Use a polarization on the perpendicular from Lemma 385.2. In the trace symplectic form the Hom spaces on distinct orthogonal source summands are orthogonal, by Definition 355.1. For one plane its columns on the two isotropic generators pair perfectly with one another and have zero internal pairings. Their direct sum with the residual polarization consequently gives positions $(x_1, \dots, x_p, z)$, with z the residual positions. By Definition 540.1, taking a zero slice sets just the corresponding x_i to zero. In this model the assertion on their composition is ordinary evaluation in independent variables. Reversing the choice on any plane uses a partial Fourier transform on that factor, with the dual coordinate and measure there, by Lemma 387.1. Such transforms leave all other factors alone and commute with evaluation there. For a given residual vector $v(z)$ take $\prod_i \exp(-\|x_i\|^2) v(z)$ in the chosen coordinates. It is Schwartz and evaluation gives v. Transport to any of the indicated polarizations preserves the Schwartz property by that lemma and Lemma 514.2. For the zero target every variable has dimension zero and all the evaluation maps are identity. □

Theorem 545.2. *Let N be even, $0 \leq r \leq N$, $b = N - r$ and $d = r - N/2$. Write $W_0 = W$, $V_0 = V$ for framed spaces of ranks N, r. For $1 \leq i \leq p$ let W_i, V_i be obtained from W_{i-1}, V_{i-1}, respectively, by adjoining one physical hyperbolic plane as in Proposition 542.2. Keep the fixed small target T_b of rank b, the admissible squarefree levels, and the fixed nonzero archimedean weight. At each rank use the large normalized period and the directly transformed normalized small period, as in Theorems 544.2 and 540.3. Let x_p, y_p be Schwartz real data at the upper rank. Suppose for each allowed ν there is $a_{\nu,p} \in \mathbb{C}$ with*

$$P^{+}_{W_p,V_p}(x_p, s_p) = a_{\nu,p} P^{-}_{W_p}(y_p, s_p)$$

as functions at every integral standard s_p. There are fixed continuous linear operations $x_p \mapsto x_0$ and $y_p \mapsto y_0$ on real Schwartz data, a fixed $D > 0$ and scalars $a_{\nu,0}$ such that

$$P^{+}_{W,V}(x_0, s) = a_{\nu,0} P^{-}_{W}(y_0, s) \qquad \text{for every integral standard } s,$$

$$|a_{\nu,0}|^2 = D |a_{\nu,p}|^2 \, \frac{\Pi^{\mathrm{int}}(d)}{\Pi^{\mathrm{int}}(d+p)}.$$

The vector y_0 is the joint zero slice for the chosen source lines. The scalar formula holds also when $a_{\nu,p} = 0$.

Proof. For the step $i \to i-1$, put $d_i = d + i$. In fact the large rank grows by two and half the ambient rank grows by one at each step. Set $\beta_i = +$ if $d_{i-1} \geq 0$ and $\beta_i = -$ otherwise, and $z_i = \max(d_{i-1} + 1, -d_{i-1})$. Take $x_{i-1} = V_{\beta_i}(z_i; x_i)$ on the indicated spaces, $y_{i-1} = y_i[0]$. Both operations are linear. For the integral operation, the seminorm majorants in the proof of Theorem 542.3 are bounded by finitely many input seminorms for fixed exponent: partial Fourier transforms are continuous, and the explicit near-identity substitution, phase and Fourier formulas in Lemma 514.2 bound these seminorms uniformly on each compact coordinate patch. A finite cover handles the compact group. For evaluation the same assertion follows directly by restricting the supremum defining each seminorm. The resulting constructions therefore have the stated continuity and depend only on the fixed weight and spaces.

Given any integral standard s_{i-1}, set $s_i = s_{i-1} \mathrm{pr}_{W_{i-1}}$ on W_i. It is integral in the split lattice on W_i. In the upper identity average both sides along the chosen source line at

$m(1,g)$, for arbitrary lower g. The two theorems in the statement give

$$\Gamma_{\nu,i} P^{+}_{W_{i-1},V_{i-1}}(x_{i-1},s_{i-1})(g) = a_{\nu,i} c_{\mathrm{sm},i} P^{-}_{W_{i-1}}(y_{i-1},s_{i-1})(g).$$

Here $\Gamma_{\nu,i} \neq 0$, $c_{\mathrm{sm},i} \neq 0$, and

$$|\Gamma_{\nu,i}|^2 = C_{\mathrm{lg},i} \Pi^{\mathrm{int}}(d_i)/\Pi^{\mathrm{int}}(d_{i-1}).$$

Thus one may take $a_{\nu,i-1} = a_{\nu,i} c_{\mathrm{sm},i}/\Gamma_{\nu,i}$. Only $\Gamma_{\nu,i}$ occurs in the denominator. Starting from p proves the lower equality at every rank, including on all tests needed for the next step. The product of the squared moduli telescopes, with $D = \prod_{i=1}^{p} |c_{\mathrm{sm},i}|^2/C_{\mathrm{lg},i} > 0$. Lemma 545.1 identifies the small datum. Empty products and identity operations give the assertion for $p = 0$. □

Corollary 545.3. *In Theorem 545.2 take $T_b = (A')^-$ and let y be the real datum on (A', W) whose opposite transform is y_0. Then for all matched residue data at the indicated squarefree level and all g,*

$$I_\Phi(g) = c_{\nu,0} I'_\Psi(g), \qquad |c_{\nu,0}| = |a_{\nu,0}|.$$

Proof. Undo the common nonzero ambient determinant character in Definition 520.1. Theorem 512.2 identifies the original small period with c_0 times the directly transformed one, where $|c_0| = 1$, uniformly on the standard tests. Its real coordinate identification is invertible, since each step is an isometric coordinate change or the conjugation identification twice in the proof there. The original comparison on standard tests therefore holds with $c_{\nu,0} = a_{\nu,0}/c_0$. For $b, r > 0$, Theorem 509.3 expresses every matched pair as a common linear combination of those tests, proving the assertion. For $b = 0$ the small matrix is unique and the large matrix runs through the invertible N-square matrices on its support. The relation of Definition 499.1 with $X_k = k^{-1}$ determines both functions by the value on the unique small matrix. The standard pair whose upper row matrix is empty has that value one. For $r = 0$ the large matrix is unique, and the same relation at $Y_k = k$ determines the pair by its value on the large matrix. The standard pair at $s = I_N$ takes value one there. This also gives the zero-dimensional case by the empty determinant convention. □

546. Changing the framed target of a theta period

Lemma 546.1. *Use the arithmetic field and physical oscillator of Theorem 395.2. Fix the framed source W'. Let T_1, T_2 be two nonsingular hermitian targets of the same rank with unimodular finite lattices and the same signatures at every infinite place. Use standard large data at the same nonempty squarefree marked set, with the same ordered coordinate convention on the operator rows. Let P_i be the normalized large period over $[U(T_i)]$. There is a fixed linear isomorphism A_∞ between the real Schwartz spaces, from the second pair to the first, a fixed $m_T > 0$, and for each twist a scalar $|u_\nu| = 1$, such that for all real v, standard s and group arguments g,*

$$P_2(v,s)(g) = m_T u_\nu \, P_1(A_\infty v, s)(g).$$

The identity holds as well for any common function of the marked matrix entries supported on surjective residue maps, using the corresponding point names in each framing.

Proof. At a split pair two unimodular hermitian lattices of the same rank are integrally isometric: choose a basis on one branch, then take its pairing-dual basis on the other. At a nonsplit finite place the residue extension is quadratic and the lifting argument in Lemma 403.1 gives the integral isometry, including at residue characteristic two. Thus the rational hermitian forms are locally isometric everywhere. The arithmetic field satisfies the hypotheses of Theorem 90.2: its cyclotomic field contains ζ_{12}, so its real subfield contains $\zeta_{12} + \zeta_{12}^{-1} = \sqrt{3}$ and the field contains $i = \zeta_{12}^3$. The CM extension of its real subfield has degree two. That theorem gives a rational isometry $j : T_1 \to T_2$. Use it to pull back the rational coordinates, forms and target group of the second pair to the first.

We explain its effect on the oscillator. Choose a rational polarization for the first Hom space and its image for the second. In transported rational bases the trace pairing and the rational position sum agree term by term. Passage from these coordinates to the stipulated physical models intertwines the group action by Theorem 395.2 and Lemma 403.1. At infinity this changes the real test by a fixed invertible Schwartz operator A_∞: it is a change of position coordinates, partial Fourier transforms and quadratic phases as in Theorem 387.2. The determinant twists are preserved under conjugation by j. These choices are independent of the finite test and of the finite characters.

Write L_1, L_2 for the two finite target lattices in these common rational coordinates. Choose $k_f \in U(T_1)(\mathbb{A}_f)$ with $k_f L_1 = L_2$, by Lemma 403.1. At each marked pair choose it to send the first ordered marked branch frame to the pulled-back second one. A map on this split branch extends by its inverse adjoint on the opposite branch, which carries the corresponding dual lattice to its prescribed lattice. Elsewhere any such lattice isometry suffices and at almost all places both lattices coincide. In particular k_f is independent of s at the given level.

Left multiplication by k_f on Hom is an isometry carrying the two scaled Hom lattices and their branch subquotients to one another. Its untwisted finite oscillator operator U_f carries the first unit vacuum to a unit multiple of the second: it intertwines the translations and the fixed line is unique by Proposition 391.1. For a residue point use the translation sum of Proposition 357.2. The isometry relabels each translation by its image, preserves the pairing in the coefficient, and induces a bijection of the finite summation groups. Since the marked bases were transported, a point with a given matrix name is carried to the same matrix name in the second frame, with precisely the vacuum phase. Adding these formulas with arbitrary coefficients proves $F_{2,s} = \eta\, U_f F_{1,s}$, $|\eta| = 1$, for the standard determinant and for the general finite datum in the statement. Replacing U_f by the normalized pair action at $(1, k_f)$ multiplies the scalar by another number of modulus one, since its target character is unitary.

In the target integral this group operator right translates the variable by k_f. Right translation preserves the quotient measure by the bi-invariance in Lemmas 161.1 and 21.4. Pullback under the rational isometry identifies the discrete subgroups and their counting measures. The two group Haar measures in common coordinates have a fixed positive ratio m_T: at infinity compare the invariant densities at the identity and at finite places use the compact normalization in Lemma 161.1. The induced quotient measures have the same ratio upon integration over a common fundamental domain. The possible choice of k_f has no effect on that ratio. The integrals converge for the stated supports by Theorem 507.3; for rank zero use the one-point interpretation in Lemma 544.1. This proves the formula. □

Proposition 546.2. *Let W, V, U be framed nonsingular rational hermitian spaces over the same field, of ranks N, r, b with $r + b = N$. Write $q_M(\tau)$ for the negative index of a space M at a real embedding τ, and assume $q_W(\tau) = q_V(\tau) + q_U(\tau)$ for every τ. Give $T_b = U^-$ the opposite frame of Lemma 505.1. Let $\mathbb{H}$ denote the rational plane on e, e' with $\langle e, e\rangle = \langle e', e'\rangle = 0$ and $\langle e, e'\rangle = 1$, with the two free finite line lattices generated by e, e'. For $p \geq 1$ put*

$$X = \mathbb{H}^p \perp W, \qquad Z = \mathbb{H}^b \perp X, \qquad T = T_b \perp X, \qquad \widetilde{T} = \mathbb{H}^{p+b} \perp V.$$

All powers here are orthogonal sums. On the common source Z the identity of Lemma 546.1 holds between T and $\widetilde{T}$. Also X has an adapted rational hyperbolic plane, and the b opposite pairs in Z and that plane together split the finite lattice as required in Definition 536.1 when $b > 0$.

Proof. At every real place a hyperbolic plane has one sign of each kind, as seen in the basis $(e + e')/\sqrt{2}, (e - e')/\sqrt{2}$. Thus the negative index of T is $b - q_U + p + q_W = b + p + q_V$, that of $\widetilde{T}$. Their dimensions are respectively $b + 2p + N$ and $2p + 2b + r$, which agree. The plane

lattice is unimodular since its pairing matrix interchanges its two basis vectors. Orthogonal sums of unimodular lattices are unimodular: the condition on a vector pairing integrally with the sum tests each component separately. The hypotheses on the two target lattices in the preceding lemma now hold. Since $p \geq 1$, reserve one of the explicit planes in X for the nested splitting. Its two lines and all the lines in $\mathbb{H}^b$ lie in the prescribed lattice as direct summands with integral duality, by its construction. □

547. Exchange at the prescribed complementary ranks

Definition 547.1. Let X be a framed even-rank space with a rational hyperbolic plane splitting its finite lattice and let z be any real Schwartz vector on $\mathrm{Hom}(X, X)$. For a nonzero fixed weight and a tame twist of the squarefree kind in Definition 516.2 write

$$R_\nu(X, z) = \frac{|J_\nu(X, z; 1)|^2}{\Pi^{\mathrm{int}}(\dim X/2)}.$$

Here $J_\nu(X, z; 1)$ is the value of the normalized equal-rank determinant period of Definition 521.1 with real datum z, at the identity. This definition uses only the space X, its own frames, measures and the twist. The finite datum is the determinant character in the split matrix coordinates and the vacuum off the marked pairs, as in Theorem 517.3. Its integral converges by Theorem 507.3, since the finite maps supporting that determinant are invertible. If an auxiliary positive-rank T_b is used to write Definition 521.1, it does not occur in the defining integral over $U(X)$. The denominator is positive by Theorem 516.3 and Corollary 515.4.

Theorem 547.2. *Use W, V, U, X, $p \geq 1$ and the hypotheses on ranks and signatures of Proposition 546.2. Assume N even and let the integer weights μ_τ be positive at all infinite places. Write $d = r - N/2$. For any prescribed Schwartz real test y on (U, W) and any Schwartz z on (X, X) there are a Schwartz real test x on (W, V) and a constant $D_* > 0$ such that for every allowed tame twist ν of weight μ and its squarefree marked set there is $c_\nu \in \mathbb{C}$ with*

$$I_\Phi(x; g) = c_\nu I'_\Psi(y; g) \qquad \text{for all matched residue tests and all adelic } g,$$
$$|c_\nu|^2 = D_* \, \Pi^{\mathrm{int}}(d) R_\nu(X, z).$$

The periods have their original orientations and determinant characters of Definition 508.1. In particular x and D_* may be fixed before the finite twist and its marked set are chosen.

Proof. Transform y to the opposite model on (W, T_b) by Theorem 512.2, and denote its real vector by $y^\flat$. Use Lemma 545.1 on $X = \mathbb{H}^p \perp W$ to choose a real y_X whose joint zero slice in those p planes is $y^\flat$. Suppose first that $b > 0$. On $Z = \mathbb{H}^b \perp X$ take the real vectors F, F' from Theorem 514.3 with residual real vectors y_X, z. On the b outside planes choose for subsequent descent the isotropic lines opposite those used for the matched frame slices in that theorem. It gives exactly y_X as the simultaneous zero slice of F' on these opposite lines.

Use the spaces $T = T_b \perp X$ and T_b on this top source. Proposition 546.2 supplies the adapted nested splitting. Every nonempty allowed S contains a branch where the finite determinant has full rank on its support, as needed in Definition 521.1. Theorem 538.3 gives the comparison for F, F'. In the directly transformed normalized periods the top scalar is $K_\nu J_\nu(X, z; 1)$: the phase c_0 in that theorem came from converting to the original small period, and the common ambient character multiplies both sides equally. It follows for every standard upper row test that the squared modulus of the top scalar is $C_*^2 |J_\nu(X, z; 1)|^2$, with $C_* > 0$ independent of ν, S. The opposite lines used for descent merely specify which upper tests to specialize next. At present the identity holds for every one of them.

If $b = 0$, then $Z = X$, $T = X$ and the small Hom space is zero. In the normalized direct orientation its source character is $\upsilon_T^0 = 1$ by Definition 540.1. The small period with the

unique standard datum is consequently $m_0 y_X$, where $m_0 > 0$ is the one-point measure times its positive vacuum value. Take $F = y_X z,\ F' = y_X$. The large period is $y_X J_\nu(X, z; 1)$ at every argument by Theorem 528.2. The required top scalar is $J_\nu(X, z; 1)/m_0$, which has the same asserted form of squared modulus. This argument also covers $y_X = 0$.

Apply Lemma 546.1 to replace T by $\widetilde{T} = \mathbb{H}^{p+b} \perp V$ on the large side. It changes the upper real vector by its fixed map, and the squared modulus of the resulting top scalar is multiplied by the fixed factor m_T^{-2}. Now Z and $\widetilde{T}$ are both obtained from W, V by adjoining $p + b$ explicit planes. Apply Theorem 545.2 to these spaces, removing first the b outside source planes in the chosen opposite orientations and then the p planes of X. Its y_0 equals $y^\flat$ by Lemma 545.1. The theorem gives a large lower test x and multiplies the squared modulus of the scalar by $D\,\Pi^{\mathrm{int}}(d)/\Pi^{\mathrm{int}}(d + p + b)$. But

$$d + p + b = r - N/2 + p + b = N/2 + p = \dim X/2.$$

Together with the top value this is the claimed size, with $D_* = D m_T^{-2} C_*^2$, where $C_* = 1/m_0$ for $b = 0$. Finally Corollary 545.3 gives the equality for the original y and every matched residue pair, preserving the modulus. The lifts, real slices and real cone operations in this proof involve only the spaces, real data and the prescribed infinite weight. Thus the same x, D_* apply to all the stated twists. □

548. Independence of the normalized size from the twist

Lemma 548.1. *Let J be a positive definite hermitian space at every real place, of positive rank, with unimodular integral finite lattice. The reduced height h_J of Definition 507.1 is bounded, and $0 < \mathrm{vol}([U(J)]) < \infty$. For every squarefree determinant datum and real Schwartz vector on (J, J), its kernel with either unitary determinant twist is absolutely integrable on $[U(J)]^2$.*

Proof. For $h \in U(J)(\mathbb{A})$ and $0 \neq v \in L(h)$ in Theorem 397.1, the rational scalar $\langle v, v\rangle$ is integral at every finite place, because $h_f^{-1} v$ is in the integral lattice. It is positive at every real place. Its product over real places is a positive integer: multiplication by this nonzero algebraic integer on an integral basis of O_{F_0} has integer determinant. Conjugating this multiplication matrix by the invertible embedding matrix in Lemma 82.2 diagonalizes it with the conjugates as diagonal entries. At least one conjugate is at least one. In the signed real frames all signs are positive, so this conjugate is the squared Euclidean length of the corresponding component of $h_\infty^{-1} v$. The real norm in the shortest length dominates a fixed positive multiple of each of these component norms by equivalence of coordinate norms in the fixed finite dimension. Thus $\lambda_J(h)$ has a fixed positive lower bound. This bounds h_J. Proposition 398.1 gives finite measure to bounded reduced height sublevels. The identity has an open neighborhood disjoint from its other rational translates by Lemma 21.4, so the quotient has positive measure. Both height variables in Theorem 507.3 are bounded. All points on the support of the finite determinant are invertible. Adding their kernel estimates gives a bounded function on this finite-measure product, proving the last claim. □

Proposition 548.2. *Let J be the space of rank $2k > 0$ with identity hermitian form and standard unimodular finite lattice. Fix $p \geq 1$, $X = \mathbb{H}^p \perp J$, and positive integer weights μ_τ. For every real Schwartz z on (X, X), $R_\nu(X, z)$ of Definition 547.1 is independent of the allowed tame twist of this weight, including its marked set.*

Proof. Take the zero-dimensional space with quotient measure $m_0 > 0$ and unit finite vacuum on the unique position. For ambient $W = J$ apply Theorem 547.2 first to $V = J,\ U = 0$, with small real datum one and residual datum z. This gives a real datum y on (J, J) and scalars b_ν with

$$|b_\nu|^2 = D_1 \Pi^{\mathrm{int}}(k) R_\nu(X, z), \qquad D_1 > 0.$$

Use its standard matched pair on the residue module of rank $2k$: the large value is $\chi^{-1}(\det)$ on the invertible matrices, extended by zero, and the small value is one on the unique

matrix. Write $K_\nu(g,h)$ for the (J,J) kernel on this determinant datum and y, with both characters $\gamma = \nu^{-1}\upsilon_T^k$. Definition 504.1 shows that the unnormalized large kernel with source denominator one is $\gamma(\det g)K_\nu(g,h)$. On the small side the action on the zero Hom space is identity and its target character gives the same $\gamma(\det g)$. Cancelling this nonzero factor gives

$$\int_{[U(J)]} K_\nu(g,h)\,\mathrm{d}h = m_0 b_\nu.$$

Apply the same theorem to $V = 0$, $U = J$, with this very y as small datum, the same X, z, and the same ordered bases on the two copies of J. Its large real datum is a scalar x on the zero Hom space independent of ν, S. This time use the standard pair with $s = I_{2k}$: the large value is one on the unique matrix and the small value is exactly the determinant function used in K_ν. The large kernel has trivial source character, while on the small kernel both characters equal γ. Thus

$$m_0 x = c_\nu \int_{[U(J)]} K_\nu(h,g)\,\mathrm{d}h, \qquad |c_\nu|^2 = D_2\Pi^{\mathrm{int}}(-k)R_\nu(X,z), \quad D_2 > 0.$$

The same zero-dimensional Haar measure occurs in the two formulas. Integrate each equality in g, using the finite positive volume in Lemma 548.1. That lemma permits the two integrations of K_ν in either order. Substitution gives $x = c_\nu b_\nu$, after cancelling $m_0\mathrm{vol}([U(J)]) > 0$. Since $\Pi^{\mathrm{int}}(k)\Pi^{\mathrm{int}}(-k) = 1$, this proves

$$R_\nu(X,z)^2 = \frac{|x|^2}{D_1 D_2}.$$

Both sides are nonnegative. The positive square root is uniquely determined when $x \neq 0$, and when $x = 0$ every such R_ν is zero. This proves the assertion for all z. □

Lemma 548.3. *Let X_1, X_2 be even positive-rank spaces having adapted hyperbolic planes as in Theorem 528.2. Suppose they have the same signatures at all real places, equal rank and unimodular finite lattices. There are an invertible linear change A of their real equal-rank Schwartz data and $m > 0$, independent of ν, S, such that*

$$R_\nu(X_2,z) = m^2 R_\nu(X_1, Az).$$

Proof. Choose a rational isometry as in the proof of Lemma 546.1, and use it simultaneously on the source and target. Conjugation identifies their rational endomorphism spaces, preserves the trace pairing and carries their rational position sums bijectively to one another in transported polarizations. Pullback at infinity and passage to these coordinates give the invertible real Schwartz change A by the same proof. On the finite part choose one k_f aligning the pulled-back lattice and the marked bases with those on X_1, using it on both sides. The resulting map on Hom is conjugation by k_f, taking both paired lattices for finite translations to the new ones. The finite point names transform with one vacuum phase by Proposition 357.2, as in the target change lemma. In particular the determinant datum is preserved in the corresponding bases. The two equal-rank determinant factors at (k_f, k_f) cancel. In the target variable k_f can be removed by right translation in the integration. In the source variable the identity is carried to k_f or its inverse, according to the direction used for aligning the lattices. Both give the value at one by Theorem 528.2. Hence the two values have absolute values in the fixed ratio m of target quotient measures, exactly as in Lemma 546.1. Their denominators in Definition 547.1 agree because their ranks agree. Squaring gives the result. □

Corollary 548.4. *In Theorem 547.2 suppose the negative index $q_W = q$ is constant over the real places and $k = N/2 - q \geq 1$. For every z the ratio $|c_\nu|^2/\Pi^{\mathrm{int}}(d)$ for the constructed scalars is independent of the allowed tame twist. It is positive if $J_{\nu_0}(X,z;1) \neq 0$ for any one such twist ν_0.*

Proof. The space $X = \mathbb{H}^p \perp W$ has the same rank and signatures as $\mathbb{H}^{p+q} \perp J$ with J the positive definite space of rank $2k$. Indeed their negative indices are $p+q$ and their total ranks are $2p + N$. Since $p \geq 1$, both have an adapted plane. Lemma 548.3 and Proposition 548.2 therefore prove independence of $R_\nu(X, z)$. Multiplication by $D_* > 0$ in Theorem 547.2 gives the claim, including positivity under the stated condition. □

549. Compact stabilizers of frames with a positive direction

Lemma 549.1. *In the real Darboux coordinates of Theorem 393.1, let* $\phi(u) = 2^{m/4} e^{-\pi \|u\|^2}$, $u \in \mathbb{R}^m$. *The complex linear span of its symmetric Heisenberg translates is dense in the real-place Schwartz space with all its usual seminorms. With* $a, l \in \mathbb{R}^m$, $\xi = a + il$, *these translates have the form*

$$\rho_a(a, l)\phi(u) = e^{-\pi \|\xi\|^2/2}\, \phi(u) \exp\bigl(-2\pi u^{\mathsf{T}} \xi - \pi \xi^{\mathsf{T}} \xi / 2\bigr).$$

The second factor after the scalar Gaussian is entire in ξ *in every Schwartz seminorm. The same statements apply on the product of all infinite places, using their direct sum of real coordinates.*

Proof. Insert ϕ in the translation formula of Lemma 392.1 and expand $(u + a)^{\mathsf{T}}(u + a)$. This gives the formula, with $\|\xi\|^2 = \bar{\xi}^{\mathsf{T}} \xi$. On a bounded complex parameter set, every fixed derivative in u, ξ of the second factor is bounded by a polynomial in $\|u\|$ times $Ce^{-\pi \|u\|^2/2}$, by completing the square. Taylor expansion of the exponentials is therefore valid in each Schwartz seminorm. More explicitly, bounds on a larger parameter disk and the scalar circle coefficient formula give a geometrically summable bound on each Taylor tail after any prescribed u-derivatives and coordinate powers, as in Proposition 408.3.

First approximate any Schwartz f by compactly supported smooth functions. Multiply by $\eta(u/M)$, where η is a smooth cutoff of compact support equal to one near zero. Such a cutoff follows by integrating the ball bump of Lemma 513.1 against the indicator of a larger ball and normalizing: convolution with a probability bump equals one sufficiently far inside the ball. In the difference from f every fixed seminorm tends to zero, by the product rule and rapid decay where $\|u\| \gg M$. For compactly supported smooth f, the function f/ϕ is compactly supported and smooth. Fourier inversion of Lemma 392.1 writes f as the integral of modulations of ϕ, with coefficient the Fourier transform of f/ϕ and the sign chosen for inversion. This coefficient is rapidly decreasing by repeated integration by parts. Each seminorm of a modulation grows at most polynomially in its frequency by the product rule. Tail truncation and finite Riemann sums on compact frequency boxes now give approximation in any finite list of seminorms. Modulations are the translates with $a = 0$, proving density. □

Proposition 549.2. *At one infinite place let the source* X *have signature* (P, Q) *and let the target* Y *have negative subspace* Y_- *in a signed orthonormal decomposition. Choose* $D \subset Y_-$ *of complex dimension at least* $Q + 1$. *Embed* $U(D)$ *in the target by the identity on its orthogonal complement. On the oscillator of* $\mathrm{Hom}(X, Y)$ *let this group act with its untwisted Weil action times* $(\det h)^a$ *for an integer* $a < P$. *Its probability mean annihilates every Schwartz vector. This conclusion is valid on the whole real product, with identity action at the other places, and for a conjugate of this subgroup in the same target group.*

Proof. For any physical Heisenberg parameter $T \in \mathrm{Hom}(X, Y)$, project the Q negative source columns of T onto D. Their span has dimension at most Q. Choose a unit vector $l \in D$ perpendicular to all those projections. For $|\eta| = 1$ let h_η act by η^{-1} on $\mathbb{C}l$ and identity on its perpendicular. The varying pieces of $h_\eta T$ involve only the positive source columns and the negative target line $\mathbb{C}l$. These are negative Hom coordinates. Thus in the Darboux coordinates of Theorem 393.1 the image of the parameter is $\xi_0 + \eta \xi_1$, where ξ_1 corresponds to those pieces and the two parts are orthogonal. This follows from conjugation of the negative physical coordinates in the formula $x + il_{\mathrm{mom}} = \sqrt{2j_0}(T_+, \overline{T}_-)$. In particular its

Hermitian squared norm is constant on the circle. The other real places contribute only to the fixed part.

On the negative Hom subspace the determinant of the physical action of h_η is η^{-P}, from the P copies of the rotated line against positive source columns. The vacuum coefficient is its inverse by Theorem 393.1. Since the operator is unitary and this coefficient has modulus one, the untwisted action on the vacuum is multiplication by η^P. Including the target twist gives η^{P-a}. Intertwining symmetric translations therefore yields for the circle orbit of the coherent vector at T the expression in Lemma 549.1 at $\xi_0 + \eta\xi_1$, multiplied by η^{P-a}. Its initial scalar Gaussian is constant on the circle. The remaining entire function of η has only nonnegative powers there and $P - a \geq 1$. Integrating its uniformly convergent Taylor series as proved in that lemma gives zero, since the mean of every positive power of η is zero. This equality holds in every Schwartz seminorm.

Right invariance of Haar probability on $U(D)$ allows insertion of this circle average before taking the group mean. Indeed for fixed η, change h to hh_η in the compact integral, then integrate in η. Thus the group mean kills every coherent vector although the chosen circle may depend on T. The compact mean is continuous in Schwartz seminorms by the finite coordinate cover argument of Theorem 542.3. Lemma 549.1 proves the assertion on all tests. Conjugation of the group conjugates the averaging operator by its invertible real pair operator, giving the last assertion. □

Corollary 549.3. *At a real place suppose Y has $b + Q$ negative directions and $x : \mathbb{C}^b \to Y$ has nonsingular Gram matrix with a positive eigenvalue. For X of signature (P, Q) and an integral target weight $a < P$, the stabilizer of x in $U(Y)$ contains a compact subgroup whose mean on $\mathrm{Hom}(X, Y)$ with this twist annihilates every real Schwartz test.*

Proof. Choose a vector in the span of x of positive norm and normalize it to norm one. A unitary transform takes it to a coordinate vector in the positive space: split off the positive line in each case and identify the complements in signed orthonormal bases as in the archimedean part of Theorem 90.2. In the new coordinates the negative projection of the column span has dimension at most $b - 1$, since that projection kills the chosen line. Its ordinary orthogonal complement inside Y_- has dimension at least $Q + 1$. The full unitary group on this complement fixes each transformed column. Apply Proposition 549.2 and conjugate the compact group back. □

550. The stabilizer integral at an incorrect signature

Lemma 550.1. *Let G be an adelic unitary group here, H the pointwise stabilizer of a nonsingular rational frame, and $\Delta = H(F_0)$. Equip G, H with their bi-invariant Haar measures and use counting measure on Δ. There is a nonnegative Borel u on G such that $\int_H u(hy)\,\mathrm{d}h = 1$ for every y. For $A \in L^1(\Delta\backslash G)$ one has*

$$\int_{\Delta\backslash G} A(g)\,\mathrm{d}g = \int_G u(y) \int_{\Delta\backslash H} A(hy)\,\mathrm{d}h\;\mathrm{d}y.$$

The inner integral is absolutely convergent for $u(y)\,\mathrm{d}y$-almost every y. If a compact subgroup $C \subset H$ has $\int_C A(hcy)\,\mathrm{d}c = 0$ for all h, y, then the integral on the left is zero.

Proof. The nonsingular frame splits off its span. Thus H is the unitary group of the perpendicular by Theorem 510.3 and it is closed in G, as fixing the frame is a closed matrix condition. Bi-invariance of the unitary measures follows as in Lemma 543.1. Take increasing compact $E_j \subset G$, $j \geq 1$, whose interiors cover G. Such sets are obtained by bounding the real coordinates and their inverses and bounding finite denominators of matrices and their inverses by $j!$. The local lattice conditions give compact products at finite places with the integral compact prescribed almost everywhere. Increasing j places each point in an interior. Put $v_j = \mathrm{vol}_H(H \cap E_j E_j^{-1}) < \infty$. For any y the set of $h \in H$ with $hy \in E_j$ has measure at

most v_j: if it contains h_0, right translation by h_0^{-1} puts it inside $H \cap E_j E_j^{-1}$. The function

$$b(y) = \sum_{j \ge 1} 2^{-j} \frac{\mathbf{1}_{E_j}(y)}{1 + v_j}$$

therefore satisfies $0 < b_H(y) := \int_H b(hy)\,\mathrm{d}h \le 1$. For positivity choose j with y interior to E_j; the corresponding set of h's contains an open neighborhood of one. By right invariance b_H is constant on Hy. Then $u = b/b_H$ has the required integral.

Take fundamental domains D_H, D_G for Δ acting on H, G as in Lemma 21.4. For a nonnegative Δ-invariant A, substitute $g = hy$ on the right. The result is the integral on G of $A(g) \int_{D_H} u(h^{-1}g)\,\mathrm{d}h$. Partition G into δD_G. The sum of the inner weights at δg is one: the sets $D_H^{-1}\delta$ partition H, and inversion and right translation preserve its measure by bi-invariance. This gives the formula for nonnegative A by Lemma 123.1. Apply it to $|A|$ and then to the real and imaginary parts to get the assertion for complex functions. For almost every fiber with finite absolute integral, right translation by $c \in C$ preserves the quotient integral on $\Delta\backslash H$. Integrating over C and using the same absolute integral bound and compact probability gives zero on that fiber if the indicated mean vanishes. The formula now proves the last assertion. □

Proposition 550.2. *Let $b > 0$, let the ambient source be $Z = \mathbb{H}^b \perp X$ with its adapted b-dimensional frame space L, and let Y be a target of rank $b + \dim X$ in the large normalized period. Use the standard upper rows projecting onto L', as in Definition 517.1. At a chosen real place let X have signature (P, Q), let Y have $Q + b$ negative directions, and suppose the residual target determinant character at this place has weight $a < P$. Here the residual character is the upper target character divided by v_T^b. At every g preserving L, the coefficient of this period at a nonsingular rational hermitian B with a positive direction at this real place is zero for every real Schwartz test.*

Proof. On the support of the standard test the operator rows restricted to $L^\perp$ form an invertible residue matrix. This follows by expanding the stacked determinant in the fixed projection rows. At a parabolic g the reduced subspace in Definition 511.1 equals the reduction of $L^\perp$ in its adapted lattice. Hence Theorem 511.2 and Corollary 511.3 apply. If no rational frame has Gram matrix B, the coefficient is zero by Theorem 510.3. Otherwise fix one x with that Gram matrix and put $G = U(Y)(\mathbb{A}_{F_0})$. The coefficient is the integral of $A(y) = t_x(g, y; f)$ on $\Delta\backslash G$, with H the stabilizer of x and $\Delta = H(F_0)$. It is absolutely convergent.

For a fixed y, move the test first by $\Omega(g, y)$ and then restrict it to x in the frame positions. On this residual vector, left application of $(1, h)$ for $h \in H$ acts by the residual pair operator, by Lemma 520.2 with $q = 1$. Indeed $h^{-1}x = x$ and the frame scalar for $q = 1$ is one. This gives $A(hy)$ by the rational residual theta sum. At the chosen real place the stabilizer of x contains the compact subgroup C of Corollary 549.3; extend it by identity elsewhere. For every h, y its average in $A(hcy)$ applies the zero operator to the real residual vector before applying $(1, h)$. The rational residual theta sum can be taken after that compact mean. At fixed h, y it is continuous in the Schwartz vector, since only a bounded-denominator lattice occurs in finite support and all sufficiently high real decay norms control its absolute sum, as in Proposition 396.2. A fixed finite test remains fixed during this real averaging. Lemma 550.1 now gives zero. □

Corollary 550.3. *Write J for a positive definite space of rank $b = 2k > 0$, $X = \mathbb{H}^p \perp J$ with $p \ge 1$, $Z = \mathbb{H}^b \perp X$, and $Y = J^- \perp X$. Use the upper normalized pair on (Z, Y) of Definition 520.1 and positive weights μ_τ. For each represented nonsingular B which fails to be negative definite at some real place, the large coefficient in Proposition 550.2 is zero.*

Proof. At the exceptional place a nonsingular hermitian matrix which is not negative definite has a positive direction, by diagonalization in Theorem 90.2. Here $P = p + 2k$, $Q = p$, and Y

has $p+b$ negative signs. The residual target character is $\nu^{-1}\upsilon_T^{\dim X/2}$ by Lemma 520.2. Its real weight is $a = p+k-\mu_\tau$ since υ_T has weight one by Theorem 374.2. Thus $P-a = k+\mu_\tau > 0$, as required in the proposition. □

551. Consequences of a zero equal-rank functional

Lemma 551.1. *Use the spaces J, X, Z, Y and characters of Corollary 550.3. Fix an allowed tame twist ν and suppose that $J_\nu(X, z; 1) = 0$ for every real Schwartz vector z. Use the finite standard test with upper rows the projection onto L' in Definition 517.1. For every real Schwartz test on (Z, Y), every parabolic argument $g = u\, m(q, g_0)$ preserving L, and every rational B for which $H_f(B, q)$ of Definition 532.1 is unimodular at all finite places, the large B-coefficient is zero.*

Proof. A unimodular transported matrix is nonsingular. If B has a positive direction at any real place apply Corollary 550.3, or use Theorem 510.3 if there is no rational frame. It remains to consider B negative definite at every real place. In a rational basis of L identify it with the underlying vector space of J^-. Write B_* for the pullback of the form of J^- by that identification. The finite lattice isometries and rational local-global argument in the proof of Theorem 532.3 depend only on the two forms: they give $c \in \mathrm{GL}(L)(E)$ with $B = c^* B_* c$ and the normalized finite frame for cq in $U(J^-)(\mathbb{A}_f)\mathrm{GL}_b(\widehat{O}_E)$. By Lemma 532.2, the coefficient in question is the B_*-coefficient at $m(c, 1)g$. This is still parabolic, and we use q, g_0, u for its Levi coordinates and radical in what follows.

The frame $x_* = \binom{I_b}{0}$ is now rational in the identified coordinates. Its stabilizer is $H = U(X)(\mathbb{A}_{F_0})$ acting in the second block of $Y = J^- \perp X$. By the full residue rank on $L^\perp$ established in Proposition 550.2, Corollary 511.3 unfolds the coefficient to $t_{x_*}(g, y; f)$ on the rational stabilizer quotient of $G = U(Y)(\mathbb{A}_{F_0})$. Apply Lemma 550.1 to integrate first on $[U(X)]$ at fixed y.

We describe the integrand on that fiber. If the finite part of y is outside the block coset in Proposition 518.3, no point on the fiber contributes, since left multiplication by H_f preserves that coset. In the coset, remove the right integral factor, which fixes the standard finite test by Lemma 518.2. Write what remains as $\mathrm{diag}(y_b, l_f)$ at finite places. At x_* the finite residual test on $\mathrm{Hom}(X, J^-) \perp \mathrm{Hom}(X, X)$ then has, up to a scalar, the form

$$f_b \otimes \Omega_{\gamma_X, \gamma_X}(g_{0,f}, l_f) F_{\mathrm{eq}}, \qquad \gamma_X = \nu^{-1}\upsilon_T^{\dim X/2}.$$

Indeed by Lemma 520.2 the frame is evaluated at $\binom{y_b^{-1} Y(q)_f}{0}$, whose upper block is an integral isomorphism by Proposition 518.3. Theorem 517.3 factors this finite slice with equal-rank factor F_{eq}. Proposition 520.3 puts the equal-rank characters on this factor. The remaining radical acts only on the first factor at the frame x_*, by Lemma 522.2. All other finite scalars and the first finite vector f_b are independent of the integration variable $h \in [U(X)]$.

At infinity move the original arbitrary real test by its given arguments and restrict to the indicated rational frame. This gives a possibly coupled Schwartz vector on the two residual summands: evaluation at fixed frame coordinates and the real Weil action preserve Schwartz class by Lemma 514.2. For $h \in H$ the same Levi formula with frame multiplier one and $hx_* = x_*$ now acts only on the equal-rank summand, with target character γ_X. Evaluate the rational theta sum on the first summand, putting its finite component f_b there. This gives a real Schwartz vector z_y on the second summand. In rational product positions the first sum is over a fixed denominator lattice with bounded finite coefficients. For each residual seminorm its absolute sum is bounded by multiplying the original Schwartz seminorm by the sum of $(1 + \|n\|)^{-M}$ on this lattice, finite when M exceeds its dimension plus one. Applying this also to derivatives proves the Schwartz assertion and the continuity of the partial theta operator. It commutes with the second group action: fixed-coordinate substitutions and phases in the second variables commute term by term, and partial Fourier integrals do so by the same Schwartz bounds, as in Lemma 387.1.

Consequently, after a scalar independent of h, this fiber integrates the equal-rank kernel for z_y, F_{eq}, with source argument $g_{0,f}$ extended by one at infinity and target variable hl_f. Translating the latter variable on the right preserves its quotient measure. The resulting integral is $J_\nu(X, z_y; g_{0,f})$, absolutely convergent by Theorem 507.3. It vanishes by Theorem 528.2 and the hypothesis on every z. This applies on all contributing fibers with arbitrary real y_∞, so Lemma 550.1 gives zero. □

Proposition 551.2. *Under the hypotheses of Lemma 551.1, every real Schwartz test on (Z, Y) gives a zero large period on every integral standard datum and at every group argument, for the chosen twist. The same holds with Y replaced by $\mathbb{H}^{p+b}$, the $p + b$ planes above the zero target.*

Proof. Form the difference of Proposition 529.3 with scalar zero, any fixed real small vector, and the given real large vector. Thus its value on a column test is just the original large period on a matching row test. The statement and proof of that proposition apply to arbitrary real data. We verify the primitive-coefficient criterion for this difference. The coefficient of the large standard test at a parabolic point vanishes whenever its transported $H_f(B, q)$ has a nonintegral entry: the finite frame integrality part of Corollary 532.4 uses only the finite test. Indeed a contributing frame is integral in the transported lattice by Lemma 517.2 and Proposition 518.3's annihilator test for frame support. Its Gram matrix would then be integral. This contradiction in the rational frame sum of Theorem 510.3 gives the vanishing without a condition on the real test.

Fix a marked primitive B and an admissible argument in Definition 536.1; in the final criterion it suffices to take arguments supported at unmarked finite places by Lemma 538.2. The index is nonsingular and its transported Gram matrix is unimodular at the marked pairs for a point trivial there, by Lemma 538.1. If the coefficient were nonzero, the proof of Lemma 537.1 would produce a point p_* with nonzero plus mean after possibly interchanging the two line labels. Its hypotheses on the period are the marked rank bound and radical invariance of its bordered coefficient, valid for the period difference by Theorems 533.3, 534.3 and 535.3, and the integrality of the projected lattice at a contributing parabolic point. The latter follows by the finite-support argument just given, exactly as in Lemma 536.3. Theorem 537.2 consequently makes the projected lattice self-dual at every unmarked place. As in the proof of Corollary 537.3, the nonzero plus mean gives a nonzero coefficient at a shift of p_* preserving L. This shift has the same Levi action on L, hence a transported Gram matrix unimodular at all unmarked places and unchanged at the marked pairs. At this parabolic point Lemma 551.1 gives zero, a contradiction. The common determinant normalization has no effect on vanishing since the center has determinant one. The primitive criterion just cited therefore makes the difference zero at every argument and on every matched residue pair, in particular every standard pair.

The proposed replacement target has the rank and real signature of $J^- \perp X$, by Proposition 546.2 with original $W = U = J$, $V = 0$. Both have unimodular lattices. The real change of tests in Lemma 546.1 is onto and its finite scalar is nonzero. This proves the same vanishing for every real test on the replacement. □

552. Hermite polynomials on the two scalar sections

Lemma 552.1. *Work at a single real place, with the real Darboux complex coordinate $\xi = x + il$ of Theorem 393.1. Denote its multiplication by i by I, its Hermitian inner product by $\langle\ ,\ \rangle_D$, conjugate linear in the first variable. On a signed hermitian Hom space, let J_{ph} mean multiplication of every physical matrix entry by i. In Darboux coordinates J_{ph} acts by i on positive Hom entries and by $-i$ on negative Hom entries. Suppose the space is an orthogonal sum of external and residual matrix blocks for both the physical hermitian form and the positive Darboux metric. On the external space take a real position subspace P which is physically complex, with complementary momentum $IP = P^\perp$ for the real Darboux*

metric. Use a real orthonormal basis of P for its position coordinates, identifying it with a standard real coordinate space. Let $z = z_E + z_R$ be a coherent parameter in Darboux coordinates and assume z_E uses only positive Hom entries. Write B_z for the coherent vector of Proposition 408.3. Evaluation in the external position $v \in P$, retaining the full residual vector, gives

$$B_z[v] = c_P e^{-\pi\|v\|_D^2} e^{2\sqrt{\pi}\,\langle v, z_E\rangle_D} B_{z_R}, \qquad c_P \neq 0.$$

Here v is measured in Darboux units and the fixed scalar includes normalization of the position model. With a fixed linear position rescaling, use its Darboux image for v.

Proof. The coordinates satisfy $J_{\mathrm{ph}} = i$ or $-i$ on the indicated entries because negative physical coordinates are conjugated when forming ξ. Choose the unitary change of external Darboux coordinates U carrying the chosen real orthonormal basis of P to the standard real basis and its I-multiples to the standard imaginary basis. These vectors have the required complex orthonormality: the imaginary pairings on P are zero since IP is orthogonal to P for the real metric. Extend U by identity on the residual. The change of real polarization carries B_z to a fixed unit multiple of $B_{Uz_E} \otimes B_{z_R}$ in these coordinates for normalized Lebesgue measures. Indeed the transformed vacuum is a scalar multiple of the standard Gaussian: its translation coefficient is the same radial Gaussian by Theorem 393.1. Equivalently use the Gaussian action of Proposition 392.2; a complex unitary Darboux matrix fixes iI in that formula. Intertwining the symmetric translations and using Lemma 549.1 then transports the coherent parameter by the same complex linear U. The relation between a translate and B_z is $z = -\sqrt{\pi}\,\xi$, times its scalar $e^{-\pi\|\xi\|^2/2}$, by the explicit formulas in that lemma and Proposition 408.3. A change of position volume by a constant only rescales the unitary coordinate identification by a fixed nonzero factor.

Let C_P be conjugation fixing the real subspace P: it is identity on P and minus identity on IP. Physical multiplication J_{ph} preserves P and commutes with I, so preserves IP and commutes with C_P as a real operator. Since C_P is antilinear in Darboux coordinates, $J_{\mathrm{ph}} C_P z_E = -iC_P z_E$. Thus $C_P z_E$ is orthogonal to z_E, whose coordinates are positive Hom entries. In the U-coordinates this gives

$$(Uz_E)^{\mathsf{T}} U z_E = \langle C_P z_E, z_E\rangle_D = 0.$$

Now evaluate the formula $B_w(u) = \phi_0(u)\exp(2\sqrt{\pi}\,u^{\mathsf{T}}w - w^{\mathsf{T}}w/2)$ from Proposition 408.3. The linear term at Uv is $2\sqrt{\pi}\langle v, z_E\rangle_D$, since Uv is real. The external quadratic term vanishes by the preceding identity. This proves the formula. □

Proposition 552.2. *At a real place take $W' = Ee \oplus W \oplus Ee'$, $V' = Ea \oplus V \oplus Ea'$ as in Definition 541.1, now viewed over $\mathbb{C}$. Use the signed orthonormal vectors $(e+e')/\sqrt{2}, (e-e')/\sqrt{2}$ and $(a+a')/\sqrt{2}, (a-a')/\sqrt{2}$ for the new positive and negative directions. Keep corresponding signed bases on W, V. Call an entry of $\mathrm{Hom}(W', V')$ a same-sign entry when its source and target signs agree. In the Hermite test $H_P = P(C)\phi_0$ of Definition 408.1 let P involve only same-sign creation labels. For either $\beta = +, -$ there are $c_\beta, a_\beta \neq 0$ and $A_\beta > 0$, independent of P, t, such that*

$$\mathcal{L}_{\beta,\infty}(t) H_P = c_\beta e^{-A_\beta |t|^2} H_{P_\beta(t)} \qquad (t \in \mathbb{C}).$$

In forming the residual polynomial $P_\beta(t)$, retain every residual label. Substitute $a_\beta \bar{t}$ for each of the two entries from new positive source to new positive target and from new negative source to new negative target. Substitute zero for every other same-sign entry involving a new row or column.

Proof. Use the external blocks $A, B_1, C_1, D, H_1, I, J, K$ in Definition 541.1; the middle matrix is residual. In the Witt bases displayed there the real positive metric is the Frobenius metric up to its common Darboux scale $2j_0$, since the real orthogonal change between the two new

signed vectors and the two Witt vectors has entries $1/\sqrt{2}$, $-1/\sqrt{2}$. In physical matrices the Darboux complex structure is

$$IM = iJ_{V'}MJ_{W'},$$

where the J's in these bases interchange each of the two Witt vectors and are diagonal sign matrices on the old spaces. This follows entrywise in the signed bases from the two rules on positive and negative Hom entries in Theorem 393.1. Consequently the plus external positions A, D, I, J span a physically complex subspace, and their I-images fill the complementary blocks K, H_1, C_1, B_1, respectively. Here I in the list of blocks is the matrix entry called I in Definition 541.1. Orthogonality for the real metric follows from disjoint matrix entries. The minus section exchanges the A and K scalar blocks, with the same properties. Indeed the trace symplectic pairing between $K = t'$ and $A = u$ is the real trace of $\jmath\overline{t'}u$, up to sign from the order of the blocks. This is the kernel defining the scalar Fourier transform at t in that definition. Thus the physical coordinate on the new position block is $t' = \pm t$. More generally an equivalent fixed scaling of the measures and the dual coordinate changes these by a nonzero constant and the corresponding linear rescaling. The trace pairing is nondegenerate, so this change is complex linear in t: both kernels pair u with a constant times $\bar{t}$.

In either section the selected external position has just one nonzero physical entry, $A = t$ or $K = t'$, in the two-by-two Witt block. If this entry is $\kappa_\beta t$ with $\kappa_\beta \neq 0$, its two same-sign entries in signed bases both equal $\kappa_\beta t/2$: for the K block the two minus signs on the new negative vectors multiply to one. All other external same-sign entries are zero. Its squared Darboux norm is a positive constant times $|t|^2$. For coherent parameters using only same-sign entries the substitution in the linear exponent of Lemma 552.1 therefore has the form $a_\beta \bar{t}(z_{++} + z_{--})$, with $a_\beta \neq 0$; the common positive scale $\sqrt{2j_0}$ multiplies both entries. Apply $P(\partial_z)$ at $z = 0$ to the coherent evaluation formula. On residual coordinates the derivatives give the corresponding creation operators by the power series for B_z in Proposition 408.3. On external coordinates they replace the label by the indicated scalar. All differentiations commute with the section by the Schwartz estimates of Lemma 549.1 and the continuity of partial Fourier transformation. This proves the asserted formula for both sections. □

553. A nonzero equal-rank functional

Definition 553.1. Let J have rank $2k > 0$, with positive identity form, and let the integral weights satisfy $\mu_\tau \geq k$ at all real places. Write $m_\tau = \mu_\tau - k$. Set $W_i = \mathbb{H}^i \perp J$, $V_i = \mathbb{H}^i$, for integers $i \geq 0$, in the reference coordinates of Proposition 552.2. Order the added planes by the order in which they are adjoined. Among the positive source columns, use just the i positive columns in these planes. Use all i negative source columns. Let $D_{i,+}$ denote the determinant of the i-by-i matrix of creation labels on the positive target rows and the selected positive source columns. Similarly let $D_{i,-}$ use the negative target rows and negative source columns. Within each sign use the same ordering of added planes in source and target. Define the Schwartz vector v_i as the product over infinite places of

$$H_{D_{i,+}^{m_\tau} D_{i,-}^{m_\tau + 2k}}.$$

The notation is that of Definition 408.1 in signed Darboux coordinates. For $i = 0$ both determinants are one and $v_0 = 1$ on the zero Hom space. In positive ranks the indicated constant-coefficient polynomial in creation operators sends a Gaussian to a polynomial times that Gaussian, which is Schwartz.

Lemma 553.2. *For $i \geq 1$ the real test v_i of Definition 553.1 is fixed by the real compact target group in the large normalized pair on W_i, V_i. For both signs and all real $\sigma > 0$ the cone operation from i to $i - 1$ in Theorem 542.3 satisfies*

$$V_\beta(\sigma; v_i) = d_{i,\beta,\sigma} v_{i-1}, \qquad d_{i,\beta,\sigma} \neq 0.$$

Proof. Work first at one real place. For the target compact block $(k_+, k_-) \in U(i) \times U(i)$, the negative physical Hom entries comprise positive target rows against the i negative source columns and negative target rows against the $2k+i$ positive source columns. The untwisted action on the vacuum is therefore $(\det k_+)^{-i}(\det k_-)^{-(2k+i)}$, by Theorem 393.1; its vacuum coefficient has modulus one. The normalized target character $\nu^{-1}\upsilon_T^{\dim W_i/2}$ adds the power $k+i-\mu_\tau = i-m_\tau$ to each exponent by Theorem 374.2. A same-sign creation label in one source column transforms by k_+ on positive rows, or k_- on negative rows. Indeed its Darboux coordinate undergoes just this complex linear action, so the compact implementer sends B_z to its vacuum character times B_{kz}, by the translation identity in Lemma 552.1. Differentiating at zero gives the asserted law on the labels. Taking the determinant of labels in i ordered columns multiplies it by the determinant of the target block: expand each new row label linearly, and the alternating sum retains only distinct indices. The polynomial therefore adds m_τ on positive rows and $m_\tau + 2k$ on negative rows. The two total exponents are

$$-i + (i - m_\tau) + m_\tau = 0, \qquad -(2k+i) + (i - m_\tau) + (m_\tau + 2k) = 0.$$

This proves compact invariance at every place.

In either scalar section the last row and last column of each determinant have zeros outside their intersection and the common value $a_\beta \bar{t}$ at that intersection, by Proposition 552.2. Thus each determinant goes to $a_\beta \bar{t}$ times its preceding determinant. Their powers sum to $2m_\tau + 2k = 2\mu_\tau$. After including the Gaussian factor, the local section of v_i equals

$$C_{i,\tau,\beta}\, e^{-A_{i,\tau,\beta}|t|^2}\, \bar{t}^{\,2\mu_\tau}\, (v_{i-1})_\tau, \qquad C_{i,\tau,\beta} \neq 0, \quad A_{i,\tau,\beta} > 0.$$

The compact variable drops out by the first assertion. The real character $(t/|t|)^{2\mu_\tau}$ in the cone integral cancels the angle of this power. For $t = re^{i\theta}$, angular integration gives one with its probability measure, and the radial integral is

$$\int_0^\infty e^{-A_{i,\tau,\beta} r^2}\, r^{\,2\sigma+2\mu_\tau-1}\, \mathrm{d}r.$$

It is strictly positive, finite near zero since $\sigma + \mu_\tau > 0$, and finite at infinity by Gaussian decay. The product of these radial integrals and the nonzero fixed constants is the asserted $d_{i,\beta,\sigma}$. The product over the places is justified by Theorem 542.3. □

Theorem 553.3. *Let $X = \mathbb{H}^p \perp J$ with $p \geq 1$, $\dim J = 2k > 0$ and the positive standard form on J. For positive integer weights $\mu_\tau \geq k$ and each allowed tame twist ν, there is a real Schwartz z on (X, X) with $J_\nu(X, z; 1) \neq 0$.*

Proof. Fix the twist and suppose the functional vanishes on every real Schwartz test. Put $b = 2k$ and $h = p + b$. By Proposition 551.2, on $(W_h, V_h) = (\mathbb{H}^{p+b} \perp J, \mathbb{H}^{p+b})$ every large period is identically zero on every integral standard test, for every upper real vector. Use the test v_h of Definition 553.1. We show this forces the large period for v_0 to vanish on every standard test. Suppose this assertion of vanishing holds at floor $i \geq 1$ with v_i. For any s at floor $i-1$, apply it at $s\,\mathrm{pr}_{W_{i-1}}$ upstairs and take the source-line constant term. Its value at every $m(1, g)$ is zero. Theorem 544.2 gives the lower period there multiplied by $\Gamma_{\nu,i} \neq 0$, with real vector $V_{\beta_i}(s_{d_{i-1}}; v_i)$. Here $d_{i-1} = \dim V_{i-1} - \dim W_{i-1}/2 = i - 1 - k$, and the exponent is at least one even when $d_{i-1} = 0, -1$. Lemma 553.2 makes this vector a nonzero scalar multiple of v_{i-1}. Hence the period for v_{i-1} vanishes at every g, s. Induction reaches $i = 0$.

At the bottom the large target is zero and the source is J. Evaluate at $g = 1$ with standard $s = I_b$ on every marked branch. The stacked determinant at the unique zero-dimensional map is one, so the finite test is the unit vacuum, and $v_0 = 1$. The theta sum on the zero Hom space has a single term one, and the determinant characters at the identities are one. Its target integral is the positive one-point mass. This contradicts the forced vanishing. The functional at the fixed twist is therefore nonzero. □

554. The nonzero complementary-period comparison

Theorem 554.1. *Let W, V, U be framed nonsingular rational hermitian spaces over the arithmetic CM field, with unimodular finite lattices, and put $N = \dim W$, $r = \dim V$, $b = \dim U$. Suppose $N = r + b$, $r = b + 2e$ with e a nonnegative integer. Write q for the number of negative signs of W, assumed the same at each infinite place. Assume*

$$q = q_V(\tau) + q_U(\tau), \qquad 2q + 2 \le N \le 2q + 2\mu_\tau \quad \text{for every } \tau,$$

where μ_τ are integer weights. Fix a real Schwartz test y on $\mathrm{Hom}(U, W)$. There are a real Schwartz test x on $\mathrm{Hom}(W, V)$ and $C > 0$ with the following property. At every nonempty allowed tame squarefree marked set S and character ν of this weight as in Definitions 516.2 and 508.1, put $\gamma = \nu^{-1}\upsilon_T^{N/2}$. There is $c_\nu \neq 0$ such that for all matched pairs (Φ, Ψ) over the residue ring and every $g \in U(W)(\mathbb{A}_{F_0})$,

$$\int_{[U(V)]} K^{W,V;\gamma,1}_{x,\Phi}(g,h)\,\mathrm{d}h = c_\nu \int_{[U(U)]} K^{U,W;\gamma,\gamma}_{y,\Psi}(h,g)\,\mathrm{d}h,$$

$$|c_\nu|^2 = C \left(\prod_{w \in S} q_w \right)^{e^2} \prod_{j=1}^{e} |L^S(j, \nu)|^2.$$

Here $K^{S_1,T_1;a_0,b_0}$ takes the target determinant character a_0 and inverse source determinant character b_0^{-1} as in Definition 504.1. The matching law is Definition 499.1. The Euler product in Definition 515.1 omits both w and $\bar{w}$, and each chosen w occurs once in the conductor product. When $e = 0$ the entire explicit factor multiplying C equals one.

Proof. Since $N = 2b + 2e$, $k = N/2 - q$ is an integer. The two inequalities in the hypotheses say $1 \le k \le \mu_\tau$ for every τ. If there is no twist in the stated family the universal assertion holds with $x = 0, C = 1$. Otherwise choose one member ν_0 and take $X = \mathbb{H} \perp W$. It has the signatures of $X_1 = \mathbb{H}^{q+1} \perp J$ with J positive definite of rank $2k$, and both have an adapted plane. Theorem 553.3 supplies real z_1 on (X_1, X_1) with $R_{\nu_0}(X_1, z_1) > 0$. The real change in Lemma 548.3 is invertible, so take z on (X, X) corresponding to this z_1. The positive quotient-measure ratio gives $R_{\nu_0}(X, z) > 0$. Proposition 548.2 and that same change show that $R_\nu(X, z) = R_{\nu_0}(X, z)$ for every member of the family.

Apply Theorem 547.2 with $p = 1$, the given y, and this single z. Its real test x has the required equality on all matched pairs and all arguments. Its squared scalar is $C\Pi^{\mathrm{int}}(r - N/2)$ with $C = D_* R_{\nu_0}(X, z) > 0$. Since $r - N/2 = (r - b)/2 = e \ge 0$, the definition in Theorem 516.3 gives exactly the displayed product. All its factors are strictly positive by Corollary 515.4. Consequently $c_\nu \neq 0$ for every twist. All choices involving ν_0 have been made once for the entire family. □

Corollary 554.2. *In Theorem 554.1 suppose $W = W_1 \perp W_2$ with the product framing, all four spaces W_1, W_2, V, U of positive rank. Let f_i be rapidly decreasing functions on $[U(W_i)]$ as in Proposition 398.1, and ξ_i be unitary rationally trivial adelic determinant characters there. In the period identity put $g = g_1 \oplus g_2$, multiply by $\prod_{i=1}^{2} f_i(g_i)\xi_i(\det g_i)$, and integrate in g_1, g_2 over their rational quotients. The resulting two triple integrals converge absolutely and satisfy the same equality with exactly c_ν and the real tests just constructed. In either integral the finite point expansion and the three integrations can be taken in any order.*

Proof. On the support of Φ the named residue point is a surjection, and on that of Ψ it is an injection, by Definition 499.1. For each point Theorem 507.3 gives the absolute triple integral bound. The added characters have modulus one. Both finite sums of these bounds are finite, and Lemma 123.1 permits all stated exchanges. Now integrate the pointwise equality of Theorem 554.1. □

555. Factoring the complementary kernels on an orthogonal sum

Lemma 555.1. *Use the pair kernels $K_{z,D}^{S,T;a,b}$ of Definitions 504.1 and 507.1. In a sum of framed spaces use product vacuum vectors, product position measures, and the concatenation of the two rational position bases. Write $\delta(g) = \det_E g$. In a source sum $W = W_1 \perp W_2$ write $X = (X_1\ X_2)$ for a residue point mapping to T. In a target sum write $Y = (Y_1; Y_2)$ for a point mapping from S into W. In either formula let z_i be Schwartz vectors on the respective Hom factors, and put $g = g_1 \oplus g_2$. With rationally trivial unitary torus characters a, a_1, a_2, b, b_1, b_2 one has*

$$K_{z_1\otimes z_2,X}^{W,T;a_1a_2,b}(g,h) = \prod_{i=1}^{2} K_{z_i,X_i}^{W_i,T;a_i,b}(g_i,h),$$

$$K_{z_1\otimes z_2,Y}^{S,W;a,b_1b_2}(h,g) = \prod_{i=1}^{2} K_{z_i,Y_i}^{S,W_i;a,b_i}(h,g_i).$$

The subscript of a kernel at a residue point denotes its point mass.

Proof. In either sum the physical Hom space, its trace symplectic pairing and its finite reference lattice are orthogonal direct sums of the two separate ones. This follows from the formula for the Hom pairing in Definition 355.1 by taking the trace in block coordinates. Both groups in the finite sum defining a name in Proposition 357.2 are direct products. Their orders multiply, their dual characters multiply, and translation by (z, z') on a tensor acts by the two translations separately because the cross pairings are zero. Thus the name of a point is the tensor of its two point names.

On the underlying block actions the untwisted oscillator is the tensor of the two separate actions. Indeed, in the real splitting of Theorem 393.1 the positive and negative diagonal blocks split after reordering coordinates. The vacuum coefficients therefore multiply. In the finite splitting of Theorem 390.2 the lattice intersections are direct sums, so their indices and determinant ideals multiply. These equalities determine the operator since tensoring the two operators implements the same translation action, with the same vacuum coefficient, as in the direct-sum calculation of Theorem 403.2. In rational position coordinates the sum at zero of a tensor is the product of its two sums. Indeed the lattice of rational positions is the Cartesian product and the individual series converge absolutely by Theorem 395.2. Finally $\delta(g) = \delta(g_1)\delta(g_2)$. The two target characters in the first product multiply to a_1a_2 at h; the two source denominators in the second multiply to b_1b_2 there. This proves both identities. □

Definition 555.2. Use the positive-rank spaces W_i, V, U and the hypotheses of Corollary 554.2. Denote their ranks by n_i, r, b, so $n_1 + n_2 = r + b = N$. For the chosen twist fix torus characters a_j, b_j at the ranks used below, satisfying $a_p b_q = \gamma$ whenever $p + q = N$ among those ranks. All are unitary and trivial on rational points. To a test on (W_1, V) attach numerator and denominator (a_{n_1}, a_r), and on (W_2, V) attach (b_{n_2}, b_r). On (U, W_1) attach (b_b, b_{n_1}), and on (U, W_2) attach (a_b, a_{n_2}).

Given rapidly decreasing f_i on $[W_i]$, define $\mathcal{B}_X(z)$ as the integral on $[W_1] \times [W_2] \times [V]$ of

$$\frac{f_1(g_1)f_2(g_2)}{a_r(\delta(g_1))b_r(\delta(g_2))} K_{z,X}^{W,V;\gamma,1}(g_1 \oplus g_2, h).$$

Define $\mathcal{D}_Y(y)$ by the same formula on $[W_1] \times [W_2] \times [U]$ with kernel $K_{y,Y}^{U,W;\gamma,\gamma}(h, g_1 \oplus g_2)$. We use $\mathcal{B}_X$ for split surjections X and $\mathcal{D}_Y$ for split injections Y. The quotient notation is that of Definition 507.1. For any single twisted kernel K on a pair (S, T) put

$$(L_K f)(h) = \int_{[S]} f(g)K(g,h)\,\mathrm{d}g, \qquad (D_K F)(g) = \int_{[T]} F(h)K(g,h)\,\mathrm{d}h$$

whenever these integrals converge. These pairings are bilinear.

Proposition 555.3. *In Definition 555.2, let K_1, K_2 be the two kernels through V with the indicated characters, point names X_1, X_2, and real data z_1, z_2. Let K_1', K_2' be the kernels through U with point names Y_1, Y_2 and real data y_1, y_2. Then*

$$\mathcal{B}_X(z_1 \otimes z_2) = \int_{[V]} (L_{K_1} f_1)(h)(L_{K_2} f_2)(h)\,\mathrm{d}h,$$
$$\mathcal{D}_Y(y_1 \otimes y_2) = \int_{[U]} (D_{K_1'} f_1)(h)(D_{K_2'} f_2)(h)\,\mathrm{d}h.$$

The two integrals converge absolutely. For any fixed $y = y_1 \otimes y_2$, the x and c_ν of Theorem 554.1 satisfy

$$\sum_X \Phi(X)\mathcal{B}_X(x) = c_\nu \sum_Y \Psi(Y)\mathcal{D}_Y(y)$$

for every matched pair, with sums over the indicated full-rank points.

Proof. For the large kernels the two numerators multiply to γ at $\delta(h)$, since $n_1 + n_2 = N$; the separate denominators supply exactly the outside character in $\mathcal{B}_X$. For the small kernels the two denominators at $\delta(h)$ multiply to γ, since $a_{n_2} b_{n_1} = \gamma$. At g_1, g_2, respectively, their numerators are $b_b = \gamma/a_r$ and $a_b = \gamma/b_r$, since $r + b = N$. Lemma 555.1 therefore identifies the two triple integrands with products of the two individual kernels and $f_1 f_2$.

For each fixed h, either kernel in its endpoint variable has at most polynomial growth in the reduced height by Lemma 421.1 and Theorem 397.1. In those estimates the other group argument is fixed. Proposition 398.1 then gives existence of each endpoint integral. The joint rank assumption gives absolute integrability of each triple product by Theorem 507.3. Integrating first in the independent endpoint variables and applying Lemma 123.1 proves the two bilinear formulas and their absolute convergence. The last identity is Corollary 554.2 with outer determinant characters a_r^{-1}, b_r^{-1}, using linearity in the finitely many point names. Its real test on the large group may depend on the whole tensor y. □

556. A simultaneous scalar on the prescribed real types

Lemma 556.1. *Let $R(u)F(h) = F(hu)$ be right real translation on $L^2([V])$ with its invariant measure. Take e, d_e, P_e^R as in Lemma 492.1. For two indices j, l, let $F_i, G_i, A_i, B_i \in L^2([V])$, $i = j, l$, with $\bar{F}_i, G_i$ fixed by P_e^R. Write $B(F, G) = \int_{[V]} FG$. Suppose that for each $\epsilon > 0$ there are finite sums $S = \sum_u s_u R(u)$, $T = \sum_v t_v R(v)$ satisfying*

$$\|A_i - SF_i\|_2 + \|B_i - TG_i\|_2 < \epsilon \quad (i = j, l).$$

Then

$$B(A_j, B_j)B(F_l, G_l) = B(A_l, B_l)B(F_j, G_j).$$

If $B(F_i, G_i) = 0$ at either index, then $B(A_i, B_i) = 0$ at that index.

Proof. Right invariance of measure and Lemma 492.1 give

$$B(R(u)F_i, R(v)G_i) = \langle \bar{F}_i, R(u^{-1}v)G_i \rangle = \overline{e(u^{-1}v)} B(F_i, G_i)/d_e.$$

For each index this gives

$$B(SF_i, TG_i) = mB(F_i, G_i), \qquad m = d_e^{-1} \sum_{u,v} s_u t_v \overline{e(u^{-1}v)}.$$

Given the stated approximation, Cauchy–Schwarz bounds $|B(A_i, B_i) - B(SF_i, TG_i)|$ by $\epsilon\|B_i\|_2 + (\|A_i\|_2 + \epsilon)\epsilon$. Multiply the equality at each index by the canonical pairing at the other index, subtract, and let ϵ tend to zero. When a canonical pairing is zero, each approximating pairing there is itself zero. Passing to the limit proves the last assertion. □

Theorem 556.2. *Fix the spaces in Definition 555.2. For each allowed twist choose the endpoint characters there. At each twist take any specified families of rapidly decreasing endpoint functions f_1, f_2. An index j consists of the twist, a split surjection X, and a choice of these functions. For fixed Hermite polynomials P_1^0, P_2^0 on the two large Hom factors let F_j, G_j be the two lifts through V in Proposition 555.3 with the respective real vectors $H_{P_1^0}, H_{P_2^0}$. For general polynomials P, Q on the separate factors write these lifts as $F_j(P), G_j(Q)$, respectively.*

Assume all these lifts lie in $L^2([V])$, and that a single kernel e as in Lemma 556.1 fixes $\bar{F}_j, G_j$ by its projection, for every j. Assume that for each P, Q and every two indices the finite translation approximations of that lemma hold with $A_i = F_i(P), B_i = G_i(Q)$. Then for every Schwartz x on $\mathrm{Hom}(W, V)$ *there exists $\kappa(x) \in \mathbb{C}$ independent of all indices such that*

$$\mathcal{B}_X(x) = \kappa(x) B(F_j, G_j).$$

Here $\mathcal{B}_X$ uses the functions and characters at j. If one $B(F_j, G_j) \neq 0$, the scalar is unique and is a continuous linear functional of x in the Schwartz topology. If all are zero one may take $\kappa(x) = 0$ for every x.

Proof. Abbreviate the left side by $\mathcal{B}_j(x)$, the canonical pairing by b_j. At $x = H_P \otimes H_Q$, Proposition 555.3 identifies $\mathcal{B}_j(x)$ with $B(F_j(P), G_j(Q))$. Lemma 556.1 gives $\mathcal{B}_j(x) b_l = \mathcal{B}_l(x) b_j$ for any two indices, and gives $\mathcal{B}_j(x) = 0$ whenever $b_j = 0$ (take the two indices equal if necessary). Both identities extend by linearity to the span of such tensors.

This span is dense in the Schwartz space. Indeed a coherent vector B_z on the full space factors over the two Hom summands by the explicit exponential in Proposition 408.3. Its Taylor polynomials along the parameter line $t \mapsto tz$ are finite sums of the indicated Hermite tensors, by multiplication of the two exponential series and the definition of the creation operators in Definition 408.1. They converge at $t = 1$ in every Schwartz seminorm by the disk estimate in Lemma 549.1. That lemma also gives density of the span of coherent vectors, since they are nonzero scalar multiples of translated vacua.

For each individual j, the absolute integral estimate of Theorem 507.3 bounds $|\mathcal{B}_j(x)|$ by a constant times a fixed Schwartz seminorm of x. The constant and seminorm may depend on j, which is sufficient for continuity of each of the preceding two-index identities. Hence they hold on the entire Schwartz space. If $b_l \neq 0$, take $\kappa(x) = \mathcal{B}_l(x)/b_l$. Its continuity, linearity and uniqueness follow from this formula. Otherwise every individual functional vanishes by the second identity. □

Corollary 556.3. *Under the hypotheses of Theorem 556.2, fix the two small polynomials P_1', P_2' and let $y = H_{P_1'} \otimes H_{P_2'}$. For all twists, endpoints and matched point functions simultaneously,*

$$\sum_Y \Psi(Y) \int_{[U]} (D_{K_1'} f_1)(D_{K_2'} f_2) = \frac{\kappa}{c_\nu} \sum_X \Phi(X) \int_{[V]} (L_{K_1} f_1)(L_{K_2} f_2).$$

The kernels here use the prescribed four polynomials P_1', P_2', P_1^0, P_2^0 and the corresponding blocks of the summation variable. The integrals use the measures of Definition 555.2. The complex number κ is independent of the twist, the finite point functions and the two endpoint forms. The nonzero c_ν has the squared modulus in Theorem 554.1.

Proof. Use the common x supplied by Theorem 554.1 for this fixed y. In the last identity of Proposition 555.3, replace each $\mathcal{B}_X(x)$ by $\kappa(x) B(F_j, G_j)$ using Theorem 556.2. Replace each $\mathcal{D}_Y(y)$ by the small bilinear pairing in that proposition. Division by $c_\nu \neq 0$ gives the formula. □

557. The real tests throughout the triangular array

Definition 557.1. Use $r, k, m, t = 5r, h_2 = r^8$, the rounded weight $W = W_\tau$ and $w_l = W + 2lh_2 + h_2$ at an infinite place τ from Definitions 12.3, 56.1 and 497.1. In particular

$r = k+m-1$ is odd, k is even, $W \ge 0$. Fix $z = 0$ or 1, put $D = t - z$, and for $0 \le j \le m$ set

$$n_j = r + z + jD, \qquad d_j = \{z - (1-z)(j \bmod 2)\}/2.$$

For offset $o \ge 0$ and $j \ge 0$ with $o + j \le m$ form a list $T_{o,j}$ of n_j parameters with signs. For the $r + z$ base indices $0 \le i < r + z$ use $c_i = \max(0, i + 1 - k)$ and the following parameter and sign:

$$\lambda_i = d_j + \begin{cases} k/2 - 1 - i, & i < k-1, \\ -W - 2(c_i+1)h_2, & i \ge k-1, \end{cases} \qquad s_i = (-1)^{o+c_i+\min(\max(c_i-o,0),j)}.$$

For $1 \le l \le j,\ 0 \le v < D$ include the parameter

$$(t-1)/2 - w_{o+l} - z - v + d_j$$

with positive sign. Identify these entries by the labels $(o+l, v)$, disjoint from the base labels. Order each list by decreasing parameter. For $-T_{o,j}$ negate every parameter, retain each sign, and order again.

For $o + j < m$ consider the following two Hom pairs. Put $n = n_j,\ H = n_{j+1}$. In the first row $w = w_{o+j+1}$; in the second $w = w_{o+1}$. The last columns are the source and target determinant exponents in the real action.

source	target	α	β
$T_{o,j}$	$T_{o,j+1}$	$w - \lfloor H/2 \rfloor$	$\lfloor n/2 \rfloor - w$
$-T_{o+1,j}$	$-T_{o,j+1}$	$-\lceil H/2 \rceil - w$	$\lceil n/2 \rceil + w$

For the first row the torus character at rank p has weight $\lfloor p/2 \rfloor - w$; for $z = 1$ both ranks are even. In the second row take the quotient of the p-th power of v_T by the first character of weight $\lfloor p/2 \rfloor - w_{o+1}$. Its weight is $\lceil p/2 \rceil + w_{o+1}$ by Theorem 374.2. In each case the numerator uses rank n and the denominator rank H, as in Definition 504.1.

Proposition 557.2. *For each row in Definition 557.1, at sufficiently large scale, put $u_i = \lambda_i + \alpha + H/2$ for a source coordinate and $v_a = \lambda'_a - \beta + n/2$ for a target coordinate. Indices i, a in these formulas denote labels in their respective ordered lists. There is exactly one target entry with the old label of each source entry. They satisfy*

$$v_{\mathrm{old(i)}} = u_i, \qquad s_i t_{\mathrm{old(i)}} u_i > D/2,$$

where t_a denotes the target sign. The remaining D entries have positive sign and, in decreasing order, centered values $(D-1)/2 - v,\ 0 \le v < D$. In particular all lie strictly between $-D/2$ and $D/2$. In both lists distinct parameters are at least one apart; parameters with opposite signs are separated by more than $500(n+H)^3$, uniformly in the row, offset, level and place.

Proof. In the first row the common shift on the old labels is $d_{j+1} - d_j = \alpha + \beta + D/2$; in the second it is its negative, again $\alpha + \beta + D/2$ for the exponents there. Indeed $H - n = D$, and $n_j \bmod 2$ is $1 - (j \bmod 2)$ if $z = 0$, zero if $z = 1$. Substituting these parities gives both identities. The new block in the first row is labeled $o + j + 1$. In the second the common added labels run from $o + 2$ through $o + j + 1$, and the new block has label $o + 1$. Its internal order is reversed by negation. Thus the new parameters, at internal position v starting at zero, are

$$\begin{cases} (t-1)/2 - w - z + d_{j+1} - v, & \text{first row}, \\ (t-1)/2 + w - d_{j+1} - v, & \text{second row}. \end{cases}$$

In both cases the constant term equals $\beta - (H+1)/2 + D$ by the same parities. This proves the two assertions on centered values.

In the first row increasing j changes the parity of the base sign exactly when $c_i > o + j$. For $i \ge k - 1$ one has $\lambda_i + w = 2h_2(o + j - c_i) + h_2 + d_j$. For $i < k - 1$ this sum is at least $h_2 - r - 1$, since $W \ge 0$ and the central base parameters before the shift have magnitude at most r. For every old added block it is at least $2h_2 - t - 2$, by the spacing $w_{s+1} - w_s = 2h_2$.

Since $u_i = \lambda_i + w + (H \bmod 2)/2$, the sign of u_i is positive on the unchanged signs and negative on the flipped signs.

In the second row the two sign exponents on a base label are $o+1+c_i+\min(\max(c_i - o-1,0),j)$ and $o+c_i+\min(\max(c_i-o,0),j+1)$. They have the same parity if $c_i \geq o+1$, and opposite parity if $c_i \leq o$. Here $u_i = \lambda_i - w - (H \bmod 2)/2$, with λ_i the negated source parameter. For $i \geq k-1$, before the last rank correction this equals $2h_2(c_i-o) - h_2 - d_j$. On the central base it is negative with magnitude at least $h_2 - r - 1$. Every old added block has label at least $o+2$, so its negated parameter exceeds w_{o+1} by at least $2h_2 - t - 2$. Thus the same description by unchanged and flipped signs holds. In either row we have $|u_i| \geq h_2 - 2t - r - 4 > D/2$ at large scale.

Within each consecutive block all signs agree and successive parameters differ by one. The base entries outside the central block before shifting lie at $-W - 2(c_i+1)h_2$. The added blocks lie within $t+1$ of $-W-(2(o+l)+1)h_2$. The central block lies within r of zero and its sign is constant. Hence different blocks, treating each outer base entry as its own block, are separated by at least $h_2 - 2t - r - 4$. Negation leaves distances unchanged. The bounds $n, H \leq r + 1 + 5r^2$ follow from $j+1 \leq m \leq r$. As $h_2 = r^8$, the stated separation exceeds $500(n+H)^3$ for all sufficiently large r. Within a list all entries have the same fractional part, since k is even and t is odd. The separation and the internal differences now prove distinctness in both orders. □

Theorem 557.3. *For every row in Definition 557.1 at sufficiently large scale there is an explicit nonzero Hermite polynomial test H_P on its Hom space which is fixed by both compact coefficient averages of Theorem 496.1. Its polynomial satisfies both second-derivative vanishing conditions in that theorem, and its projection by the conjugate source kernel is nonzero. At all infinite places the tensor of these choices works for every finite twist with the prescribed weights.*

Proof. In the source order, the condition $u_i > 0$ picks out a prefix, since u_i differs from λ_i by a fixed scalar. Proposition 557.2 places the new block just after this prefix in the target order. Old signs are preserved in the prefix and reversed in the suffix. The common shift on old parameters and the formula on the new block proved there are exactly those of Theorem 496.1.

Before negation the fractional part $d_j \bmod 1$ equals $(n_j - 1)/2 \bmod 1$ by the rank parity, and negation preserves this congruence. Thus the compact weights in Definition 96.1 are integral. On a list of size s, the formula there gives $|\Lambda_i - \lambda_i| \leq 3s/2$ by bounding the half sum of all differences of basis weights and the same-sign count separately. For two consecutive positions $i < i'$ of the same sign within its sign set the weight difference is $\lambda_i - \lambda_{i'} + i' - i - 2$, by the count of preceding equal signs in Theorem 496.1. Successive parameters in the whole list differ by at least one, so this difference is nonnegative. This verifies dominance on both lists.

The source integer $b_i = \Lambda_i + \alpha + h_{-s_i}$ of Theorem 495.1 differs from u_i by at most $3n/2 + H/2$, since $0 \leq h_{-s_i} \leq H$. The lower bound $h_2 - 2t - r - 4$ in Proposition 557.2 exceeds that error. Hence b_i is positive in the prefix and negative in the suffix. Apply Theorem 496.1 using each old label and its target copy. Its construction is the product of the determinants on the initial and final segments defined by the positive and negative integer levels of b_i. It supplies both compact assertions and both differential identities.

For either list the μ in Definition 96.1 differs from its parameter by the decreasing full half sum. At an adjacent change of sign its gap is therefore at least the parameter gap. The bound in Proposition 557.2 supplies the moment condition of Theorem 96.3 with $B = 0$. A definite group is compact and has the same integrability directly. The last assertion of Theorem 495.1 now proves nonvanishing of the source projection at each place. The product projection acts on the tensor by the tensor of the projected vectors, by absolute integration

of the kernel as in Lemma 492.1. Its norm is the product of their positive norms. All lists and polynomials depend only on the infinite data. □

558. Uniqueness of the polynomial at a triangular insertion

Lemma 558.1. *Let I be a finite ordered set and J_+, J_- two disjoint finite sets. On the complex polynomial algebra with variables Z_{ai} for $a \in J_+ \sqcup J_-$, $i \in I$, write $\partial_{ai} = \partial/\partial Z_{ai}$ and put*

$$A_{ij} = \sum_{a \in J_+} Z_{ai}\partial_{aj}, \qquad B_{ij} = \sum_{b \in J_-} Z_{bi}\partial_{bj}.$$

Additional polynomial variables may be present. Suppose $A_{ij}P = B_{ji}P$ for $i < j$, and $\sum_{i \in I} \partial_{ai}\partial_{bi}P = 0$ for $a \in J_+, b \in J_-$. Then $A_{ij}P = B_{ji}P = 0$ for $i < j$ and $\partial_{ai}\partial_{bi}P = 0$ for every a, b, i in the indicated sets. If also $(A_{ii} - B_{ii})P = m_i P$, then each monomial of P has degree $|m_i|$ in column i when $P \neq 0$. For $m_i > 0$ only J_+ variables occur there; for $m_i < 0$ only J_- variables occur; for $m_i = 0$ that column is absent.

Proof. Give the monomials the orthogonal inner product with squared norm $\alpha! = \prod_\ell \alpha_\ell!$ on Z^α. Multiplication by a variable is adjoint to its partial derivative: this follows by comparing the two factorials when the two multi-indices differ by its unit multi-index, with zero on both sides otherwise. Define $u_i^{ab} = \partial_{ai}\partial_{bi}P$ and $N_{ij} = \sum_{a,b} \langle u_i^{ab}, u_j^{ab} \rangle$. Moving the multiplication operators across the inner product and commuting derivatives in distinct variables gives

$$N_{ij} = \langle B_{ji}P, A_{ij}P \rangle.$$

Indeed the individual summand on the right is the pairing of $Z_{bj}\partial_{bi}P$ with $Z_{ai}\partial_{aj}P$; the variables with first indices a and b are distinct. For $i < j$ this is, after summation, the squared norm of $A_{ij}P$ by hypothesis. Since $N_{ji} = \overline{N_{ij}}$ and each N_{ii} is a sum of squares, all entries are nonnegative real numbers. Their sum is $\sum_{a,b} \| \sum_i u_i^{ab} \|^2 = 0$. Each entry therefore vanishes. This proves both differential assertions. For fixed a, b, i, distinct monomials on which the second derivative is nonzero give distinct monomials after differentiation. Consequently no monomial contains variables from both sets in column i. The diagonal operator counts the degree in J_+ minus that in J_-, separately on each monomial. This proves the assertions about the support and degree. □

Definition 558.2. For an insertion as in Theorem 496.1 use the decreasing orders on both lists. On the variables Z_{ai} in its target rows and source columns put, for each sign $\delta \in \{+1, -1\}$,

$$C_{ij}^{\delta} = \sum_{a : t_a = \delta} Z_{ai}\partial_{aj}, \qquad R_{ab}^{\delta} = \sum_{i : s_i = \delta} Z_{ai}\partial_{bi}.$$

Let $\mathcal{H}$ be the space of complex polynomials P satisfying the following linear equations. For source indices of sign s and target indices of sign t require

$$\begin{aligned} (C_{ii}^{s} - C_{ii}^{-s})P &= b_i P, \\ C_{ij}^{s}P &= C_{ji}^{-s}P \quad (i < j,\ s_i = s_j = s), \\ R_{ab}^{t}P &= R_{ba}^{-t}P \quad (a < b,\ t_a = t_b = t). \end{aligned}$$

In the first line $s = s_i$. Require also both sets of summed Laplacian equations

$$\sum_{i : s_i = s} \partial_{ai}\partial_{bi}P = 0 \quad (t_a \neq t_b), \qquad \sum_{a : t_a = t} \partial_{ai}\partial_{aj}P = 0 \quad (s_i \neq s_j)$$

for each possible sign of the summation set. Here b_i is the integer source weight relative to the vacuum, with the sign convention in Theorem 495.1.

Proposition 558.3. *In Definition 558.2 the first three equations are the diagonal and compact raising equations on the source of weight $-\Lambda$, and the compact raising equations on the target, respectively. Source raising uses the decreasing order of $-\lambda$, target raising*

the decreasing order of λ'. No diagonal weight condition on the target is required in the definition.

Proof. In a source compact sign block s, same-sign creation labels transform by $\bar{k}_s$ and opposite-sign ones by k_s, by the compact computation in Theorem 495.1. The vacuum contributes exponent $a_s = \alpha + h_{-s}$. For a unitary matrix k, $\bar{k} = (k^{-1})^{\mathsf{T}}$. Thus the differential of its action on the conjugate labels, extended complex linearly, sends E_{ij} to $-E_{ji}$, whereas the ordinary labels give E_{ij}. Here E_{ij} is the matrix with a single one in position (i, j). This extension follows by writing each matrix as a sum of a skew-hermitian matrix and i times another such matrix. On polynomials in creation labels apply the product rule, or differentiate the substitution on each monomial. At E_{ii} the resulting operator is $a_s + C_{ii}^{-s} - C_{ii}^{s}$, giving weight $-\Lambda_i$ exactly when the first equation holds. For $i < j$ the raising matrix for the reversed parameter is E_{ji}; its action is $C_{ji}^{-s} - C_{ij}^{s}$. On target rows of sign t the ordinary labels have source sign t, and the conjugate labels source sign $-t$, by Theorem 496.1. The corresponding off-diagonal action of E_{ab}, $a < b$, is $R_{ab}^{t} - R_{ba}^{-t}$. Hermite creation is injective by the squared norm formula in Theorem 408.2, so these computations on polynomials give the equations for the tests. □

Theorem 558.4. *For the data of Theorem 496.1 the space $\mathcal{H}$ of Definition 558.2 is one-dimensional. Its generator is the determinant polynomial P_0 constructed there. More precisely, let Z_* have ones at $(\mathrm{old}(i), i)$ and zeros elsewhere. Then every $P \in \mathcal{H}$ satisfies $P = P(Z_*)P_0$. This applies in particular to both lists of Theorem 557.3.*

Proof. Apply Lemma 558.1 to each source sign s, with J_+ the target rows of sign s and J_- those of sign $-s$. It gives separate vanishing of the two column operators in the second equation, and each column has degree $|b_i|$ and uses only rows of sign $s_i \operatorname{sign}(b_i)$. All b_i are nonzero by the insertion hypotheses. Apply the same lemma to the transposed array for each target sign t, now using source signs $t, -t$ for its two sets of variables. The third equation and the second system of Laplacians give separate vanishing of the two target raising operators. This application uses only the off-diagonal part of the lemma.

For each target sign t the active entries separate into two rectangular matrices. One uses the columns $s_i = t, b_i > 0$ in increasing order of position, and all target rows of sign t in increasing order. The other uses the columns $s_i = -t, b_i < 0$ and all rows of sign t, both in reverse order. Each rectangle has at least as many rows as columns because all its columns have distinct old target copies. The four rectangles have disjoint variables. In either rectangle P is invariant under adding any multiple of an earlier row to a later row and under adding any multiple of an earlier column to a later column. For rows this is exactly the separate equation $R_{ab}^{t} P = 0$ in the first rectangle and $R_{ba}^{-t} P = 0$ in the second, with $a < b$. In these operators the active columns of the indicated source sign belong only to the chosen rectangle. For columns use $C_{ij}^{s} P = 0$ in the first and $C_{ji}^{-s} P = 0$ in the second. The polynomial after each shear is constant in its parameter: differentiating it gives the corresponding vanishing operator evaluated at the sheared matrix. These shears act independently on the four rectangles.

In a rectangle with p columns assume the first l rows and columns have nonzero determinant for $1 \le l \le p$. The allowed operations take it to a rectangular diagonal matrix. At the first pivot clear the column below by row additions, then the row to the right by column additions. All subsequent leading minors in the residual submatrix are the original next leading minors divided by that pivot, by expansion of the first row and column. Induction clears every remaining pivot, including entries below row p. The first p row labels in the order of each rectangle are the old copies of its columns in the same order: the unchanged prefix comes first for the positive multiplicities, the reversed suffix first in reverse order for the negative ones. Thus when all pivots are nonzero the reduced array on the active entries is Z_* with its columns multiplied by the individual nonzero pivots. Inactive entries do not affect P. The column degrees just proved determine P there from $P(Z_*)$.

The polynomial P_0 has the required column degrees. Its determinants are leading square minors in these rectangle orders (simultaneously reversing rows and columns leaves the determinant unchanged). The shears preserve each such minor: if the changed row or column occurs, so does the earlier one being added. Each source column and each target row in P_0 uses only one opposite-side sign, by Theorem 496.1. All individual mixed second derivatives in the Laplacians therefore vanish. Together with the degrees and shears this verifies all equations for P_0; for a column operator differentiating in the other sign block its derivative is zero. One has $P_0(Z_*) = 1$.

It follows that $P - P(Z_*)P_0$ vanishes whenever all the indicated leading minors are nonzero. For any matrix Z restrict to $Z_* + u(Z - Z_*)$, $u \in \mathbb{C}$. Each required minor is a polynomial in u nonzero at $u = 0$, so all are nonzero outside a finite set. A univariate polynomial vanishing there is identically zero, since division by each distinct root gives a linear factor. Evaluate at $u = 1$. This proves the identity at every matrix and hence the polynomial identity, by applying the same root test in each variable. □

559. Infinitesimal actions on the triangular tests

Lemma 559.1. *In the Hom oscillator of Definition 558.2 identify P with H_P. For source indices i, j of opposite signs and target indices a, b of opposite signs put*

$$D_{ij} = \sum_{c:t_c=s_i} \partial_{ci}\partial_{cj}, \qquad E^{\partial}_{ab} = \sum_{l:s_l=t_a} \partial_{al}\partial_{bl}.$$

Use * for the Fischer adjoint on polynomials from Lemma 558.1. Let S_{ij} and T_{ab} be the complexified differentials of the matrix units on the source and target, respectively. Then

$$S_{ij} = D_{ij} - D^*_{ji}, \qquad T_{ab} = (E^{\partial}_{ab})^* - E^{\partial}_{ba}.$$

On equal-sign indices the two actions are

$$S_{ij} = C^{-s}_{ij} - C^{s}_{ji} + \mathbf{1}_{i=j}(\alpha + h_{-s}), \quad s = s_i = s_j,$$
$$T_{ab} = R^{t}_{ab} - R^{-t}_{ba} + \mathbf{1}_{a=b}(\beta - n_{-t}), \quad t = t_a = t_b.$$

Here n_δ, h_δ count source and target indices of sign δ. All real derivatives used to define these actions exist in Hilbert norm at each polynomial test.

Proof. The equal-sign calculations were made in Proposition 558.3, including the target vacuum exponent in Theorem 496.1. The compact action preserves the finite polynomial space of each bounded degree.

On a pair of opposite signs the real Lie algebra contains $X = E_{ab} + E_{ba}$ and $Y = i(E_{ab} - E_{ba})$; their sum $X - iY$ equals $2E_{ab}$. Use the corresponding i, j for the source. In Darboux coordinates the differential of either underlying real action exchanges pairs of coordinates by $(w_1, w_2) \mapsto (v\bar{w}_2, v\bar{w}_1)$. For a target pair its coefficients in column l are $v = 1$ for X and $v = i\,t_a s_l$ for Y. Indeed physical matrices are multiplied on the left, and a negative Hom coordinate is conjugated in the Darboux model. For a source pair the coefficients in row c are $v = -1$ for X and $v = i\,t_c s_i$ for Y. Right multiplication by the inverse changes the physical matrix derivative to $-MX$ or $-MY$, giving these signs after the same conjugations.

In Lemma 494.1 the complex-linear block U consequently has zero derivative at the identity. In the antilinear block the two off-diagonal derivatives are v in this pair. The derivative of the vacuum coefficient is zero by Theorem 393.1, since the two physical diagonal blocks have zero derivative there. The determinant twist also has zero derivative because X, Y are traceless. Differentiation of that coherent coefficient thus gives

$$e^{z^{\mathsf{T}}y} \sum_{\text{pairs}} (v z_1 z_2 - \bar{v} y_1 y_2),$$

where the subscripts in each summand refer to that pair. The moment formula of Theorem 408.2 shows this is precisely the coefficient of $\sum(v Z_1 Z_2 - \bar{v}\partial_1\partial_2)$ between the two coherent generating vectors. For example, multiplication by $Z_1 Z_2$ differentiates the second vector in its two parameters and hence multiplies $e^{z^{\mathsf{T}} y}$ by $z_1 z_2$. Extracting Taylor coefficients proves the claimed weak derivatives on all polynomial tests.

For Hilbert norm differentiation, let $U_0(t)$ denote the one-parameter group for X or Y, let $v_0 = H_P$, and $f(t) = \langle v_0, U_0(t) v_0 \rangle$. The same coherent formula gives f twice continuously differentiable near zero: the matrix exponential is analytic, the inverses there are regular and only finitely many coherent coefficients are required. For nonzero real h, k the inner product of the two difference quotients is

$$\frac{f(k-h) - f(-h) - f(k) + f(0)}{hk} = -\int_0^1 \int_0^1 f''(kv - hu)\,\mathrm{d}u\,\mathrm{d}v.$$

It tends to the same real number $-f''(0)$ as $h, k \to 0$, since $f(-t) = \overline{f(t)}$. Expanding the squared norm of the difference gives the Cauchy property of the quotients. Their limit lies in the Hilbert closure of the polynomial tests: each translated test is Schwartz, and the Hermite density proved in Theorem 556.2 also implies Hilbert norm approximation, by integrating a sufficiently high inverse polynomial weight. Hence the weak derivatives just computed identify this limit uniquely. Finally combine X, Y with coefficients $1/2, -i/2$. In the target this selects multiplication where $s_l = t_a$ and minus differentiation on the other columns. In the source it selects differentiation where $t_c = s_i$ and minus multiplication on the other rows. This gives the formulas. □

Lemma 559.2. *Consider the restriction of the preceding oscillator to a source compact group K. On Hermite polynomials of bounded degree it is a finite-dimensional unitary action. Order the indices inside the two sign blocks in any chosen order. If $v \neq 0$ has integral diagonal weight ν and is killed by all E_{ab} with a before b in the same sign block, then ν is weakly decreasing in each chosen order and*

$$\langle v, \Omega(k) v \rangle = \|v\|^2 m_\nu(k) \qquad (k \in K)$$

with those orders for the leading minors. The compact projection $(\dim T_\nu) \int_K \overline{m_\nu(k)} \Omega(k)\,\mathrm{d}k$ *fixes v.*

Proof. On the finite polynomial space the compact action extends to a rational representation of the product of complex general linear groups by replacing conjugate matrices by inverse transposes in the action on labels and taking the same integer determinant powers for the vacuum. The substitution formula in Proposition 558.3 verifies this extension. Unitarity on K gives $d\Omega(E_{ab})^* = d\Omega(E_{ba})$, by differentiating skew-hermitian generators and extending complex linearly. The commutator of these two matrix units is $E_{aa} - E_{bb}$, also in the differential representation by differentiating the matrix group law. Therefore $\|d\Omega(E_{ba})v\|^2 = (\nu_a - \nu_b)\|v\|^2$ for a before b, proving dominance.

Upper triangular unipotents fix v: each elementary one-parameter subgroup acts by the exponential of a raising matrix in the representation, by differentiation of the group law, and elimination with unit diagonal generates the whole upper unipotent group. The conjugate transpose of each lower unipotent is upper; adjunction for the rational representation extends to these exponentials by the differential identity. Thus the functional $\langle v, \cdot \rangle$ is fixed on the left by lower unipotents. The diagonal complex torus acts on v by its integral weight, since the polynomial action on labels is diagonal in monomials. Wherever all leading pivots of k are nonzero, write it as lower unipotent times diagonal times upper unipotent. Successive leading minors are the partial products of the diagonal entries, so evaluation gives exactly the stated minor product. For any k write it as the exponential of a block skew-hermitian matrix A by the diagonalization used in Theorem 98.2. The leading minors of $\exp(tA)$ are analytic in t and nonzero at zero, so values with all pivots nonzero approach $t = 1$. Continuity gives the formula everywhere. The compact convolution computation in Theorem 100.3 applies

to any unitary action and makes the displayed compact operator an orthogonal projection. Its expectation on v is $\|v\|^2$, since the same computation gives $\int_K |m_\nu|^2 = 1/\dim T_\nu$. It therefore fixes v. □

Proposition 559.3. *In either case of Theorem 557.3 let e be the source coefficient kernel for λ at one real place, $Q = P_{\bar{e}}^{\Omega}$ its conjugate source projection as in Lemma 492.1. Write $i < j$ for source positions in decreasing λ order. Suppose a polynomial test v is compact highest in the reverse order, with source diagonal weight $\eta = -\Lambda + \mathbf{e}_i - \mathbf{e}_j$, where $\mathbf{e}_i$ denotes the i-th standard coordinate vector. Then*

$$Q\Omega(g,1)v = 0 \qquad (g \in U(S)_\infty)$$

at this place, and $QS_{ab}v = 0$ for every source matrix unit. For every polynomial v one also has

$$Q(S_{aa} + \Lambda_a)v = 0, \qquad QS_{ab}v = 0 \quad (a < b,\ s_a = s_b).$$

Proof. If $u \neq 0$ lies in the range of Q, the coefficient of its unit normalization is $\bar{\psi}_\lambda$ by Lemma 492.1. Its cyclic space is isometric to the complex conjugate of the cyclic row space in Definition 100.1: mapping the translates of the conjugate unit generator to the corresponding translates of $u/\|u\|$ preserves every Gram entry by Lemma 100.2, and extends by density to an intertwining isometry. The compact orbit of this generator has the same Gram matrices as that of the distinguished unit vector of $\overline{T_\Lambda}$, since $\psi_\lambda|_K = m_\Lambda$ by Definition 96.1 and the compact minor convolution of Theorem 100.3. In a tensor of the initial exterior wedges in Theorem 77.2, the distinguished vector has weight Λ and is killed by the ordinary upper units (a replacement gives a repeated wedge entry or zero). Thus u is compact highest of weight $-\Lambda$ in reverse order, conjugation interchanging the two orders. Pair any polynomial with u, and move the equal-sign operators across by compact adjunction as in Lemma 559.2. This proves the last two annihilations.

Suppose $v \neq 0$ has the stated weight η. Lemma 559.2 gives dominance and the compact coefficient projection fixing it. Under conjugation this projection becomes the one for $\nu = -\eta = \Lambda - \mathbf{e}_i + \mathbf{e}_j$ in the original order. Indeed conjugation of a unitary diagonal character negates the weight and conjugation of the upper raising operators yields the lower ones; equivalently apply the minor coefficient calculation of that lemma to the conjugate finite tensor representation. Its dimension is unchanged. The prefix ending at i has $\sum_{a \le i}(\nu_a - \Lambda_a) = -1$. By Theorem 100.3 this compact projection vanishes on the whole original cyclic row space. After conjugating back it vanishes on every translate of u. Orthogonality of the compact projection now gives $\langle \Omega(g,1)u, v\rangle = 0$ for all g. As u was arbitrary in the range of Q, unitarity proves the claimed equality for every translate of v. Differentiate it on real one-parameter groups and take complex linear combinations by Lemma 559.1. This proves the remaining source annihilations, including for opposite-sign matrix units. □

560. Generating the source-projected polynomial tests

Theorem 560.1. *At a real place take either insertion of Theorem 557.3, with its source projection $Q = P_{\bar{e}}^{\Omega}$ and its determinant polynomial P_0. Write $\mathcal{P}$ for the complex polynomial algebra in all the Hom coordinates. For every $P \in \mathcal{P}$ there is a finite complex linear combination D of words in the target operators T_{ab} of Lemma 559.1 such that*

$$QH_P = QH_{DP_0}.$$

Words of length zero act as the identity. The equality holds in the oscillator Hilbert space. The coefficients involve only the fixed real data and P.

Proof. Let $\mathcal{M} \subset \mathcal{P}$ be the sum of the kernel of $P \mapsto QH_P$ and the span of all the indicated words applied to P_0. The source projection commutes with every real target translation because the two factors of the twisted Hom representation commute by Theorem 396.1;

the integral defining Q is norm convergent on vectors by Lemma 492.1. Differentiating by Lemma 559.1 shows that the kernel in this definition is stable under all T_{ab}. Thus $\mathcal{M}$ is also stable under them.

Suppose there is a least $d \geq 0$ for which the polynomials V_d of total degree at most d are not all in $\mathcal{M}$. Write $V_h = 0$ for $h < 0$. Take the nonzero orthogonal complement L of $V_d \cap \mathcal{M}$ inside V_d, for the Fischer inner product in Lemma 558.1. All degrees less than d belong to $\mathcal{M}$. The equal-sign target operators preserve V_d and $V_d \cap \mathcal{M}$, and satisfy $T_{ab}^* = T_{ba}$ there by Lemma 559.1. They therefore preserve L. The T_{aa} are diagonal on monomials with integral eigenvalues. Inside L one may choose a simultaneous eigenvector, because projection onto an eigenvalue on V_d is a polynomial in the diagonal operator by interpolation on its finite list of eigenvalues. Among the weights occurring in L take one maximizing $-\sum_a a\nu_a$ in the decreasing order of the target parameters. Choose a nonzero vector $H \in L$ of that weight. For $a < b$, $t_a = t_b$, the operator T_{ab} shifts weight by $\mathbf{e}_a - \mathbf{e}_b$ if its result is nonzero, by the two label actions in Proposition 558.3. Such a shift strictly increases the chosen score. Hence $T_{ab}H = 0$ for all these pairs.

By Proposition 559.3, $(S_{ii} + \Lambda_i)V_d$ and $S_{ij}V_d$ for $i < j$ of equal source sign are contained in $\mathcal{M}$. They lie in V_d. Taking Fischer adjoints gives $S_{ii}H = -\Lambda_i H$ and $S_{ji}H = 0$ for these pairs: the vectors on the left after subtraction are themselves in V_d and orthogonal to every vector there. Thus H is source compact highest of weight $-\Lambda$ in reverse order.

For opposite target signs, write $J = E_{ab}^{\partial} H$ with the operators of Lemma 559.1. Then $J \in V_{d-2} \subset \mathcal{M}$, so $T_{ab}J \in \mathcal{M}$. Also $E_{ba}^{\partial} J \in V_{d-4} \subset \mathcal{M}$. Their sum $(E_{ab}^{\partial})^* J$ belongs to $V_d \cap \mathcal{M}$. Pairing it with H gives $\|J\|^2 = 0$. Varying the order a, b gives both target Laplacian equations in Definition 558.2. Lemma 558.1 on each source sign, using the source compact equations just proved, now implies that column i of H uses only rows of sign $s_i \operatorname{sign}(b_i)$.

For opposite source signs let $J_{ij} = D_{ij}H$. By the column restriction this is zero except possibly when i is in the unchanged prefix and j is in the reversed-sign suffix. Indeed its derivatives have target sign s_i; this requires $b_i > 0$ and $b_j < 0$. In particular $i < j$ for every possibly nonzero J_{ij}. Each such polynomial has compact source weight $-\Lambda + \mathbf{e}_i - \mathbf{e}_j$. For a compact source matrix unit S_{uv} the explicit differential operators give

$$[S_{uv}, D_{ij}] = \mathbf{1}_{v=i} D_{uj} - \mathbf{1}_{u=j} D_{iv} \qquad (s_u = s_v).$$

A term on the right whose indicator is zero is omitted. This follows from $[Z_{al}\partial_{ak}, \partial_{bp}] = -\mathbf{1}_{a=b}\mathbf{1}_{l=p}\partial_{ak}$ and the two signs of the column label actions. In particular the span of the J_{ij} is preserved by compact source raising operators since H is killed by them.

If some $J_{ij} \neq 0$, choose one whose weight η maximizes $\sum_l l\eta_l$ among the nonzero J_{ij}. A compact raising S_{uv} has $u > v$ in this source order and hence strictly increases that score on any nonzero image. The commutator formula forces its image on the chosen J_{ij} to vanish. This J_{ij} is therefore compact highest. Proposition 559.3 gives $QH_{S_{ji}J_{ij}} = 0$, so $S_{ji}J_{ij} \in \mathcal{M}$. Since $D_{ji}J_{ij} \in V_{d-4} \subset \mathcal{M}$, Lemma 559.1 yields

$$D_{ij}^* J_{ij} = D_{ji}J_{ij} - S_{ji}J_{ij} \ \in \mathcal{M} \cap V_d.$$

Its pairing with H gives $\|J_{ij}\|^2 = 0$, a contradiction. Thus all source Laplacians also vanish.

The polynomial H now satisfies all equations of Definition 558.2. Theorem 558.4 places it in $\mathbb{C}P_0 \subset \mathcal{M}$, contradicting the choice of nonzero $H \in L$. No exceptional degree exists. By the definition of $\mathcal{M}$, its equality with $\mathcal{P}$ gives the asserted finite linear combination. □

Corollary 560.2. *For the product of the real places at a fixed triangular arrow use the product source projection Q and the product P_0 of the polynomials in Theorem 560.1. Every polynomial P in the joint variables has the same projected Hermite vector as DP_0, for a finite linear combination D of words of the product target Lie algebra. The choice is independent of the finite level, residue data and finite character of the prescribed weight.*

Proof. A monomial is a product of monomials in the separate places. The product source projection acts on the tensor of their Hermite vectors by the separate projections, by

integrability of the coefficient kernels in Lemma 492.1. Replace each factor by the finite linear combination from Theorem 560.1. Distributing the finite products gives the corresponding words in operators acting on separate factors; they commute by their definitions. Summing the finitely many monomials proves the claim together with its dependence on data. □

561. The projected polynomial identity under a global lift

Definition 561.1. Let S, T be a triangular source and target of ranks n, H. Put $d = nH$ and $J = 2d + 1$. In real position coordinates at each infinite place take the $2d$ operators of multiplication by each position coordinate and differentiation in each such coordinate. A word means their ordered composition, including the identity. On the real Schwartz space put

$$p_J(z) = \sum_{(A_\tau)} \left\| \prod_\tau A_\tau z \right\|_2 ,$$

where A_τ runs over words of length at most J at that place. Give its completion the name $\mathcal{B}_J$. Use the symmetric translation gauge in Lemma 392.1; multiplication and differentiation span the complexified infinitesimal translations.

Lemma 561.2. *The natural map $\mathcal{B}_J \to L^2$ in Definition 561.1 is continuous and injective. For real source and target elements u, v put $M(u_\tau) = 1 + \|u_\tau\| + \|u_\tau^{-1}\|$ in their signed orthonormal bases, and similarly for v. Their action satisfies*

$$p_J(\Omega_\infty(u, v)z) \le C\, p_J(z) \prod_\tau M(u_\tau)^J M(v_\tau)^J.$$

For a fixed finite Schwartz datum B and adelic arguments g, h, there are constants $C_B, A > 0$, independent of z, g, h, with

$$|K_{z,B}(g, h)| \le C_B p_J(z) H(g)^A H(h)^A.$$

Here H is the height of Proposition 396.2. For any rapidly decreasing f on the rational source quotient, the functional $z \mapsto (L_{K_{z,B}} f)(h)$ thus extends continuously to $\mathcal{B}_J$ at every fixed h.

Proof. The norm contains the Hilbert norm. If a Cauchy sequence z_k in p_J tends to zero in Hilbert norm, every word Az_k has a Hilbert limit. Its pairing against a smooth compactly supported ϕ is zero because $\langle \phi, Az_k \rangle = \langle A^* \phi, z_k \rangle$ by integration by parts. This already forces that limit to vanish: each Az_k is Schwartz and can be approximated in Hilbert norm by smooth compact cutoffs as in the proof of Lemma 549.1. Hence its limit is itself in their closed span. This proves injectivity.

Heisenberg covariance in Theorem 393.1 carries each infinitesimal translation past $\Omega_\infty(u, v)$ by applying the inverse real symplectic matrix to its direction, as in Proposition 396.2. The Hom action and its inverse have operator norm at most a fixed constant times $M(u_\tau)M(v_\tau)$ at that place by left and right matrix multiplication. Expand each word and use unitarity in the Hilbert norm. Operators at distinct places commute, and there are at most J factors per place. This gives precisely the displayed estimate. It also proves that changing a fixed real polarization separately at each place only changes this norm by bounded factors, using Heisenberg covariance for the fixed intertwiner and its inverse. Strong continuity on Schwartz vectors in p_J follows by the same word expansion and strong continuity on their Hilbert translates. The latter follows first on Gaussian translates from the coherent coefficients of Lemma 494.1 and unitarity, and then by the density in Lemma 549.1.

For the theta value choose rational position and momentum spaces by restriction of scalars from $F = E^+$. Indeed the form $\mathrm{Tr}_{E/F}(\jmath H(\cdot, \cdot))$ in the order of Definition 385.1 is a nondegenerate alternating F-form. Alternation follows by conjugation, and a nonzero Hermitian functional takes all values in E. Repeatedly choose a pair with pairing one and take its orthogonal complement as in the construction following that definition. This gives

dual F-spaces whose real coordinates split over the real places. In rational coordinates this is a rational polarization of the trace alternating form. The rational partial Fourier change preserves theta evaluation with its self-dual measures by Lemma 387.1 and Theorem 395.2. Thus we may use these placewise real coordinates, with a fixed change on the finite datum and the above equivalent real norm.

For every Schwartz z in these coordinates, the mixed derivative estimate in Proposition 396.2 gives

$$|z(x)| \prod_{\tau} (1 + \|x_\tau\|)^{d+1} \le C p_J(z).$$

Indeed expand the weight bound as a finite sum using at each place 1 and $|x_{\tau,l}|^{d+1}$ for its coordinates l. For the function after multiplication by the chosen monomial integrate the derivative of its squared absolute value once in every real coordinate from x to infinity. In the product rule each of the two factors has at most d derivatives per place. Cauchy–Schwarz therefore uses words of length at most $d + (d+1) = J$.

In rational position coordinates the finite argument restricts the sum to positions in $b^{-1}\mathbb{Z}^{d[F:\mathbb{Q}]}$, with finite supremum bounded by $b^{d[F:\mathbb{Q}]/2}\|B\|_2$, by the finite translation and modulation argument in Proposition 396.2. Here b can be taken to be a fixed multiple of the clearing integer of the Hom action. In placewise coordinates the unscaled lattice is a fixed invertible real linear image of the integer lattice. Distinct points of the scaled lattice are separated by at least c/b for $c > 0$, by the norm of the inverse linear change. Packing disjoint balls of radius $c/(3b)$ in a fixed enlargement bounds the number in each unit cube by $C(1+b)^{d[F:\mathbb{Q}]}$. The suprema on these cubes of $\prod_\tau (1 + \|x_\tau\|)^{-(d+1)}$ are summable: in a single place the cubes with maximum index magnitude l number $O((1+l)^{d-1})$ by subtracting two successive cube counts, and the resulting series is $O(\sum (1+l)^{-2})$. Multiply the one-place sums. Applying the pointwise word estimate to the moved real test and using the first inequality in the statement now bounds the absolute rational sum. The clearing integer and archimedean norms of the Hom action are bounded by fixed powers of $H(g)H(h)$, by the matrix formula and the fixed bases as in Theorem 396.1. Unit determinant characters preserve the bound in absolute value.

On the rational quotient use representatives with $H(g) \ll h_S(g)$ by Theorem 397.1. Proposition 398.1 bounds $\int |f| h_S^A$. Thus the absolute integral defining the lift has a bound by a constant times $p_J(z)$, and its extension to the completion is unique. □

Theorem 561.3. *Take either triangular insertion with its torus characters and real product e of the source kernels. Let f be rapidly decreasing on the source quotient, fixed in L^2 by the real convolution P_e^R. For every joint Hermite polynomial P there is a finite complex linear combination D of words in right target Lie derivatives such that for every finite datum B, such f, and adelic target argument h,*

$$(L_{K_{H_P,B}} f)(h) = D(L_{K_{H_{P_0},B}} f)(h).$$

Here P_0 is the product determinant polynomial of Corollary 560.2. The same D works for all finite twists with the given archimedean characters.

Proof. Write $F_h(z) = (L_{K_{z,B}} f)(h)$. The integral $Qz = \int \bar{e}(u) \Omega_\infty(u,1) z \, du$ converges in $\mathcal{B}_J$ on each polynomial test. Indeed Lemma 561.2 bounds the norm by an integrable product, using Theorem 96.3 at each place with $B = J$ as its moment exponent. The sign gaps from Proposition 557.2 exceed $J + 1 + 4n^2$ at sufficiently large scale, also after the decreasing half sum is added; definite places are compact. For a continuous vector function with integrable norm the integral in the completion is the limit of finite sums: truncate to compact sets with arbitrarily small integral of the norm on their complements, partition the compact sets into measurable pieces with small oscillation, and take the norm limit by completeness. Its image in Hilbert space is the original projected vector by continuity of the inclusion, which is injective by that lemma.

For Schwartz z, representation covariance in Theorem 396.1 gives $K_{\Omega_\infty(u,1)z,B}(g,h) = K_{z,B}(gu,h)$, where u has trivial finite part. The bound in Lemma 561.2 for the absolute source integral gives absolute integrability after multiplication by $\bar{e}(u)$ and integration in u, using the same product moment estimate. Thus for a polynomial test

$$F_h(Qz) = \int_{[S]} \left(\int \bar{e}(u) f(gu^{-1}) \, \mathrm{d}u \right) K_{z,B}(g,h) \, \mathrm{d}g = F_h(z).$$

Here right invariance of quotient measure and Lemma 123.1 justify the change and interchange. By inversion symmetry of e, the inner integral is $P_e^R f$. Its pointwise definition agrees almost everywhere with the Hilbert integral: test on compactly supported L^2 functions, use Cauchy–Schwarz and $\|e\|_1 < \infty$ to interchange, and exhaust the quotient by compact sets as in Theorem 491.2. Thus it equals f almost everywhere by hypothesis.

Use D from Corollary 560.2. The two projected polynomial tests there agree also in $\mathcal{B}_J$ by injectivity. Hence $F_h(H_P) = F_h(H_{DP_0})$. For the target derivatives note first that $F_{hv}(z) = F_h(\Omega_\infty(1,v)z)$ by the same covariance. On a polynomial test the derivatives of Lemma 559.1 hold also in p_J. For each word in that norm commute it past the one-parameter target operator. The resulting word coefficients are differentiable functions of the real symplectic matrix and all the word vectors at identity are Hermite polynomials, by the creation and annihilation formulas in Definition 408.1. Their Hilbert derivatives exist by that lemma. The limit for each word is the word on the Hilbert derivative, as is seen by testing against smooth compactly supported functions and moving the word to its adjoint. Since there are only finitely many words in the norm, this proves the asserted differentiability. Apply the continuous functional F_h, and then iterate on polynomials for each word in D. This identifies the right side with the stated derivatives everywhere. The coefficients arose entirely from the real polynomial identity. □

562. The two Casimir eigenvalues at an insertion

Lemma 562.1. *On $U(S)$ of rank n write $D_X f(g)$ for the right derivative obtained from $g\exp(tX)$, extended complex linearly from the real Lie algebra. For its complex matrix units set*

$$\mathcal{C}_n = \sum_{a,b=1}^{n} D_{E_{ab}} D_{E_{ba}}, \qquad c(\lambda) = \sum_{i=1}^{n} \lambda_i^2 - n(n^2-1)/12.$$

For the coefficient of Definition 96.1, $\mathcal{C}_n \psi_\lambda = c(\lambda)\psi_\lambda$, and the same holds for $\overline{\psi_\lambda}$.

Proof. At a fixed g_0 take its block diagonal part a. The accretive formula in Theorem 97.3 expresses $\psi_\lambda(g)$ near g_0 as a compact integral of fixed factors depending on a, k, times

$$F_k(g) = 1/m_{-\mu}(k^{-1}a^{-1}gk), \qquad \mu_i = \lambda_i + \rho_i, \quad \rho_i = (n+1)/2 - i.$$

The denominators are nonzero on a single complex neighborhood of g_0, uniformly in k, as in the analytic continuation argument of Theorem 98.2. Cauchy bounds on a smaller neighborhood permit two differentiations inside the integral. The operator $\mathcal{C}_n$ is unchanged when all units are conjugated by k: the bases E_{ab}, E_{ba} are dual for the nondegenerate form $(X,Y) \mapsto \operatorname{tr}(XY)$, which conjugation preserves. Expanding in the original units contracts the two inverse change-of-basis matrices to the identity coefficients.

The sequence $-\mu$ is increasing in the full parameter order. Hence its principal minors are trailing minors. Right multiplication of their matrix by a lower triangular matrix b multiplies each trailing minor by the product of the corresponding diagonal entries of b. Multiplication of the powers in the minor formula yields

$$F_k(gkbk^{-1}) = F_k(g) \prod_i b_{ii}^{\mu_i}$$

near the identity in the lower triangular group. Right derivatives in directions $kE_{ij}k^{-1}$ for $i > j$ thus annihilate F_k everywhere locally, while the diagonal derivatives give $\mu_i F_k$. For the

pair $i < j$ the term with the lower unit on the right is zero. In the opposite order only the commutator contributes, with value $(\mu_j - \mu_i)F_k$. Indeed right differentiation in $g\exp(tX)$ differentiates the representation $f(g) \mapsto f(gu)$, so its commutator is differentiation by the matrix commutator, as follows by expanding two matrix exponentials through their mixed term. The Casimir value on F_k is therefore $\sum_i(\mu_i^2 - 2\rho_i\mu_i)$. Substitution of $\mu_i = \lambda_i + \rho_i$ gives $c(\lambda)$, since summing the squares of the arithmetic progression ρ_i gives $n(n^2-1)/12$ by induction on n. The scalar $c(\lambda)$ is real. Complex conjugation of the complexified real Lie algebra sends E_{ab} to $-s_a s_b E_{ba}$, because the real algebra is given by $X^*S + SX = 0$. The two signs cancel in each term of the Casimir and their order reversal preserves the sum. Thus conjugation of functions commutes with this operator, proving the last claim. □

Lemma 562.2. *For the source projection Q of Proposition 559.3 and every polynomial P, one has*

$$QH_{C_S P} = c(\lambda) Q H_P, \qquad C_S = \sum_{i,j} S_{ij} S_{ji}.$$

For any vector fixed by P_e^R or by $P_{\bar{e}}^R$ in a continuous unitary representation, its real orbit is smooth in Hilbert norm. Each word in right real Lie derivatives on that orbit is a Hilbert norm limit of finite linear combinations of real translates with coefficients depending only on the word and the difference step.

Proof. For a fixed such vector u, unitarity and Lemma 492.1 identify $\langle R(v)u, R(w)u\rangle$ with $\|u\|^2$ times $\psi_\lambda(v^{-1}w)$ or its conjugate. This function is real analytic by the local integral in Lemma 562.1. For clarity, a Hilbert-valued function with smooth Gram function in two variables is smooth. In local real coordinates take iterated forward differences of any order in the first vector and the same in the second vector of the inner product, dividing each difference by its step. The result is the corresponding scalar difference of the Gram function, hence an average of the mixed derivative on a product of intervals by the fundamental theorem of calculus. As all steps tend to zero these averages converge continuously in the base points. Expanding the squared norm of the difference of two vector difference quotients in the resulting inner products shows they are Cauchy, including for steps tending to zero at different rates. This defines continuous vector limits of all orders. Each limit is a derivative of the previous one: integrate the next divided difference in its last base variable over a compact interval, pass to the limit using local uniform Cauchy convergence, and use the fundamental theorem of calculus on the two endpoint values. For a product of one-parameter subgroups this gives the asserted ordered word by differentiating its successive parameters at zero. Its divided differences are finite combinations of the corresponding group translates. Complex linear combinations give the complexified derivatives.

Apply this to $R = \Omega$ on the source. For $u = Qu$, the coefficient between $\Omega(g,1)u$ and $\Omega(h,1)u$ is $\|u\|^2\overline{\psi_\lambda(g^{-1}h)}$. Applying the Casimir in h at identity and using Lemma 562.1 shows that the Casimir derivative of u equals $c(\lambda)u$ when paired with all its translates. The derivative is in their closed span by the preceding difference argument, so this is an equality of vectors. On smooth vectors the adjoint relation for infinitesimal units is $S_{ij}^* = s_i s_j S_{ji}$ as a pairing identity, by unitarity on real one-parameter groups and the real conjugation computed in the preceding lemma. Applying this identity twice moves the whole Casimir to the other vector unchanged. Pair $H_{C_S P}$ with every $u = Qu$; its derivatives are the polynomial ones by Lemma 559.1. This gives the claimed equality after orthogonal projection. □

Theorem 562.3. *Under Theorem 560.1 the nonzero vector $v = QH_{P_0}$ is compact highest of target weight Λ' and satisfies*

$$\sum_{a,b} d\Omega_T(E_{ab}) d\Omega_T(E_{ba}) v = c(\lambda')v.$$

Furthermore the same target Casimir scalar acts on the projected image of every polynomial H_P.

Proof. On the polynomial algebra put $A_{ai} = Z_{ai}, B_{ai} = \partial_{ai}$ when $t_a = s_i$, and $A_{ai} = -\partial_{ai}, B_{ai} = Z_{ai}$ otherwise. Thus $[B_{ai}, A_{bj}] = \mathbf{1}_{a=b}\mathbf{1}_{i=j}$; the A's commute among themselves, as do the B's. Put $Z_0 = \sum_{a,i} A_{ai}B_{ai}$. All formulas of Lemma 559.1 become

$$T_{ab} = \sum_i A_{ai}B_{bi} + \beta\mathbf{1}_{a=b}, \qquad S_{ij} = -\sum_a A_{aj}B_{ai} + \alpha\mathbf{1}_{i=j}.$$

In particular the sum of the source diagonals is $-Z_0 + n\alpha$. Writing $C_T = \sum_{a,b} T_{ab}T_{ba}$ gives the polynomial operator equality

$$C_T - C_S = 2\Delta Z_0 + H\beta^2 - n\alpha^2, \qquad \Delta = \alpha + \beta + (H-n)/2.$$

In fact, moving the two creation symbols before the annihilation symbols in the quadratic products gives the same quartic sum on both sides, after swapping the two source indices in one sum. The commutator contributes HZ_0 on the target and nZ_0 on the source; the cross terms in the two scalar twists contribute $2\beta Z_0$ and $-2\alpha Z_0$.

On P_0 the source weights are $-\Lambda_i$, and the sum of the compact corrections to λ_i is zero by Definition 96.1. Thus Z_0 acts there by $n\alpha + \sum_i \lambda_i$. The n old target parameters are $\lambda_i + \Delta$; the $D = H - n$ new ones are $\beta - n/2 + (D-1)/2 - l$ for $0 \le l < D$, by Proposition 557.2. The last centered progression has sum zero and sum of squares $D(D^2-1)/12$. Substitution in Lemma 562.1 therefore gives

$$\begin{aligned} c(\lambda') - c(\lambda) &= 2\Delta\sum_i \lambda_i + n\Delta^2 + D(\beta - n/2)^2 - nHD/4 \\ &= 2\Delta(n\alpha + \sum_i \lambda_i) + H\beta^2 - n\alpha^2. \end{aligned}$$

The second equality follows by inserting $\Delta - \alpha = \beta + D/2$ and $H = n + D$. The source Casimir after projection is $c(\lambda)$ by Lemma 562.2. The target derivatives commute with Q on these tests, by the same differentiation as in Theorem 560.1. The displayed differences prove the eigenvalue on v. Its nonvanishing is Theorem 557.3. Its target compact weight is Λ' by the weight identities in Theorem 496.1, and its target raising operators annihilate it by the equations for P_0 in Theorem 558.4.

Finally C_T commutes with all target infinitesimal units. Indeed the invariance of the contraction of trace-dual bases in Lemma 562.1, differentiated under conjugation, gives zero commutator with any matrix unit. Every projected polynomial is a finite sum of the derivatives of v by Theorem 560.1. It therefore has the same Casimir eigenvalue. □

563. The compact Casimir on successive derivatives

Definition 563.1. At one real place write $G = U(S)$, $S = \mathrm{diag}(s_i)$, $s_i \in \{1, -1\}$, of rank $n \ge 1$, and $K = G \cap U(n)$. Use the decreasing parameter order and the weights λ, ρ, Λ of Definition 96.1 with $t = \lambda$. Let $2\rho_c$ at i count the indices of the same sign after i minus those before i, and put $\rho_n = \rho - \rho_c$. Thus $\Lambda = \lambda + \rho_n - \rho_c$. Write $\mathbf{e}_i$ for the i-th coordinate vector and $|\xi|^2 = \sum_i \xi_i^2$ for real weights. Put

$$\kappa(\mu) = |\mu|^2 + 2\langle\rho_c, \mu\rangle, \qquad C_K = \sum_{s_i = s_j} E_{ij}E_{ji}, \qquad C_G = \sum_{i,j} E_{ij}E_{ji}.$$

Products of matrix units in these two operators mean products of the complexified infinitesimal actions on smooth vectors of a strongly continuous unitary representation π of G. Smoothness means that the orbit map is smooth in Hilbert norm. Call a smooth nonzero w compact highest of weight μ if $E_{ii}w = \mu_i w$ for all i and $E_{ij}w = 0$ for all $i < j$ of equal sign.

Suppose the span M_0 of the compact orbit of w is finite dimensional. Put $M_{j+1} = M_j + \sum_{a,b} E_{ab}M_j$, and $L_j = M_j \cap M_{j-1}^{\perp}$ for $j \ge 1$. These spaces are taken in the Hilbert space of π.

Lemma 563.2. *Suppose w of Definition 563.1 is compact highest of weight Λ, $C_G w = c(\lambda)w$ with $c(\lambda)$ as in Lemma 562.1, and $|\lambda_a - \lambda_b| > 500n^3$ whenever $s_a \neq s_b$. Every M_j is finite dimensional, consists of smooth vectors and is K-invariant. On these spaces $C_G = c(\lambda)$. On M_0 one has $C_K = \kappa(\Lambda)$. Every eigenvalue of C_K on L_1 is at least $\kappa(\Lambda) + 4n + 1$.*

Proof. Differentiating the group law gives $[E_{ab}, E_{cd}] = \mathbf{1}_{b=c}E_{ad} - \mathbf{1}_{a=d}E_{cb}$ as in Lemma 562.1. The same calculation gives $\pi(k)d\pi(X)\pi(k)^{-1} = d\pi(kXk^{-1})$. Derivatives of a smooth orbit have smooth orbits, by differentiating its translate in exponential coordinates. Induction on j now proves the first three claims about M_j. Conjugation by K preserves C_K, by the trace-dual basis calculation of Lemma 562.1 on its two diagonal blocks. The same calculation on the full matrix algebra shows that C_G commutes with the group and its derivatives. This proves its asserted scalar action. As pairing identities on smooth vectors, $E_{ab}^* = s_a s_b E_{ba}$ by Lemma 562.2. In particular C_K is self-adjoint and nonnegative on a finite-dimensional compact invariant subspace. On a compact highest vector of weight μ, commute the raising units to the right. Each equal-sign pair $a < b$ contributes $\mu_a - \mu_b$, so C_K there has value $\kappa(\mu)$. Commutation with K proves the formula throughout M_0.

Take an eigenspace for C_K in L_1. It is K-invariant, because K acts unitarily and preserves M_0, M_1, C_K. The commuting self-adjoint diagonal units are simultaneously diagonalizable there by successive orthogonal eigenspace decompositions. Choose a nonzero weight vector y whose weight μ maximizes $-\sum_i i\mu_i$ among the weights in this eigenspace. Each compact raising unit strictly increases this quantity if its image is nonzero, by the commutator. Thus y is compact highest.

We claim that $\langle y, E_{ab}w\rangle \neq 0$ for some opposite-sign pair a, b. Indeed M_1 is generated under compact derivatives by w and the $E_{ab}w$. To see this, the compact derivative span of each K-finite vector equals its compact orbit span. It contains the exponentials of compact infinitesimal matrices applied to that vector by the convergent series in finite dimension. Every block unitary matrix is one such exponential by unitary diagonalization, as in Theorem 98.2. The reverse inclusion follows by differentiating in the finite-dimensional orbit span. Commuting a single E_{ab} past compact derivatives using the matrix commutator proves the claim about generators. In a word of compact derivatives move the lower units to the left, diagonals next, upper units to the right: each interchange gives the swapped word and a commutator of smaller length, so induction on length and the number of inversions terminates. The space spanned by $w, E_{ab}w$ is stable under compact raising units and diagonals by the same commutator and the highest equations on w. A nonempty left string of lower units pairs to zero with y by adjunction. Since $y \neq 0$ pairs nontrivially with M_1 and is orthogonal to M_0, this proves the claim. Diagonal self-adjointness then gives $\mu = \Lambda + \delta$ with $\delta = \mathbf{e}_a - \mathbf{e}_b$ and $s_a \neq s_b$.

For any compact highest y in these spaces the opposite-sign part of C_G gives, in each pair $i < j$, the expectation $-2\|E_{ji}y\|^2 - (\mu_i - \mu_j)\|y\|^2$: replace the two ordered products by $2E_{ij}E_{ji} - (E_{ii} - E_{jj})$. Consequently

$$c(\lambda) \leq |\mu|^2 + 2\langle \rho_c - \rho_n, \mu\rangle = |\lambda + \delta|^2 - |\rho_n - \rho_c|^2.$$

Since $c(\lambda) = |\lambda|^2 - |\rho|^2$, this gives $2(\lambda_a - \lambda_b) \geq -2 - |\rho|^2$. The separation hypothesis and $|\rho|^2 = n(n^2-1)/12$ force $\lambda_a - \lambda_b > 500n^3$. Finally

$$\kappa(\Lambda + \delta) - \kappa(\Lambda) = 2(\lambda_a - \lambda_b) + 2(\rho_{n,a} - \rho_{n,b}) + 2 \;\geq\; 4n + 1,$$

because each $|\rho_{n,i}| \leq (n-1)/2$. This applies to every nonzero eigenspace considered, including all possibilities when $L_1 \neq 0$. □

Theorem 563.3. *Under Lemma 563.2, every eigenvalue of C_K on L_j, $j \geq 1$, is at least $\kappa(\Lambda) + j$. In particular a smooth vector a with $C_K a = \kappa(\Lambda)a$ orthogonal to M_0 is orthogonal to every derivative word applied to w.*

Proof. Let $\mathfrak{a} = M_n(\mathbb{C})$ with the Hilbert–Schmidt inner product, on which K acts by conjugation, and let P be orthogonal projection to M_0. Put $A = \mathfrak{a}^*$ with the dual unitary

action. For a smooth vector z form the multilinear function

$$J_j z(X_1,\dots,X_j) = P\, d\pi(X_1)\cdots d\pi(X_j)z \quad \in M_0.$$

Regard it as a vector of $M_0 \otimes A^{\otimes j}$ with the tensor Hilbert norm. It intertwines the K-actions by the conjugation formula of Lemma 563.2 and $PK = KP$, meaning that P commutes with the representation of each $k \in K$. If $z \perp M_{j-1}$, the analogous expressions of all orders less than j vanish by adjunction: the reversed adjoint words send every vector of M_0 into M_{j-1}. An adjacent interchange of arguments of $J_j z$ changes it only by an expression of order $j-1$, by the commutator. Hence $J_j z$ is symmetric. Moreover J_j is injective on L_j, since vanishing there makes z orthogonal to all words of length at most j applied to M_0, which span M_j.

Let $B = J_1(L_1) \subset M_0 \otimes A$. For $z \in L_j$, fix the other $j-1$ arguments of $J_j z$, leaving the first slot free. The resulting vector of $M_0 \otimes A$ belongs to B. In fact $y = d\pi(X_2)\cdots d\pi(X_j)z$ is orthogonal to M_0 by the lower-order vanishing. In $J_1 y$ we may replace y by its orthogonal projection to M_1, since the adjoint of each $d\pi(X_1)$ sends M_0 into M_1. That projection belongs to L_1. Symmetry now gives the same conclusion in every slot. In particular, after expanding the other slots in any orthonormal tensor basis, every remaining $M_0 \otimes A$ coefficient belongs to B.

Number the M_0 slot by 0 and the other slots by $1,\dots,j$. For any set I of slots let C_I be the compact Casimir on their diagonal action, extended by the identity on complementary tensor factors. Each such operator is positive semidefinite by the compact adjunction in Lemma 563.2. On B, the two-slot operator has eigenvalues at least $\kappa(\Lambda)+4n+1$: J_1 maps a C_K-eigenbasis of L_1 to eigenvectors with the same eigenvalues because of its K-equivariance. Thus, for $v = J_j z$,

$$\langle v, (C_{\{0,i\}} - C_{\{0\}})v\rangle \geq (4n+1)\|v\|^2.$$

On A, the one-slot Casimir has norm at most $4n$. Indeed on $\mathfrak{a}$ the sum of the squared norms of its compact infinitesimal actions is bounded by

$$\sum_{c,d} \|E_{cd}M - ME_{cd}\|_{\mathrm{HS}}^2 \leq 2\sum_{c,d}(\|E_{cd}M\|_{\mathrm{HS}}^2 + \|ME_{cd}\|_{\mathrm{HS}}^2) = 4n\|M\|_{\mathrm{HS}}^2.$$

The last two sums each count every squared entry n times. On the dual the same estimate follows from the nondegenerate invariant trace pairing $(M,N) \mapsto \operatorname{tr}(MN)$, which identifies the orthonormal matrix-unit basis with the transposed dual basis.

Expanding the quadratic operator on the diagonal action gives

$$C_{\{0,\dots,j\}} = C_{\{0\}} + \sum_{i=1}^{j}(C_{\{0,i\}} - C_{\{0\}}) + C_{\{1,\dots,j\}} - \sum_{i=1}^{j} C_{\{i\}}.$$

The right side contains each one-slot square and each two-slot cross term with its original coefficient. Positivity of $C_{\{1,\dots,j\}}$ and the preceding bounds give $\langle v, C_{\{0,\dots,j\}}v\rangle \geq (\kappa(\Lambda) + j)\|v\|^2$. For a nonzero C_K-eigenvector $z \in L_j$, the vector $v = J_j z$ is nonzero with the same eigenvalue. This proves the bound. Finally $M_j = M_0 \oplus L_1 \oplus \cdots \oplus L_j$ orthogonally. Distinct compact Casimir eigenvalues on smooth vectors are orthogonal by compact adjunction. The stated a therefore annihilates all of M_j in the inner product, at every j, proving the last assertion. □

564. Identification of the real projections on both groups

Lemma 564.1. *Let $u(t)$ be a Hilbert-valued function of a real parameter on an open interval. If its Gram function $K(s,t) = \langle u(s), u(t)\rangle$ is real analytic near every diagonal point, the function u is real analytic in Hilbert norm. In particular for any polynomial H_P in the oscillator of Lemma 494.1 its orbit under either real unitary factor, on each path*

$$t \longmapsto g(t) = \exp(tY)\exp(tX)$$

with X, Y in the real Lie algebra, is analytic in that norm. The same holds after a bounded projection commuting with that unitary factor.

Proof. On a sufficiently small interval the proof by mixed differences in Lemma 562.2 gives norm derivatives of every order with $\|u^{(j)}(t)\|^2 = \partial_s^j \partial_t^j K(s,t)|_{s=t}$. A convergent real power series in two variables extends by the same series to a complex bidisk of smaller radius. On a smaller real interval the Cauchy estimates of Lemma A.3, applied in each variable on circles of a fixed radius $r > 0$, give $\|u^{(j)}(t)\|^2 \le M(j!)^2 r^{-2j}$, where M bounds the extension there. Iteration of the fundamental theorem of calculus bounds the remainder after order $j-1$ in the vector Taylor formula by $\sup \|u^{(j)}\| \, |h|^j/j!$ on the segment of length $|h|$. For continuous vector derivatives this follows by taking Riemann sums in norm in each integral, or by pairing with a unit vector in the direction of the remainder. Thus for $|h| < r$ the remainder tends to zero. This proves the first assertion.

For the oscillator its two-variable Gram function is the coefficient of H_P at $g(s)^{-1}g(t)$. Extract finitely many polynomial coefficients in the two formal labels of Lemma 494.1. The entries of U, V there are real analytic in s, t; U is invertible. The vacuum coefficient is the reciprocal block determinant of Theorem 393.1 with the determinant character of the arrow. Both are real analytic here, since the block is invertible and the character has integral weight on the determinant circle. Conjugates of these entries are real analytic in the two real variables as well. This proves the Gram condition around every diagonal point. A bounded operator can be applied to a convergent vector power series term by term. □

Proposition 564.2. *Use the one-place data of Lemma 563.2, with its separation condition. Let v be a unit smooth vector in a strongly continuous unitary representation π of G with*

$$C_G v = c(\lambda) v, \qquad \langle v, \pi(k) v \rangle = m_\Lambda(k) \quad (k \in K).$$

Suppose its scalar coefficient is real analytic on each path of Lemma 564.1. Then

$$\langle v, \pi(g) v \rangle = \psi_\lambda(g) \quad (g \in G), \qquad P^\pi_{e_\lambda} v = v, \qquad e_\lambda = d_\lambda \overline{\psi_\lambda}.$$

Here the last equality involving the projection uses the moment hypothesis of Lemma 492.1.

Proof. Take the unit vector $v_0 = \sqrt{d_\lambda}\psi_\lambda$ in the right translation representation π_0 of Lemma 100.2; its coefficient is ψ_λ. For the coefficient comparison the required model is available by Theorem 96.3, since the stated separation implies its integrability condition with moment zero. In the definite case compactness gives it directly. The compact coefficient is m_Λ by the compact convolution in Theorem 100.3. Consequently the maps from the K-translates of either v or v_0 to the corresponding translates of the distinguished unit vector of T_Λ preserve all Gram matrices, by evaluating the common coefficient at $k^{-1}k'$. They extend linearly and isometrically to the compact orbit spans. These spans are finite dimensional and the two vectors are compact highest of weight Λ, by the exterior wedge construction of Theorem 77.2. The reference vector is smooth because its Gram function $\psi_\lambda(g^{-1}h)$ is analytic in the real group coordinates by Lemma 562.1 and the mixed-difference argument in Lemma 562.2. Its Casimir derivative equals $c(\lambda)v_0$: pair the derivative with each translate and differentiate the reference coefficient by Lemma 562.1. The derivative is a limit of finite combinations of translates by Lemma 562.2, so those pairings determine it.

In the direct sum take $w = (v, v_0)$ and $a = (v, -v_0)$. The compact orbit of w is finite dimensional and $\langle a, (\pi \oplus \pi_0)(k) w \rangle = 0$ for $k \in K$. Both vectors are compact highest of weight Λ; in particular $C_K a = \kappa(\Lambda) a$ by the computation in Lemma 563.2. The full Casimir has value $c(\lambda)$ on w. Theorem 563.3 now gives $\langle a, Dw \rangle = 0$ for every derivative word D, including the identity.

For any $g \in G$ take its polar decomposition $g = kP$ in Definition 96.1. The positive matrix P satisfies $SPS = P^{-1}$, and $k \in K$ by Theorem 96.3 and the isometry equation. Choose Y block skew-hermitian with $\exp Y = k$ by unitary diagonalization, and $X = \log P$ by the real logarithms of its positive eigenvalues. Then $SXS = -X$ by the same diagonalization, hence

$X^*S + SX = 0$. Along $g(t) = \exp(tY)\exp(tX)$ the j-th derivative of the difference of the two coefficients at zero is the sum of the pairings against $\binom{j}{i} Y^i X^{j-i} w$, $0 \le i \le j$. Indeed apply the ordinary product differentiation rule to the two independent parameters and then set them equal, using smoothness of the orbit. All these derivatives vanish. Both scalar coefficients are real analytic along the path, the reference one by its local integral above. Their difference vanishes on the whole interval from zero to one: the set where it and all its derivatives vanish is closed and is open by its local power series. Thus the coefficients agree at g. Finally integrating this equality against $e_\lambda = d_\lambda \overline{\psi_\lambda}$ gives expectation one, since $d_\lambda \int |\psi_\lambda|^2 = 1$ by Lemma 100.2. The operator is an orthogonal projection by Lemma 492.1; its expectation at a unit vector equals one only if it fixes that vector. □

Theorem 564.3. *At any one real place in either row of Definition 557.1, let $z = H_{P_0}$ be the determinant test of Theorem 557.3 and put*

$$Q = P^{\Omega_S}_{\bar{e}_\lambda}, \qquad Q' = P^{\Omega_T}_{e_{\lambda'}}.$$

Then $Qz = QQ'z = Q'z \neq 0$. The normalized self-coefficient of this vector is $\overline{\psi_\lambda}$ on the source and $\psi_{\lambda'}$ on the target. The projection equalities and the product formulas for the two coefficients hold on the tensor product over all the infinite places.

Proof. The projections commute, since the two real actions commute and their defining integrals converge absolutely on vectors by Lemma 492.1. Let $v = Qz \neq 0$. Theorem 562.3 gives $C_T v = c(\lambda')v$ and the compact highest equations on the target. The span under the target compact group of z is isometric up to scale to $T_{\Lambda'}$: its compact coefficient is the normalized minor coefficient by Lemma 559.2 applied to the target compact action with its two row blocks as in Proposition 558.3. The proof of that lemma only uses the finite-dimensional rational compact action and its raising and weight equations. Now Q intertwines that compact action, and its restriction to the compact orbit span has an invariant positive Gram form. Relative to the original norm the form is a scalar, since each eigenspace of its matrix is invariant and this finite-dimensional representation is irreducible by Theorem 77.2. The scalar is positive because $v \neq 0$. Thus the compact coefficient of $v/\|v\|$ is $m_{\Lambda'}$. Its orbit is smooth by commutation of Q with the target derivatives on polynomial tests, as in Theorem 561.3, or directly by the matrix Gram formula and mixed differences before applying Q. It is analytic along polar paths by Lemma 564.1. Proposition 557.2 supplies the required separation and moment on the target. Apply Proposition 564.2 to obtain $Q'Qz = Qz$.

In particular $v' = Q'z \neq 0$, since $Qv' = Qz \neq 0$. The source compact normalized coefficient of v' is $\overline{m_\Lambda}$ by the same finite compact orbit argument, using the source compact highest equations with reversed order in Theorem 558.4 and the conjugate compact representation. The target Casimir projection identity of Lemma 562.2 gives $Q'C_T z = c(\lambda')Q'z$; its proof applies equally to P_e and $P_{\bar{e}}$ because both coefficients have the same real Casimir value. Before projection the calculation of Theorem 562.3 gives $(C_T - C_S)z = (c(\lambda') - c(\lambda))z$. Since Q' commutes with the source derivatives, subtraction gives $C_S v' = c(\lambda)v'$.

Pass to the conjugate Hilbert representation on the source. Conjugation commutes with the full Casimir by Lemma 562.1. The normalized compact coefficient there is m_Λ; smoothness and analyticity follow as for v with the factors interchanged. Proposition 564.2 on that representation gives $Qv' = v'$ after conjugating back. Hence $Qz = Q'Qz = QQ'z = Q'z$. The two prescribed coefficients follow also from Lemma 492.1. Finally the product projections factor on elementary tensors by integration of their integrable kernels. The local Gram values multiply, and the tensor norm is the product of the positive local norms. This gives both claims at all the real places. □

565. Reproducing the canonical lift and its transpose

Lemma 565.1. *At a place of Definition 557.1, negate a list λ, retain the physical sign of each entry and put the entries back in decreasing order. In the same physical coordinates,*

$$\Lambda(-\lambda) = -\Lambda(\lambda), \quad \psi_{-\lambda}(u) = \overline{\psi_\lambda(u)}, \quad e_{-\lambda} = \overline{e_\lambda}.$$

Proof. Reversing the full order negates each of ρ, ρ_c in Definition 563.1. This proves the assertion on compact weights. In the conjugate of the cyclic representation of Lemma 100.2 the reference unit vector has coefficient $\bar{\psi}_\lambda$ and full Casimir $c(\lambda) = c(-\lambda)$ by the real conjugation computation in Lemma 562.1. Its compact coefficient is the minor coefficient for $-\Lambda$ in reverse order. Indeed on each compact sign block a unitary matrix k has $\det k[I^c, I^c] = (\det k)\overline{\det k[I, I]}$. For invertible principal block I^c this follows by block elimination and $k^{-1} = k^*$; continuity gives the other cases by approximating $k = \exp Y$ along $\exp(tY)$ where both minor polynomials in the entries are nonzero near $t = 0$ and have isolated zeros unless identically zero. In the product of principal minors for $-\Lambda$ every initial segment of the reverse order is a complement of an initial segment of the original order. Substitution in Definition 96.1 gives exactly $\overline{m_\Lambda}$, including the full determinant power. The conjugate reference orbit is smooth and its coefficient is analytic along polar paths by the local formula in Lemma 562.1 and the Gram argument in Lemma 562.2. Proposition 564.2 for $-\lambda$ identifies this coefficient with $\psi_{-\lambda}$. Their squared absolute integrals then give the same formal degree by Lemma 100.2. □

Theorem 565.2. *Fix either triangular pair of ranks n, H at sufficiently large arithmetic scale. Denote by e_S, e_T the products over the infinite places of the kernels for its signed source and target lists, respectively. Let z_0 be the product determinant test of Theorem 564.3, B any fixed finite Schwartz datum, and $K = K_{z_0,B}$ the twisted arrow kernel. Use L_K, D_K of Definition 555.2.*

(1) *For rapidly decreasing f on $[S]$ with $P^R_{e_S} f = f$ in L^2, the integral $F = L_K f$ satisfies, at every adelic h,*

$$\int e_T(v) F(hv)\,\mathrm{d}v = F(h).$$

(2) *For rapidly decreasing g on $[T]$ with $P^R_{\bar{e}_T} g = g$ in L^2, the integral $G = D_K g$ satisfies, at every adelic s,*

$$\int \bar{e}_S(u) G(su)\,\mathrm{d}u = G(s).$$

All the displayed integrals converge absolutely. The kernel measures are those of Lemma 492.1; the real translations have trivial finite component.

Proof. For general real test z write $l_h(z) = (L_{K_{z,B}} f)(h)$ and $d_s(z) = (D_{K_{z,B}} g)(s)$. Both are continuous functionals of z for the norm p_J of Definition 561.1, $J = 2nH + 1$. For l_h this is Lemma 561.2. For d_s integrate its bound $C_B p_J(z) H(s)^A H(h)^A$ over reduced representatives in the second variable against $|g|$, using Proposition 398.1. The same estimates bound the absolute integrals defining each functional.

The vectors $Qz_0 = \int \bar{e}_S(u)\Omega_S(u) z_0\,\mathrm{d}u$ and $Q'z_0 = \int e_T(v)\Omega_T(v) z_0\,\mathrm{d}v$ both exist in the completion for p_J. For the second integral use the target factor of Lemma 561.2 and at each place the moment J from Theorem 96.3. In signed orthonormal coordinates $\|v^{-1}\| = \|v\| \geq 1$ by $v^{-1} = Sv^*S$ with the target sign matrix in this formula. The separation in Proposition 557.2 exceeds $J + 1 + 4H^2$ and $J + 1 + 4n^2$ and remains sufficient after adding the decreasing half sum. The source calculation is the one in Theorem 561.3. The inclusion of this completion into the Hilbert space is injective by Lemma 561.2. Hence Theorem 564.3 gives $Qz_0 = Q'z_0$ in this completion as well.

By covariance in Theorem 396.1, $F(hv) = l_h(\Omega_T(v) z_0)$. The preceding integrable norm bounds show that its integral against e_T is absolutely convergent and equals $l_h(Q'z_0)$, by

continuity of the functional. The proof of Theorem 561.3 gave $l_h(Qz_0) = l_h(z_0)$ from the source equation on f. This proves the first identity.

Similarly $G(su) = d_s(\Omega_S(u)z_0)$, so its integral against $\bar{e}_S$ is $d_s(Qz_0) = d_s(Q'z_0)$. To evaluate the latter, substitute the integral for $Q'z_0$, use covariance in the target, and change h to hv^{-1} under its invariant quotient measure. The integral in v on g is then $\int e_T(v)g(hv^{-1})\,\mathrm{d}v = P^R_{\bar{e}_T}g(h)$ by inversion symmetry. Fubini applies by the absolute-integral bound for d_s just proved and the convergent p_J-moment. The inner convolution represents the Hilbert convolution almost everywhere, by testing on compactly supported L^2 functions as in Theorem 561.3. It equals g almost everywhere, giving $d_s(Q'z_0) = d_s(z_0)$. □

566. The determinant pairing in the large middle group

Lemma 566.1. *For a split surjection X in Definition 555.2 fix rapidly decreasing endpoint functions. Write*

$$b_X(z,w) = \mathcal{B}_X(z \otimes w).$$

Let Ω_1, Ω_2 denote the target actions on the two twisted oscillator tests for the large middle group V. For all real target elements v one has

$$b_X(\Omega_1(v)z, w) = b_X(z, \Omega_2(v^{-1})w).$$

On polynomial tests and words D in complexified target Lie directions one therefore has

$$b_X(d\Omega_1(D)z, w) = b_X(z, d\Omega_2(\check{D})w).$$

Here $\check{D} = (-1)^j X_j \cdots X_1$ for $D = X_1 \cdots X_j$, extended complex linearly to finite sums.

Proof. Proposition 555.3 expresses b_X as the integral of the product of the two individual lifts on $[V]$. Translations of each real test translate its individual lift in the same target argument by Theorem 396.1. Both product integrals are absolutely convergent by Theorem 507.3, applied to the translated Schwartz tests and the same finite point. Right invariance of the quotient measure proves the group identity.

That convergence theorem also bounds $|b_X(z,w)|$ by a fixed joint Schwartz seminorm of $z \otimes w$ times a constant for the given endpoints and X. A polynomial orbit can be differentiated in every Schwartz seminorm. Indeed the word argument at the end of Theorem 561.3 applies with any fixed word-length cutoff when differentiating the polynomial orbit itself: commute each Heisenberg word past the target translate and differentiate the resulting smooth matrix coefficients and Hilbert-valued polynomial orbits. For each prescribed Schwartz seminorm these word norms with sufficient cutoff dominate it. To see this apply the mixed integration of the squared absolute value in the proof of Lemma 561.2 to each $x^\alpha \partial^\beta z(x)$ involved. Cauchy–Schwarz uses finitely many multiplication and differentiation words whose lengths depend only on the seminorm and the fixed dimension. The derivatives in Hilbert norm are those of Lemma 559.1, hence are polynomials in this realization. We can therefore differentiate the preceding group identity once and iterate it on polynomial tests. Each transfer of a real direction supplies its minus sign, and successive transfers reverse the order of the word on the second vector. Bilinearity proves the complex-linear version. □

Theorem 566.2. *In Definition 555.2 let the two pairs through V be insertions of Definition 557.1 with opposite target parameters in the same physical frame at each place. This holds for the two ascending pairs from lists $T_{o,j+1}$ and $-T_{o+1,j+1}$ into the large middle node of the square $o + j + 2 \le m$. Use their assigned finite characters and require each rapidly decreasing endpoint function to be fixed in L^2 by its prescribed source kernel. Let z_1, z_2 be the respective product determinant tests, and write $b^0_X = \mathcal{B}_X(z_1 \otimes z_2)$.*

For every real Schwartz test x on the full Hom space there is $\kappa(x) \in \mathbb{C}$, independent of the residue point, finite twist and endpoint functions at these fixed real data, such that for every split surjection X

$$\mathcal{B}_X(x) = \kappa(x) b^0_X.$$

If some $b_X^0 \neq 0$ in this entire collection then κ is uniquely determined and continuous in the Schwartz topology. If all the canonical pairings vanish then all the functionals $\mathcal{B}_X$ vanish. For polynomial tensors $x = H_A \otimes H_B$ one possible scalar is

$$((\check{D}_A D_B)\psi_2)(1).$$

Here D_A, D_B are words or finite sums given by Corollary 560.2 for their separate factors. The right derivatives act on the product target coefficient ψ_2 of the second determinant after source projection, with the convention of Lemma 562.1.

Proof. The two rows of the specified square have targets $T_{o,j+2}$ and $-T_{o,j+2}$ by substituting $(o, j+1)$ into both rows of Definition 557.1. Write Q_1, Q_2 for the two source projections on the oscillators. In a product pairing we can replace any Hermite test by another polynomial with the same Q_i-image. Indeed at each fixed middle argument its endpoint integral agrees by the identity $l_h(Q_i z) = l_h(z)$ and the injectivity of the completed word space proved in Theorem 561.3. For the whole pairing integrate the two equal functions as in Proposition 555.3. First replace the two test polynomials by $d\Omega_1(D_A)z_1, d\Omega_2(D_B)z_2$ using Corollary 560.2. Then Lemma 566.1 moves the first derivatives to the second side. Put $D = \check{D}_A D_B$.

On the compact target group K let m_i be the product normalized compact coefficient of z_i, and s the common dimension of these two dual compact representations. The compact coefficients are conjugates by Lemma 565.1, restricted to K. Their dimensions agree by conjugating the compact orbit isometry in the proof of Theorem 564.3. Write $p_i = s \int_K \overline{m_i(k)}\Omega_i(k)\,\mathrm{d}k$. Then $p_1 z_1 = z_1$ by the compact projection in Theorem 557.3. The group identity of Lemma 566.1, integrated over the compact group, gives

$$b_X(z_1, d\Omega_2(D)z_2) = b_X(z_1, p_2 d\Omega_2(D)z_2).$$

In fact after inversion the first compact scalar $\overline{m_1(k^{-1})}$ becomes $m_1(k) = \overline{m_2(k)}$. The compact integration is in finite-dimensional polynomial spaces, since the compact actions preserve each degree cutoff, as in Proposition 558.3.

By Theorem 564.3 the nonzero vector $v_2 = Q_2 z_2$ has normalized target coefficient ψ_2. The cyclic target space of its unit normalization is isometric to the tensor product of the reference cyclic row spaces, by the equality of Gram matrices of translates in Lemma 100.2. On that cyclic space p_2 projects onto the line of v_2 by Theorem 100.3, place by place. The derivative $d\Omega_2(D)v_2$ belongs to the space by the difference quotients of Lemma 562.2. Its coefficient along v_2 is $(D\psi_2)(1)$ by differentiating the normalized group coefficient. Since Q_2 commutes with the target group and derivatives on polynomial vectors, we obtain

$$Q_2 p_2 d\Omega_2(D)z_2 = (D\psi_2)(1) Q_2 z_2.$$

Replacing the second polynomial inside its lift now gives the stated scalar for every polynomial tensor simultaneously at all the allowed indices.

Finite sums of polynomial tensors are dense in the full Schwartz space by the coherent-vector argument in Theorem 556.2. At every index i the functional $b_i(x) = \mathcal{B}_X(x)$ is continuous there by Theorem 507.3. For any two indices i, j the identity $b_i(x) b_j^0 = b_j(x) b_i^0$ and the implication $b_i^0 = 0 \Rightarrow b_i(x) = 0$ just proved on the dense span therefore hold for all x. If $b_j^0 \neq 0$, choose $\kappa(x) = b_j(x)/b_j^0$. Otherwise every functional is zero and $\kappa(x) = 0$ is sufficient. This proves all independence, continuity and uniqueness assertions. □

Corollary 566.3. *Under Theorems 554.1 and 566.2, take any fixed small test $y = y_1 \otimes y_2$, in particular the product determinant tests on the orientations through U. With K_i, K_i' the four corresponding individual kernels, every matched pair at the squarefree levels in those theorems satisfies*

$$\sum_Y \Psi(Y) \int_{[U]} (D_{K_1'} f_1)(D_{K_2'} f_2) = \frac{\kappa}{c_\nu} \sum_X \Phi(X) \int_{[V]} (L_{K_1} f_1)(L_{K_2} f_2).$$

The scalar κ depends on the fixed real choices and c_ν has the positive squared modulus of Theorem 554.1.

Proof. Use that theorem's common large test x for y. Apply Theorem 566.2 to its integrated large period for each summation point and use both factorizations of Proposition 555.3. The terms satisfy the joint rank conditions and hence have absolutely convergent integrals by Theorem 507.3. Dividing by $c_\nu \neq 0$ gives the formula with $\kappa = \kappa(x)$. □

567. Higher powers of the marked primes

Lemma 567.1. *In Definition 529.1, choose integers $h_w \geq 1$, write $\delta = (\pi_w^{h_w})_{w\in S} \in K^\times$, $\mathfrak{l} = \delta O$, $R_\mathfrak{l} = O/\mathfrak{l}$. Let $\chi_\mathfrak{l}$ be the character on $R_\mathfrak{l}^\times$ inflated from the product of the tame characters on the residue fields, and denote its value on $O^\times$ also by χ. For $Y = \binom{A}{C} \in M_{N,b}(K)$ with row blocks of sizes b, d, define $p(Y) = \chi(\det A)^{-1}$ if $A \in K_b$ and $C \in M_{d,b}(O)$, and zero otherwise. Let $\mathcal{N}$ be a linear subspace of functions of Y which contains p and is stable under precomposition by $Y \mapsto qY$ for $q \in G_N$. For every matched pair (Φ, Ψ) over $R_\mathfrak{l}$ in Definition 499.1, the lift*

$$\widetilde{\Psi}(Y) = \mathbf{1}_{M_{N,b}(O)}(Y)\, \Psi(Y \bmod \mathfrak{l})$$

belongs to $\mathcal{N}$, with the expression defined as zero off the indicated set.

Proof. First take the basis pair of Proposition 499.2 for the standard submodule $y_0 R_\mathfrak{l}^b$. Its lifted column function is supported where $A \in K_b$, $C \in \delta M_{d,b}(O)$, and there takes the value $\chi(\det A)^{-1}$. Indeed reducing to a basis of that submodule means exactly that the lower block is zero and the first block is invertible over $R_\mathfrak{l}$. An integral square matrix is invertible over O exactly when its determinant is a unit at each branch, equivalently a unit modulo $\mathfrak{l}$, by the adjugate identity. Hence this entire function is $p(d_\mathfrak{l} Y)$ where

$$d_\mathfrak{l} = \operatorname{diag}(I_b, \delta^{-1} I_d) \in G_N.$$

It belongs to $\mathcal{N}$. For any other summand use its basis-pair representative $k \in \mathrm{GL}_N(R_\mathfrak{l})$ and lift its entries to $\widehat{k}$ over O. Its determinant is still a unit at every branch, so $\widehat{k} \in K_N$. The lifted column basis function takes value $\chi_\mathfrak{l}(\det j)^{-1}$ on $ky_0 j$ modulo $\mathfrak{l}$. It is exactly $p(d_\mathfrak{l} \widehat{k}^{-1} Y)$. These basis functions span the column functions of all matched pairs by Proposition 499.2. Stability and linearity give the result. □

Theorem 567.2. *In Theorem 554.1, keep the allowed nonempty tame set S and character ν and prescribe arbitrary positive integers h_w at its members. Replace the residue ring there by $R_\mathfrak{l} = \prod_{w\in S} O_w/\pi_w^{h_w} O_w$ and its character by the inflation $\chi_\mathfrak{l}$. Use the finite point names at these depths with their normalized finite Fourier sums. For every matched pair at these depths, the same tests x, y and the same c_ν as at depth one give*

$$I_\Phi(g) = c_\nu I'_\Psi(g) \qquad (g \in U(W)(\mathbb{A}_{F_0})).$$

In particular the squared modulus of c_ν depends on the tame conductor and weight as in that theorem, with no additional factor depending on the integers h_w.

Proof. In the split position model of Theorem 360.1 a finite residue function has as its test its literal lift by reduction on the integral box and zero outside. To check normalization for a point B, the Fourier sum of Proposition 357.2 becomes the vacuum indicator times the normalized sum over the characters of the finite position lattice modulo $\mathfrak{l}$ of their values at $X - B$. At these depths the dual lattice quotient is $\bar{\delta}^{-1} L_-/L_-$ by the self-dual pairing of Lemma 355.2, or the finite duality computation of Proposition 501.2. The character sum is one on the residue point and zero elsewhere by translation in the character group. Linearity proves the assertion for any residue function.

A lifted deep matched pair satisfies $B^-_{\widetilde{\Psi}}(u) = \omega(u) B^+_{\widetilde{\Phi}}(u)$ for all $u \in G_N$, in the notation of Definition 529.1. For $u \in K_N$, a nonzero row or column integrand requires the changing

frame matrix to be in K_d or K_b, respectively. Indeed its base frame is primitive, and integrality with primitive reduction at every marked branch is equivalent to carrying the corresponding full lattice onto itself. Primitivity modulo π_w already suffices for this test, since an invertible maximal minor has unit determinant. On those compact groups the determinant transformation from the basis calculation of Proposition 499.2 cancels ω^{-1}, and the measures have mass one. The remaining condition is $\Psi(uy_0) = \chi_{\mathfrak{l}}(\det u)^{-1}\Phi(x_0u^{-1})$ on reductions, which is the assumed matching. For all u it follows by multiplying on the right by an upper block parabolic element, using the two transformation laws and the factorization $G_N = K_N P$ proved in Proposition 529.3. Here P is the stabilizer of the standard b-plane. In particular the lifted column lies in that proposition's space $\mathcal{M}$, and the functional D there at this column is the actual deep period difference for the chosen row. The absolute convergence needed on both full-frame loci follows from Lemma 529.2.

Apply that proposition to x, y, c_ν of Theorem 554.1. Its kernel is a linear space stable under the full G_N. It contains the function p of Lemma 567.1, since p is the lifted depth-one standard column determinant of Lemma 509.2 and the corresponding standard pair obeys the depth-one period identity at every argument. The lemma therefore puts $\widetilde{\Psi}$ in the kernel of D, which is precisely the desired equality. The chosen scalar was fixed before the depths were prescribed. □

Corollary 567.3. *The integrated identity of Corollary 566.3 holds at all the marked depths of Theorem 567.2. A single scalar κ works at all these depths and tame twists with fixed real data. In particular the identity applies at depth b_1 at the original marked primes with trivial additional twist whenever the stated signature and character conditions hold.*

Proof. A split surjection or injection over $R_{\mathfrak{l}}$ remains so modulo each π_w, by reducing its one-sided inverse. The rank detection of Lemma 507.2 is stated using this first-layer condition: in its proof all other translations fix the named vector by the actual dual lattice at depths h_w. Thus the absolute triple integral and its continuity estimate in Theorem 507.3 apply to every point in the deep matched sums. The constants in those bounds can depend on the level. Apply Theorem 567.2, multiply by the two endpoint functions and determinant characters as in Corollary 554.2, and factor the point kernels using Lemma 555.1. In its proof the two dual boxes and their orders factor at every depth.

For polynomial tensors the scalar of Theorem 566.2 uses only the two real representations, also at these depths. Indeed the individual lift equalities there apply to every finite datum, and its derivative transport uses precisely the joint convergence estimate verified above. For a general test take the two-index cross products in the proof of that theorem with indices from the union of all depths and twists. Each index still defines a continuous functional. Thus density proves one scalar for the whole union, including the case when every canonical pairing vanishes. The same x serves for all depths by Theorem 567.2. Substitution gives the asserted formula. □

568. The arithmetic parameters in the first square relation

Lemma 568.1. *In Definition 557.1 fix $z = 0$ or 1 and $o, j \geq 0$ with $o + j + 2 \leq m$. Put $D = t - z$, $b = n_j$, $n = n_{j+1}$, $d = n_{j+2}$. Let W_1, W_2, V, U have the framed spaces at $(o, j+1), (o+1, j+1), (o, j+2), (o+1, j)$, respectively. Negation of a parameter list retains the space and its signs. Write $\vartheta_z = \vartheta$ for $z = 0$ and $\vartheta_z = 1$ for $z = 1$, with ϑ an m-fold twist as in Proposition 501.3. Use the arithmetic character χ of Definition 376.1 and put*

$$\eta = \upsilon_T^{-h}\chi\vartheta_z, \quad h = 2h_2, \qquad \nu = \eta^{-(j+1)}, \qquad f_z(p) = \lfloor (p+z)/2 \rfloor.$$

At rank p define the two families of torus characters

$$a_p = \mu^{\flat}\eta^{o+j+2}\upsilon_T^{f_z(p)}, \qquad b_p = (\mu^{\flat}\eta^{o+1})^{-1}\upsilon_T^{p-f_z(p)}.$$

The notation b_p denotes a character, and b the rank of the small middle group. For $N = \dim(W_1 \perp W_2)$ and q its number of negative signs, one has

$$N = 2n = b + d, \qquad d = b + 2D, \qquad q = q_U + q_V,$$

$$1 \le n - q \le (j+1)h, \qquad a_p b_{p'} = \nu^{-1} \upsilon_T^{N/2} \quad (p + p' = N).$$

The weight of ν is $(j+1)h$ at every place. Its finite unit character is tame of conductor the product of the chosen marked primes and, for $z = 0$, the twist primes. It is nontrivial on each residue-field factor in this conductor.

Proof. The rank equalities follow from $n_l = r + z + l(t - z)$. All nonbase blocks in the four spaces have positive sign. At a base index write $c = c_i$ in Definition 557.1. If $c \le o + j + 1$, raising the stage from $j+1$ to $j+2$ at offset o, or from j to $j+1$ at offset $o+1$, preserves the sign. If $c \ge o + j + 2$, the sign exponents of $(o, j+2)$ and $(o+1, j+1)$ are both $o + c + j + 2$, and those of $(o+1, j)$ and $(o, j+1)$ are both $o + c + j + 1$. In either case the two sums of negative-sign counts are equal. Since $q \le 2(r + z)$, $n \ge 6r$ and $n \le r + 1 + 5r^2$, the bounds $1 \le n - q \le 2r^8 = h \le (j+1)h$ hold at sufficiently large scale.

By Definition 376.1 and Theorem 376.2, $\mu^\flat \eta^l$ has weight $-w_l$ and finite unit character $(\omega_w \vartheta_{z,w})^l$. Dividing the formulas for consecutive $l = 1, 2$ shows that η has weight $-h$, since $w_2 - w_1 = h$, and finite unit character $\omega_w \vartheta_{z,w}$. All these characters are unitary and trivial on rational points by their construction in Theorem 372.2. Proposition 501.3 applies since $1 \le j + 1 \le m$, including its assertion for the inverse characters; the twist factor is absent for $z = 1$. This proves the claims about ν.

Finally p, p' of sum $2n$ have the same parity. Checking that common parity gives $f_z(p) + p' - f_z(p') = n$ for either z. The two $\mu^\flat$ factors cancel and the remaining power of η is $j + 1$. This proves the product identity. At the actual ranks for $z = 1$ all ranks are even, so $f_z(p) = p/2$. The archimedean factors on both large arrows thus have exactly the numerator and denominator weights in Definition 557.1. □

Theorem 568.2. *Use the data of Lemma 568.1 and the four kernels of Definition 555.2. On the large ascending pairs take their product determinant tests, of respective target parameters $T_{o,j+2}$ and $-T_{o,j+2}$. On the two small pairs take the product determinants for $-T_{o+1,j} \to -T_{o,j+1}$ and $T_{o+1,j} \to T_{o+1,j+1}$, respectively. At $z = 0$ take the squarefree marked and twist level. At $z = 1$ take depth b_1 at the marked primes and trivial additional twist, with $b_1 \ge 1$ as in Definition 12.3. Let R be the resulting finite residue ring and $G_\ell(R) = \mathrm{GL}_\ell(R)$. At the endpoints take any rapidly decreasing forms of the prescribed source types $T_{o,j+1}$ and $-T_{o+1,j+1}$, respectively.*

For any function $p : M_{n,b}(R) \to \mathbb{C}$ take p^+ and the matrices $Y(y), X(Z)$ of Theorem 499.3 with its dimensions $a = b$, $t = D$. Write $L(X)$ for the large bilinear pairing of the four-arrow factorization and $L'(Y)$ for the small one. Then

$$|G_b(R)| \sum_y p(y) L'(Y(y)) = \frac{\kappa}{c_\nu} |G_d(R)| \sum_Z p^+(Z) L(X(Z)).$$

The scalar κ is independent of the finite twist, residue function and endpoint forms, at fixed triangular position, z, and real choices. The nonzero c_ν satisfies

$$|c_\nu|^2 = C \Big(\prod_{w \in S} q_w \Big)^{D^2} \prod_{\ell=1}^{D} |L^S(\ell, \nu)|^2$$

with $C > 0$ independent of the tame twist. Here κ may be zero in the bilinear identity.

Proof. The character weights and the rank, signature and conductor conditions required in Theorem 554.1 are precisely those in Lemma 568.1, with $e = D$. The two ascending pairs have the opposite target parameters required in Theorem 566.2 by Lemma 565.1. The smaller through-U orientations use the second and first rows of Definition 557.1 at (o, j) and $(o+1, j)$, respectively. Their characters are b and a of that square: the indices for the

underlying η-powers are $o+1$ and $(o+1)+j+1 = o+j+2$. Hence Corollary 567.3 gives the matched-sum identity $\sum_Y \Psi(Y)L'(Y) = (\kappa/c_\nu)\sum_X \Phi(X)L(X)$.

Let ξ denote the finite character of ν on $R^\times$. We also have $L(uX) = \xi(\det u)L(X)$ for $u \in G_d(R)$ and $L'(Yv) = \xi(\det v)L'(Y)$ for $v \in G_b(R)$. Indeed before integration over the endpoints these are the second-line identities of Lemma 508.2 on the orthogonal sum, with the separate determinant kernels identified by Lemma 555.1. At the deeper level lift each integral residue matrix to the local integers, where its determinant is a unit. The split operator substitution of Theorem 360.1 then permutes the residue boxes at this depth. Its ideal phase and modulus are one, and the same determinant character is inflated from the residue fields. The identical quotient change of variables thus proves both transformations. Multiplication by the endpoint factors and integration preserve them by absolute convergence of the joint rank integrals. We may therefore apply Theorem 499.3 with $c = \kappa/c_\nu$. This gives the displayed graph sum with its precise group orders even if $\kappa = 0$. The squared-modulus formula and the independence statements come from Theorem 567.2 and Corollary 567.3. □

569. Constant terms of the prescribed real types

Lemma 569.1. *Let $G = \prod_\tau U(S_\tau)$ be a product of real signed unitary groups. At each place choose a totally isotropic complex subspace A_τ, with at least one of positive dimension. Write N_τ for the subgroup acting by the identity on A_τ, $A_\tau^\perp/A_\tau$, and $\mathbb{C}^n/A_\tau^\perp$, preserving this flag, and $N = \prod_\tau N_\tau$. A zero-dimensional choice gives the trivial subgroup at that place. Suppose e is a continuous integrable kernel on G with $e * e = e$, and, at one place τ_0 of positive chosen dimension, suppose $\int_G |e(v)| \log \|v_{\tau_0}\|_{\mathrm{op}} \, \mathrm{d}v < \infty$. Every bounded continuous function F on G satisfying*

$$F(ux) = F(x) \quad (u \in N), \qquad F(x) = TF(x), \quad TF(x) = \int_G e(v)F(xv)\,\mathrm{d}v$$

vanishes identically.

Proof. At τ_0 put $l = \dim A_{\tau_0} > 0$, $A = A_{\tau_0}$. The compact group $K_0 = U(S_{\tau_0}) \cap U(n)$ is transitive on the isotropic l-planes. Indeed the positive and negative projections of an ordinary orthonormal basis of such a plane, scaled by $\sqrt{2}$, are each orthonormal: the differences of their Hermitian products vanish and the sums are the original ordinary products. Complete each set to an orthonormal basis of its sign space and apply the unitary sending one completed basis to another in each block. Thus $U(S_{\tau_0}) = PK_0$, where P stabilizes A, by applying transitivity to the inverse image of A.

Choose an opposite isotropic plane A' by changing the sign of the negative projection of all vectors in A. The signed pairing of the two planes is nondegenerate by the same basis calculation. Scale A by $t > 0$, A' by t^{-1}, and their signed orthogonal complement by one. This defines an isometry a_t at τ_0; extend it by the identity at the other places. It normalizes N since it preserves the three-step flag. On the three quotients it acts by scalars, so at the chosen place $a_t p a_t^{-1} p^{-1} \in N_{\tau_0}$ for $p \in P$. For every $x \in G$, writing $x_{\tau_0} = pk$ shows that $a_t x = u x b_x$ with $u \in N$ and b_x conjugate to a_t by a member of K_0. These b_x tend to the identity uniformly in x as $t \to 1$, since compact unitaries preserve the ordinary operator norm.

Let $\mathcal{X}$ be the vector space of bounded continuous left N-invariant functions fixed by T, with the supremum norm. Right convolution by e is bounded and commutes with left translation; it is idempotent by $e * e = e$ and absolute integration of the bounded function. If $F = TF$, then

$$\sup_x |F(xb) - F(x)| \le \|F\|_\infty \int_G |e(b^{-1}v) - e(v)|\,\mathrm{d}v.$$

The integral tends to zero as $b \to 1$. For a continuous compactly supported kernel this follows from uniform continuity on a compact enlargement of its support. Approximate e in L^1 by multiplying by continuous cutoffs in the matrix norm that increase to one, as

in Theorem 96.3; left invariance of Haar measure gives the estimate for the difference of the tails. Fix $t > 1$ so close to one that left translation $UF(x) = F(a_t x)$ on $\mathcal{X}$ satisfies $\|I - U\| \le 1/2$, by the reduction to b_x.

Choose a nonzero exterior product v_A of a basis of A and define $H(x) = \log \|(\bigwedge^l x_{\tau_0}^{-1}) v_A\|$ with the ordinary exterior norm. This is left N-invariant, and $H(a_t x) = H(x) - l \log t$. Moreover

$$|H(xv) - H(x)| \le l \log \|v_{\tau_0}\|_{\mathrm{op}}.$$

In fact the operator norm of $\bigwedge^l v_{\tau_0}^{\pm 1}$ is at most $\|v_{\tau_0}^{\pm 1}\|_{\mathrm{op}}^l$ by its action on the alternating tensor subspace, and the norms of a signed unitary and its inverse agree by $v^{-1} = S v^* S$.

For an integer $m \ge 1$ put $\epsilon = \pi/(ml \log t)$, $M(x) = \exp(i\epsilon H(x))$, and $BF = T(MF)$. The operator B maps $\mathcal{X}$ to itself by idempotence and left invariance. Since $|e^{iu} - e^{iv}| \le |u - v|$ for real u, v, integrating the derivative of the exponential,

$$\|BF - MF\|_\infty \le \epsilon l \, \|F\|_\infty \int_G |e(v)| \log \|v_{\tau_0}\|_{\mathrm{op}} \, \mathrm{d}v.$$

Choose m sufficiently large that the multiplier is less than one. Then B is injective since $|M| = 1$. The formula for $H(a_t x)$ gives $UB = e^{-i\pi/m} BU$, hence $UB^m = -B^m U$. Consequently

$$2B^m = (I - U)B^m + B^m(I - U).$$

The operator norm bound on $I - U$ implies $2\|B^m\| \le \|B^m\|$. Thus $B^m = 0$, and injectivity of B forces $\mathcal{X} = \{0\}$. □

Proposition 569.2. *For any framed group $U(Y)$ of the triangular array let e be the product kernel of either prescribed real type, or its conjugate. Suppose F is a bounded continuous automorphic function fixed pointwise by right convolution with e. For every nonzero rational totally isotropic E-subspace $A \subset Y$, let N_A be the unitary subgroup preserving its flag and inducing the identity on all three quotients of Lemma 569.1. With invariant quotient probability one has*

$$\int_{N_A(F_0)\backslash N_A(\mathbb{A}_{F_0})} F(ug) \, \mathrm{d}u = 0 \quad \big(g \in U(Y)(\mathbb{A}_{F_0}),\ F_0 = E^+\big).$$

Proof. Choose a rational basis of A and vectors dual for the hermitian form by solving the independent equations of pairing with that basis. Subtract from the dual vectors the corresponding linear combinations of the basis of A given by half their Gram matrix. Their new Gram matrix is zero, since the two subtracted cross terms each equal half the old matrix, and the corrections pair to zero with one another. This gives a rational opposite plane and a nonsingular perpendicular. Definition 522.1 therefore gives coordinates $u(C, Z)$ on N_A, with Z skew-hermitian. Multiplication adds the two C's and adds the two Z's with the commutator term $(C' J_0^{-1} C^* - C J_0^{-1} C'^*)/2$. Thus the quotient has compact representatives and invariant probability by the iterated additive means of Lemma 539.2: its proof uses rational bases of the additive C, Z spaces, which here have any finite dimensions. In each coordinate of $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ first make the finite part integral and then reduce the real coordinate into $[0, 1)$. Reduction of Z follows that of C; both left and right changes translate the central coordinate for each fixed C, so preserve its inner mean.

For fixed g form the bounded function of $x \in G_\infty$ given by the displayed mean at gx. It is continuous by compact representatives, boundedness and Lemma A.2. Right convolution T with e fixes it, by Fubini and $\|F\|_\infty \|e\|_1 < \infty$. It is left invariant under $g_\infty^{-1} N_A(F_0 \otimes \mathbb{R}) g_\infty$, by right invariance of quotient probability, inserting identity at all finite places. This is exactly the product of the radicals for the real isotropic subspaces $g_\tau^{-1} A_\tau$, each of positive dimension. The integrable kernel e is idempotent by Theorem 99.1. Its one-place logarithmic moment is finite by Theorem 96.3 and the gaps of Proposition 557.2; use $0 \le \log u \le u$ for $u \ge 1$ and integrate the product of the remaining absolute kernel factors. Conjugating the

kernel preserves these properties. Lemma 569.1 makes the function of x zero, in particular at identity. □

570. Small representatives for a rational unipotent quotient

Lemma 570.1. *Use the framed rank n hermitian space V, lattice Λ_x, and real and finite frames X_∞, X_f of Definition 91.1. Write $F_0 = E^+$ and $d = [E : \mathbb{Q}]n$. For sufficiently small $\lambda = \lambda_1(x)$ there are a nonzero rational totally isotropic E-plane A and a basis $B_1, \dots, B_s, Z_1, \dots, Z_c$ over $\mathbb{Q}$ of the Lie algebra over $\mathbb{Q}$ of $N_A(F_0)$ in Proposition 569.2. The Z_j span the subspace of operators killing $A^\perp$ with image in A. Every commutator belongs to this subspace, and its products in either order with the entire algebra are zero. For each basis operator D,*

$$D\Lambda_x \subseteq \Lambda_x, \qquad \max_\tau \|X_\tau^{-1} D_\tau X_\tau\|_{\mathrm{op}} \le C\lambda^{1/(2d)}$$

with C depending only on the fixed framed space.

Proof. Take A and the two integral bases a_i, v_j from Lemma 108.1. Its choice by the first gap in the successive minima gives $\max_i \|a_i\|_x \le C_1\sqrt{\lambda}$, since the last minimum of the first sublattice is at most the minimum just below the gap. For $a \in A,\ v \in A^\perp$ put

$$D(a,v)u = a\,h(v,u) - v\,h(a,u).$$

Our hermitian form is linear in its second variable. These operators are skew-adjoint, send V into $A^\perp$, send $A^\perp$ into A, and annihilate A. The product of any two operators with these three mapping properties kills $A^\perp$ and has image in A. Commutators are again skew-adjoint. A product of three such operators is zero.

Choose the rational opposite plane and perpendicular V_0 as in Proposition 569.2, with dual bases on the opposite planes. In the order A, V_0, A' the skew-adjoint operators with the three mapping properties are exactly the matrices

$$\begin{pmatrix} 0 & C & Z \\ 0 & 0 & -J_0^{-1}C^* \\ 0 & 0 & 0 \end{pmatrix}, \qquad Z^* = -Z,$$

where J_0 is the Gram matrix of V_0. Indeed matrix multiplication with the Gram matrix having identity opposite corner blocks gives precisely these two adjoint conditions. Exponentiation $I + D + D^2/2$ gives the unipotent matrices of Definition 522.1, with inverse coordinate $D = (g - I) - (g - I)^2/2$. This also identifies the stated Lie algebra over $\mathbb{Q}$.

The classes of $D(a,v)$ with $v \in V_0$ span all the blocks C. Their restrictions to V_0 are $ah(v,\cdot)$. Every E-linear functional on V_0 occurs, by invertibility of J_0, and maps supported on a single row and column span the rectangular matrices. Any scalar can be absorbed in a. The $D(a,a')$ with $a, a' \in A$ span all the skew-hermitian corner blocks over $\mathbb{Q}$. In the dual bases their matrices are $a(a')^* - a'a^*$. Taking $a = ze_i,\ a' = e_j$ realizes any off-diagonal pair $z, -\bar{z}$. With $i = j$ this gives any diagonal entry $b = -\bar{b}$ by taking $z = b/2$. These descriptions cover a zero-dimensional V_0 as well. Since $D(a,v)$ is $\mathbb{Q}$-bilinear, the two spanning assertions still hold using only the pairs (a_i, v_j) modulo the corner and $(a_i, a_{i'})$ on the corner: the integral bases span $A, A^\perp$ over $\mathbb{Q}$. Select independent members first in the quotient, then on the corner.

Each selected pair consists of lattice vectors. Their pairings with Λ_x are in O_E because the finite frame has integral Gram matrix. In a signed real frame the norm of $D(a,v)$ is at most $2\|a\|_x\|v\|_x$ by the ordinary Cauchy–Schwarz inequality applied to the sign matrix. For the quotient generators use Lemma 108.1. For corner generators use $2C_1^2\lambda \le 2C_1^2\lambda^{1/(2d)}$ for $\lambda \le 1$. □

Theorem 570.2. *Under the hypotheses of Lemma 570.1 fix a compact open finite level K in the standard finite frame. There exist $C_K, \delta_K > 0$ such that for $\lambda_1(x) < \delta_K$ the plane A of*

that lemma has the following property. Every coset in $N_A(F_0)\backslash N_A(\mathbb{A}_{F_0})$ has a representative u satisfying

$$X_f^{-1}u_fX_f\in K,\qquad \max_{\tau}\|X_\tau^{-1}u_\tau X_\tau-I\|_{\mathrm{op}}\le C_K\,\lambda_1(x)^{1/(2d)}.$$

Proof. Choose a positive integer M such that the integral isometries congruent to I modulo $M\widehat{O}_E$ lie in K. Indeed a neighborhood of identity in the restricted product contains the full integral group outside finitely many primes, and at the remaining primes entrywise congruence to identity of sufficiently high order gives a neighborhood basis. Multiply every generator of Lemma 570.1 by $2M$, retaining the names B_i, Z_j for this basis. For D in the new basis put $\exp(tD)=I+tD+t^2D^2/2$. If $t_f\in\widehat{\mathbb{Z}}$, this element in the finite frame is integral and congruent to I modulo M, as is its inverse $\exp(-t_fD)$. The finite matrices of the original operators are integral because the completions of Λ_x are the lattices of X_f by Lemma 20.2.

Use ordered coordinates

$$(t,z)\longmapsto\prod_{i=1}^{s}\exp(t_iB_i)\prod_{j=1}^{c}\exp(z_jZ_j).$$

They give a bijection on rational points for $t_i,z_j\in\mathbb{Q}$, and on adelic points for $t_i,z_j\in\mathbb{A}_{\mathbb{Q}}$. To see this, take the polynomial logarithm in the preceding lemma. Since all triple products vanish, the logarithm is $\sum_i t_iB_i+\sum_j z_jZ_j+\frac12\sum_{i<k}t_it_k[B_i,B_k]$. The last term is entirely in the corner. Reading the quotient basis coordinates first and then the corner coordinates recovers all entries uniquely. Rational change of coordinates and these polynomials are integral away from finitely many primes for each fixed x, so this bijection respects the restricted products. Equivalently, extending a rational basis to $\mathbb{Q}_p$ or $\mathbb{R}$ gives the indicated local matrix algebra by the same linear skew-adjoint and flag equations.

In every additive coordinate any adele is congruent modulo $\mathbb{Q}$ to one with finite coordinate in $\widehat{\mathbb{Z}}$ and real coordinate in $[0,1)$. Clearing finite denominators and solving the finitely many congruences uses Lemma A.1, after which an integer puts the real coordinate in this interval. First multiply on the left by the inverse of the rational product of horizontal exponentials removing the rational parts of the t_i. This adds or subtracts corner terms but changes the horizontal coordinates only by the specified subtractions. Next multiply by rational corner exponentials to reduce each new z_j. The resulting representative has each coordinate in $[0,1)\times\widehat{\mathbb{Z}}$ and its finite part satisfies the required condition by closure of K under products.

In each real frame every scaled generator has norm at most $L=2MC\lambda_1(x)^{1/(2d)}$. For a parameter in $[0,1]$, the norm of its exponential minus identity is at most $L+L^2/2$. The difference of the ordered product from identity has norm at most $(1+L+L^2/2)^{s+c}-1$, by expanding the product in its given order and using submultiplicativity. For $L\le1$ this is bounded by a constant times L, because $s+c$ is bounded by $[E:\mathbb{Q}]n^2$ independently of the plane. Decreasing δ_K to ensure $L\le1$ proves the claim. □

571. Pointwise projections and decay at any finite level

Theorem 571.1. *Fix a framed triangular group $G=U(Y)$ of positive rank, a product kernel e as in Proposition 569.2, and a compact open subgroup K of its finite group. Write $[G]=G(F_0)\backslash G(\mathbb{A}_{F_0})$, $F_0=E^+$. Let $\mathcal{V}(e,K)$ be the space of bounded continuous functions on $[G]$, invariant on the right under K and fixed pointwise by*

$$R(e)F(x)=\int_{G_\infty}e(t)F(xt)\,\mathrm{d}t.$$

With $h(x)=\max(1,\lambda_1(x)^{-1})$ measured in the sum-of-squares real norm of Definition 91.1, for every real $A\ge0$ there is a constant $C_{A,e,K,G}$ such that

$$\sup_{x\in[G]}h(x)^A|F(x)|\le C_{A,e,K,G}\|F\|_\infty\qquad(F\in\mathcal{V}(e,K)).$$

The space is finite dimensional. All its members are rapidly decreasing in the sense of Proposition 398.1 and belong to $L^1([G]) \cap L^2([G])$.

Proof. Intersect K with the stabilizer of the standard integral lattice. It suffices to prove the assertions at this smaller compact open level since the given space is a subspace. The rational and finite frames and signed real coordinates are those of Definition 91.1. At a point with sufficiently small $\lambda_1(x)$, take A_x and representatives for the unipotent quotient as in Theorem 570.2. Their constant term vanishes by Proposition 569.2. For each such representative u, its finite displacement from x lies in the level, and its real displacement is $b = X_\infty^{-1} u_\infty X_\infty$. Subtracting $F(x)$ inside the probability average gives

$$|F(x)| \le \sup_{\max_\tau \|b_\tau - I\| \le C_K \lambda_1(x)^{1/(2d)}} |F(xb) - F(x)|, \qquad d = [E:\mathbb{Q}] \dim_E Y.$$

Here the supremum is over the real isometry group. It is enough that each coset has a representative satisfying the bound, since $F(ux)$ itself descends to the quotient. If the group has no rational isotropic plane, Theorem 570.2 makes the region of sufficiently small λ_1 empty.

The compactly supported continuous β with $\beta * e = e$ comes from Theorem 99.1 by taking the product over the places and complex conjugating if needed. Thus $F = R(\beta)F$ by absolute integration. For b in a fixed relatively compact identity neighborhood,

$$F(xb) - F(x) = \int_{G_\infty} [\beta(b^{-1}v) - \beta(v)] F(xv) \, \mathrm{d}v.$$

The differences of translates have supports in one compact set D, and their L^1-norm tends to zero as $b \to 1$ by uniform continuity and finite Haar measure on D. For $v \in D$ and $T \ge 1$, put $h_T = \min(h, T)$. Then $h_T(x) \le C_D h_T(xv)$ for some $C_D \ge 1$. Indeed a shortest vector for x has length in the new frame at most $\max_\tau \|v_\tau^{-1}\|$ times its old length and the finite lattice stays fixed. Taking reciprocals, maxima with one, and truncations gives the claim.

Set $M_T = \sup_x h_T(x)^A |F(x)|$, which is finite. The integral bound on $h_T(x)^A |F(xb) - F(x)|$ is $M_T C_D^A \|\beta(b^{-1}\cdot) - \beta\|_1$. Choose H_A large enough that at $h(x) > H_A$ all b in the oscillation bound give a multiplier at most $1/2$. Thus

$$M_T \le \max\{M_T/2, H_A^A \|F\|_\infty\}.$$

This gives $M_T \le H_A^A \|F\|_\infty$ for every T, and the pointwise limit yields the estimate on h. The maximum norm over places defining the first minimum in Proposition 398.1 lies between the sum-of-squares norm and that norm divided by $\sqrt{[F_0:\mathbb{Q}]}$, with the upper bound given by the sum-of-squares norm. The resulting two heights differ by at most this fixed factor. That proposition gives integrability on the group quotient. Boundedness gives square integrability.

For finite dimension, use the estimate for $A = 1$ to put $|F(x)| \le \|F\|_\infty / 2$ outside $h(x) \le 2C_{1,e,K,G} + 1$. This thick set lies in the image of a compact set at the finite level by Proposition 107.2. On every sufficiently small right real neighborhood the integral difference above is at most $\|F\|_\infty / 2$ uniformly in x, F. Finitely many such neighborhoods cover the compact set, since the finite coordinates modulo the level are discrete. If F vanishes at all their centers, its supremum on the thick and the thin parts alike is at most $\|F\|_\infty / 2$. These finitely many evaluations therefore give an injective linear map to a finite-dimensional coordinate space. □

Proposition 571.2. *In Theorem 571.1, suppose f is any finite-valued measurable automorphic function on $[G]$, pointwise right invariant under K, such that $t \mapsto e(t) f(xt)$ is absolutely integrable for every x. Then the pointwise function $R(e)f$ is in $\mathcal{V}(e, K)$. In particular a function satisfying these hypotheses and $R(e)f = f$ everywhere is continuous and rapidly decreasing.*

Proof. For $j \geq 1$ put $f_j = f\,\mathbf{1}_{\{|f| \leq j\}}$. Then $u_j = R(e)f_j$ is bounded by $j\|e\|_1$, automorphic, and right invariant under K. Left translation of e is continuous in L^1, as proved in Lemma 569.1. Hence u_j is continuous in the real variable by the translated-kernel integral bound, and is continuous in the finite variable by its open level. It is fixed pointwise by $R(e)$, by $e * e = e$ from Theorem 99.1 and Fubini with bound $j\|e\|_1^2$. Thus all the u_j belong to the same finite-dimensional $\mathcal{V}(e, K)$. At each x, dominated convergence in Lemma A.2, with dominating function $|e(t)f(xt)|$, gives $u_j(x) \to R(e)f(x)$.

For completeness, write $v_1, \ldots, v_l$ for a basis of $\mathcal{V}(e, K)$. If $l = 0$ all the limits vanish. Otherwise the finite evaluation matrix from Theorem 571.1 has column rank l. Choose l independent rows. Multiplying the vector of values of u_j there by the inverse matrix gives its l coefficients, so they converge. The resulting linear combination has value $R(e)f(x)$ at every x by taking the finite sum of the coefficient limits. This identifies the stated pointwise function with a member of $\mathcal{V}(e, K)$. □

572. The first square as an equality of automorphic forms

Proposition 572.1. *For a prescribed triangular real list λ on S write $\mathcal{V}_S(\lambda)$ for the union over compact open K of $\mathcal{V}(e_\lambda, K)$ from Theorem 571.1. Intersections of levels show that this is a complex vector space. For either pair of signed lists (λ, λ') on (S, T) in Definition 557.1, let $K_{z,B}$ be the twisted theta kernel attached to the canonical determinant z and any fixed finite Schwartz datum B. The bilinear integrals of Definition 555.2 give maps*

$$L_{K_{z,B}} : \mathcal{V}_S(\lambda) \longrightarrow \mathcal{V}_T(\lambda'),$$
$$D_{K_{z,B}} : \mathcal{V}_T(-\lambda') \longrightarrow \mathcal{V}_S(-\lambda).$$

Each image is contained in a space at a finite level depending on the datum and characters alone. For $f \in \mathcal{V}_S(\lambda)$ and $g \in \mathcal{V}_T(-\lambda')$ one has

$$\int_{[T]} g\, L_{K_{z,B}} f = \int_{[S]} f\, D_{K_{z,B}} g.$$

Proof. The finite datum has an open stabilizer under each finite unitary action. In fact in the finite model of Theorem 358.1 and Proposition 391.1 it is a finite linear combination of translated integral vacua. Integral isometries fix the vacuum, including the ideal phase, by Theorem 390.2. The Heisenberg covariance carries a translate by v to that by kv. For k sufficiently close to identity, $kv - v$ is in the vacuum lattice and the scalar from its composition with the translate by v equals one. This follows from the finite Heisenberg multiplication law, since the additive character is locally constant, the phase is continuous, and its value at the identity is one. Intersect the finitely many neighborhoods and pull them back by the Hom action. Also restrict the determinant characters to their open kernels on finite units, as in Theorem 376.2. Taking a compact open subgroup in each resulting neighborhood gives the claimed levels. The rational invariance of the theta sum with its torus factors follows from Theorem 396.1. Thus both integrals, when defined, descend to their rational quotients.

The sources are in L^2 and rapidly decreasing by Theorem 571.1. Their pointwise real projections agree with the Hilbert projections: absolute integration of the translated L^2 norm is bounded by $\|e\|_1\|f\|_2$, and testing against any L^2 function identifies the two by Cauchy–Schwarz and Fubini as in Lemma 492.1. Use Lemma 565.1 on the negative lists. Theorem 565.2 now gives both pointwise projection identities with absolute convergence everywhere. The functions to which they apply are measurable, since their theta kernels are continuous on the real components and locally constant in finite coordinates by Theorem 396.1. Their defining integrals converge by the height bound of Lemma 561.2 and Proposition 398.1. Applying Proposition 571.2 in each target proves both range assertions.

On the reduced domains of the last height proposition the absolute integrand in the claimed pairing is bounded by

$$C\,|f(s)|H(s)^{a}\,|g(t)|H(t)^{a}$$

for some a, C, again by the pointwise theta bound. Each factor is integrable. Lemma 123.1 therefore identifies the two integrations in either order. □

Theorem 572.2. *In Theorem 568.2 use the same real determinants, levels, characters, residue ring R, function p, and finite names $X(Z), Y(y)$. Denote the individual kernels for the two large blocks of $X(Z)$ by $K_1(X_1), K_2(X_2)$, and for the two small blocks of $Y(y)$ by $K_1'(Y_1), K_2'(Y_2)$, in the order of Proposition 555.3. For $f \in \mathcal{V}_{W_1}(T_{o,j+1})$ define*

$$A_Z f = D_{K_2(X_2)} L_{K_1(X_1)} f,$$
$$B_y f = L_{K_2'(Y_2)} D_{K_1'(Y_1)} f.$$

They belong to $\mathcal{V}_{W_2}(T_{o+1,j+1})$. In that space one has the pointwise identity

$$|\mathrm{GL}_b(R)| \sum_y p(y) B_y f = (\kappa/c_\nu)\,|\mathrm{GL}_d(R)| \sum_Z p^{+}(Z) A_Z f$$

with the same scalars, uniformities and p^{+} as in Theorem 568.2. It holds at every square $o + j + 2 \le m$, also at the deeper untwisted companion level.

Proof. On the large route the first arrow sends $T_{o,j+1}$ to $T_{o,j+2}$. The second kernel is the one for the pair $(-T_{o+1,j+1}, -T_{o,j+2})$. Its transpose takes $T_{o,j+2}$ to $T_{o+1,j+1}$ by Proposition 572.1. On the small route the first kernel is for the pair $(-T_{o+1,j}, -T_{o,j+1})$. Its transpose sends f to type $T_{o+1,j}$. The second kernel sends that type to $T_{o+1,j+1}$. Thus the finite sums on both sides belong to the asserted space, taking the intersection of their finite levels.

Let v be the left side minus the right side. For any $g \in \mathcal{V}_{W_2}(-T_{o+1,j+1})$, the composition pairing in Proposition 572.1 identifies $\int_{[W_2]} g\, A_Z f$ and $\int_{[W_2]} g\, B_y f$ with the two middle integrals of Proposition 555.3. In the small pairing use the identity with the just obtained intermediate form on U. The graph matrices of Theorem 499.3 are split on the support of the sums. Theorem 568.2 at the endpoints f, g gives $\int g v = 0$. Take $g = \bar{v}$, which belongs to the opposite real type by Lemma 565.1 and conjugation of the pointwise defining equation. Then $\int |v|^2 = 0$. A nonzero value of the continuous v would give a nonempty open quotient neighborhood with $|v|$ bounded below by a positive constant. Such a neighborhood has positive measure by Lemma 21.4. Hence v vanishes at every adelic argument. □

573. The joint projected line for either order of a pair

Theorem 573.1. *For either triangular insertion with ordered groups (S, T) and lists (λ, λ'), use the product determinant z, projections $Q = P^{\Omega_S}_{\bar{e}_\lambda}$ and $Q' = P^{\Omega_T}_{e_{\lambda'}}$, and nonzero $v = Qz = Q'z$ of Theorem 564.3. For every real Schwartz test w, with inner product linear in the second argument,*

$$QQ'w = \frac{\langle v, w\rangle}{\|v\|^2}\, v.$$

The same formula holds for the reversed pair (T, S) with lists (λ', λ), the same character at each rank, and the signed conjugate transpose $z^{\#}$ of z from Theorem 502.3. There $v^{\#} = C_\infty v$ is the common nonzero image of $z^{\#}$ under the two reversed projections.

Proof. Both projections are orthogonal by Lemma 492.1. They commute by the two group actions and absolute integration, so their product is an orthogonal projection. For a Hermite polynomial w, Corollary 560.2 expresses $Qw = d\Omega_T(D)v$ for a finite linear combination of target words D. Indeed Q commutes with target translations and hence with their derivatives on z; smoothness on $v = Qz$ follows by differentiating the bounded operator Q on that orbit. Since $Q'v = v$, Lemma 492.1 gives $Q'\Omega_T(t)v = \overline{e_{\lambda'}(t)}v/e_{\lambda'}(1)$. Consequently all its derivatives at identity, and thus $Q' d\Omega_T(D)v$, lie in $\mathbb{C}v$. One can see this directly by

iterated differences of the projected multivariable orbit $\Omega_T(\exp(t_1X_1)\cdots\exp(t_lX_l))v$, whose values after Q' all lie on that closed line. The Hermite polynomials are dense in the real Schwartz test space by Lemma 549.1, and Schwartz convergence implies convergence in L^2 by bounding the pointwise difference by a sufficiently high inverse power of $1+\|x\|$. Boundedness of the projections extends the inclusion of the image in the line to every such test. Since the product fixes v, self-adjointness gives $\langle v, QQ'w\rangle = \langle v, w\rangle$, proving the formula.

Write C_∞ also for the product over the real places of the antiunitary signed permutations of Theorem 502.3. Reversing the ordered groups interchanges the numerator and denominator of the unitary character ratio, hence conjugates it. This is the precise twist intertwined by C_∞. Its antilinearity changes Q' into the reversed source projection $P_{\bar{e}_{\lambda'}}$, and Q into the reversed target projection P_{e_λ}. Apply C_∞ to the original formula at $C_\infty^{-1}w$. Norms are preserved and inner products conjugated, giving the displayed formula in the reversed order. Applying it and the separate projection identities to $z^{\#} = C_\infty z$ gives the common vector there. □

Proposition 573.2. *Use either order in Theorem 573.1 and write $(S,T;\lambda,\lambda';z,v)$ in that order, including the signed permutation on the test when appropriate. Let B be any finite Schwartz datum, let $f\in\mathcal{V}_S(\lambda)$, $g\in\mathcal{V}_T(-\lambda')$, and put*

$$\mathcal{I}(w)=\int_{[S]\times[T]} f(s)g(t)K_{w,B}(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

Then $\mathcal{I}(w)$ converges absolutely for each real Schwartz test and satisfies

$$\mathcal{I}(w)=\frac{\langle v,w\rangle}{\|v\|^2}\mathcal{I}(z).$$

In either order the canonical $L_{K_{z,B}}, D_{K_{z,B}}$ have the two mapping and pairing properties of Proposition 572.1, with the lists taken in that order.

Proof. Use the completed word norm p_J in Definition 561.1 with $J = 2\dim_E S\dim_E T+1$. The estimate for the absolute theta kernel and the translation estimate in Lemma 561.2 apply also in the reversed order. For precision, the bound on translated Heisenberg words uses only the matrix norm of $X\mapsto tXs^{-1}$ and its inverse. Both are bounded by the products of the source and target matrix norms in either order. The Schwartz bound on the theta evaluation there uses J for the same number of real position variables and the finite support and denominator bounds of its fixed finite test. None of these bounds requires an ordering of the ranks. This gives a constant times $p_J(w)H(s)^aH(t)^a$ in either order. By Proposition 398.1 and Theorem 571.1, the double integral of the absolute value against the two forms is at most a constant times $p_J(w)$. Thus $\mathcal{I}$ extends continuously to the completed word space.

The integrals defining Q,Q' act boundedly on that completion. Indeed the bound on p_J of a translate uses a J-th operator-norm moment at each place, finite for both real kernels by the gap and moment argument of Theorem 565.2. Integration on Schwartz tests and extension by density yield these bounded operators; their images in L^2 agree with the Hilbert projections by the continuous injective inclusion of Lemma 561.2. For Q, covariance followed by $s\mapsto su^{-1}$ in the source integral transfers $\bar{e}_\lambda(u)$ to $\int \bar{e}_\lambda(u)f(su^{-1})\,\mathrm{d}u = f(s)$. Inversion uses the conjugate symmetry of the kernel in Theorem 99.1. The same operation for Q' transfers $e_{\lambda'}$ to $R(\bar{e}_{\lambda'})g = g$. Fubini in both changes follows from the absolute bound just obtained for translated tests and the J-th moments. Consequently $\mathcal{I}=\mathcal{I}Q=\mathcal{I}Q'$ on the completion, first on Schwartz tests and then by continuity. The joint projection formula there, valid by its injective inclusion into L^2, gives the assertion since $QQ'z=v$.

For the range assertions in the reversed order, fix the target argument t in the lift from f. The resulting functional of w has the same p_J bound, now with the fixed factor $H(t)^a$. The preceding source change of variables proves that it is unchanged by Q. Convolving its value at z in the target by $e_{\lambda'}$ replaces z by $Q'z=Qz$, so reproduces the value. This integral

converges absolutely by the same moment bound. In the transpose fix s, first transfer Q' to g, then use $Qz = Q'z$ for the convolution in s. Rational invariance and an open finite level follow in both orders by the finite vacuum proof in Proposition 572.1. Proposition 571.2 then gives the asserted mapping properties. Integrating the absolutely convergent double integral in the two orders gives the stated pairing identity. □

574. The comparison through middle groups of equal rank

Lemma 574.1. *In Definition 555.2 suppose the two orders $(W_1, V), (W_2, V)$ through the middle group V are respectively a triangular insertion and the reversal of a triangular insertion, with opposite lists on V. Take endpoint forms of the indicated source types at their finite levels. Denote by z_1, z_2 the two product determinant tests from Theorem 573.1 in those orders. For each real Schwartz test w on the joint Hom space, there exists a scalar $\kappa(w)$ depending only on the real data such that at every split surjection X, all compatible finite twists, and the indicated endpoint forms,*

$$\mathcal{B}_X(w) = \kappa(w)\mathcal{B}_X(z_1 \otimes z_2).$$

If all the canonical values vanish in this family then every $\mathcal{B}_X$ vanishes.

Proof. First take $w = H_A \otimes H_B$. By Theorem 561.3 the first endpoint lift has the same value at every middle argument as the lift of $d\Omega_1(D)z_1$, for a finite sum of target words D depending only on A and the real data. Lemma 566.1 moves these derivatives to the second real test, replacing it by $w_2 = d\Omega_2(\check{D})H_B$. Its proof uses the simultaneous right translation law and the Schwartz bound at a split surjection, which apply equally to the second Hom space in the reversed order. In particular its polynomial derivatives exist in Schwartz norms by signed permutation and conjugation in Theorem 502.3.

The remaining first lift $F = L_{K_{z_1,X_1}} f_1$ is a rapidly decreasing form on V by Proposition 572.1. Its type is opposite to the target type of the second kernel. The middle pairing with its arbitrary test w_2 is therefore $\mathcal{I}(w_2)$ of Proposition 573.2 with $g = F$, source form f_2, and finite test X_2. That proposition applies by absolute integration. If v_2 is the common projection of z_2, it gives the required scalar $\langle v_2, w_2\rangle/\|v_2\|^2$. This is the same number for every finite choice and both endpoint forms. Finite sums of polynomial tensors are handled by linearity. Such sums are dense by the coherent polynomial expansion in Lemma 549.1. Each joint evaluation on a split surjection is continuous by Theorem 507.3. Cross-multiplication by the canonical values at any two indices, followed by passage to a Schwartz limit, shows that all evaluations are proportional to their canonical values whenever one of the latter is nonzero. At an index whose canonical value is zero the functional vanishes on the dense span and hence everywhere. This also proves the last assertion. □

Theorem 574.2. *Use a primary square $z = 0$, $o + j + 2 \le m$ of Lemma 568.1, with the same $b = n_j$, $n = n_{j+1} = b + D$, $d = n_{j+2} = b + 2D$, characters a_p, b_p, ν, and squarefree residue ring R of order Q. Write*

$$O = (o+1, j), \qquad P = (o, j+1), \qquad M = (o+1, j+1), \qquad T = (o, j+2)$$

for the four framed spaces, and λ_Z for the list $T_{i,l}$ at a node $Z = (i, l)$. Use $(W_1, W_2, V, U) = (O, T, M, P)$ in the four-arrow construction. The a-kernels from O to M and from P to T have their positive-list determinants. The b-kernels from T to M and from P to O have the signed reversals of the negative-list determinants from M to T and from O to P, respectively.

For $f \in \mathcal{V}_O(\lambda_O)$ and $g \in \mathcal{V}_T(-\lambda_T)$, let $L(X)$ denote the integral of the two individual lifts on M, and $L'(Y)$ the integral of the two individual transposes on P, with blocks of X, Y in endpoint order. These definitions use any point names of the required sizes, including

singular ones. Set

$$X(Z) = \left(\begin{pmatrix} I_b \\ 0 \end{pmatrix} \quad Z \right), \qquad Y(y) = \begin{pmatrix} y \\ I_n \\ 0 \end{pmatrix}, \quad Z \in M_{n,d}(R), \quad y \in M_{b,n}(R).$$

Take $p : M_{n,b}(R) \to \mathbb{C}$, and $p^+ : M_{d,n}(R) \to \mathbb{C}$ as in Theorem 572.2. For the normalized transforms $\mathcal{U}$ of Definition 501.1 one has

$$\sum_y (\mathcal{U}_{n,b}p)(y) L'(Y(y)) = s_\nu \sum_Z (\mathcal{U}_{d,n}p^+)(Z) L(X(Z)).$$

There is a constant $C_{\mathrm{eq}} \geq 0$ depending only on the fixed triangular position and real choices such that

$$|s_\nu|^2 = C_{\mathrm{eq}}\, Q^{D^2}.$$

The scalar is independent of the endpoint forms and of p.

Proof. All four mapping properties, including those for the reversed orders, follow from Proposition 573.2. The two factors in each middle pairing have conjugate types. Their product is integrable at every residue point by Cauchy–Schwarz and Theorem 571.1. At split surjections or split injections these definitions agree with the three-group integral of Proposition 555.3.

Both middle ranks are n, both endpoint ranks sum to $2n$, and $q_O + q_T = q_M + q_P$ by Lemma 568.1. The common sign count q, constant across the real places by the sign formula in Definition 557.1, is the same as for the first square. Thus $1 \leq n - q \leq (j+1)h$. The torus products $a_l b_{2n-l} = \nu^{-1} \upsilon_T^n$ and the conductor and weight conditions are also those of that lemma. On the reversed arrows the b-ratio in the new order is the conjugate of the ratio in the old order by unitarity. Theorem 554.1 with rank excess $e = 0$ now supplies a common real test opposite the prescribed through-P tensor, and a nonzero c_ν with $|c_\nu|^2 = C > 0$ independent of the finite twist. Apply Lemma 574.1 to this common test on the through-M side. The integrated exchange of Proposition 555.3 therefore gives the matching equation of Definition 499.1 for L, L' with $c = \kappa / c_\nu$. The scalar κ is fixed by the real choices and may be zero.

Write ξ for the finite unit character of ν on $R^\times$. For every $k \in \mathrm{GL}_n(R)$,

$$L(kX) = \xi(\det k) L(X), \qquad L'(Yk) = \xi(\det k) L'(Y).$$

To see that this holds even at singular names, lift k to an integral isometry in the split factors, identity at other finite places. Such a lift exists by lifting the entries of the invertible matrices, whose determinants remain units, and using inverse conjugate transpose at the partner. In the position model of Theorem 360.1, acting on a point indicator by the integral middle element sends the point to kX_i on the large side. On the small side the inverse middle element sends it to $Y_i k$. The change of variables has modulus one and the untwisted vacuum phase is one. For each side, removing these two actions from the two kernels by a common right translation of the middle argument multiplies the product by the inverse of the product of the two endpoint-rank torus characters evaluated at $\det k$. For a source action here the determinant is inverted both by the inverse middle element and by its position in the character denominator. Thus the product before inversion is $(\nu^{-1} \upsilon_T^n)(\det k)$. The splitting character is trivial on integral units by Theorem 374.2. Right invariance of measure in the two convergent middle integrals proves both formulas.

The multiplicative character ξ is primitive in Definition 500.1. If $v \neq 0$ in the product of fields R, choose a factor with $v_w \neq 0$. There the character is nontrivial by Lemma 568.1. Choose $t_w \neq 0$ on which it differs from one, and take $z_w = (t_w - 1)/v_w$, with all other components of z zero. Then $1 + zv$ exhibits primitivity. The additive character ψ_R is primitive by Proposition 501.2. Apply Theorem 500.3 with its n, t, a equal here to b, D, n. The two general linear group orders cancel because the two middle ranks agree. For the

unnormalized transforms it gives the scalar $\gamma\kappa/c_\nu$, where $|\gamma|^2 = Q^{-D(D+2b)}$. Passing to $\mathcal{U}$ multiplies this scalar by $Q^{(dn-nb)/2} = Q^{nD}$. Hence

$$|s_\nu|^2 = (|\kappa|^2/C)Q^{2nD-D(D+2b)} = (|\kappa|^2/C)Q^{D^2},$$

as asserted. □

575. The Petersson identity for the two descents

Lemma 575.1. *For an ordered triangular kernel K of Proposition 573.2, let c_n, c_N be the numerator and denominator characters at the source and target ranks n, N. For a finite norm-one idele u with real component one, write u_S, u_T for its central scalar matrices and $R(u)F(x) = F(xu)$. On the spaces of forms of Proposition 572.1 one has*

$$R(u_T)L_K f = \frac{c_n(u^N)}{c_N(u^n)} L_K R(u_S) f.$$

Moreover for $z \in \mathcal{V}_T(\lambda')$ the form $x = \overline{D_K \bar{z}}$ belongs to $\mathcal{V}_S(\lambda)$ and, for all $l \in \mathcal{V}_S(\lambda)$,

$$\langle z, L_K l\rangle = \langle x, l\rangle.$$

All inner products here are $\langle F, H\rangle = \int \bar{F}H$.

Proof. The simultaneous central action on $\mathrm{Hom}_E(S,T)$ is $X \mapsto uXu^{-1} = X$. The oscillator representation therefore acts as identity before the determinant characters are included. The definition of the twist in Definition 504.1 gives $K(su_S, tu_T) = c_n(u^N)K(s,t)/c_N(u^n)$. Changing the source variable by u_S in $L_K f(tu_T)$ proves the identity. Central right translations preserve the finite level and boundedness and commute with the real projection. Thus they are bijections on each indicated space by taking the inverse translation. For the second assertion use Lemma 565.1 to put $\bar{z}$ in the negative target type. The mapping and bilinear pairing assertions of Proposition 573.2 give the required statements on taking the displayed inner product convention. □

Theorem 575.2. *Keep O, P, M, T, their lists and ranks, the finite function p and its extension p^+ as in Theorem 574.2. Let a_O and a_P denote the positive-list lifts $O \to M$ and $P \to T$, each with the standard inclusion indicator. Use the negative-list canonical kernels on $O \to P$ and $M \to T$ with b-characters and finite functions p, p^+. Denote their bilinear transposes on the positive-list spaces by $d_p : \mathcal{V}_P(\lambda_P) \to \mathcal{V}_O(\lambda_O)$ and $d_{p^+} : \mathcal{V}_T(\lambda_T) \to \mathcal{V}_M(\lambda_M)$. There exists κ_ν, independent of p, such that for every $z \in \mathcal{V}_T(\lambda_T)$ the form $x = \overline{D_{K_{a_P}} \bar{z}} \in \mathcal{V}_P(\lambda_P)$ satisfies*

$$\langle z, a_P l\rangle = \langle x, l\rangle \qquad (l \in \mathcal{V}_P(\lambda_P)),$$
$$\langle d_p x, y\rangle = \kappa_\nu \langle d_{p^+} z, a_O y\rangle \quad (y \in \mathcal{V}_O(\lambda_O)).$$

Here K_{a_P} means the individual inclusion kernel defining a_P, and $|\kappa_\nu|^2 = C_{\mathrm{eq}} Q^{D^2}$ with the constant of Theorem 574.2.

Proof. The first equation is Lemma 575.1. Use for both Fourier transforms the units $\epsilon_w = (-\varpi \jmath \alpha \bar{\alpha})_w$ of Theorem 503.2. The scale α of Lemma 348.1 and the finite conductor idele δ of Definition 501.1 are common to the two Hom spaces. Therefore $u = \overline{\alpha\delta}/(\alpha\delta)$, with identity real component, is the same torus element in both applications of Theorem 504.2. Write $B_0(o,l)$ for the negative kernel on $O \to P$ with datum p, and $B_1(m,t)$ for the one on $M \to T$ with datum p^+. Write $B_0^{\mathrm{rev}}, B_1^{\mathrm{rev}}$ for their signed reversals, with finite data $\mathcal{U}_{n,b}p, \mathcal{U}_{d,n}p^+$. That theorem provides unit scalars c_0, c_1, independent of p, with

$$B_0^{\mathrm{rev}}(l,o) = c_0^{-1}\overline{B_0(ou_O^{-1}, l)},$$
$$B_1^{\mathrm{rev}}(t,m) = c_1^{-1}\overline{B_1(mu_M^{-1}, t)}.$$

The real substitutions are exactly the ones used in Theorem 574.2.

Apply its Fourier identity with endpoint functions $f, \bar{z}$, where $f \in \mathcal{V}_O(\lambda_O)$. On its left the second block of $Y(y)$ is the standard inclusion for $P \to T$; on its right the first block of $X(Z)$ is the standard inclusion for $O \to M$. Linearity in the other finite block sums to B_0^{rev} and B_1^{rev}. The first middle integral becomes

$$\int_{[P]} \overline{x(l)} \int_{[O]} f(o) B_0^{\mathrm{rev}}(l,o)\,\mathrm{d}o\,\mathrm{d}l = c_0^{-1}\langle d_p x, R(u_O) f\rangle.$$

Indeed insert the first reversal formula, integrate first in l, then change o to ou_O. In the other middle integral the corresponding expression is

$$\int_{[M]} (a_O f)(m) \int_{[T]} \overline{z(t)} B_1^{\mathrm{rev}}(t,m)\,\mathrm{d}t\,\mathrm{d}m = c_1^{-1}\langle d_{p^+} z, R(u_M) a_O f\rangle.$$

These integrations are absolutely convergent: the intermediate $x, a_O f$ are rapid forms, and the single-kernel double integral bound in Proposition 573.2 applies after the central changes of variable as well. Using the scalar s_ν of the Fourier identity and then Lemma 575.1 for a_O gives the claimed second equation with

$$y = R(u_O) f, \qquad \kappa_\nu = s_\nu (c_0/c_1)\, a_b(u^n)/a_n(u^b).$$

The translate y ranges over the whole stated space and all extra factors in κ_ν have absolute value one. This proves the uniformity and size assertions. □

576. The framed nodes and the first descent in each row

Proposition 576.1. *For both $z = 0, 1$ and all $o, j \geq 0,\ o + j \leq m$ the signed lists $T_{o,j}$ of Definition 557.1 have framed rational hermitian realizations $S_{o,j}$ of rank n_j with integral finite frames for the identity matrix. One may take $S_{0,j} = Y_j$ with its actual period lattice for $z = 0$, $S_{0,0} = Y_0 \perp \langle 1\rangle$ for $z = 1$, and $S_{o+1,0} = S_{o,0}^-$ in both arrays. At $j = 0$ the identity on the successive underlying unitary groups gives isometries*

$$j_o : \mathcal{V}_{S_{0,0}}(T_{0,0}) \longrightarrow \mathcal{V}_{S_{o,0}}(T_{o,0})$$

onto the indicated spaces, preserving every finite level in the given frames. Here j_0 is identity. The two arrays have their own base spaces and maps.

Proof. On the primary row the initial k labels have positive sign. Of the remaining base indices $c_i = 1, \ldots, m-1$, those with $c_i \leq j$ are positive and those with $c_i > j$ have sign $(-1)^{c_i + j}$. The other jt coordinates are positive. Hence the number of negative signs is $\lceil (m - 1 - j)/2 \rceil$ for $j < m$, and zero for $j = m$. Theorem 348.3 gives precisely these signatures and the required actual finite frames. Choose an ordered real orthonormal basis of each sign subspace corresponding to the list labels. At $z = 1,\ j = o = 0$ the old signs are the same and the new last label $i = r$ has $c_i = m$, which is even by $r = k + m - 1$, r odd and k even. Appending a positive line with standard finite frame realizes the required base.

Passing from o to $o + 1$ at $j = 0$ reverses every sign and keeps every parameter. Take the opposite space with the finite frame multiplied by the integral unit of norm -1 from Lemma 505.1. Its group, quotient measure and real kernel are unchanged under the indicated identification, and its finite scalar frame change preserves all matrix levels. This gives the iterated isometries. At every as yet unassigned node take the rational diagonal form with entries the listed signs in label order: the sign attached to a label is independent of τ, even if the parameter ordering uses τ. At infinity use the indicated permutation of the basis. At finite places keep each positive basis vector and multiply each negative vector by the same finite unit of norm -1. The resulting Gram matrix is the identity. All these choices are independent of the finite twist. □

Theorem 576.2. *Fix $z \in \{0, 1\}$, $o < m$, and one allowed twist and level from Theorem 568.2. Write $S = S_{o,0},\ S^- = S_{o+1,0},\ T = S_{o,1}$, and let I be the point indicator of the standard inclusion at all marked residue places, with vacuum elsewhere. Write a for the canonical lift*

$S \to T$ with I in the positive row, and $j : \mathcal{V}_S(T_{o,0}) \to \mathcal{V}_{S^-}(T_{o+1,0})$ for the opposite-source identification. There is a single finite function $p_{o,0}$ on $M_{n_1,n_0}(R)$ such that the negative-list canonical kernel on $S^- \to T$ with this function has transpose $d_{o,0}$ satisfying

$$\langle d_{o,0} z', jy \rangle = \langle z', ay \rangle \quad (z' \in \mathcal{V}_T(T_{o,1}),\ y \in \mathcal{V}_S(T_{o,0})).$$

On the full space W_S of this source type at principal level on the marked set of the chosen modulus and integral level elsewhere, $d_{o,0} a y = j H_o y$, where

$$H_o = C_o T_{n_1}, \qquad C_o = \frac{\operatorname{vol}[U(T)]\,|R|^{-n_1 n_0} \|P_{\bar{e}}^{\Omega_S} z_a\|^2}{|\mathrm{GL}_{n_0}(R)|\, e(1)} > 0.$$

Here z_a is the real determinant of a, e the source kernel, and T_{n_1} the unmarked operator of Theorem 478.1. This endomorphism is injective for all sufficiently large arithmetic scales. It preserves any further prescribed invariance under integral subgroups or finite group-algebra projections at the marked pairs.

Proof. At offset o the positive base arrow uses the characters

$$c_p = \mu^\flat \eta^{o+1} \upsilon_T^{\lfloor (p+z)/2 \rfloor},$$

where η is as in Lemma 568.1, also with the trivial extra twist for $z = 1$. By the character formula of Definition 557.1 the negative base arrow uses $c_p^{-1} \upsilon_T^p$ at both ranks. In particular it is the opposite-source character pair of Theorem 505.2. Write z_- for the polynomial test obtained there by conjugation and relabeling of z_a, and z_b for the prescribed determinant on the negative pair. That theorem gives

$$\overline{K^a_{z_a,I}(s,t)} = c_- K^b_{z_-,I^-}(js,t)$$

for a fixed unit scalar c_-. Here js denotes the identification of the group argument, and I^- is the indicator of the inclusion multiplied by the residue unit from the opposite finite frame. This use of the theorem works at either indicated depth since an integral unit acts invertibly on each matrix residue ring.

Let v_b be the common nonzero real projection of z_b. Put $p_{o,0} = c_- \langle v_b, z_- \rangle\, I^- / \|v_b\|^2$. For every y, z' as in the theorem, test the conjugated kernel on the right against $\overline{jy}$ on S^- and z' on T. These are the source and opposite target types for the negative row. Proposition 573.2 replaces the real test z_- by the displayed scalar $\langle v_b, z_- \rangle / \|v_b\|^2$ times z_b. The remaining factors c_- and I^- are precisely those in $p_{o,0}$, since the theta value is linear in the finite datum. Absolute integration and the unchanged quotient measure now give $\langle jy, d_{o,0} z' \rangle = \langle ay, z' \rangle$. Taking conjugates proves the asserted pairing. Equivalently $d_{o,0} z' = j\, \overline{D_{K^a_{z_a,I}} \bar{z}'}$, since the difference belongs to the opposite-source form space by Proposition 572.1 and can itself be tested in the pairing.

At stage zero the positive determinant is the initial minor test of Theorem 497.2: the first list and inserted block agree by the base formulas in Definition 497.1, and the leading and trailing minors are chosen by the same positive and negative column multiplicities. Theorem 506.2 therefore gives the formula on W_S. More explicitly it computes $a^* a = C_o T_{n_1}$ on this level in L^2; testing the preceding pairing at $z' = ay$ identifies its opposite with the actual form $d_{o,0} a y$. The finite norm is $N_F = |R|^{-n_1 n_0}$ by the one-point support of I in Proposition 498.1, and its level index is $|\mathrm{GL}_{n_0}(R)|$ by the same proposition. The positive lower bound and invariance under the additional marked actions are those in Theorem 506.2. Equality almost everywhere of the continuous representatives is pointwise by Lemma 21.4. □

577. The terminal projection on the primary row

Lemma 577.1. *In the primary array $z = 0$, put $d_i = k + i(t+1)$ and $n = d_m - 1 = \dim_E Y_m$. At a real place arrange $T_{0,m} = (\lambda_1, \dots, \lambda_n)$ in decreasing order, and let Λ be its compact*

weight. Use $e_m = \lfloor d_{m-1}/2 \rfloor - w_m$ and the diagram β of Definition 416.3, at this same place. If $\beta(h)$ is the number of its columns of height h, then

$$\Lambda_i = e_m + \sum_{h=i}^{n} \beta(h) \qquad (1 \le i \le n).$$

This identity holds for every infinite place at sufficiently large arithmetic scale.

Proof. All signs are positive by Proposition 576.1. Thus Definition 96.1 gives $\Lambda_i = \lambda_i + i - (n+1)/2$. Write $H = h_2$ and $W = W_\tau$ for the rounded weight. Since m is even, the common half shift in Definition 557.1 is zero at $(0, m)$. The initial $k-1$ entries have values $k/2 - i$ in the present one-based indexing. For each $0 \le l < m$ they are followed, in order of l, by a singleton and a block of t entries with respective values

$$-W - 2(l+1)H, \qquad -w_{l+1} + (t-1)/2 - v, \quad 0 \le v < t, \qquad w_{l+1} = W + (2l+3)H.$$

These are in decreasing order. Within the blocks this follows from the unit step and between them from $H > t + k$ at large scale. The initial entries precede the first singleton since $W \ge 0$. The singleton for l is at position d_l, and its block ends at $d_{l+1} - 1$. In particular $\lambda_n = -w_m - (t-1)/2$, which gives $\Lambda_n = (n-t)/2 - w_m = e_m$.

The differences $\Lambda_i - \Lambda_{i+1}$ equal $\lambda_i - \lambda_{i+1} - 1$. They vanish within every consecutive block. Immediately before the first singleton, at $i = k-1$, the value is $W + 2H - k/2$. At each $i = d_l,\ 0 \le l < m$, it is $H - (t+1)/2$, by subtracting the top of the following block from the singleton. At each $i = d_l - 1,\ 1 \le l < m$, subtract the next singleton from the end of the preceding block; the value is again $H - (t+1)/2$.

We compare with the columns, keeping $k > 2$, which holds at the scales of Proposition 10.5. Definition 26.1 gives $w_1 - d_1/2$ columns at height k, $2H - (t+1)$ at each d_l for $1 \le l < m$, and the height-one columns. All d_l are even. At each height d_l the shortening operation of Definition 416.3 retains $H - (t+1)/2$ columns and moves the others to $d_l - 1$. For $l \ge 1$ that remainder is again $H - (t+1)/2$. At $l = 0$ it is $w_1 - d_1/2 - H + (t+1)/2 = W + 2H - k/2$. Every height-one column shortens to zero and contributes nothing in positive height. This verifies $\beta(i) = \Lambda_i - \Lambda_{i+1}$ for $i < n$ and $\beta(n) = 0$. Summing up from the bottom proves the result. □

Theorem 577.2. *On the primary terminal space Y_m the right real convolution $P^R_{e_{T_{0,m}}}$ equals the compact projection Π_Y of Corollary 419.3. In particular, choose all real tests on the primary row $o = 0$ to be the positive-list canonical determinants. For any source form y of its prescribed real type and initial level $K_0(g)$, the weighted forward chain of Definition 422.1 satisfies*

$$C_{U_\vartheta y} = \mathcal{F}_{m-1} \cdots \mathcal{F}_0 y$$

as continuous automorphic functions at the terminal group, for every allowed twist. These fixed real tests can be used for the rows of Theorem 469.2.

Proof. At a real place the definite group is compact. Its polar positive part is identity. The compact coefficient identity in Theorem 100.3, or the compact minor integral of Definition 96.1, therefore gives $\psi_{T_{0,m}}(k) = m_\Lambda(k)$. In the ordered unit flag of Corollary 419.3 the inner product of a unit h-fold initial wedge with its image under $\bar{k}$ is the determinant of the initial h-by-h block of $\bar{k}$. This is the alternating multilinear formula of Lemma 414.2. Multiply these factors for the columns of β and include $(\det \bar{k})^{e_m}$ in R_Y. The preceding lemma identifies the resulting flag coefficient, before taking the product over places, with $\overline{m_\Lambda(k)}$. Ties in Λ have zero minor exponent, so the indicated parameter order gives the same coefficient on them.

Over the product group K, the probability integral of the squared modulus of this flag coefficient is $1/\dim E_Y$ by the compact orthogonality in Theorem 418.2. Equivalently, average the rank-one orthogonal projection onto the unit flag over its irreducible compact representation; the average is scalar and its trace is one. Changing from Haar probability to

the measures defining e multiplies the integral by their total mass on K. Consequently the measure $e_{T_{0,m}}(k)\,\mathrm{d}k$ there is exactly

$$\dim(E_Y)\langle u_0, R_Y(k)u_0\rangle\,\mathrm{d}k_{\mathrm{prob}}.$$

The translated-kernel formula in Proposition 472.1 now identifies the two operators pointwise.

Every unweighted canonical inclusion lift carries its real source type to its prescribed target type by Proposition 572.1. The positive finite index multiplying it in Definition 422.1 preserves that projection equation. Iterating makes the terminal function fixed by $P^R_{e_{T_{0,m}}} = \Pi_Y$. Definition 468.1 thus yields the displayed equality. Theorem 422.3 permits any fixed real tests on these arrows; our determinants are independent of finite twists by Theorem 557.3. The source rows have the required type and level by Theorem 498.2. This verifies the specialization to the index estimate. □

578. The canonical chains on their initial level

Theorem 578.1. *Use the framed spaces $S_{o,j}$ of Proposition 576.1, fix either array and one twist allowed there, and put $w_j = |\mathrm{GL}_{n_j}(R)|$. On the full spaces of positive-list forms take $a_{o,j}$ to be the positive-list inclusion lift. In each row take $p_{o,0}$ from Theorem 576.2 and put $p_{o,j+1} = p^+_{o,j}$ successively, using the finite extension of Theorem 499.3. Let $d_{o,j}$ be the transpose of the negative-list kernel on $S_{o+1,j} \to S_{o,j+1}$ with datum $p_{o,j}$.*

Let W be the space of initial positive-list forms at principal level on the marked modulus and integral elsewhere. One may also take a subspace invariant under every transported base return H_o of Theorem 576.2. Use j_o to identify each such base space with the initial one and write $A_{o,l} = a_{o,l-1}\cdots a_{o,0}$, including $A_{o,0} = 1$. For every $o+l \le m$, $A_{o,l}j_o$ is injective on W. The same holds for its weighted version $X_{o,l} = (\prod_{j<l} w_j)A_{o,l}j_o$. For the primary array at $o = 0$, $l = m$, the canonical map U_ϑ of Definition 468.1 is injective on the initial $K_0(g)$-level forms. In particular the terminal class Gram matrix on the short rows of Corollary 470.2 is positive definite.

Proof. The base space is finite dimensional by Theorem 571.1; all arrows act on the asserted types by Proposition 572.1. The base returns are injective endomorphisms of the stated spaces by Theorem 576.2. In a square $o + j + 2 \le m$, use the graph of Theorem 572.2. Its first block in $X(Z)$ and its second block in $Y(y)$ are precisely the two positive inclusion tests. Summing by the other block gives the defining finite function of each descent. Consequently

$$w_j a_{o+1,j} d_{o,j} = (\kappa^{\mathrm{up}}_{o,j}/c_{\nu,o,j})\, w_{j+2} d_{o,j+1} a_{o,j+1}.$$

Here $\kappa^{\mathrm{up}}_{o,j}$ is that theorem's κ, and $\nu = \eta^{-(j+1)}$ there. The finite functions may depend on the twist; the same identity applies by its quantification over every p. Thus all the hypotheses of Theorem 473.2 hold, including when a square scalar might be zero. It gives both injectivity assertions. In the primary array the initial level $K_0(g)$ imposes full integral invariance at twist places, so its forms belong to the indicated principal level. The weights w_j are the indices $[K_j : J_j]$ of Theorem 406.1: at all chosen split places every invertible residue matrix lifts to an integral one, with its partner determined by inverse conjugate transpose, and the kernel is the specified level. Theorem 577.2 identifies the resulting scalar terminal chain with $C_{U_\vartheta y}$. Independence of these scalar functions implies the positive-definite assertion by Corollary 472.2. □

Theorem 578.2. *For the primary array use any nonzero W as in Theorem 578.1. Write $S_o = w_0 H_o$ for the positive first-return endomorphisms transported to W. For each primary square set*

$$\beta_{o,j} = \overline{c_{\nu,o,j}/\kappa^{\mathrm{up}}_{o,j}}, \qquad D_{o,j} = \kappa_{\nu,o,j}\beta_{o,j},$$

where $\kappa_{\nu,o,j}$ is the Petersson scalar of Theorem 575.2. These quotients are defined and $D_{o,j} \neq 0$. Put $L_{o,l} = \prod_{h=0}^{l-2}\prod_{u=0}^{l-h-2} D_{o+h,u}$, with empty products equal to one. Then for

$x, y \in W$, $o + l \le m$,

$$\langle X_{o,l}x, X_{o,l}y\rangle = L_{o,l}\,\frac{w_0 w_1^{\,l}}{w_l}\,\langle x, S_o \cdots S_{o+l-1}y\rangle.$$

All S_o commute on W, and $L_{o,l} > 0$. At fixed arithmetic data there is $C_{o,j} > 0$, independent of the finite twist, such that, writing S for the full tame set and $Q_R = |R|$, one has

$$|D_{o,j}|^2 = C_{o,j}\,Q_R^{2D^2} \prod_{v=1}^{D} |L^S(v, \eta^{-(j+1)})|^2, \qquad D = t.$$

The size formula compares all twists for which the initial space under consideration is nonzero.

Proof. In the identity of Theorem 473.2 at length $j+1$, the left side $A_{o+1,j+1}j_{o+1}H_o$ is injective on the nonzero space W, by the preceding theorem and the base return. Thus none of the square scalars in the product on the right is zero. In particular $\kappa_{o,j}^{\mathrm{up}} \neq 0$. The first square has exactly multiplier $w_{j+2}/(w_j\bar{\beta}_{o,j})$. The base inner product identity is Theorem 576.2. For each adjacent pair of descents the datum is p, p^+ by construction. Theorem 575.2 therefore gives the adjoint inner product identities, including the required intermediate form for each terminal test. Theorem 474.1 now gives the displayed equality. Corollary 474.2 gives commutativity and positivity for every interval. The product on an interval starting at o of length $j+2$ contains $D_{o,j}$, so positivity implies that this factor is nonzero.

Theorem 568.2 gives $|c_{\nu,o,j}|^2 = CQ_R^{D^2}\prod_{v=1}^{D}|L^S(v,\nu)|^2$ because at the squarefree level $Q_R = \prod_{w\in S} q_w$. The modulus of $\kappa_{o,j}^{\mathrm{up}}$ is independent of the finite twist. Theorem 575.2 gives $|\kappa_{\nu,o,j}|^2 = C_{\mathrm{eq}}Q_R^{D^2}$. Multiplication gives the asserted $C_{o,j} = CC_{\mathrm{eq}}/|\kappa_{o,j}^{\mathrm{up}}|^2$; it is positive since the square factor is nonzero and all the Hecke factors are nonzero by Theorem 554.1 and the nonzero formula for $c_{\nu,o,j}$. □

579. Relative determinants at two twist levels

Lemma 579.1. *For each allowed primary twist ϑ, let P_ϑ be its twist places, $S_\vartheta = M_g \cup P_\vartheta$ and $\eta_\vartheta = \upsilon_T^{-h}\chi\vartheta$ as in Lemma 568.1. Here the trivial twist 1 has $P_1 = \varnothing$. For $1 \le l < m$ and real $s \ge 1$, put*

$$B_{l,\vartheta}(s) = L^{S_\vartheta}(s, \eta_\vartheta^{-l})$$

using Definition 515.1 and its continuation at 1. Then $B_{l,\vartheta}(s) > 0$.

Proof. For the associated unitary idele character $\lambda(x) = \eta_\vartheta^{-l}(x/\bar{x})$ one has $\lambda(\bar{x}) = \overline{\lambda(x)}$. The omitted places are stable under conjugation after adjoining $\bar{S}_\vartheta$. At conjugate distinct unmarked primes choose conjugate uniformizers. At real $s > 1$ their combined denominator is $|1 - \lambda_f(\pi_w)q_w^{-s}|^2 > 0$. If conjugation fixes the prime, the conjugate uniformizer differs by an unmarked unit. Its character value therefore equals its conjugate and is either 1 or -1. The denominator is again positive. The weight is $lh > 0$ in every component and the residue character is tame by Lemma 568.1 applied with $j = l - 1$. Theorem 515.3 gives absolute convergence and nonvanishing at $s > 1$. The limit of finite positive products is consequently positive. Continuity from the right at 1 and nonvanishing there, both from the same theorem, give the endpoint. □

Theorem 579.2. *Fix primary data at a scale where the base returns are positive definite. Put $H = r + t$, and use the fixed real tests of Theorem 577.2. Take a nonzero finite-dimensional initial space W at level $K_0(g)$, invariant under the transported operators $T_{o,\vartheta}$ and $T_{o,1}$ from Theorem 576.2, $0 \le o < m$. The same W and the same source inner product*

are used for both twists. Put $d = \dim W$, $Q = \prod_{w \in P_\vartheta} q_w$ *and*

$$J_\vartheta = \prod_{w \in P_\vartheta} \prod_{a=t+1}^{H} (1 - q_w^{-a}),$$

$$\Xi_\vartheta = J_\vartheta^m \prod_{l=1}^{m-1} \prod_{v=1}^{t} \left(\frac{B_{l,\vartheta}(v)}{B_{l,1}(v)} \right)^{m-l}.$$

Write $Z_\vartheta(W)$ *for the determinant of the terminal class Gram matrix of* U_ϑ *on an orthonormal basis of* W*, with class weights as in Proposition 471.2. Then*

$$\frac{Z_\vartheta(W)}{Z_1(W)} = \Xi_\vartheta^d \prod_{o=0}^{m-1} \frac{\det(T_{o,\vartheta}|_W)}{\det(T_{o,1}|_W)}.$$

The same ratio is obtained by evaluating both Gram matrices on any one ordered basis of W*.*

Proof. The source level $K_0(g)$ is integral at the twist primes, so its forms belong to the principal level required in Theorem 578.1 for either twist. The assumed invariance gives its other subspace hypothesis. Let $\mathcal{G}_\vartheta(x, y)$ denote the terminal weighted inner product on the two images of x, y. Theorem 577.2 identifies the terminal functions in Proposition 471.2 with the canonical forward images themselves. Theorem 578.2 then expresses their Petersson pairing as the scalar $L_{0,m} w_0 w_1^m / w_m$ times the source pairing with $\prod_{o=0}^{m-1} S_o$. The terminal index of Theorem 471.1 cancels this weight quotient. More precisely its product with $w_0 w_1^m / w_m$ is $CQ^{-k_m} \delta(P_\vartheta)^m$. Thus the class form, before determinants, is

$$\mathcal{G}_\vartheta(x, y) = b_\vartheta \left\langle x, \left(\prod_{o=0}^{m-1} T_{o,\vartheta} \right) y \right\rangle, \qquad b_\vartheta > 0.$$

Indeed $S_o = w_0 H_o = c_o |R_\vartheta|^{-Hr} T_{o,\vartheta}$ by Theorem 576.2. Here $c_o = \mathrm{vol}[U(S_{o,1})] \| P_{\bar{e}}^{\Omega_S} z_a \|^2 / e(1)$ is positive and independent of the finite twist. The rational spaces, quotient measures, real tests and real actions in this factor are fixed by Proposition 576.1 and Lemma 568.1. The terminal factor d_Y / b_H is fixed as well. Since $|R_\vartheta| / |R_1| = Q$, division gives

$$\frac{b_\vartheta}{b_1} = Q^{-k_m - mHr} \left(\frac{\delta(P_\vartheta)}{\delta(\varnothing)} \right)^m \frac{L_{0,m}(\vartheta)}{L_{0,m}(1)}.$$

There are exactly $m - l$ factors with $j = l - 1$ in the triangular product for $L_{0,m}$, one for each $o = 0, \dots, m - l - 1$. Its positivity for both twists in Theorem 578.2 permits taking absolute values of the individual square factors. Their fixed constants cancel, and the positive root from Lemma 579.1 gives

$$\frac{L_{0,m}(\vartheta)}{L_{0,m}(1)} = Q^{t^2 \binom{m}{2}} \prod_{l=1}^{m-1} \prod_{v=1}^{t} \left(\frac{B_{l,\vartheta}(v)}{B_{l,1}(v)} \right)^{m-l}.$$

Finally the auxiliary index calculation in Theorem 471.1 gives $\delta(P_\vartheta) / \delta(\varnothing) = A_H / A_t$, where $A_j = \prod_{w \in P_\vartheta} |\mathrm{GL}_j(\mathbf{F}_w)|$. The same count gives $A_H / A_t = Q^{H^2 - t^2} J_\vartheta$. Using $k_m = mrt + \binom{m}{2} t^2$ and $H^2 - t^2 = r(H + t)$ yields $b_\vartheta / b_1 = \Xi_\vartheta$. Taking the determinant in an orthonormal source basis proves the assertion. A common change of basis with matrix A multiplies both determinants by $|\det A|^2$. □

580. Uniform real estimates on the selected rows

Proposition 580.1. *Under the hypotheses and notation of Theorem 579.2, let* $y = (y_1, \dots, y_e)$ *be any independent list in* W*,* $e \geq 1$*. Write* $Z_\vartheta(y)$ *for its terminal class Gram determinant. With* $\epsilon_H = \exp(24 f \, 2^{-H/2+2r}) - 1$*, assume* $H/2 - 2r \geq 2$ *and* $\epsilon_H < 1$*. Then*

$$\left| \frac{1}{e} \log \frac{Z_\vartheta(y)}{Z_1(y)} - \log \Xi_\vartheta \right| \leq m \log \frac{1 + \epsilon_H}{1 - \epsilon_H}.$$

This applies to the common list of short independent source rows of Corollary 472.2.

Proof. Use the identity of forms in the proof of Theorem 579.2. For each fixed twist the transported operators commute and are self-adjoint by Theorems 578.2 and 478.1. Take an orthonormal basis diagonalizing this finite family, by the successive eigenspace argument of Lemma 79.2. Each eigenvalue is in $[1-\epsilon_H, 1+\epsilon_H]$ by the latter operator bound. The product has spectrum in the interval with these endpoints raised to the m-th power. Its Hermitian form satisfies the corresponding two quadratic inequalities on all of W, hence on the span of y. In an orthonormal basis of this span, the eigenvalues of the restricted form lie in the same interval by evaluation on its unit eigenvectors. Its determinant lies between $(1-\epsilon_H)^{me}$ and $(1+\epsilon_H)^{me}$. Changing to the list y contributes the determinant of its source Gram matrix for both twists. Divide the two estimates, use $b_\vartheta/b_1 = \Xi_\vartheta$, and take logarithms. For the stated short rows take the entire initial space of level $K_0(g)$, which the returns preserve by Theorem 576.2. □

Theorem 580.2. *Keep these hypotheses, with $f = [E:\mathbb{Q}]/2$. For any integer $2 \le k \le t+1$, define*

$$V_{\vartheta,k} = \prod_{l=1}^{m-1} \prod_{v=1}^{k-1} \left(\frac{B_{l,\vartheta}(v)}{B_{l,1}(v)} \right)^{m-l}.$$

Uniformly in the twist, source list and excluded places,

$$\left| \frac{1}{e} \log \frac{Z_\vartheta(y)}{Z_1(y)} - \log V_{\vartheta,k} \right| \le m \log \frac{1+\epsilon_H}{1-\epsilon_H} + 12mf\, 2^{-t} + 48 \binom{m}{2} f\, 2^{-k}.$$

When $k = t+1$ the last term can be replaced by zero. For $k = 2$ the product on the left involves only Hecke values at 1.

Proof. Above a rational prime p there are at most $2f$ primes of E, each of norm at least p, by the degree calculation in Theorem 477.2. For an integer $v \ge 2$, monotonicity of u^{-v} gives

$$\sum_w q_w^{-v} \le 2f \sum_{n=2}^{\infty} n^{-v} \le 2f \left(2^{-v} + \int_2^{\infty} u^{-v}\, \mathrm{d}u \right) \le 6f\, 2^{-v}.$$

If $|z| \le 1/2$, the inequalities $1 - |z| \le |1-z| \le 1 + |z|$ and integration of $1/(1-u) \le 2$ on $0 \le u \le |z|$ give $|\log|1-z|| \le 2|z|$. Apply this to the absolutely convergent Euler product of Definition 515.1. The positivity in Lemma 579.1 then yields $|\log B_{l,\vartheta}(v)| \le 12f\, 2^{-v}$. The same inequality holds for the untwisted character, so the absolute logarithm of the ratio is at most $24f\, 2^{-v}$. Summing for $k \le v \le t$, and summing the multiplicities $m-l$, gives at most $48f\, 2^{-k}\binom{m}{2}$, and gives zero for the empty range. The same estimate for $\log(1-q_w^{-a})$ over the subset P_ϑ gives $-\log J_\vartheta \le \sum_{a=t+1}^{H} 12f\, 2^{-a} \le 12f\, 2^{-t}$. Insert these two bounds into Proposition 580.1. □

581. The determinant test on the auxiliary unitary pair

Definition 581.1. Fix $1 \le j \le m$. Put $o = j-1$, $p = k+o$, $q = r-p$, $H = r+t$. Let W_j be the rational diagonal hermitian space with p entries 1 and q entries -1 in the order of Definition 497.1, and let $V_j = E^H$ have identity form. Use the unitary finite frames for the identity lattice constructed in the proof of Proposition 576.1. Choose the r old coordinates of V_j as a copy of E^r, with ordered complement of rank t. Denote by i the inclusion and by L the projection onto these r coordinates, so that $Li = 1_r$. For a fixed primary twist ϑ take $\mu = \mu_j$ from Definition 376.1. Write ψ_w for its residue characters, $M = M_g$, $P = P_\vartheta$, $S = M \cup P$, $R = \prod_{w \in S} \mathbf{F}_w$, and $Q = \prod_{w \in P} q_w$. Use the pair action of Theorem 396.1 with denominator character $\mu \upsilon_T^{\lfloor H/2 \rfloor}$ on the source determinant and numerator character υ_T^q on the target determinant. Equivalently the latter is obtained from the earlier action by the automorphic determinant character $\upsilon_T^q / (\mu \upsilon_T^{\lfloor r/2 \rfloor})$ on the target.

On H-by-r matrices X over R, define

$$I(X) = \mathbf{1}_{X_w=i\ (w\in M)}\mathbf{1}_{LX_w=1_r\ (w\in P)},$$
$$F(X) = \mathbf{1}_{X_w=i\ (w\in M)}\prod_{w\in P}\psi_w(\det(LX_w)).$$

Extend each multiplicative character by zero at zero, including the trivial character. For either test use its inflation on integral matrices, zero elsewhere at the split pairs of S, and the usual vacuum off these pairs. Let J_{aux} be the target level of Theorem 471.1 of index $I_{\mathrm{aux}}(P)$. At $w \in P$, an element preserving the complement of i has a well-defined invertible quotient block a_w on the old coordinates.

Proposition 581.2. *Let k_s, k_t be integral source and target elements. At $w \in M$ suppose that $\bar{k}_{t,w} i = i\bar{k}_{s,w}$, with bars here denoting residue matrices. At $w \in P$ suppose that $\bar{k}_{t,w}$ preserves the complement of i, with quotient block a_w. Then on the finite datum F the pair action is multiplication by*

$$\prod_{w\in M}\psi_w(\det \bar{k}_{s,w})^{-1}\prod_{w\in P}\psi_w(\det a_w)^{-1}.$$

In particular the target level J_{aux} acts on F by the second product. Let $\gamma_P = \prod_{w\in P}|\mathrm{GL}_r(\mathbf{F}_w)|$. On every source form of level $K_0(g)$, the forward lift with F agrees with the forward lift with $\gamma_P I$, for the same real test, whenever its integral exists absolutely. For the finite vector $v_{\gamma_P I}$ the squared norm and the finite factor in Theorem 491.2 are respectively

$$\|v_{\gamma_P I}\|^2 = |R|^{-Hr}\gamma_P^2 Q^{tr}, \qquad \frac{\|v_{\gamma_P I}\|^2}{|\mathrm{GL}_r(R)|} = \frac{|R|^{-Hr}}{I_{\mathrm{aux}}(P)}\,\delta(P).$$

Proof. For integral elements all Jacobians and ideal multipliers are one. The character v_T is trivial on finite integral units by Theorem 374.2. Thus in the split realization of Theorem 360.1 the action evaluates the residue function at $\bar{k}_t^{-1} X \bar{k}_s$, with the scalar $\prod_{w\in S}\psi_w(\det \bar{k}_{s,w})^{-1}$. At M the marked point condition is preserved in both directions. At P the top block of the substituted matrix is $a_w^{-1}(LX_w)\bar{k}_{s,w}$, since the top right block of $\bar{k}_{t,w}^{-1}$ is zero. Taking determinants cancels the source character at P, including when the determinant vanishes. Off these pairs the integral groups fix the vacuum by Theorem 390.2. This proves the transformation law.

For $A = (A_w)_{w\in P}$ in the product of the residue groups, take an integral source lift k_A of A^{-1} at P and identity elsewhere, using split matrix lifting as in Theorem 471.1. The vector $\Omega(k_A, 1)I$ is the indicator of the same condition at M and $LX_w = A_w$ at P, multiplied by $\prod_{w\in P}\psi_w(\det A_w)$. Summing over A gives F exactly. Covariance of the kernel and right translation in the source integral replace the lift of each term by the lift of I, since the source form is fixed by k_A. This proves the asserted equality. At each $w \in P$, the lower t-by-r block of I is arbitrary, giving Q^{tr} simultaneous possibilities. At M there is only one. Every occupied matrix satisfies $LX = 1_r$. Apply Theorem 476.1 with this common left inverse to $\gamma_P I$. Its norm formula and the definition $\delta(P) = I_{\mathrm{aux}}(P)\gamma_P^2 Q^{tr}/|\mathrm{GL}_r(R)|$ in Theorem 471.1 give both equalities. □

582. The class Gram form of the auxiliary lifts

Lemma 582.1. *Use the groups and characters of Definition 581.1, and set $w = w_j(\tau)$ at an infinite place τ. Write λ_i, $0 \le i < r$, for the source parameters of Definition 497.1 at $z = 0$. Let z_j be the real tensor of the auxiliary determinant tests from Theorem 497.2. For the pair action of Definition 581.1 the target compact weight λ_τ^+, in its decreasing order, consists of $b_0, \dots, b_{p-1}$, followed by t zeros, followed by $b_p, \dots, b_{r-1}$, where*

$$b_i = \begin{cases} \lambda_i + w + (1-t)/2 + i - p & (0 \le i < p), \\ \lambda_i + w + (t+1)/2 + i - p & (p \le i < r). \end{cases}$$

Every prefix entry is positive and every suffix entry is negative. The target compact projection with coefficient $\prod_\tau \overline{m_{\lambda_\tau^+}}$, divided by its squared norm for probability Haar measure, fixes z_j. The source projection $P_{\bar e}^{\Omega} z_j$ is nonzero, where e is the source kernel with parameter λ.

Proof. The source phase and source action on the polynomial have not changed: the ratio of the pair characters is evaluated on the target determinant alone. In Theorem 496.1 one has $\alpha = w - H/2$ and $b_i = \Lambda_i + \alpha + h_{-s_i}$, where the target sign counts are $h_+ = H,\ h_- = 0$. The source compact weight on equal-sign prefixes satisfies $\Lambda_i = \lambda_i + (r+1)/2 - i - n_{s_i} + 2B_i$, where B_i is the number of earlier indices of that sign. For $i < p$ we have $n_{s_i} = p,\ B_i = i$. For $i \ge p$ we have $n_{s_i} = q,\ B_i = i - p$. Substitution gives exactly the two formulas in the statement. Their signs and weak decrease in each block are those proved for this auxiliary signature in Theorem 497.2.

The target determinant exponent in the new action is q. Its vacuum weight in the positive target block is thus $q - n_- = 0$ by the target vacuum calculation in Theorem 496.1. The old positions have this vacuum weight plus b_i, and the new positions have the vacuum weight. More explicitly, changing the determinant exponent by an integer u multiplies the target action by $(\det k)^u$ and its coefficient m by the same factor, by Definition 96.1. Their factors cancel in the average with $\bar m$. Its squared coefficient norm is unchanged. The fixed-vector assertion follows from the original compact average and nonvanishing of the source projection from Theorem 497.2. □

Theorem 582.2. *Use the notation of Lemma 582.1. Let $G = U(W_j)$, $H_j = U(V_j)$ and let $\mathcal{P}_j$ consist of $y \in L^2([G])$ fixed by the source projection P_e^R and by $K_0(g)$. For each real place take the irreducible representation of weight λ_τ^+ from Theorem 77.2, composed with entrywise conjugation on the compact unitary group. Let E_j be their tensor product, R_j its action, $d_j = \dim E_j$ and u_j its unit flag tensor. Write $\epsilon_{\mathrm{aux}}(k) = \prod_{w\in P} \psi_w(\det a_w)^{-1}$ for $k \in J_{\mathrm{aux}}$. There is a linear map $v_{j,\vartheta} : \mathcal{P}_j \to \{\xi : (H_j)_f \to E_j\}$ with*

$$\xi(\gamma_f a k) = \epsilon_{\mathrm{aux}}(k) R_j(\gamma_\infty)\xi(a) \quad (\gamma \in H_j(E^+),\ k \in J_{\mathrm{aux}}).$$

Its scalar coefficient $\langle R_j(u)u_j, \xi(a)\rangle$ is the theta lift with $z_j \otimes F$ of Definition 581.1, with the bounded extension of Theorem 491.2. Let $(\ ,\ )_{\mathrm{cl}}$ be the sum of the target coefficient pairings over the finite double classes at J_{aux}, each divided by its rational stabilizer order. There is a positive constant a_j, independent of the finite twist, such that

$$(v_{j,\vartheta}x, v_{j,\vartheta}y)_{\mathrm{cl}} = a_j\, |R|^{-Hr} \delta(P) \langle x, T_{j,\vartheta}^+ y\rangle_{[G]} \qquad (x, y \in \mathcal{P}_j).$$

Here $T_{j,\vartheta}^+ = T_H$ of Theorem 478.1, acting on G in the specified finite frame with source denominator $\mu v_T^{\lfloor H/2 \rfloor}$. For sufficiently large scales $v_{j,\vartheta}$ is injective, and $\mathcal{P}_j$ is finite dimensional.

Proof. First use bounded compactly supported functions on $[G]$ invariant under $K_0(g)$. Theorem 491.2 applies with $h = H = 6r$, V_j definite with standard self-dual finite lattice, and finite test $\gamma_P I$, which has the left inverse on its support proved in Proposition 581.2. The target character change multiplies its lifted functions by a unitary automorphic character and preserves their Petersson products. The finite translate calculation of that proposition identifies these lifts with the corresponding integrals for F. Such source functions are dense in the subspace fixed by $K_0(g)$: truncate an L^2 function on compact sets and in amplitude and average over the compact group. Averaging has norm at most one by the integral triangle inequality and the product of a compact set with this group is compact. The bounded extension therefore defines the lift on that subspace. The J_{aux} transformation and the compact projection identity persist in L^2, by covariance of the kernel, Proposition 581.2 and Lemma 582.1.

The coefficient of u_j in $R_j(k)u_j$ is precisely $\prod_\tau \overline{m_{\lambda_\tau^+}(k_\tau)}$ by Theorem 77.2. Averaging the orthogonal projection onto the translates of u_j gives $1/d_j$ times the identity by the compact

orthogonality proof in Theorem 418.2. The product version follows by applying that identity at each place. On a projected function A the vector

$$\xi(a) = d_j \int_{(H_j)_\infty} R_j(u) u_j A(u,a)\,\mathrm{d}u$$

has coefficient A, by the integral kernel computation in Corollary 419.3. Conversely that integral recovers ξ from its coefficient. The finite classes are finite in number by Lemma 411.2. On each class the finite transformation law and compact projection give a continuous representative, with the stated rational covariance by changing u by γ_∞ in the integral. This defines $v_{j,\vartheta}$, including on L^2 classes.

The same compact average and the two quotient measures of Lemma 411.2 express the class pairing as $d_j I_{\mathrm{aux}}(P)/b_{H_j}$ times the Petersson pairing of the lifted functions. In the latter pairing Theorem 491.2 gives the finite factor of Proposition 581.2. At infinity Theorem 480.2 gives integrability. The compact source average from Theorem 496.1 permits Lemma 506.1 to act on $y = P_e^R y$. The real factor is $\|P_{\bar{e}}^{\Omega} z_j\|^2/e(1)$. Thus

$$a_j = \frac{d_j \operatorname{vol}[H_j]}{b_{H_j}} \frac{\|P_{\bar{e}}^{\Omega} z_j\|^2}{e(1)} > 0.$$

Its data are independent of the finite twist by the preceding lemma and the fixed quotient measures. The unmarked T_H commutes with the real projection and with the integral actions at $M \cup P$, since it acts in the complementary coordinates. It preserves the other unmarked integral conditions by Theorem 478.1. Its lower bound $1 - \epsilon_H > 0$ holds at sufficiently large scales by Theorem 477.2. The class norm of the image of a nonzero source vector is therefore positive. Finally evaluation on the finite class representatives embeds the target coefficient space in finitely many copies of E_j, proving the dimension assertion. □

583. The auxiliary determinants on the good subspaces

Lemma 583.1. *Let $o = j - 1$, $1 \le j \le m$. On the principal source level $K_0(g)$, the two operators $T_{o,\vartheta}$ of Theorem 579.2 and $T^{+}_{j,\vartheta}$ of Theorem 582.2 are operator-norm limits of the actions of the same finite tests in the rank r group, identified in the integral finite frames. Those tests are compactly supported and bi-$K_0(g)$-invariant, act by the identity factor at the principal marked level and commute with the marked finite groups.*

Proof. In the initial and auxiliary finite frames the integral lattices on the source and the target are the standard ones. In passing to an opposite base at another offset the finite frame is multiplied by a scalar unit by Proposition 576.1. Conjugation by this scalar does not change any matrix of a finite group element. The rank H, the tame places S and the denominator character $\mu^{\flat}(\upsilon_T^{-h}\chi\vartheta)^j \upsilon_T^{\lfloor H/2 \rfloor}$ are the same in the two source actions by Theorem 576.2 and Definition 581.1. The formula for ρ_H in Theorem 476.1 depends just on these data and the row lattice in the source frame. The target numerator character is evaluated at the identity. Thus the cosets, weights and integral stabilizer K_* off the marked and twist pairs agree for the two sums.

Truncate to finitely many double K_*-cosets. Each is a finite union of right cosets because the intersection of K_* with its conjugate is open of finite index in the compact group. On one such right coset the integral of translation against the indicator, divided by the volume of K_*, acts on fixed vectors by any representative. The coefficient ρ_H is bi-invariant by Theorem 478.1. At the omitted coordinates use the probability indicators of the principal subgroup at M_g and the integral subgroup at P_ϑ. These act as identity on the common prescribed level. The resulting tests have all the stated support and commutation properties. Their tails in operator norm are at most the tails of the common absolutely summable series of $|\rho_H|$ by unitarity and Theorem 477.2. □

Theorem 583.2. *On the primary base and on W_j take the good images W, W_j^{+} of e_0^{good} in their principal-level source spaces, in the notation of Definition 110.1 and Theorem 111.2. Use for the joint eigensystems all split spherical tests off the marked level. Assume for each pair of these groups (the primary group and W_j) the following condition. For every joint eigensystem and every good tuple there is a primitive coordinate over g_0 at which one cubic alternative occurs on neither space of that eigensystem in the two $\mathcal{E}$-images of Definition 110.1. Suppose $d = \dim W > 0$. Then $\dim W_j^{+} = d$ for every j. Write $Z_\vartheta = Z(U_\vartheta|_W)$ and $Z_{j,\vartheta}^{+} = Z(v_{j,\vartheta}|_{W_j^{+}})$ with the class pairings of Theorems 579.2 and 582.2. For every allowed primary twist,*

$$\frac{Z_\vartheta / Z_1}{\prod_{j=1}^{m} (Z_{j,\vartheta}^{+} / Z_{j,1}^{+})} = \left\{ Q^{-mrt} \prod_{l=1}^{m-1} \prod_{v=1}^{t} \left(\frac{B_{l,\vartheta}(v)}{B_{l,1}(v)} \right)^{m-l} \right\}^{d}.$$

Proof. Both groups have constant signature over the real places, the same row parameters and standard finite frames. In both, the first $k-1$ signs coincide within the list, by Definition 497.1. Thus Corollary 106.3 and Theorem 110.3 evaluate a given marked elliptic test, supported in $\mathcal{E}$ on both sides under convolution, by the same definite trace. When a group is already definite this is its compact trace by Proposition 80.2. All the marked spaces are finite dimensional by Theorem 109.3. Apply the construction of Theorem 111.2 directly to the two $\mathcal{E}$-images. For precision, on each joint system ω take its finite spherical polynomial projector from Lemma 79.2. For each good tuple subtract from its label projector the one obtained by substitution of its missing cubic alternative. Their sum D_ω acts on both ω-spaces as the good projection. Its support has irreducible reduction at some marked coordinate by Corollary 75.3. Moreover $\mathcal{E} D_\omega = D_\omega \mathcal{E} = D_\omega$, since every tuple in the differences has a primitive coordinate. Convolving any of the finite tests in Lemma 583.1, or any power of that test, with the spherical projector and D_ω retains that support condition. Indeed the former tests and the spherical projector have principal reduction at all the marked coordinates and commute with their group algebra. The trace equality just proved therefore applies. Sum over ω to obtain equality of traces on the two good spaces. The same argument with the identity gives their common dimension.

Applying the trace equalities to the positive powers and the power-sum identities in Theorem 79.3 gives equality of determinants of each truncation. The good projections commute with the truncations, so restriction to their ranges also converges. A determinant in dimension d is a finite polynomial in the entries, which converge by the norm bound in the preceding lemma. The normalized return determinants on the two spaces thus agree, for the chosen twist and for the trivial one.

By Theorem 582.2 the auxiliary volume ratio is

$$\left(Q^{-Hr} \frac{\delta(P)}{\delta(\varnothing)} \right)^{d} \frac{\det(T_{j,\vartheta}^{+}|_{W_j^{+}})}{\det(T_{j,1}^{+}|_{W_j^{+}})}.$$

Its scalar in parentheses equals $Q^{rt} J_\vartheta$ by the index computation of Theorem 579.2. In the quotient with the primary formula of that theorem all normalized return determinants now cancel, as does J_ϑ^{md}. This gives the displayed formula. □

584. The definite one-vector mean on Schwartz functions

Lemma 584.1. *Let $a \geq 1$, $f \geq 1$ be integers, let $y = (y_\tau)_{\tau=1}^{f}$ belong to $(\mathbb{C}^a)^f$, and write $H(y_\tau) = \sum_i |y_{\tau,i}|^2$. There is a finite sum $p_a(\phi)$ of Schwartz seminorms, with coefficients allowed to depend on a, f, such that for every real tuple t and every Schwartz function ϕ,*

$$\left| \int \phi(y) \exp\left(-2\pi i \sum_\tau t_\tau H(y_\tau) \right) \, \mathrm{d}y \right| \leq p_a(\phi) \prod_\tau (1 + |t_\tau|)^{-a}.$$

Here each complex coordinate has ordinary two-dimensional Lebesgue measure.

Proof. For $|t_\tau| \geq 1$ multiply the corresponding phase by $\exp(-\pi\delta H(y_\tau))$, $0 < \delta \leq 1$. In $2a$ real dimensions the Fourier transform of this damped phase at ξ, with phase convention $e^{-2\pi i x\cdot\xi}$, is

$$(\delta + 2it_\tau)^{-a} \exp(-\pi|\xi|^2/(\delta + 2it_\tau)).$$

This is the product Gaussian transform from Lemma 4.2, continued in the right half-plane by Lemma A.3. Its modulus is at most $(2|t_\tau|)^{-a}$. Let J be the set of these large coordinates. Insert Fourier inversion of ϕ in these coordinates alone, leaving the other phases of modulus one unchanged. For each fixed δ absolute integration is justified by integrability of the damped Gaussian and of the partial transform in the remaining variables and frequencies. In detail, integration by parts in the original large coordinates bounds $(1 + 4\pi^2|\xi_J|^2)^N|\widehat{\phi}_J(\xi_J, y_{J^c})|$ by the integral in y_J of $|(1 - \Delta_J)^N\phi|$. For $N > af$ the inverse weight is integrable in ξ_J; the integral of the derivatives over all y is bounded by finitely many Schwartz seminorms using a power of $1 + |y|$ larger than $2af$. The inversion is the real Fourier inversion of Lemma 392.1. The resulting bound is uniform in δ. Let δ decrease to zero by dominated convergence in the original integral. Bounding the finitely many choices of J together, and using $1 + |t| \leq 2$ for $|t| < 1$, proves the assertion. □

Theorem 584.2. *Let $a \geq 3$, $V = E^a$ with the standard positive definite hermitian form H in the arithmetic CM extension E/F. For every Schwartz test ϕ on $V(\mathbb{A}_F)$, with the measures of Definition 62.1, one has*

$$\frac{1}{\operatorname{vol}[U(V)]} \int_{[U(V)]} \sum_{y \in E^a} \phi(b^{-1}y)\, \mathrm{d}b = \phi(0) + \sum_{t \in F} \int_{V(\mathbb{A}_F)} \phi(y)\Psi(-tH(y))\, \mathrm{d}y.$$

The left integral of the absolute theta sum and the series of absolute values of the right integrals are finite. At fixed finite test both sides are continuous in the real Schwartz topology.

Proof. It suffices by linearity to fix a finite test and let the real factor ϕ_∞ vary. The definite quotient has finitely many finite classes at its level by Lemma 411.2. Their finite supports restrict the rational vectors to finitely many translates of full real lattices, by the lattice correspondence in Lemma 13.4. Over the compact real unitary group, a sufficiently large power-decay seminorm bounds the sum uniformly. Indeed the number of points of a fixed lattice translate in a ball of radius $R \geq 1$ is $O((1 + R)^{2af})$, by disjoint translates of a small ball about the lattice points. Sum over dyadic shells with a larger decay exponent. This proves continuity and absolute integration on the left.

For $t \in F$, the finite quadratic integral is bounded by $CD(t)^{-a}$ by the coset calculation in Proposition 63.3, with $D(t)$ of Lemma 63.1. The preceding lemma bounds the archimedean factor by its seminorm times $\prod_\tau (1 + |\tau(t)|)^{-a}$. Lemma 63.1 at $R = 1$ sums the product for $a > 2$. Evaluation at zero is also continuous.

Both sides are unchanged by averaging the real test over $\prod_\tau U(a)$ with probability Haar measure: use right translation in the quotient integral and preservation of H and Lebesgue measure in each quadratic integral. The interchange with sums follows from the estimates just given, uniform over this compact group. Polynomials times $\exp(-\pi\sum_\tau H(y_\tau))$ are dense in the real Schwartz space by Lemma 549.1: the Taylor polynomials of a modulated vacuum have this form and converge in its Schwartz seminorms, and these vacua have dense span. A fixed real dilation permits this normalization of the Gaussian. The compact average of any such polynomial is a polynomial in $H(y_\tau)$. In fact the averaged polynomial is invariant, and the unitary group takes each y_τ to its length times the first coordinate vector by orthonormal completion. Restriction to those real coordinate axes gives a polynomial even in each length, since a sign change belongs to the group. Thus only integral powers of the squared lengths occur.

For a product of radial Gaussians $\exp(-\pi\sum_\tau z_\tau H(y_\tau))$, $z_\tau > 0$, the asserted identity is Theorem 64.2; the normalized definite class sum there equals the given integral by

Lemma 411.2, evaluated also on the constant one to normalize its mass. Partial differentiation in the real parameters z_τ is valid in the Schwartz topology, by the Gaussian bound for the derivatives on compact subsets of the positive orthant. At $z_\tau = 1$ these derivatives give all the radial polynomial multipliers. Continuity and the density just proved complete the argument. □

585. The doubling identity for a positive line

Lemma 585.1. *Let $S = E$ have form $\bar{x}y$ and $V = E^a$ have identity form. Put $D = S \oplus S^-$. Take the rational hyperbolic basis $e = (1, 1)$, $f = (1/2, -1/2)$ of D, and write $x = X(e)$, $z = X(f)$ for $X \in \mathrm{Hom}_E(D, V)$. Use configuration x and modulation z for its rational Schrödinger realization. For a test ϕ in this realization write $Z_A(\phi)$ for the section mean over $X|_A = 0$ of Definition 483.1. For $A_\infty = Ee$, $Z_{A_\infty}(\phi) = \phi(0)$. The other rational isotropic lines A give, once each,*

$$\int_{V(\mathbb{A}_F)} \phi(y)\Psi(-tH(y))\,\mathrm{d}y, \qquad t \in F.$$

The additive measure here has rational covolume one.

Proof. In this basis $H_D(e, f) = H_D(f, e) = 1$ and both diagonal entries are zero. The line A_∞ has modulation space $x = 0$, whose compact additive quotient kills every nonzero rational frequency by the calculation in Theorem 385.3. For the line Ef the space of integration is $z = 0$. Both terms of the quadratic correction q in that theorem vanish there, since the configuration space is totally hermitian isotropic. Thus the integrand is $\sum_{\xi \in E^a} \phi(\xi + x)$. Integrating over its rational additive quotient unfolds to $\int \phi$, with the measure of Lemma 62.2. Absolute convergence follows by lattice decay at infinity and compact finite support.

Every line other than Ee has a unique generator $f + ce$ with coefficient of f equal to one. Isotropy reads $c + \bar{c} = 0$. Its integration space is $z = -cx$. The rational shear $(x, z) \mapsto (x, z - cx)$ carries the configuration space to this graph. It preserves the hermitian form $H_V(x, z') + H_V(z, x')$, hence preserves the alternating pairing and the rational additive quotient measure. Transfer the graph mean back to $z = 0$ by acting on the test with the inverse shear from Theorem 386.1. Its linear substitution on the configuration is identity, and its multiplier at y is $\mathbf{e}(-a((y, 0), (0, cy))/2)$, where a in this expression denotes the alternating form of Definition 385.1. In that definition $H((y, 0), (0, cy)) = cH(y)$ and its conjugate is $-cH(y)$. Since $\jmath c \in F$ and $\omega + \bar{\omega} = 1$, it gives

$$-a((y, 0), (0, cy))/2 = \mathrm{Tr}_{F/\mathbb{Q}}(\jmath cH(y)).$$

For an adele s of F, $\mathbf{e}(\mathrm{Tr}_{F/\mathbb{Q}} s) = \Psi(-s)$ by the real signs and finite trace formula of Definitions 385.1 and 62.1. Thus the asserted frequency is $t = \jmath c$. Multiplication by the nonzero anti-invariant $\jmath$ takes the anti-invariant line of E bijectively onto F, proving the parametrization. □

Theorem 585.2. *Let $a \geq 3$, keep S, V of the preceding lemma, and write $G = U(S), H_a = U(V)$. Take any two Schwartz oscillator tests f_1, f_2 for this pair, with the same unitary splitting characters as in Theorem 396.1. Any further unitary automorphic determinant twists on the source and target are allowed. For $g_1, g_2 \in G(\mathbb{A}_F)$ write $\Theta_f(g, b) = \theta(\Omega(g, b)f)$. Then*

$$\frac{1}{\mathrm{vol}[H_a]} \int_{[H_a]} \overline{\Theta_{f_2}(g_2, b)}\,\Theta_{f_1}(g_1, b)\,\mathrm{d}b = \sum_{\gamma \in G(F)} \langle f_2, \Omega(g_2^{-1}\gamma g_1, 1)f_1 \rangle.$$

The series converges absolutely, uniformly for g_1, g_2 in any fixed compact set.

Proof. Set $\xi_i = \Omega(g_i, 1)f_i$. The doubled test $\xi_1 \otimes \overline{\xi_2}$ uses the opposite alternating form on its second factor. In the product polarization its theta value is the product of the two theta values by multiplication of the absolutely convergent lattice sums. Change to the rational

polarization of the preceding lemma by Theorem 387.2, preserving the section under this coordinate change. The resulting test ϕ is Schwartz. Simultaneous target translation acts in this polarization by $\phi(y) \mapsto \phi(b^{-1}y)$. Indeed this substitution preserves volume and implements the Heisenberg change, so it differs from the doubled action at most by a unit scalar as in the intertwiner calculation of Theorem 491.1. To determine the scalar, Ee is the graph with $\gamma = -1$ in Lemma 479.2; one can instead use Ef, the graph with $\gamma = 1$. Its mean on a tensor of a nonzero test with its conjugate is the squared norm of the test. The doubled action preserves this number by unitarity, and the ordinary substitution preserves its integral by the preceding lemma at frequency zero. Consequently the scalar is one for every b. Any extra target determinant factors cancel in the tensor; source determinant characters are trivial on rational γ.

Apply Theorem 584.2 to ϕ and use Lemma 585.1 to express its right side as $\sum_A Z_A(\phi)$. Since S is positive definite, each such A is the graph of a unique rational isometry by Theorem 479.3. Its term is $\langle \xi_2, \Omega(\gamma, 1)\xi_1 \rangle$ by Lemma 479.2. This proves the formula with its absolute convergence. On compact sets of the g_i the real factors have bounded Schwartz seminorms by continuity of the action in Theorem 393.1, including the fixed polarization change. The finite factors range in a finite list up to unit scalars, by their open stabilizers as in Theorem 483.3. Hence the summable bound from the proof of Theorem 584.2 is uniform on these compact sets. □

586. Positive-line lifts for the critical integers above one

Lemma 586.1. *Fix $1 \le l < m$, $2 \le v \le t$, set $a = 2v$, $\mu = \eta_\vartheta^l$ and let $S = S_\vartheta$ be as in Lemma 579.1. Use the pair $U(1), U(a)$ of standard definite groups, with source denominator μv_T^v and trivial target numerator on determinants. On the unmarked finite source modulo the integral subgroup let ρ_a be the vacuum coefficient of Theorem 476.1 for source rank one. Its series converges absolutely, and*

$$\sum_{gK_*} \rho_a(g) = B_{l,\vartheta}(v) \prod_{\mathfrak{u} \notin S_F} (1 - (N\mathfrak{u})^{-2v}),$$

where S_F is the set of primes of F below S.

Proof. At a split $\mathfrak{u}$ off S_F, choose w above it and write $q = N\mathfrak{u}$. The unitary line is the multiplicative group of the local field at w. Modulo its units it is generated by $b = \pi_w/\bar{\pi}_w$, with π_w inserted at w only. On b^n the row lattice ideal in Theorem 476.1 has exponents $\max(n, 0), \max(-n, 0)$ at $w, \bar{w}$, respectively. The conjugation identity for the splitting character in Theorem 374.2 gives $\beta(\bar{w}) = \beta(w)^{-1}$ and $v_T(b) = \beta(w)^2$, writing $\beta(w)$ for the value of its finite ideal character on the prime ideal. Consequently $\beta(\mathfrak{a}(b^n))^{2v} = v_T(b^n)^v$. The coefficient is $\mu(b)^{-n} q^{-v|n|}$. Summing the two geometric tails, including the identity once, gives

$$\frac{1 - q^{-2v}}{(1 - \mu(b)^{-1} q^{-v})(1 - \mu(b) q^{-v})}.$$

The two denominators are exactly those of Definition 515.1 at $w, \bar{w}$. At a nonsplit prime the local extension is unramified by Lemma 57.1. The norm-one line is integral there, and its only coset contributes one. A uniformizer of the real field is also a uniformizer of E there and is fixed by conjugation. The E-Euler denominator is $1 - q^{-2v}$, since the residue size in E is q^2. This proves the formula locally. At a split prime the absolute local sum is $(1 + q^{-v})/(1 - q^{-v})$. The product of these bounds is finite since $v \ge 2$ and $\sum_{\mathfrak{u}} (N\mathfrak{u})^{-v} < \infty$ by the degree count in Theorem 580.2. Every finite coset is nonintegral at finitely many places. Exhaustion by finite sets of places now multiplies the local formulas, with absolute convergence. □

Theorem 586.2. *With the preceding notation put $c = lh$ and $b = c - v$. Use the positive-line source form 1, and at every infinite place use the Hermite polynomial test with b copies of*

the creation operator of the first target coordinate applied to the vacuum. At the finite places use F from Definition 581.1 with source rank 1, target rank a, first-coordinate inclusion and residue characters of μ. Let $\xi_{l,v,\vartheta}$ be the coefficient form of this forward lift with class weights at the target auxiliary level as in Theorem 582.2. Then this form exists, is nonzero, and has at every infinite place the conjugated compact coefficient representation of highest weight $(b,0,\dots,0)$. Its squared class norm satisfies

$$\frac{B_{l,\vartheta}(v)}{B_{l,1}(v)} = Q^{-(2v-1)}\,\frac{\|\xi_{l,v,\vartheta}\|_{\mathrm{cl}}^2}{\|\xi_{l,v,1}\|_{\mathrm{cl}}^2}.$$

Proof. Here $b \ge 0$, since $h = 2r^8,\ v \le t = 5r$ at our sufficiently large scales. On the source circle the determinant twist exponent is $c - v$, and the compact weight of the constant form is zero. Thus the polynomial construction of Theorem 495.1 gives exactly the specified monomial and fixes its test under source averaging. A projection onto invariants is the identity on its range, so the real test is source invariant. Its target vacuum weight is zero since the target numerator is trivial and the source has no negative signs. In the target compact action each of the b factors is a first-coordinate vector. Taking its inner product with its unit translate gives the coefficient k_{11}^b after normalization, by the symmetric tensor calculation of Theorem 496.1. Thus the conjugated flag representation and recovery integral in Theorem 582.2 give the indicated coefficient form. On the compact adelic quotients the integrals exist absolutely by lattice decay, locally uniformly in the compact arguments as in Theorem 584.2. The finite transformation law is the rank-one instance of the matrix substitution in Proposition 581.2; its proof uses only the sizes of the two blocks.

Write $M_R = |R|$, $W_R = \prod_{w\in S}(q_w - 1)$, $\gamma_P = \prod_{w\in P}(q_w - 1)$. For the norm of the scalar lift one may use $\gamma_P I$ as finite test by that same translation identity. Integrate Theorem 585.2 with this test against the two constant source forms. For fixed first argument unfolding the rational sum in the second gives the integral of the vacuum matrix coefficient over the whole source, with the source volume as the remaining multiplier. The unfolding is absolutely integrable: at the real places the group is compact, at S the coefficient is supported on the principal subgroup, and on the rest its absolute coset sum is finite by Lemma 586.1. Use quotient measure and right invariance as in Theorem 481.1. The real integral is a positive fixed number, the norm square of the real test times the real compact volume. The finite marked factor, including subgroup volume, is $M_R^{-a}\gamma_P^2 Q^{a-1}/W_R$ by the same support count as in Proposition 581.2.

Passing to the class norm multiplies by the target index $I_{\mathrm{aux}}(P)$ and by a fixed positive factor, by compact averaging and Lemma 411.2. In the notation of the index calculation with ranks $1, a-1$, the marked factor times the index is $M_R^{-a}\delta(P)$. For each new prime with residue size q_w its ratio to the untwisted contribution is

$$q_w^{-a}\,\frac{|\mathrm{GL}_a(\mathbf{F}_w)|}{|\mathrm{GL}_{a-1}(\mathbf{F}_w)|} = q_w^{a-1}(1 - q_w^{-a}).$$

Indeed the block count of Theorem 471.1 works for these two block sizes by choosing the freely variable rectangular block and the two invertible diagonal blocks. In the ratio of the unmarked Euler products from Lemma 586.1, removing this prime contributes $(1 - q_w^{-2v})^{-1}$ in addition to the ratio of the two B-values. Since $a = 2v$, the factors cancel. The remaining conductor power is Q^{a-1}. Both norms are positive by the lemma, positivity of the critical values in Lemma 579.1, and positivity of every real, index and volume factor used above. □

587. The determinant period at the endpoint

Lemma 587.1. *At one infinite place use the positive oscillator on complex two-by-two matrices, with the positive target $U(2)$ acting by the usual left action times $(\det k)^{-b}$, $b \ge 0$ an integer. Its fixed Schwartz subspace consists exactly of the multiples of the Hermite vector $H_{(\det Z)^b}$, where Z is the matrix of creation variables.*

Proof. The central rotation $k = e^{i\alpha} I_2$ acts on Hermite degree s by $e^{i(s-2b)\alpha}$ by Theorem 408.2. Invariance thus leaves only the finite Hermite degree $2b$. For its polynomial P, the same theorem on the two column factors gives $P(k^{\mathsf{T}} Z) = (\det k)^b P(Z)$. All unitary matrices occur as transposes, with their determinants unchanged. This transformation law holds on complex diagonal matrices with nonzero entries: at diagonal unitary matrices it is a polynomial identity in the two entries, and a polynomial in one variable with infinitely many zeros vanishes. Apply this argument successively in the two entries. Every invertible complex matrix is a unitary matrix times a positive diagonal matrix times a unitary matrix, by diagonalizing $g^* g$ as in Lemma 79.2 and dividing the orthogonal image columns by their positive lengths. Hence $P(g) = (\det g)^b P(I_2)$ for every invertible g. For any singular Z, the matrices $Z + tI_2$ are invertible for all but finitely many t; taking $t \to 0$ extends the identity. The determinant power itself satisfies the required law and has nonzero Hermite vector by Theorem 408.2. □

Theorem 587.2. *Fix $1 \le l < m$, set $c = lh$, $\nu = \eta_\vartheta^{-l}$, $\gamma = \nu^{-1} \upsilon_T$, and let χ_S be the finite unit character of ν on $R^\times = \prod_{w \in S_\vartheta} \mathbf{F}_w^\times$. Use the pair of standard definite planes, the real tensor $z = \bigotimes_\tau H_{(\det Z)^{c-1}}$, and the finite residue datum $\Phi(X) = \chi_S^{-1}(\det X)$ with zero value on singular residue matrices. Let $K^{2,2;\gamma,1}$ be the kernel of Definition 504.1 for these two planes. With the same Haar measure for all twists define*

$$D_{l,\vartheta} = \int_{[U(2)]} K_{z,\Phi}^{2,2;\gamma,1}(1, b)\, \mathrm{d}b.$$

The integrand is invariant under the full target compact real group and the full integral finite level K^0. Thus the integral equals $b_{U(2)}$ times its sum over the finite K^0-classes, each divided by its rational stabilizer order, with K^0 of finite volume one. The period is nonzero, and

$$\frac{|D_{l,\vartheta}|^2}{|D_{l,1}|^2} = Q \left(\frac{B_{l,\vartheta}(1)}{B_{l,1}(1)} \right)^2.$$

Proof. Use Theorem 554.1 with $W = V = E^2$ positive definite and $U = 0$. Then $N = r = 2$, $b = 0$, $e = 1$, $q = q_U = q_V = 0$. The weight $c = lh \ge 1$ gives $2 \le 2c$, and all the finite ramification requirements, including nontriviality at each marked prime, hold by Lemma 568.1. At the rank-zero side use the scalar test 1 and probability measure on the trivial group. The standard finite pair of Lemma 509.2 consists precisely of Φ on the large side and 1 on the small side. Evaluated at $g = 1$ the rank-zero kernel is 1 by Definition 504.1. The theorem therefore gives a fixed Schwartz test x, independent of the finite twist, for which the left period at the identity is c_ν and $|c_\nu|^2 = C(\prod_{w \in S_\vartheta} q_w) B_{l,\vartheta}(1)^2$ with $C > 0$.

Right translating the target and using Haar invariance allows us to average x over its compact real target action. The integral and average commute by compactness of the definite quotient at each fixed level and the lattice estimate in Theorem 584.2; that estimate for a theta sum on compact sets uses only power decay, in every rank. At each real place the actual target action is the positive action times $(\det k)^{1-c}$. The projection is thus onto the line in Lemma 587.1 with $b = c - 1$. Applying the central rotation formula also with the other real variables present proves that their product projection has range $\mathbb{C} z$. It sends x to $a_l z$ with a_l independent of ϑ, since the real characters are fixed. Our family includes the trivial finite twist with $S = M_g \ne \varnothing$; its nonzero c_ν implies $a_l \ne 0$. Hence $a_l D_{l,\vartheta} = c_\nu$ for every twist. Division of the squared moduli proves the ratio, since $\prod_{w \in S_\vartheta} q_w / \prod_{w \in M_g} q_w = Q$ and the B's are positive by Lemma 579.1. The chosen vector is fixed by the real target as proved above. The finite datum and its target phase cancel on K^0 by Lemma 518.2, here with no upper rows in the standard determinant. The class evaluation now follows from Lemma 411.2. □

588. An integral normalization for residue moments of varying level

Lemma 588.1. *For an integer $Q \geq 1$, write $\mathcal{I}(Q)$ for the complex numbers integral over $\mathbb{Z}[1/Q]$, meaning roots of monic polynomials with coefficients in this ring. A number $x \in \mathbb{C}$ belongs to $\mathcal{I}(Q)$ if and only if $|j(x)|_\ell \leq 1$ for every prime $\ell \nmid Q$ and every field isomorphism $j : \mathbb{C} \to K_\ell$ of Lemma 16.2. This is a subring stable under complex conjugation. Every finite subset of $\overline{\mathbb{Q}} \subset \mathbb{C}$ is contained in $N^{-1}\mathcal{I}(1)$ for a common integer $N \geq 1$.*

Proof. For the forward implication all coefficients of a monic equation are integral at $\ell \nmid Q$. If $|j(x)|_\ell > 1$, its leading term strictly dominates all its lower terms, contrary to that equation and the ultrametric inequality of Lemma 16.2. Conversely choose a prime ℓ dividing $Q+1$, or $\ell = 2$ when $Q = 1$, and one such j. If x were transcendental, include it in a transcendence basis of $\mathbb{C}/\mathbb{Q}$ as in Proposition 16.4. For any integer $k \geq 1$ the replacement $x \mapsto \ell^{-k}x$ fixing all other basis elements gives an automorphism of the rational function field. It extends to $\mathbb{C}$ by the algebraic-closure argument of Lemma 16.2. Composing with j gives values of absolute size $\ell^k|j(x)|_\ell$, contradicting the bound. For algebraic x, every root of its monic rational minimal polynomial in K_p is the image of x under one of the isomorphisms there, by the same lemma. For $p \nmid Q$ all these roots have size at most one. The coefficients, their signed elementary symmetric functions, are rational numbers with denominator prime factors only among the divisors of Q, by Lemma A.1. This proves the converse. The valuation condition is closed under addition, negation and multiplication. Precomposing all the isomorphisms with complex conjugation gives the same condition. For finitely many algebraic numbers choose N divisible by every coefficient denominator of their monic rational equations. If x satisfies an equation of degree d_0, substitution $X \mapsto X/N$ and multiplication by N^{d_0} gives a monic integral equation for Nx. □

Theorem 588.2. *Fix the arithmetic data, a positive standard oscillator on E^n, finitely many finite adelic isometries G_i of its hermitian space, and an integer $D \geq 0$. Put $M = M_g$. For every allowed finite set P of split places disjoint from $M, \bar{M}, \bar{P}$, use $Q = \prod_{w \in P} q_w$ and $R = \prod_{w \in M \cup P} \mathbf{F}_w$. For $X \in R^n$ let f_X be the inflated indicator of the single residue X, with reference vacuum off these split pairs. There is $A \in \mathbb{C}^\times$, depending only on the fixed data, such that for all these P, X, every isometry k of the finite reference lattice and every homogeneous polynomial U of total degree D with integer coefficients in the creation variables at infinity,*

$$A\,\theta(H_U \otimes \mathcal{V}_{G_i k} f_X) \;\in\; \mathcal{I}(Q).$$

Here H_U is the Hermite vector of Proposition 408.3, and $\mathcal{V}$ is the finite splitting of Theorem 390.2.

Proof. Write $L_0 = \alpha \widehat{O}_E^n$, which is the completion of $\mathcal{A}^n$ by Theorem 425.2. In the paired residue modules of Proposition 357.2 each of the n entries contributes one copy of the additive residue ring and its dual. Thus its finite Fourier formula gives $f_X = |R|^{-n} \sum_z \chi_z(-X) D_z v_0$. The representatives z may be taken to vanish off the conjugates of $M \cup P$ and to lie locally in the inverse uniformizer times L_0 there. Indeed changing a representative by the local lattice fixes the summand by that proposition. If $\ell \nmid Q$, the components at ℓ thus satisfy $g_0 g_1 z_\ell \in L_{0,\ell}$. At each member of M the underlying rational prime divides $g_0 g_1$, by Lemma 13.2. At the other places above ℓ the representative is already integral.

The integral k fixes v_0. By Definition 375.1 the image of $D_z v_0$ under $\mathcal{V}_{G_i k}$ is the product of $\Delta(G_i)/|\Delta(G_i)|^2$ with a sum of translates $D_{G_i k z + b} v_0$, each multiplied by a root of unity. Here b ranges over fixed representatives of $G_i L_0/(G_i L_0 \cap L_0)$. This follows by intertwining displacements and then by the product law in Lemma 356.2. The phase in that law is a finite additive character at a finite adele, hence a root of unity by Proposition 18.5. There is an integer $c_0 \geq 1$ with $c_0 G_i L_0 \subset L_0$ for every i. All matrices are integral outside a finite set of places, and a sufficiently large power of each underlying rational prime clears the denominators of their finitely many matrix entries at the other places. The same c_0 clears

all the b, chosen inside G_iL_0. Put $\delta = c_0g_0g_1$. For $\ell \nmid Q$ it follows that $\delta(G_ikz + b)_\ell \in L_{0,\ell}$, uniformly in k.

Write $v_n^\circ = \kappa_n v_0$ with $\kappa_n \neq 0$ as in Corollary 427.2. All the finitely many scalars $\Delta(G_i)/|\Delta(G_i)|^2$ are algebraic by their definition and Theorem 374.2. By Lemma 588.1, an integer $N \geq 1$ clears these scalars divided by $(\prod_{w\in M} q_w)^n$. The remaining factor Q^{-n} and every root of unity are integral at each $\ell \nmid Q$. Thus $N\kappa_n\mathcal{V}_{G_ik}f_X$ belongs to $\mathcal{D}_\ell(\delta)$ in Definition 428.1 for every such ℓ and every isomorphism there.

Take u of that definition with $s_1 = s_0$, and take the positive real fourth root $d^{1/4}$. Theorem 429.1 makes $U(\partial)$ at zero of the corresponding section at scale $\bar{\delta}d^{1/4}u$ integral at each such isomorphism. Indeed it is an integer combination of the mixed derivatives. By Lemma 425.1 its section uses the negative coherent variable. Homogeneity and Proposition 408.3 identify this derivative with the asserted moment times $A = N\kappa_n(-\bar{\delta}d^{1/4}u)^D$, independent of P. Apply Lemma 588.1. □

589. An integral group-ring element for the determinant period

Definition 589.1. Use $T_f, K_T, T(E)$ of Definition 372.1. For P as in Theorem 588.2 write $K_T(P)$ for the subgroup of K_T having residue one at every $w \in P$, and put

$$C(P) = T_f/(T(E)K_T(P)).$$

A tame twist at P is a complex unitary character τ_0 of $C(P)$, inflated to the adelic torus with trivial real part. For a unit residue $z \in \prod_{w\in P}\mathbf{F}_w^\times$ write $u_z \in K_T$ for any lift equal to one off $P \cup \bar{P}$: lift each residue to a local unit at w, and use its inverse conjugate at $\bar{w}$. For $P = \varnothing$ take $u_z = 1$. For an abelian group C the group ring $\mathcal{I}(Q)[C]$ consists of formal finite sums $\sum c_a[a]$ with $[a][b] = [ab]$. Its evaluation at τ_0 substitutes $\tau_0(a)$ for $[a]$. Its involution is $(\sum c_a[a])^* = \sum \bar{c}_a[a^{-1}]$.

Lemma 589.2. *The group $C(P)$ is finite, and its characters are exactly the weight-zero torus characters with finite unit parts factoring through the indicated residues. Evaluation at any of them maps $\mathcal{I}(Q)[C(P)]$ to $\mathcal{I}(Q)$ and takes Y^*Y to $|Y(\tau_0)|^2$.*

Proof. The proof of Theorem 372.2 gives finiteness of $T_f/(K_TT(E))$ via the finite ideal class group. The kernel of the natural map from $C(P)$ to this quotient is an image of $K_T/K_T(P)$, a subgroup of the finite product of residue unit groups. Hence $C(P)$ is finite. A weight-zero character is trivial at infinity and on $T(E)$, so the assertion on finite unit parts is exactly triviality on $K_T(P)$. Conversely inflation has all these properties and is continuous since this latter subgroup is open. A character of a finite group has roots of unity as values. These and their inverses belong to $\mathcal{I}(Q)$, the conjugate of a unitary character value is its inverse, and Lemma 588.1 gives the ring and conjugation assertions about the coefficients. Multiplying the two evaluated finite sums gives the squared modulus formula. □

Theorem 589.3. *Fix a torus character μ_0 of weight $(-c_\tau)_\tau$, where $c_\tau \geq 1$ are integers. Suppose its finite unit character factors through $\prod_{w\in M}\mathbf{F}_w^\times$. For P and τ_0 of Definition 589.1 put $\mu = \mu_0\tau_0$, $\gamma = \mu\upsilon_T$, and let ψ_μ be the residue unit character of μ at $S = M \cup P$. Take*

$$U = \prod_\tau (\det Z_\tau)^{c_\tau - 1}, \qquad \Phi_\mu(X) = \mathbf{1}_{X\in \mathrm{GL}_2(R)}\psi_\mu(\det X) \quad (X \in \mathrm{Mat}_2(R)),$$

where the value is defined to be zero on the complement. Define $D(P, \tau_0)$ by the integral of Theorem 587.2 with this real Hermite vector, residue datum and characters $(\gamma, 1)$. There exist $A_1 \in \mathbb{C}^\times$, independent of P, τ_0, and, for each P, a $Y_P \in \mathcal{I}(Q)[C(P)]$ such that

$$A_1D(P, \tau_0) = Y_P(\tau_0).$$

In particular, for $\mu_0 = \eta_1^l,\ 1 \le l < m$, the nonzero number $F_1(\varnothing, 1) = |A_1 D(\varnothing, 1)|^2$ is an algebraic integer. On the primary twists one has

$$\frac{(Y_P^* Y_P)(\vartheta^l)}{F_1(\varnothing, 1)} = Q \left(\frac{B_{l,\vartheta}(1)}{B_{l,1}(1)} \right)^2 \quad (P = P_\vartheta).$$

In this formula the positive ratio at one is algebraic.

Proof. The real determinant transforms by $(\det k)^{c_\tau - 1}$ under the untwisted positive oscillator at τ, by Theorem 408.2. The target determinant character there is $(\det k)^{1 - c_\tau}$, so the whole vector is fixed. At finite integral target elements the substitution $X \mapsto k^{-1} X$ preserves the invertible residue locus and contributes $\psi_\mu(\det k)^{-1}$. The character γ contributes its inverse by Theorem 374.2, since υ_T is trivial on units. These calculations apply even when a residue character is trivial. The kernel is rationally invariant by Definition 504.1. Its integral is a finite class sum by Lemma 411.2, with fixed representatives h_i at the full finite integral level K^0, stabilizers Γ_i, and measure factor $b_H > 0$. All are independent of P.

For an invertible X put $u_X = u_{(\det X_w)_{w \in P}}$. Let $\psi_{0,M}$ be the fixed marked unit character. The value of the residue datum on X equals $\psi_{0,M}((\det X_w)_{w \in M}) \tau_0(u_X)$. Let G_i act on matrices by $Z \mapsto h_i Z$. At h_i the scalar outside this finite oscillator is $(\mu_0 \upsilon_T)(\det h_i) \tau_0(\det h_i)$, by Definition 504.1. Expanding into the point indicators of Theorem 588.2 therefore expresses $D(P, \tau_0)$ as the sum over i, X of

$$b_H \, \frac{(\mu_0 \upsilon_T)(\det h_i)}{|\Gamma_i|} \, \psi_{0,M}((\det X_w)_{w \in M}) \, \theta(H_U \otimes \mathcal{V}_{G_i} f_X) \, \tau_0((\det h_i) u_X).$$

By Theorem 372.2 the character values at these fixed determinants are algebraic. Choose an integer $N_1 \ge 1$ clearing the finitely many quotients by $|\Gamma_i|$ into $\mathcal{I}(1)$, using Lemma 588.1. The values of $\psi_{0,M}$ are roots of unity. Apply Theorem 588.2 with $n = 4,\ D = 2 \sum_\tau (c_\tau - 1)$, and call its scalar A. Taking $A_1 = N_1 A / b_H$ makes every coefficient of the displayed character values integral away from Q. Replace each value by the corresponding group element of $C(P)$ to obtain Y_P. The coset of u_X is independent of its lifts by Definition 589.1.

For the stated primary specialization the new integral is $D_{l,\vartheta}$ since $\mu = \eta_1^l \vartheta^l = \eta_\vartheta^l$. Apply Lemma 589.2 and Theorem 587.2. At the untwisted empty level that theorem gives nonvanishing, and $\mathcal{I}(1)$ consists of algebraic integers. Finally the square of the positive ratio is algebraic, as a quotient of algebraic numbers by a nonzero algebraic number. A square root of an algebraic number is algebraic by transitivity of algebraicity in Lemma 16.2. □

590. Integral interpolation of the class norm on a definite group

Lemma 590.1. *Let H be a definite group of Lemma 411.2, K its full integral level, $J \subset K$ an open subgroup, and h_i fixed representatives of the K-classes. Write $\Gamma = H(F)$, $\Gamma_i = \Gamma \cap h_i K h_i^{-1}$. Suppose a function $\xi : H_f \to V$, with V a finite dimensional Hermitian space, transforms on the left by unitary operators of Γ and on the right by a unitary character of J. Its class norm with stabilizer weights at J is*

$$\|\xi\|_{\mathrm{cl},J}^2 = \sum_i \frac{1}{|\Gamma_i|} \sum_{k \in K/J} \|\xi(h_i k)\|^2.$$

Proof. The cosets above h_i form the finite set K/J acted on from the left by $h_i^{-1} \Gamma_i h_i$. Its orbits correspond bijectively to J-classes above h_i: two representatives $h_i k, h_i k'$ are in the same double class precisely when $k' J = h_i^{-1} \gamma h_i k J$ for some $\gamma \in \Gamma_i$. The stabilizer at kJ has order $|\Gamma \cap h_i k J k^{-1} h_i^{-1}|$. An orbit has size $|\Gamma_i|$ divided by that order, since the elements sending one point to another form a coset of the stabilizer. The squared norm is constant on each orbit by the two unitarity assumptions. Regrouping the finite sum gives precisely the class norm. □

Theorem 590.2. *Fix an integer $v \geq 2$, put $a = 2v$, and let μ_0 be a torus character as in Theorem 589.3 with $c_\sigma \geq v$ at all real places σ of F. Set $b_\sigma = c_\sigma - v$. For P, τ_0 as in Definition 589.1 take $\mu = \mu_0 \tau_0$. Use the source $U(1)$, target $U(a)$, constant source form, source denominator μv_T^v, and trivial target numerator. Use the product of the real tests $H_{Z_1^{b_\sigma}}$, and at finite places use F of Definition 581.1 for the first-coordinate line and the residue characters of μ. Let $\xi(P, \tau_0)$ be its coefficient form as in Theorem 586.2, at the auxiliary target level $J = J(P)$. There are $C_v > 0$, independent of P, τ_0, and $Y_{v,P} \in \mathcal{I}(Q)[C(P)]$ such that*

$$Y_{v,P}(\tau_0) = C_v Q^{-(a-1)} \|\xi(P, \tau_0)\|_{\mathrm{cl},J}^2.$$

For $\mu_0 = \eta_1^l,\ 1 \leq l < m,\ 2 \leq v \leq t$, its value at $(\varnothing, 1)$ is a positive algebraic integer and

$$\frac{Y_{v,P}(\vartheta^l)}{Y_{v,\varnothing}(1)} = \frac{B_{l,\vartheta}(v)}{B_{l,1}(v)} \in \overline{\mathbb{Q}}_{>0} \qquad (P = P_\vartheta),$$

where $\overline{\mathbb{Q}}_{>0} = \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$.

Proof. The integrals converge at fixed finite level: the two definite quotients are compact by Lemma 411.2 and their lattice theta sums converge uniformly on compact sets by the decay estimate in Theorem 584.2. Let $U_\alpha = \prod_{\sigma,i} Z_{\sigma i}^{\alpha_{\sigma i}}$ where $\sum_{i=1}^a \alpha_{\sigma i} = b_\sigma$. For a finite target point h', denote by $A_\alpha(h')$ the integral of the kernel with this real Hermite vector and the given finite test over the source. On the source circle at σ, the monomial contributes exponent $-b_\sigma$ from the inverse scalar action and the denominator contributes $c_\sigma - v = b_\sigma$ by Theorem 408.2. Thus all these functions are invariant under the compact real source. On the finite source they are invariant under $K_T(M)$, the integral subgroup of residue one at M. Indeed at M the point e_1 is then unchanged, and at P the multiplier from the first residue coordinate cancels the source determinant character by Proposition 581.2. This calculation applies to the trivial local character, extended by zero on the nonunits, as well.

For each σ index the ordered tensor basis of $(\mathbb{C}^a)^{\otimes b_\sigma}$ by words on $1, \ldots, a$. Define a vector at h' by assigning the same coordinate $A_\alpha(h')$ to each tuple of words with letter counts α. This vector is symmetric at every place. Its coefficient against the tensor of $(\bar{u}_\sigma e_1)^{\otimes b_\sigma}$, $u_\sigma \in U(a)$, is the theta lift at the real group argument u. In fact $\langle\ ,\ \rangle$ is conjugate-linear in the first entry. Expansion over words gives the coefficients of $\prod_\sigma (\sum_i (u_\sigma)_{i1} Z_{\sigma i})^{b_\sigma}$, and this is exactly the transformed creation polynomial by Theorem 408.2. These coefficients determine the symmetric vector: a homogeneous polynomial vanishing on all unit columns vanishes on all columns by scaling separately at each place, and taking monomial derivatives at the origin recovers its coefficients. Unit columns extend to unitary matrices by successively choosing orthonormal vectors in their orthogonal complements. Thus the vector is $\xi(P, \tau_0)(h')$. Its squared norm is

$$\sum_\alpha n_\alpha |A_\alpha(h')|^2, \qquad n_\alpha = \prod_\sigma \frac{b_\sigma!}{\prod_i \alpha_{\sigma i}!}.$$

The integers n_α count the words. The rational covariance follows from Definition 504.1 by the same real transformation law and uniqueness of the symmetric vector. The right action of J is a unit character by Proposition 581.2. This verifies the hypotheses of Lemma 590.1.

Choose fixed finite source class representatives g_s at $K_T(M)$, and fixed full-level target representatives h_i. Let Γ_s and Γ_i be their respective stabilizers. For target arguments take $h_i k$ with $k \in K/J$. The source class formula of Lemma 411.2 has a positive constant b_s^0 common to all source classes, independent of the varying level, times the stabilizer weights $1/|\Gamma_s|$. The residue matrices in the support of F are $X_w = e_1$ at M and $X_{w,1} \neq 0$ at P. For each such X put $u_X = u_{(X_{w,1})_{w \in P}}$. Its residue coefficient is $\tau_0(u_X)$, because μ_0 is unramified at P. Set $G_{is}(Z) = h_i Z g_s^{-1}$, $k'(Z) = kZ$. The finite translations are $G_{is} k'$, with G_{is} in a

fixed finite list and k' integral. Consequently $A_\alpha(h_i k)$ is the sum over s, X of

$$\frac{b_s^0}{|\Gamma_s|(\mu_0 \upsilon_T^v)(g_s)}\,\theta(H_{U_\alpha}\otimes \mathcal{V}_{G_{is}k'}f_X)\,\tau_0(u_X g_s^{-1}).$$

This follows by expanding the point indicators in the finite test in Definition 504.1. The source exists at rank one, so its determinant equals the source point.

Apply Theorem 588.2 with $n=a,\ D=\sum_\sigma b_\sigma$. The displayed fixed character values are nonzero algebraic by Theorem 372.2. Hence a fixed $A'\neq 0$ makes every $A'A_\alpha(h_i k)$ the evaluation of some $Z_{i,k,\alpha}\in \mathcal{I}(Q)[C(P)]$. Explicitly multiply the scalar of that integrality theorem by an integer clearing the finitely many quotients $1/(|\Gamma_s|(\mu_0\upsilon_T^v)(g_s))$, and divide by b_s^0. Let $N_t\geq 1$ be divisible by every $|\Gamma_i|$. Take

$$Y_{v,P}=Q^{-(a-1)}\sum_{i,k,\alpha}\frac{N_t n_\alpha}{|\Gamma_i|}Z_{i,k,\alpha}^{*}Z_{i,k,\alpha}.$$

The factor $Q^{-(a-1)}$ belongs to $\mathcal{I}(Q)$. Lemma 589.2, the norm computation and the class refinement prove the claim with $C_v=N_t|A'|^2$. For the primary specialization the coefficient identification gives exactly the form in Theorem 586.2. Its nonzero norm at the empty level and its ratio there yield both remaining conclusions. □

591. Adjoining an unramified split place to the determinant period

Lemma 591.1. *For a finite abelian group C, evaluations at its complex unitary characters distinguish elements of $\mathbb{C}[C]$.*

Proof. If $x\neq 1$, assign a primitive root of unity of order $\operatorname{ord}(x)$ to x on the cyclic subgroup it generates. This character extends to C by adjoining one generator at a time and choosing an appropriate root for its value, as in Theorem 372.2. Thus at least one character has value different from one on x. There are finitely many characters, since every value is a root of unity of order dividing $|C|$, by the coset count for a cyclic subgroup. Multiplying all characters by the one just constructed permutes their set. Thus $\sum_{\tau_0}\tau_0(x)=0$ for $x\neq 1$, while at the identity this sum is the positive number of characters. Multiply the evaluations of $\sum c_x[x]$ by $\tau_0(y)^{-1}$ and sum over all characters to recover this number times c_y. □

Theorem 591.2. *Use the notation of Theorem 589.3. Let $w,\bar w$ be a split pair disjoint from $M,\bar M,P,\bar P$ for which $P'=P\cup\{w\}$ is allowed, put $q=q_w$ and $b_w=\pi_w/\bar\pi_w\in T_f$ for a uniformizer inserted only at w. Inflate τ_0 from $C(P)$ to $C(P')$, and set $\lambda=(\mu_0\tau_0)(b_w)^{-1}$. Then*

$$D(P',\tau_0)=(1-q^{-1}\lambda)(1-\lambda)\,D(P,\tau_0).$$

In particular this formula holds when $\lambda=1$. Under the natural projection $\pi_*:\mathbb{C}[C(P')]\to\mathbb{C}[C(P)]$ one has, putting $T=\mu_0(b_w)^{-1}[b_w^{-1}]$,

$$\pi_* Y_{P'}=(1-q^{-1}T)(1-T)Y_P.$$

For the squared-period elements the multiplying factor in this last identity is the product of $(1-q^{-1}T)(1-T)$ and its involution.

Proof. At the new split place the old test is the indicator of $\mathrm{Mat}_2(O_w)$ in the normalized coordinates of Theorem 360.1. The new test retains matrices whose reduction is invertible. Its character coefficient on those matrices is one, because the inflated character and μ_0 are trivial on the units there. For a subspace $U\subset k^2,\ k=\mathbf{F}_w$, put $L_U=\{x\in O_w^2:\bar x\in U\}$ and let f_U be the indicator that both columns belong to L_U. The indicator of invertible reduction equals

$$f_{k^2}-\sum_{\dim U=1}f_U+qf_0.$$

A full-rank reduction has no line containing its image, a rank-one reduction has one, and the zero reduction is contained in every one of the $q+1=(q^2-1)/(q-1)$ lines. These counts give the identity on integral matrices. Off the integral lattice every term vanishes.

Put $d_U=2-\dim U$. There is $A_U\in \mathrm{GL}_2(E_w)$ with $A_UO_w^2=L_U$ and $v_w(\det A_U)=d_U$. Take a basis of k^2 beginning with a basis of U, lift it to an integral basis using its unit determinant, and multiply the last d_U basis vectors by π_w. Complete A_U to a finite adelic unitary h_U by the inverse conjugate transpose at $\bar w$ and identity elsewhere. Left multiplication by A_U on the two columns has determinant $(\det A_U)^2$, so by Theorem 360.1 the positive-phase transport of the old local vacuum is $q^{d_U}f_U$. Write $\beta(w)$ for the finite ideal character at w. The additional splitting phase is $\beta(w)^{-2d_U}$ by Theorem 375.2. The finite target numerator has value $(\mu_0\tau_0)(b_w)^{d_U}\beta(w)^{2d_U}$ on $\det h_U$: its determinant differs from $b_w^{d_U}$ by a local norm-one unit, and $\upsilon_T(b_w)=\beta(w)^2$ by Theorem 374.2. Hence the full pair transport of the old test by h_U equals $q^{d_U}\lambda^{-d_U}f_U$ in the new local factor.

The full target quotient measure is right invariant by Lemma 411.2. Definition 504.1 therefore gives the same period for the transported test and the old test, keeping all other places fixed. The period with local factor f_U is consequently $(q^{-1}\lambda)^{d_U}D(P,\tau_0)$. The indicator identity gives multiplier $1-(q+1)q^{-1}\lambda+q(q^{-1}\lambda)^2=(1-q^{-1}\lambda)(1-\lambda)$. The argument is linear and uses no division by the period. Inflated characters evaluate $\pi_*Y_{P'}$ at the original characters of $C(P)$. Under each such character T takes the value λ. The group-ring equality follows by Lemma 591.1 and the common normalization of Theorem 589.3. Projection commutes with the involution in Definition 589.1, giving the last assertion. □

592. Distribution laws at the critical integers

Definition 592.1. Fix $1\le v\le t$ and a torus character μ_0 of weight $(-c_\sigma)_\sigma$ with $v\le c_\sigma\le 4b$ at every real place. Here b is the arithmetic parameter of Definition 12.3. Its finite unit character factors through the residues at $M=M_g$. Call the pair ramified at M if this character is nontrivial at every member of M. A twist τ_0 of Definition 589.1 is primitive at P if its residue component is nontrivial at every $w\in P$. This holds vacuously when $P=\varnothing$. For $S=M\cup P$, $\mu=\mu_0\tau_0$, put

$$B(P,\tau_0)=L^S(v,\mu^{-1}),\qquad \rho(P,\tau_0)=B(P,\tau_0)/B(\varnothing,1).$$

At $v=1$ take the holomorphic value of Theorem 515.3.

Theorem 592.2. *For each pair of Definition 592.1 ramified at M, set $m_v=\mathbf{1}_{v=1}$ and $K_v=\{0\}$ for $v\ge 2$, $K_1=\{0,1\}$. There are group-ring elements $F_P\in\mathcal{I}(Q)[C(P)]$ for every allowed P, with $F_\varnothing(1)>0$. For every τ_0 primitive at P,*

$$F_P(\tau_0)=F_\varnothing(1)\,Q^{m_v}\rho(P,\tau_0)^{m_v+1}.$$

For $v\ge 2$ this formula holds for every τ_0 on $C(P)$. All $B(P,\tau_0)$ are positive real, and the indicated primitive ratios are algebraic. For every enlargement $P'=P\cup\{w\}$ as in Theorem 591.2, let $z=\mu_0(b_w)[b_w]\in\mathbb{C}[C(P)]^\times$. Then

$$\pi_*F_{P'}=\prod_{k\in K_v}\prod_{\epsilon\in\{-1,1\}}(1-q_w^{k-v}z^\epsilon)\,F_P.$$

In particular the same families cover inverse twists by evaluation at τ_0^{-1}.

Proof. At a split pair $u,\bar u$ outside $S,\bar S$, write $b_u=\pi_u/\bar\pi_u$. The two unmarked local denominator factors at real $s>1$ multiply to $|1-\mu(b_u)^{-1}q_u^{-s}|^2>0$. At a nonsplit prime a uniformizer of F is a uniformizer of E by Lemma 57.1, and its image divided by its conjugate equals one. Thus these factors are positive as well. Absolute convergence and nonvanishing in Theorem 515.3 give positivity for $s>1$. The weights of μ^{-1} are nonzero. Taking a right limit and using its nonzero holomorphic value at one from that same theorem gives positivity at the endpoint.

Suppose first $v \ge 2$. Use $F_P = Y_{v,P}$ of Theorem 590.2. We compute its norm at every tame character. In the absolute coset integral over the unmarked source line, the factor at a split prime u of F off S_F is $\sum_{n\in\mathbb{Z}} \mu(b_u)^{-n} q_u^{-v|n|}$, and a nonsplit prime contributes one. Here S_F denotes the primes of F below S, and a split prime in this expression is represented by either branch. Indeed the row lattice has exponents $\max(n,0), \max(-n,0)$ on the two branches. Its splitting character to power $2v$ cancels the factor υ_T^v of the denominator exactly as in Lemma 586.1. The two geometric series give the product over all unmarked primes as

$$B(P,\tau_0) \prod_{\mathfrak{u}\notin S_F} (1 - (N\mathfrak{u})^{-2v}).$$

The absolute series converges by the same lemma's norm count, which uses only $v \ge 2$ and $|\mu| = 1$.

By Proposition 581.2 the lift of the constant agrees with the lift of the minor indicator times $\prod_{w\in P}(q_w - 1)$. Apply Theorem 585.2 to the latter test and unfold the rational source sum against the constant on the two copies of the source quotient. The integrals are absolute by the preceding series, the compact real groups and support on the principal subgroup at the marked places, exactly the conditions of Theorem 481.1. For every σ the monomial is source fixed by the proof of Theorem 590.2, and has nonzero norm by Theorem 408.2. Thus the real multiplier is positive and independent of P, τ_0. Compact target averaging converts the scalar norm to the class norm by the dimension of the fixed symmetric tensor type and the target level index as in Theorem 582.2. The marked finite factor including that index equals $|R|^{-a}\delta(P)$, $a = 2v$, by the count of Proposition 581.2 for block sizes $1, a-1$. For each new place its quotient by the empty-level contribution is $q^{-a}|\mathrm{GL}_a(\mathbf{F}_q)|/|\mathrm{GL}_{a-1}(\mathbf{F}_q)| = q^{a-1}(1-q^{-a})$, by the column count in Theorem 471.1. This cancels the missing $(1-q^{-2v})$ from the unmarked product and leaves Q^{a-1}. The calculation uses every P as a residue condition whether its local unit character is trivial or nontrivial. After the normalization in Theorem 590.2 this proves $F_P(\tau_0)/F_\varnothing(1) = \rho(P,\tau_0)$, with $F_\varnothing(1) > 0$. On inflating from P to P' the two unmarked Euler factors over w are removed, contributing $\prod_{\epsilon=\pm1}(1 - q_w^{-v}(\mu_0\tau_0)(b_w)^\epsilon)$. Evaluation at all characters gives the asserted projection by Lemma 591.1.

For $v = 1$ use $F_P = Y_P^* Y_P$ of Theorem 589.3. If τ_0 is primitive at P, the unit character of μ^{-1} is nontrivial on every marked residue group: at M this is the ramification hypothesis, and M, P are disjoint. Apply Theorem 554.1 to $W = V = E^2$ positive definite, $U = 0$, $\nu = \mu^{-1}$, and the rank-zero test 1. Here $N = r = 2,\ e = 1,\ q = q_V = q_U = 0$; at each σ the required inequalities are $2 \le 2 \le 2c_\sigma$. The matched finite test on the plane is $\psi_\mu(\det X)$ on the invertible locus by Lemma 509.2. The comparison provides a fixed real test, depending on the weights, with period of squared modulus $CQ_S B(P,\tau_0)^2,\ C > 0$. Average this test over the compact target at infinity. Lemma 587.1 gives the fixed real line spanned by $\bigotimes_\sigma H_{(\det Z_\sigma)^{c_\sigma - 1}}$. The product projection is onto its tensor product by applying the central degree projection at each place as in Theorem 587.2. Haar invariance preserves the period and compact lattice decay justifies the integration. The fixed proportionality constant is nonzero because the comparison at $(\varnothing, 1)$ gives a nonzero period. Dividing the two squared period evaluations now gives $Q\rho(P,\tau_0)^2$, since $Q_S/Q_M = Q$. For an arbitrary inflated character the projection formula follows from Theorem 591.2 using $z^* = z^{-1}$, which holds since μ_0 is unitary. Finally evaluations lie in $\mathcal{I}(Q)$ by Lemma 589.2. Division by the nonzero empty-level value and, at the endpoint, taking a square root as in Theorem 589.3 prove algebraicity in the asserted range. □

593. Simultaneous polynomial values on prime-power roots of unity

Lemma 593.1. *Fix a prime ℓ, the algebraically closed valued field K_ℓ of Lemma 16.2, with $|\ell|_\ell = \ell^{-1}$, and its ring of integers O_ℓ. For $s > 0$ and $F(X) = \sum_i a_i X^i$ set $\|F\|_s =$*

$\max_i |a_i|_\ell s^i$, *with norm zero for the zero polynomial. For* $F(X) = a\prod_j (X - \alpha_j) \neq 0$,

$$\|F\|_s = |a|_\ell \prod_j \max(s, |\alpha_j|_\ell).$$

For $j \geq 1$, *put* $h_j = (\ell - 1)\ell^{j-1}$. *Every primitive* ℓ^j*-th root of unity* ζ *satisfies* $|\zeta - 1|_\ell = s_j = \ell^{-1/h_j}$. *For any fixed such* ζ_*,

$$\sum_{\substack{\eta^{\ell^j}=1 \\ \eta \neq \zeta_*}} -\log|\eta - \zeta_*|_\ell = j\log\ell.$$

Proof. For products of two polynomials the upper bound by the product of the norms follows from the ultrametric inequality. If both are nonzero, take the largest indices i_0, j_0 attaining their respective norms. In the coefficient of $X^{i_0+j_0}$ every other pair of indices has either its first index larger than i_0 or its second larger than j_0. Its weighted absolute value is strictly smaller. The ultrametric inequality gives equality of the coefficient norm with the product of the two maxima. Iteration proves the formula for the roots.

The polynomial $X^{\ell^j} - 1$ has ℓ^j distinct roots since it splits and is coprime to its derivative in characteristic zero. At most ℓ^{j-1} of its roots have order dividing ℓ^{j-1}, so a root has order ℓ^j and its powers exhaust the roots. Exactly h_j of them have full order. For an integer $t \geq 1$, the finite geometric sum gives $|\zeta^t - 1|_\ell \leq |\zeta - 1|_\ell$, because $|\zeta|_\ell = 1$. When t is prime to ℓ, choose an inverse modulo ℓ^j to get the reverse inequality. All the primitive roots therefore have the same distance from one. Their monic polynomial is $\sum_{i=0}^{\ell-1} X^{i\ell^{j-1}}$, by division of the two monic polynomials for successive orders. At one it has value ℓ, proving the formula for s_j. Differentiating the product for $X^{\ell^j} - 1$ at ζ_* shows that the product of its differences from all other roots is $\ell^j \zeta_*^{\ell^j - 1}$. Taking absolute values and real logarithms gives the stated sum. □

Theorem 593.2. *Fix* $0 < r < 1$, *a finite set* I, *thresholds* $0 < y_i \leq 1$ *for* $i \in I$, *and an integer* $j_{\min} \geq 1$. *There is* $N_0 \geq j_{\min}$, *depending only on these data and* ℓ, *such that for every* $N \geq N_0$ *and all* $F_i \in O_\ell[X]$ *with* $\|F_i\|_r \geq y_i$, *there exists a root of unity* ζ *of order* ℓ^j *with* $j_{\min} \leq j \leq N$ *and*

$$|F_i(\zeta - 1)|_\ell \geq r y_i \qquad (i \in I).$$

In particular N_0 *is uniform over the degrees and the coefficient fields of the polynomials.*

Proof. Put $D = -\log r > 0$ and choose $J \geq j_{\min}$ such that $s_J \geq r$, where s_j is as in Lemma 593.1. Choose $N_0 \geq J$ with

$$\sum_{j=J}^{N_0} \frac{D\log\ell}{D + j\log\ell} > \sum_{i\in I}(-\log y_i).$$

Such an index exists: for $j \geq D/\log\ell$ each term is at least $D/(2j)$; summing $1/j$ on each integer interval $[2^k, 2^{k+1})$ gives at least $1/2$.

Factor $G = \prod_{i\in I} F_i$ in $K_\ell[X]$ and count roots with multiplicity. Write n_j for its number of roots of absolute value s_j. Multiplicativity in the lemma and integrality of the coefficients give $\|G\|_r \geq \prod_i y_i$, $\|G\|_1 \leq 1$. For any root of absolute value $s_j \geq r$, its contribution to $\log\|G\|_1 - \log\|G\|_r$ is $-\log s_j = (\log\ell)/h_j$. Every other contribution is nonnegative. The radii are distinct, so

$$\sum_{j=J}^{N_0} n_j \frac{\log\ell}{h_j} \leq \log\|G\|_1 - \log\|G\|_r \leq \sum_i (-\log y_i).$$

Suppose that at a fixed $j \geq J$ each primitive ζ fails at least one of the required inequalities. Put $x = \zeta - 1$. At $|x|_\ell = s_j$, for each root α of G define

$$d_\alpha(x) = \log\frac{\max(s_j, |\alpha|_\ell)}{|x - \alpha|_\ell},$$

with $+\infty$ for $x = \alpha$. The terms are nonnegative and vanish for $|\alpha|_\ell \neq s_j$, by the ultrametric inequality. For a failing polynomial F_i, their sum over its roots is $\log(\|F_i\|_{s_j}/|F_i(x)|_\ell) > D$, with the same convention at zeros, because $\|F_i\|_{s_j} \geq y_i$. Consequently $\sum_\alpha \min(D, d_\alpha(x)) \geq D$. This remains valid at a zero because its one summand contributes D.

For a fixed root α of size s_j, choose a primitive ζ_* minimizing $|\zeta - 1 - \alpha|_\ell$. For every other primitive η the ultrametric inequality yields $|\eta - \zeta_*|_\ell \leq |\eta - 1 - \alpha|_\ell$. Thus $d_\alpha(\eta - 1) \leq -\log|\eta - \zeta_*|_\ell$, since $s_j \leq 1$. These upper bounds are nonnegative. Summation over all the other primitive roots is bounded by the sum over all other ℓ^j-th roots, namely $j \log \ell$ by the lemma. At ζ_* the truncated summand is at most D. Summing the two bounds for the truncated contributions gives $h_j D \leq n_j(D + j \log \ell)$. If every $j = J, \dots, N_0$ failed, inserting all these lower bounds for n_j in the preceding estimate would contradict the choice of N_0. Hence some j in this range supplies the required ζ, also for every larger N. □

594. Specializing several variables and several powers

Theorem 594.1. *On K_ℓ put $v_\ell(x) = -\log|x|_\ell/\log \ell$ for $x \neq 0$ and $v_\ell(0) = +\infty$. Fix integers $m, a \geq 1$, $q_0, L \geq 0$, a real $\delta > 0$, a finite set I and real bounds $B_i \geq 0$ for $i \in I$. There is N_0 with the following property for every $N \geq N_0$. Let $0 \leq q \leq q_0$. For $i \in I$, $1 \leq d \leq m$ take a polynomial $F_{i,d} \in O_\ell[X_1, \dots, X_q]$ and a point $u_{i,d} \in O_\ell^q$ such that*

$$v_\ell(u_{i,d,j} - 1) \geq 1/a \quad (1 \leq j \leq q), \qquad v_\ell(F_{i,d}(u_{i,d})) \leq B_i.$$

Take any finite family of monic polynomials $D_t \in O_\ell[X]$ of total sum of degrees at most L. There are ℓ^N-th roots of unity $\zeta_1, \dots, \zeta_q$ with

$$v_\ell(F_{i,d}(\zeta_1^d, \dots, \zeta_q^d)) \leq B_i + \delta,$$
$$v_\ell(D_t(\zeta_j)) \leq \delta \qquad (1 \leq j \leq q, \ \textit{all } t).$$

Proof. Choose $0 < r < 1$ sufficiently close to one that $r^{am} > \ell^{-1}$, $r^{q_0+1} > \ell^{-\delta}$ and $r^{L+1} > \ell^{-\delta}$. For each $s = 0, \dots, q_0$ use Theorem 593.2 with the thresholds $\ell^{-B_i} r^s$ indexed by pairs (i, d), and one extra threshold r^L. Take for N_0 the largest of its finitely many bounds with $j_{\min} = 1$. This choice precedes the polynomials and points.

After replacing the first s coordinates of every $u_{i,d}$ by ζ_j^d, maintain absolute values of their polynomials at least $\ell^{-B_i} r^s$. This holds at $s = 0$ by hypothesis. If $s < q$, substitute $(1 + X)^d$ in coordinate $s + 1$, keeping the other coordinates at their current values. This gives a univariate polynomial $G_{i,d} \in O_\ell[X]$. There is a root w of $w^d = u_{i,d,s+1}$ with $|w - 1|_\ell \leq \ell^{-1/(ad)} < r$: the product of $1 - w$ over all roots with multiplicity equals $1 - u_{i,d,s+1}$, whose absolute value is at most $\ell^{-1/a}$. If this product vanishes choose $w = 1$. Otherwise at least one factor is at most the d-th root of that bound. Evaluation at $w - 1$ recovers the preceding point. By the ultrametric inequality, evaluation anywhere in the closed disc of radius r is bounded by the Gauss norm there. Hence $\|G_{i,d}\|_r \geq \ell^{-B_i} r^s$.

The extra polynomial is $H(X) = \prod_t D_t(1 + X)$. It is monic of degree at most L, including the empty product, so $\|H\|_r \geq r^L$. The same root-of-unity theorem now gives a single ζ_{s+1} with $\zeta_{s+1}^{\ell^N} = 1$ at which all the new absolute values are at least $\ell^{-B_i} r^{s+1}$ and $|H(\zeta_{s+1} - 1)|_\ell \geq r^{L+1}$. Every factor $D_t(\zeta_{s+1})$ has absolute value at most one, and so each has absolute value at least r^{L+1}. Repeat until $s = q$. The inequalities defining r and monotonicity of the real logarithm give the result, including $q = 0$. □

Corollary 594.2. *Fix ℓ, m, δ as in Theorem 594.1, a positive integer h_*, a finite set I, and bounds $B_i \geq 0$. For all sufficiently large N the following holds. Let C be a finite abelian group with a homomorphism $e : C \to \mathbb{Z}/\ell^N\mathbb{Z}$, and $c_* \in C$ with $e(c_*) = h_* \bmod \ell^N$. Let $Y_i \in O_\ell[C]$ have $v_\ell(Y_i(1)) \leq B_i$; here 1 denotes evaluation at the trivial character. There is a character $\tau : C \to K_\ell^\times$ of ℓ-power order such that for $1 \leq d \leq m$ and $\epsilon = \pm 1$,*

$$v_\ell(Y_i(\tau^{\epsilon d})) \leq B_i + \delta, \qquad 0 < v_\ell(1 - \tau(c_*)^d) \leq \delta.$$

Proof. Write $Y_i = \sum_{c \in C} a_{ic}[c]$. For each sign form $F_{i,\epsilon}(T) = \sum_c a_{ic} T^{[\epsilon e(c)]}$, where brackets in the exponent take the representative between zero and $\ell^N - 1$. Their values at $T = 1$ have the prescribed valuations. In the theorem take $q = q_0 = a = 1$, use these polynomials for every d with all witness points one, and add the monic polynomials $D_d(X) = X^{h_* d} - 1$. Their degrees sum to $h_* m(m+1)/2$, independent of N and C. The resulting ζ defines $\tau(c) = \zeta^{[e(c)]}$, a character since $\zeta^{\ell^N} = 1$. Evaluation of the polynomials at ζ^d gives the first bound. Each $D_d(\zeta)$ has finite valuation by the second bound in the theorem, so $\zeta^{h_* d} \neq 1$. Its order is a positive power of ℓ. Lemma 593.1 gives $0 < v_\ell(1 - \zeta^{h_* d})$, completing the proof. □

595. Prime generators with specified residues

Lemma 595.1. *Let E be any totally imaginary number field, $f = [E : \mathbb{Q}]/2$, and $m_0 \geq 2$ a rational integer. For the ray group C_{m_0} of Definition 219.1, write $Z_A(s)$ for the sum of $(N\mathfrak{a})^{-s}$ over nonzero integral ideals prime to m_0 in the class A. Every Z_A extends meromorphically to $\Re s > 0$ with only a simple pole at one. Its positive residue is the same for all A.*

Proof. Choose a nonzero integral ideal $\mathfrak{b}$ prime to m_0 in A^{-1} as in Proposition 219.2. In $\mathbb{C}^f$, using one embedding per conjugate pair, let L be the image of $m_0\mathfrak{b}$. The generators $x \in \mathfrak{b},\ x \equiv 1 \bmod m_0 O_E$ lie on a single translate $a + L$ avoiding zero by that proposition. Their orbits under $U = \{\epsilon \in O_E^\times : \epsilon \equiv 1 \bmod m_0\}$ correspond to the ideals being counted by $\mathfrak{a} = (x)\mathfrak{b}^{-1}$.

Let $\mathcal{H} = \{z \in \mathbb{R}^f : \sum z_i = 0\}$. By Lemma 83.1, choose $U_0 \subset U$ of finite index generated by units whose logarithms form a real basis of $\mathcal{H}$; the logarithm is injective on U_0. Let $B \subset \mathcal{H}$ be a bounded half-open parallelepiped for the lattice $(2\log|\sigma_i\epsilon|)_i,\ \epsilon \in U_0$. Use Lebesgue measure $\mathrm{d}z$ for which $\prod_i \mathrm{d}t_i/t_i = (\mathrm{d}t/t)\,\mathrm{d}z$ under the real linear substitution $\log t_i = \log t + z_i$. This fixes its constant also in dimension zero. Set

$$\Theta(t, z) = \sum_{x \in a + L} \exp\left(-\pi t \sum_{i=1}^{f} e^{z_i} |x_i|^2\right).$$

For $\Re s > 1$,

$$\int_B \int_0^\infty \Theta(t, z) t^{fs} \frac{\mathrm{d}t}{t}\,\mathrm{d}z = [U : U_0]\pi^{-fs}\Gamma(s)^f (N\mathfrak{b})^{-s} Z_A(s).$$

Indeed multiplying x by $\epsilon \in U_0$ translates z by its twice-logarithm; its phases do not change a Gaussian in the absolute values. For each U_0-orbit the translated parallelepipeds tile $\mathcal{H}$ up to boundaries of measure zero. Each ideal has exactly $[U : U_0]$ such orbits since units act freely on nonzero elements. The resulting integrals over independent $t_i > 0$ are $\Gamma(s)(\pi|x_i|^2)^{-s}$ by scaling in Lemma 4.3. Their product gives the displayed result since $\prod |x_i|^2 = N\mathfrak{a}\, N\mathfrak{b}$. For real $s > 1$ the interchange follows by increasing convergence. For complex s take the real part of the exponent as a dominating integral by Proposition 219.2 and Lemma A.2.

For z in the closure of B, the two Gaussian quadratic forms with weights e^{z_i} and e^{-z_i} both dominate a fixed positive multiple of the Euclidean squared norm. The real lattice Poisson formula proved in Lemma 62.2 gives

$$\Theta(t, z) = \frac{t^{-f}}{\operatorname{covol}(L)} \sum_{y \in L^*} e^{2\pi i a \cdot y} \exp\left(-\pi t^{-1} \sum_i e^{-z_i} |y_i|^2\right),$$

where $L^* = \{y : y \cdot L \subset \mathbb{Z}\}$ uses the real Euclidean dot product. The Fourier determinant is t^{-f} because each complex coordinate has two real dimensions and $\sum_i z_i = 0$. Both lattices are discrete. The number of points in any ball of radius $R \geq 1$ in either $a + L$ or L^* is $O((1 + R)^{2f})$, by packing disjoint balls of a fixed small radius about their points. The positive minimum lengths on $a + L$ and $L^* \setminus \{0\}$, together with this count, give $O(e^{-ct})$ for

the original sum at $t \geq 1$ and $O(t^{-f}e^{-c/t})$ for the terms $y \neq 0$ at $t \leq 1$, uniformly in z, for some $c > 0$. For example, factor out one half of the Gaussian exponent at the minimum length and sum the other half at scale one.

Split the Mellin integral at one and subtract $t^{-f}/\operatorname{covol}(L)$ on the lower interval. Both remainders are entire, by the displayed bounds after multiplication by any fixed power of $\log t$, using Lemma A.3. The subtracted term integrates to $\operatorname{vol}(B)/(f \operatorname{covol}(L)(s-1))$. On $\Re s > 0$, $\Gamma(s)$ is holomorphic and nonzero by Lemma 4.3. Finally $\operatorname{covol}(L) = m_0^{2f} N\mathfrak{b} \operatorname{covol}(O_E)$ by the lattice index in Proposition 82.3. The factor $N\mathfrak{b}$ cancels at the residue. All remaining factors there are positive and depend only on E, m_0. □

Theorem 595.2. *For any totally imaginary number field E and invertible residue $a \bmod m_0 O_E$, $m_0 \geq 2$, there exist infinitely many prime ideals $w = (\pi)$ of rational prime norm with $\pi \in O_E$, $\pi \equiv a \bmod m_0 O_E$. Any fixed finite set of rational primes may be excluded.*

Proof. For $C = C_{m_0}$ and $\lambda \in \widehat{C}$ use $L_{m_0}(s, \lambda) = \sum_A \lambda(A) Z_A(s)$. The preceding lemma gives on $\Re s > 0$ a single simple pole at one for the trivial character and holomorphy for the others, since their class sums vanish by Lemma 7.1. Each nontrivial factor has nonzero value at one. To prove this, consider the product of the series over all λ. At a prime of ideal class of order e, the local product is $(1 - (Nw)^{-es})^{-|C|/e}$: the values on that class run through the e-th roots equally often by extending each character of its cyclic subgroup as in Lemma 7.1. The Euler products in Proposition 219.2 therefore give a Dirichlet series with nonnegative coefficients. Locally it dominates the nonnegative series $(1 - (Nw)^{-|C|s})^{-1}$ coefficientwise, as $e \mid |C|$. Multiplication and increasing limits preserve the domination. Its abscissa of convergence lies between $1/|C|$ and 1, since the trivial ray zeta function at $|C|s$ tends to infinity as real $s \downarrow 1/|C|$. Lemma 7.2 applies to the discrete measure at logarithms of integer norms with these nonnegative coefficients. A nontrivial zero at one would cancel the product's only possible pole on $\Re s > 0$, contradicting the singularity at its positive real abscissa.

By Lemma 7.1, expanding the Euler logarithms against the characters gives for any $A \in C$

$$\sum_{\substack{[w]=A \\ w \nmid m_0}} (Nw)^{-s} = \frac{1}{|C|} \log \frac{1}{s-1} + O_{E,m_0}(1) \qquad (s \downarrow 1).$$

Indeed terms of degree at least two have absolute sum bounded by $2 \sum_w (Nw)^{-2}$, finite by Corollary 83.4. The logarithms of the nontrivial series remain bounded along this real interval: a local logarithm exists near their nonzero values, and its difference with the Euler logarithm is a locally constant multiple of $2\pi i$, as in Corollary 7.5. Removal of primes of nonprime norm changes the sum by a bounded amount, since above a rational prime p there are at most $2f$ primes and those of degree at least two have norm at least p^2, by Proposition 82.3. Removing finitely many rational primes costs a bounded amount as well. The remaining sum is unbounded.

Choose an integral lift a_0 of the prescribed residue. It is prime to m_0. Apply the preceding assertion to the class of (a_0). Each resulting ideal is $(a_0 y)$ with $y \equiv 1 \bmod m_0 O_{E,\mathfrak{p}}$ for all $\mathfrak{p} \mid m_0$. Its generator $\pi = a_0 y$ belongs to O_E because its principal ideal is integral. At every prime over m_0 the difference $\pi - a_0$ belongs to $m_0 O_{E,\mathfrak{p}}$; prime factorization gives $\pi - a_0 \in m_0 O_E$. □

596. A split prime for all the ramified pairs

Lemma 596.1. *Let $h_* = |C(\varnothing)|$, and let u_E denote the order of the finite group of roots of unity of E. Let $w \neq \bar{w}$ be an allowed split place and choose a generator t_w of the cyclic group $\mathbf{F}_w^\times$. If $\ell^{N+v_\ell(u_E)} \mid q_w - 1$, $N \geq 1$, there is a homomorphism $e : C(\{w\}) \to \mathbb{Z}/\ell^N\mathbb{Z}$ such that $e([u_{t_w}]) = h_* \bmod \ell^N$.*

Proof. For $x \in T_f$, write $x^{h_*} = ak$, $a \in T(E)$, $k \in K_T$, using Definition 589.1 and finiteness in Lemma 589.2. The ambiguity in k is multiplication by an element of $T(E) \cap K_T$, precisely the finite group of roots of unity by Theorem 372.2. Thus the residue of k at w in $\mathbf{F}_w^\times$ modulo the residues of those roots is intrinsic. It multiplies under products of x because the torus is abelian. Multiplying x by a rational point changes only a, and multiplying by $K_T(\{w\})$ preserves that residue.

The residue group is cyclic by Lemma 13.2. Send t_w^b to $b \bmod \ell^N$. This is well defined since $\ell^N \mid q_w - 1$. A residue of an E-root of unity has order dividing u_E, hence $q_w - 1 \mid bu_E$ if it is t_w^b. The extra divisibility hypothesis gives $\ell^N \mid b$, so the map descends to the quotient of residue groups. The composite just constructed on $C(\{w\})$ takes $[u_{t_w}]$ to h_*, by choosing $a = 1$ for that finite unit. □

Theorem 596.2. *Fix a rational prime ℓ and choose either $g_i \neq \ell$ of Theorem 11.4. Choose a marked place $w_* \mid g_i$ and a generator t_* of its residue unit group. Put $e_\ell = v_\ell(g_i - 1)$. For every integer $K > e_\ell$ there are infinitely many allowed w of rational prime norm $q \equiv 1 \bmod \ell^K$, avoiding any prescribed finite set of rational primes, with the following property. For every ramified pair (μ_0, v) of Definition 592.1, put ψ_* equal to its unit character at w_*. Then for every isomorphism $j : \mathbb{C} \to K_\ell$, every $n \in \mathbb{Z}$, and each $\epsilon = \pm 1$,*

$$v_\ell(1 - q^n j(\mu_0(b_w))^\epsilon) = v_\ell(1 - j(\psi_*(t_*))^{-\epsilon}) \ \leq e_\ell.$$

The same w works for all these parameters.

Proof. Impose on a generator π the congruences $\pi \equiv 1 \bmod \ell^K O_E$, $\pi \equiv t_* \bmod w_*$, and $\pi \equiv 1$ at all other places of $M \cup \bar{M}$. The marked residue fields above g_i are $\mathbf{F}_{g_i}$ by Lemma 13.2. Since $w_* \nmid \ell$, any overlaps with ℓ^K prescribe the same residue one. There is one invertible class modulo $m_0 O_E$ implementing these conditions, with $m_0 = \operatorname{lcm}(\ell^K, g_0, g_1)$. Indeed distinct prime ideal powers here are relatively prime by Proposition 82.3. For two relatively prime ideals I, J, write $1 = a + b$, $a \in I$, $b \in J$. The linear combination $bx + ay$ has residues $x \bmod I$ and $y \bmod J$; iteration gives all the conditions.

Theorem 595.2 gives infinitely many principal primes $w = (\pi)$ with the desired congruences, excluding the specified primes, all those dividing the fixed levels, and all ramified rational primes by Lemma 219.3. Such a prime is split over $F = E^+$: an unramified nonsplit quadratic place has norm equal to the square of the underlying residue size, by Lemma 57.1. The norm of π is positive since E is totally imaginary, and equals q by Proposition 82.3. Multiplication by π on an integral basis is the identity modulo ℓ^K. Its determinant gives the congruence for q.

Let $a = \pi/\bar{\pi} \in T(E)$, viewed also at infinity and at the finite places. Choose the local generator π at w for b_w. Now b_w/a_f is a norm-one integral unit everywhere: both elements have valuations $1, -1$ at $w, \bar{w}$ and zero elsewhere. At the marked places its residues are t_*^{-1} at w_* and one at every other member of M, since the conjugate residue over w_* was prescribed to be one. Therefore rational invariance and the weight in Definition 372.1 give

$$\mu_0(b_w) = \left(\prod_{\sigma \in \Phi} \sigma(a)^{c_\sigma} \right) \psi_*(t_*)^{-1}.$$

Changing the chosen uniformizer changes b_w by units at the new pair and leaves this formula for the character unchanged. For each embedding of E into K_ℓ, $\pi, \bar{\pi}$ are congruent to one modulo $\ell^K O_\ell$, since they both belong to $1 + \ell^K O_E$ and algebraic integers have integral images by Lemma 588.1. Thus the product of archimedean weight factors in this equality is carried by every j to $1 + \ell^K O_\ell$. This multiplicative subgroup is closed under integer powers because all its members are units. The same holds for q^n.

The root $\psi_*(t_*)$ is nontrivial by ramification and its order divides $g_i - 1$. For any $H \geq 1$ the product of $1 - \xi$ over all nontrivial H-th roots of unity in K_ℓ is H, by evaluating the finite geometric sum at one. All terms have nonnegative valuation. Every such root therefore

satisfies $v_\ell(1-\xi) \le v_\ell(H)$. Apply this to $H = g_i - 1$ and either sign. Replacing ξ by its product with an element of $1 + \ell^K O_\ell$ changes $1 - \xi$ by a quantity of valuation at least $K > e_\ell$. The ultrametric inequality in Lemma 16.2 gives equality of the resulting valuations as asserted. □

597. One-place twists for the inverse critical values

Theorem 597.1. *Fix the arithmetic data, an integer $m \ge 1$, a prime ℓ, a field isomorphism $j : \mathbb{C} \to K_\ell$ and $\delta > 0$. Choose $g_i \ne \ell$, set $e_\ell = v_\ell(g_i - 1)$, and exclude any fixed finite set of rational primes from the auxiliary level. There is a split place w of rational prime norm $q_w \ne \ell$ outside these exclusions and a tame complex character ϑ of $C(\{w\})$ of ℓ-power order such that all the powers ϑ^d, $1 \le d \le m$, are primitive at w. On an integral idele u_{t_w} lifting a generator of $\mathbf{F}_w^\times$, one has $0 < v_\ell(1 - j(\vartheta^d(u_{t_w}))) \le \delta$. For every ramified pair $p = (\mu_0, v)$ of Definition 592.1, $1 \le d \le m$ and $\epsilon = \pm 1$,*

$$v_\ell\Big(j\big(\rho_p(\{w\}, \vartheta^{\epsilon d})\big)\Big) \le 2e_\ell + \delta.$$

Here the subscript p specifies which pair defines ρ.

Proof. There are only finitely many pairs for the fixed arithmetic data. The integer v and the finitely many integer weights c_σ range over bounded intervals. There are finitely many characters of the product of the marked residue groups by Lemma 7.1. For fixed weights and unit character, any two possibilities for μ_0 differ by a weight-zero character trivial on K_T, and hence by a character of the finite group $C(\varnothing)$ by Lemma 589.2. For each ramified pair use the integral elements $F_{p,P}$ of Theorem 592.2. Their untwisted values $F_{p,\varnothing}(1)$ are nonzero algebraic integers by Lemma 589.2. Define finite bounds

$$B_p = v_\ell(j(F_{p,\varnothing}(1))) + 2|K_v|e_\ell.$$

With $h_* = |C(\varnothing)|$, choose N sufficiently large for Corollary 594.2 with the bounds B_p and tolerance δ. This choice is independent of w. Choose $K > e_\ell$ with $K \ge N + v_\ell(u_E)$, where u_E is from Lemma 596.1. Invoke Theorem 596.2 at this K, excluding ℓ also, and take any resulting w. Lemma 596.1 supplies $e : C(\{w\}) \to \mathbb{Z}/\ell^N\mathbb{Z}$ and the required generator image h_*.

Apply j to the coefficients of $F_{p,\{w\}}$; the result is in $O_\ell[C(\{w\})]$ since $\ell \nmid q_w$ by Lemma 588.1. Its trivial evaluation has valuation at most B_p. Indeed the projection formula in Theorem 592.2 at the trivial character is the product of $2|K_v|$ Euler factors times $F_{p,\varnothing}(1)$. Each factor has valuation at most e_ℓ by Theorem 596.2. Corollary 594.2 gives a common ℓ-power character τ with all the asserted generator bounds and all evaluations of these group-ring elements at $\tau^{\epsilon d}$ of valuation at most $B_p + \delta$. Transport its values back through j^{-1}; being roots of unity they are unitary, so the resulting ϑ is a tame character in Definition 589.1. Its indicated powers have nontrivial unit character at w since the generator differences have finite valuation. For these primitive powers apply the value formula of Theorem 592.2. Since q_w is an ℓ-unit, this gives

$$(m_v + 1)v_\ell(j(\rho_p(\{w\}, \vartheta^{\epsilon d}))) \le 2|K_v|e_\ell + \delta.$$

For $v = 1$, $|K_v| = 2 = m_v + 1$, and for $v \ge 2$ both quantities equal one. Division proves the asserted uniform bound. □

Corollary 597.2. *Use the actual length m of Proposition 10.5. For every $\eta > 0$ one can choose one-place twists ϑ_ι of the preceding theorem for all $\iota = (\ell, j)$ in Definition 16.3. They satisfy $q_{w_\iota} \ne \ell$, and, for every ramified pair p, both signs and every $1 \le d \le m$,*

$$\mathrm{ht}\Big(\rho_p(\{w_\iota\}, \vartheta_\iota^{\epsilon d})^{-1}\Big) \;\le\; 2\log((g_0 - 1)(g_1 - 1)) + \eta \;\le\; 44\log b + \eta.$$

For fixed η and sufficiently large x this bound is at most x^{999}. For the primary ratios of Lemma 579.1, the same choices give

$$\operatorname{ht}\left(\prod_{l=1}^{m-1}\prod_{v=1}^{t}\left(\frac{B_{l,1}(v)}{B_{l,\vartheta_\iota}(v)}\right)^{m-l}\right) \;\le\; t\binom{m}{2}(44\log b+\eta).$$

Proof. For each ℓ choose $g_{i(\ell)}\neq\ell$ and use the theorem at every j with $\delta_\ell=\eta 2^{-\ell}/\log\ell$; the choices exist simultaneously by Definition 16.1. The witness in Definition 16.3 is the scalar one, with archimedean exponent zero. The finite exponents $(2v_\ell(g_{i(\ell)}-1)+\delta_\ell)\log\ell$ suffice by the definition of v_ℓ. Their sum over any finite set of primes is at most the first claimed bound since

$$\sum_{\ell} v_\ell((g_0-1)(g_1-1))\log\ell=\log((g_0-1)(g_1-1))$$

by Lemma A.1, and $\sum_\ell 2^{-\ell}\le 1$. Theorem 11.4 gives $g_0,g_1\le b^{11}$, and Proposition 10.5 gives $\log b=x^{\varepsilon}$ with $0<\varepsilon<1/100$, proving the numerical claims.

For the last assertion take the base character η_1^l with weight $-lh$ at every place. Its unit character at M is nontrivial by Lemma 568.1. It has $t\le lh\le 4b$ for $1\le l<m$ at sufficiently large scales: $t=5r,\ h=2r^8$, while the upper bound follows from $m\le 2\sqrt{x},\ r\le 3x^5,\ b=\exp(x^{\varepsilon})$ in Proposition 10.5. Thus it gives an eligible ramified pair at every $1\le v\le t$. The twist in its ratio is ϑ_ι^l by Lemma 579.1. Use Proposition 16.4 and $\sum_{l=1}^{m-1}(m-l)=\binom{m}{2}$ to obtain the result. □

598. Imposing further polynomial conditions on the tame characters

Definition 598.1. Fix $\ell,N\ge 1$ with ℓ prime, and a finite nonempty allowed set P above rational primes distinct from ℓ. A system of coordinates here consists of homomorphisms $e_w : C(\{w\})\to\mathbb{Z}/\ell^N\mathbb{Z}$, one for each $w\in P$, also viewed on $C(P)$ by inflation. Given $y_w^{\ell^N}=1$ in K_ℓ, define a character

$$\tau_y(c)=\prod_{w\in P} y_w^{[e_w(c)]},$$

where $[t]$ denotes the integer representative in $[0,\ell^N)$. For elements of the torus put $e_w(b)=e_w([b])$. Choose residue generators $t_w\in\mathbf{F}_w^{\times}$. Fix $w_0\in P$ and an integer $H\ge 1$. Require

$$e_w(u_{t_w})=a_w,\quad 1\le a_w\le H;\qquad e_{w_0}(b_w)=s_w,\quad 1\le|s_w|\le H\quad(w\in P\setminus\{w_0\})$$

as congruences modulo ℓ^N, with the first condition for all $w\in P$. In the second condition b_w is taken at a place away from the modulus of e_{w_0}, so the choice of uniformizer is immaterial.

Theorem 598.2. *Fix the arithmetic data, a prime ℓ, an isomorphism $j:\mathbb{C}\to K_\ell$, a choice $g_i\neq\ell$, and put $e_\ell=v_\ell(g_i-1)$. Fix positive integers m,J,H,A and real $\delta>0$. Fix a finite index set I with real $B_i\ge 0$ for $i\in I$. For all sufficiently large N the following holds. Take a system of Definition 598.1 with $|P|\le J$ and suppose w_0 satisfies the Euler-factor conclusions of Theorem 596.2. For each $i\in I,\ 1\le d\le m$ let $G_{i,d}$ be a polynomial over O_ℓ in the coordinates indexed by P. Suppose there exist points $u_{i,d}\in O_\ell^P$ with*

$$v_\ell(u_{i,d,w}-1)\ge 1/A,\qquad v_\ell(G_{i,d}(u_{i,d}))\le B_i.$$

There exists a tuple y of ℓ^N-th roots of unity, putting $\vartheta=j^{-1}\circ\tau_y$, such that for all $1\le d\le m$,

$$v_\ell(G_{i,d}(y^d))\le B_i+\delta\quad(i\in I),$$
$$v_\ell(j(\rho_p(P,\vartheta^{\epsilon d})))\le 2e_\ell+\delta\quad(\epsilon=\pm1,\ \text{every ramified pair } p),$$
$$\sum_{w\in P} v_\ell(1-j(\vartheta^d(u_{t_w})))\le\delta.$$

All these powers are primitive at P. The bound for N is independent of P, of the individual coordinate maps and of the degrees of the $G_{i,d}$.

Proof. Use the finitely many pairs in the proof of Theorem 597.1. Write $b_p = v_\ell(j(F_{p,\varnothing}(1)))$, $D_p = b_p + 2|K_v|e_\ell$ for a pair $p = (\mu_0, v)$, and put $\eta = \delta/(8J+8)$. For each sign ϵ first evaluate the coefficients of $F_{p,\{w_0\}}$ under j, substituting $X^{[\epsilon e_{w_0}(c)]}$ for $[c]$ in its group ring. This gives an integral polynomial since $q_{w_0} \neq \ell$. Its value at one has valuation at most D_p, by the projection formula of Theorem 592.2 and the assumption at w_0.

For each $w \neq w_0,\ k \in K_v,\ \lambda = \pm 1$ and sign ϵ, take the function on units $X \in O_\ell^\times$

$$1 - q_w^{k-v} j(\mu_0(b_w))^\lambda X^{\lambda \epsilon s_w}.$$

Its valuation on those units is the valuation of a monic polynomial $X^{|s_w|} - a,\ a \in O_\ell^\times$. Indeed the coefficient on the displayed monomial is a unit by Theorem 372.2 since b_w is a unit above ℓ. For positive exponent divide by its negative coefficient. For negative exponent multiply by the positive power of X. Both changes are by units. The sum of the degrees of all these monic polynomials is at most $8JH$ times the number of pairs.

Theorem 594.1, applied in one variable, with power one, point one, and tolerance η, supplies $k_0 \geq 1$ independent of the system, such that some $z^{\ell^{k_0}} = 1$ meets the bounds $D_p + \eta$ for all the first polynomials and η for all the monic polynomials. More explicitly use the theorem's order bound to fix k_0 first, then on any system of level $N \geq k_0$ apply it with exponent k_0. Put $u = (1, \dots, 1)$ except for $u_{w_0} = z$. For its character τ_u^ϵ, inflate successively from $\{w_0\}$ to P. At each added w, $\tau_u^\epsilon(b_w) = z^{\epsilon s_w}$. By the same projection formula the increase in valuation is at most $2|K_v|\eta$. Consequently

$$v_\ell\Big(j\big(F_{p,P}(j^{-1} \circ \tau_u^\epsilon)\big)\Big) \leq D_p + (1+4J)\eta.$$

All the quantities on the left are finite. Moreover every coordinate of u satisfies $v_\ell(u_w - 1) \geq \ell^{-k_0}$, by Lemma 593.1, including $u_w = 1$.

At the full level make integral polynomials from $F_{p,P}$ by applying j to coefficients and replacing $[c]$ by $\prod_w X_w^{[\epsilon e_w(c)]}$. At u their values are the ones just estimated. Use these polynomials for every power index d, together with the given $G_{i,d}$ at their own witness points. Apply Theorem 594.1 with at most J variables, depth integer $\max(A, \ell^{k_0})$, and tolerance η. Use also the monic tests $X^{hd} - 1$ for $1 \leq h \leq H,\ 1 \leq d \leq m$. Their sum of degrees is fixed. All the parameters determining this second order bound are independent of the system, so N can be chosen large enough for both applications.

For the resulting y, each group-ring polynomial gives the evaluation at $\tau_y^{\epsilon d}$, of valuation at most $D_p + (2+4J)\eta$. The coordinate $e_{w'}$ for $w' \neq w$ is trivial on u_{t_w}, by the modulus defining it. Thus $j(\vartheta^d(u_{t_w})) = y_w^{a_w d}$. The monic tests give valuation at most η for its difference from one, and in particular this difference is nonzero. This proves primitivity and the bound on the sum of at most J terms. Use the primitive value identity in Theorem 592.2, subtract b_p and divide by $m_v + 1 = |K_v|$; the conductor is an ℓ-unit. Since $(2+4J)\eta \leq \delta$, the claimed bound follows for the ratios. The second specialization itself gives the assertion about every $G_{i,d}$. □

599. Combining residue prescriptions with complete splitting

Lemma 599.1. *Let A/E be a finite extension of number fields unramified at a prime $\mathfrak{p}$ of E, meaning that every exponent in $\mathfrak{p}O_A$ is one. Let $n \geq 1$ be prime to its residue characteristic and $a_1, \dots, a_t \in A^\times$ units at every prime of A above $\mathfrak{p}$. The splitting field over A of $(X^n - 1)\prod_i (X^n - a_i)$ is unramified over E at $\mathfrak{p}$.*

Proof. It suffices to adjoin roots one at a time. In an intermediate field K, localize at a prime $\mathfrak{q}$ over $\mathfrak{p}$, with discrete valuation ring O by Proposition 82.3. For a root x of $X^n - a$, where a is one of the listed units or one, its monic minimal polynomial D over K and the

monic quotient $(X^n - a)/D$ lie in $O[X]$. Indeed their coefficients are symmetric polynomials in integral roots and belong to K; an element of K integral over a valuation ring has nonnegative valuation, since otherwise the leading term of its monic equation dominates. To see integrality of sums and products here, the algebra over O generated by finitely many integral roots is finite as module by successive power bases. Multiplication by any of its elements satisfies a monic equation by the determinant of a relation matrix on these generators, as in Proposition 82.3. Modulo $\mathfrak{q}$, $X^n - a$ is separable since n, a are units. Thus the discriminant of D is a unit, by its product of squared root differences in a splitting field.

The integral closure of O in $K(x)$ is $O[x]$. In fact write an integral b in the power basis $1, x, \ldots, x^{d-1}$ where $d = \deg D$. The traces $\mathrm{Tr}_{K(x)/K}(bx^i)$ are integral by the conjugate-sum argument above. The matrix $(\mathrm{Tr}(x^{i+j}))_{0 \le i,j < d}$ has unit determinant: it is the transpose of the Vandermonde matrix of the roots times that matrix, so its determinant is the discriminant. Its inverse has entries in O by the adjugate identity. All coordinates of b are therefore integral. Modulo $\mathfrak{q}$, $O[x]$ is the quotient by the squarefree reduction of D, a product of fields by the polynomial Euclidean algorithm. No prime above $\mathfrak{q}$ has exponent greater than one: localizing the integral closure at such a prime would give a discrete valuation ring whose quotient by $\mathfrak{q}$ had a nonzero nilpotent uniformizer. The localized ring of integers is indeed the integral closure used here: clearing denominators outside $\mathfrak{q}$ in a monic equation makes an appropriate multiple of its root integral over the ring of integers. This is the same scaling argument as in Lemma 588.1. Thus no step introduces an exponent above one. Exponents in a tower multiply by extending the ideal factorization from one ring of integers to the next. This proves the claim. □

Theorem 599.2. *Let E be totally imaginary, L/E a finite Galois extension unramified at a prime $\mathfrak{s}$ of E, and $a \in (O_E/\mathfrak{s})^\times$. Let $\mathfrak{m}$ be a nonzero integral ideal prime to $\mathfrak{s}$, allowing the unit ideal. Outside any finite set of rational primes there are infinitely many primes $w = (\pi)$ of rational prime norm in E splitting completely in L/E, with*

$$\pi \equiv a \bmod \mathfrak{s}, \qquad \pi \equiv 1 \bmod \mathfrak{m}.$$

Proof. Write $k = O_E/\mathfrak{s}$. Localizing O_L at the complement of $\mathfrak{s}$ in O_E gives a finite free module over the resulting discrete valuation ring, of rank $[L : E]$. Indeed the integral ring is finitely generated by its $\mathbb{Z}$-basis as in Lemma 82.2. Clear the common denominators of its coordinates in an E-basis. A submodule of a finite free module over a discrete valuation ring is free: in a nonzero coordinate its image is a principal ideal, choose a vector mapping to a generator, split off its multiples, and induct on the number of remaining coordinates. The rank follows after tensoring with E.

Computing the determinant of multiplication by an integral element on this free module and reducing at $\mathfrak{s}$ gives the norm on $O_L/\mathfrak{s}O_L$. By unramifiedness this algebra is $\prod_{\mathfrak{S}|\mathfrak{s}} O_L/\mathfrak{S}$ by the ideal Chinese remainder argument of Theorem 596.2. The norm on this product is the product of the finite-field norms. Each residue norm is onto: for an extension of a field of size q of degree d, a generator of the multiplicative group exists by Lemma 13.2. It generates the field, and its d distinct Frobenius conjugates give its minimal polynomial since the q-power has order d there. Indeed $q^j - 1 < q^d - 1$ for $0 < j < d$, and all elements satisfy $X^{q^d} = X$. Its norm is thus the generator raised to $(q^d - 1)/(q - 1)$, of order $q - 1$. Prescribe a unit with norm a at one $\mathfrak{S} \mid \mathfrak{s}$ and one at all others. Prescribe one modulo $\mathfrak{m}O_L$.

Choose a rational $m_0 \ge 2$ with $m_0 O_L \subset \mathfrak{m}\mathfrak{s}O_L$, for example a multiple of the additive index of this integral ideal. The prescriptions extend to an invertible class modulo $m_0 O_L$: lift the units at the prescribed prime powers and put one at any further primes, using relatively prime ideal powers. The totally imaginary field L qualifies for Theorem 595.2. It supplies infinitely many principal primes $\mathfrak{W} = (\alpha)$ in that class of generators, of rational prime norm and avoiding the excluded primes and the finite exceptions of Lemma 219.3 for L/E. Their contraction w has the same residue field, since this field is already the prime field. The residue-action lemma gives complete splitting because the residue degree at $\mathfrak{W}$ is

one. Put $\pi = N_{L/E}\alpha$. Its absolute norm is the rational prime $N\mathfrak{W}$ by multiplicativity of field norms in a tower, obtained by multiplying the embedding products. It belongs to w since the conjugate product defining it has factor α. Thus $(\pi) = w$ by Proposition 82.3. The determinant computation gives $\pi \bmod \mathfrak{s} = a$. All conjugates of α equal one modulo $\mathfrak{m}O_L$, so their product does too. Contraction back to O_E gives $\mathfrak{m}O_L \cap O_E = \mathfrak{m}$, by extending prime factorizations and comparing the positive exponents in the extensions of the individual primes. Finally only finitely many primes of L lie above any one w, proving infinitude in E. □

600. Independent coordinates at several split primes

Lemma 600.1. *Fix $\ell, N \geq 1$ with ℓ prime, a split prime $\mathfrak{s} \nmid \ell$ of E/F, and a finite list $\mathcal{B} \subset T_f$ of ideles equal to one off finite sets and integral at $\mathfrak{s}, \bar{\mathfrak{s}}$. There are a finite Galois extension L_0/E unramified at $\mathfrak{s}$, and a finite excluded set of primes, with the following property. At any other split $w \nmid \ell$ splitting completely in L_0/E, each choice of generator $t_w \in \mathbf{F}_w^\times$ gives a homomorphism $e_w : C(\{w\}) \to \mathbb{Z}/\ell^N\mathbb{Z}$ satisfying*

$$e_w(u_{t_w}) = 1, \qquad e_w(b) = 0 \quad (b \in \mathcal{B}).$$

On arbitrary $k \in K_T$ it is the discrete logarithm of $k_w \bmod w$ to base t_w, reduced modulo ℓ^N.

Proof. For each $A \in C(\varnothing)$ choose a finite idele x_A representing it, with $x_1 = 1$. The list is finite by Lemma 589.2. Arrange that all representatives are one off finite sets and at $\mathfrak{s}, \bar{\mathfrak{s}}$. For the latter, choose z with valuation one at $\mathfrak{s}$ and valuation zero at $\bar{\mathfrak{s}}$: prescribe a nonzero class in $\mathfrak{s}/\mathfrak{s}^2$ and one modulo $\bar{\mathfrak{s}}$ by the ideal Chinese remainder argument of Theorem 596.2. The class exists by Proposition 82.3. A power of $z/\bar{z} \in T(E)$ corrects both valuations. Multiplying by a norm-one integral idele then makes every unit component outside the finite support, and at this pair, equal to one. This does not change the class.

Choose factorizations

$$x_A x_B = x_{AB} a_{AB} k_{AB}, \qquad b = x_{[b]} a_b k_b \quad (b \in \mathcal{B})$$

with $a_{AB}, a_b \in T(E)$, $k_{AB}, k_b \in K_T$; $[b]$ here is its class at empty level. The rational factors are units at $\mathfrak{s}$ since all the other factors are. Adjoin to E all the ℓ^N-th roots of all the a_{AB}, a_b, of one and of each root of unity of E. This is a finite splitting field in characteristic zero. The roots of unity of E form a finite set by Theorem 372.2. By Lemma 599.1 the field L_0 is unramified at $\mathfrak{s}$. Exclude supports of the x_A, b, the prime divisors of the rational factors, and primes above ℓ.

At a completely split remaining w, every one of those radicands is an ℓ^N-th power in $k_w^\times$ by reducing a root at a prime with residue $k_w = \mathbf{F}_w$; the roots are units because the radicands are. All the ℓ^N roots of unity are distinct after reduction, since the polynomial $X^{\ell^N} - 1$ is separable there. Thus $\ell^N \mid q_w - 1$. The discrete logarithm to base t_w reduced modulo ℓ^N is a homomorphism $\lambda : k_w^\times \to \mathbb{Z}/\ell^N\mathbb{Z}$ vanishing on all these powers. For $x = x_A a k$, with $a \in T(E)$, $k \in K_T$, define $e_w(x) = \lambda(k_w \bmod w)$. For fixed A the ambiguity is a rational integral norm-one unit, hence a root of unity. This ambiguity is killed by λ. In multiplying two such decompositions the extra finite unit is k_{AB}, with component a_{AB}^{-1} at w, since all x_A were chosen to equal one there. This too is killed, proving the homomorphism law. It vanishes on $T(E)K_T(\{w\})$, restricts to λ on K_T, and vanishes on each $b \in \mathcal{B}$ since $k_{b,w} = a_b^{-1}$ at w. This proves all assertions. □

Theorem 600.2. *Fix $\ell, N, J \geq 1$ with ℓ prime, $g_i \neq \ell$, and a marked place $\mathfrak{s} \mid g_i$. Let L/E be any finite Galois extension and $\mathcal{B}$ as in Lemma 600.1. There exist J split places P of pairwise distinct rational prime norms, a distinguished $w_0 \in P$, and coordinates $e_w : C(\{w\}) \to \mathbb{Z}/\ell^N\mathbb{Z}$ satisfying Definition 598.1 with $H = 1$. The norms may avoid any prescribed finite set and the fixed levels. The prime w_0 satisfies the conclusion of*

Theorem 596.2. Every $w \neq w_0$ in P splits completely in L/E. For all $u \in P$ one has $e_u(\mathcal{B}) = 0$. One can choose the local uniformizers defining b_w so that

$$e_u(u_{t_w}) = \mathbf{1}_{u=w} \quad (u, w \in P), \qquad e_u(b_w) = \mathbf{1}_{u=w_0} \quad (u \in P,\ w \in P \setminus \{w_0\}).$$

In particular the character τ_y takes the value y_w on u_{t_w} and y_{w_0} on b_w for $w \neq w_0$.

Proof. Choose $K \geq N + 1$ larger than $v_\ell(g_i - 1)$. Construct L_0 in the lemma at $\mathfrak{s}$ and use Theorem 599.2 in L_0/E, prescribing a generator t_* modulo $\mathfrak{s}$ and one modulo the product of $\ell^K O_E$ with the other primes of $M \cup \bar{M}$. Their product is prime to $\mathfrak{s}$. Choose an output $w_0 = (\pi_0)$ avoiding the rational primes below the exclusions of the lemma, all fixed levels and the finitely many exceptions of Lemma 219.3 for LL_0/E. It is split over F as in Theorem 596.2 since it has prime norm, and that proof gives its Euler-factor bounds from the congruences of π_0. It splits completely in L_0 and is unramified in LL_0. Choose a residue generator t_{w_0} and the coordinate of the lemma there.

Suppose a proper initial set P_1, containing w_0, has been chosen. For every ordinary $u \in P_1 \setminus \{w_0\}$ retain its integral generator π_u, and put $a_u = \pi_u/\bar{\pi}_u$. Enlarge LL_0 by all ℓ^N-th roots of these a_u, giving L'. It is Galois over E as the compositum of splitting fields, and is unramified at w_0 by Lemma 599.1: each a_u is a unit there since its only nonzero valuations are at $u, \bar{u}$. Apply Theorem 599.2 to L'/E at w_0, with prescribed residue $t_{w_0}^{-1}$ and with the congruence to one modulo

$$\ell^K \bar{w}_0 \prod_{u \in P_1 \setminus \{w_0\}} u\bar{u}.$$

Here ℓ^K denotes the principal ideal. Select an output $w = (\pi_w)$ avoiding the previously used rational primes and all the original exclusions. It is split over F, splits completely in L', and hence splits in L and L_0. Choose its residue generator and apply the lemma there as well.

Initially take b_w from the local uniformizer π_w. In T_f the quotient $b_w/(\pi_w/\bar{\pi}_w)_f$ is an integral unit. At w_0 its residue is t_{w_0} and at every ordinary earlier u it is one, by the congruences. Thus the older coordinates take the required values at b_w. In the other direction, $e_w(b_u) = 0$ for every earlier ordinary u, because at w the integral factor of $b_u/(a_u)_f$ is a_u^{-1}, an ℓ^N-th power in the residue field by splitting in L'. This proves the cross equalities by induction until $|P| = J$. For any ordinary w adjust its local uniformizer by a unit so that $e_w(b_w) = 0$, possible because its unit map is onto. This changes b_w only by an integral idele supported at $w, \bar{w}$ and hence preserves every other coordinate. The same unit calculation gives $e_u(u_{t_w}) = \mathbf{1}_{u=w}$ and the assertion about τ_y follows directly from its definition. □

601. Finite coordinates for cocycles under scalar twists

Definition 601.1. Fix an algebraic closure Ω of a number field E. Write $G_E = \mathrm{Aut}(\Omega/E)$ with the topology of agreement on finite normal extensions. A place on Ω over a finite prime of E is an extension of its nonarchimedean valuation. Its inertia subgroup consists of those $g \in G_E$ preserving the valuation and acting as the identity on its residue field. To vanish on inertia outside S means to vanish on every such subgroup over every finite place not in S. For a discrete commutative ring R, a map $A : G_E \to \mathrm{Mat}_r(R)$ and homogeneous linear relations $\mathcal{L}$ in the values of a function $z : G_E \to R^r$, consider the R-module of continuous functions satisfying these inertia conditions, the relations $\mathcal{L}$, and

$$z(gh) = z(g) + A(g)z(h) \quad (g, h \in G_E).$$

Each relation takes a finite linear combination of coordinates at specified group elements. For an action A the displayed identity is the usual 1-cocycle identity.

Lemma 601.2. *Let K be a number field containing all roots of unity of order dividing $n \geq 1$. For any finite set S of finite places of K there are only finitely many continuous characters $G_K \to \mu_n$ trivial on inertia outside S.*

Proof. Use Ω also as an algebraic closure of K. Each character c factors through a finite Galois extension B/K: its kernel is open, hence contains the group fixing some finite normal extension. For $\Gamma = \mathrm{Gal}(B/K)$ choose $t \in B$ with $b = \sum_{\gamma\in\Gamma} c(\gamma)^{-1}\gamma(t) \neq 0$. Such t exists because distinct field embeddings are linearly independent as functions with coefficients in their common target field. Indeed in a nonzero dependence with the fewest terms, there must be at least two terms by evaluation at one. Choose x on which two of the embeddings differ. Apply the dependence to xt and subtract its evaluation at t times the first image of x; this gives a shorter nonzero dependence. Since $\mu_n \subset K$, reindexing the sum gives $\gamma b = c(\gamma)b$. Thus $a = b^n \in K^\times$. The class of a modulo $K^{\times n}$ is intrinsic and determines c: any two choices for the same character have quotient fixed by G_K. Conversely if $b_1^n/b_2^n = t_0^n$ with $t_0 \in K$, then $b_1/(t_0 b_2) \in \mu_n \subset K$. The fixed field assertion follows since a nontrivial separable conjugate extends to Ω by Lemma 16.2.

There is a finite enlargement S' depending only on K, S such that $v_\mathfrak{p}(a) \in n\mathbb{Z}$ for every $\mathfrak{p} \notin S'$. To see this, take a finite normal closure $K'/\mathbb{Q}$, an integral primitive generator α of K' as in Lemma 82.2, and exclude the rational primes dividing the discriminant of its minimal polynomial or the lattice index $[O_{K'} : \mathbb{Z}[\alpha]]$, as well as the exceptional primes in Lemma 219.3 for $K'/\mathbb{Q}$. At a remaining prime $\mathfrak{p} \mid p$ outside S, lift the residue embedding at a prime of K' above $\mathfrak{p}$ to an embedding $K' \hookrightarrow U$, the union of the unramified local fields of Lemma 113.1 inside K_p. Indeed the simple residue root of the polynomial of α lies in some finite residue field; the successive digit lifting in that lemma produces a root with the prescribed residue in the corresponding finite extension. Since the index is prime to p, the reduction agrees on the localizations of the full ring of integers. A prime above $\mathfrak{p}$ exists by extending its proper ideal to the integral closure: after localizing, the latter is a nonzero finite free module by the discrete valuation ring argument in Theorem 599.2, so reduction gives a nonzero finite algebra with a maximal ideal. Extend the embedding to $\Omega \hookrightarrow K_p$ by the algebraic extension argument of Lemma 16.2.

Every automorphism of K_p/U preserves the valuation and acts trivially on its residue field, as U already realizes all the residues by Lemma 113.1. It preserves the embedded Ω, which consists of all elements of K_p algebraic over $\mathbb{Q}$, and restricts to inertia over $\mathfrak{p}$. Thus it fixes the image of b. The fixed field of $\mathrm{Aut}(K_p/U)$ is U, again by separability and extension of embeddings, since K_p is algebraic over U. The valuation of b, normalized at p, is therefore an integer. Since p is unramified in K', its valuation is one in the normalization of Proposition 82.3 at $\mathfrak{p}$; both valuations on K agree. This proves the assertion on $v_\mathfrak{p}(a)$.

The classes in $K^\times/K^{\times n}$ with these valuation conditions form a finite group. The list of valuations modulo n at S' has finite range. In its kernel one may write $(a) = \mathfrak{a}^n$ with a unique fractional ideal $\mathfrak{a}$ by Proposition 82.3. Its ordinary ideal class is well defined modulo replacing a by an n-th power times a, and takes finitely many values by Lemma 82.5. In the kernel of this class map, division by an n-th power leaves a unit. The units modulo n-th powers are finite by the finite-index free subgroup of Lemma 83.1. The injection of characters into this finite set proves the lemma. □

Theorem 601.3. *Let S be a finite set of finite places of E, let ℓ be prime, and let $M, N \geq 1$. Let O be the valuation ring of a finite extension of $\mathbb{Q}_\ell$ inside K_ℓ. Fix a representation $\rho : G_E \to \mathrm{GL}_r(O)$ whose reduction modulo ℓ^M factors through a finite continuous quotient. Fix a finite list of continuous homomorphisms $h_a : G_E \to \mathbb{Z}/\ell^N\mathbb{Z}$. There is a finite quotient D of G_E, independent of the following O', y and $\mathcal{L}$. For any valuation ring $O' \subset K_\ell$ of a finite extension containing O, put $R = O'/\ell^M O'$. For $y \in (O')^{\{a\}}$ use the matrix*

$$A_y(g) = \rho(g) \prod_a y_a^{[h_a(g)]} \pmod{\ell^M},$$

with $0^0 = 1$. Every function in Definition 601.1 for A_y factors through D. On $R^{r|D|}$ these functions are precisely the solutions of the induced homogeneous linear equations at cosets.

The equations for the cocycle law have coefficients obtained by evaluating polynomials over O in the variables y_a.

Proof. The finite normal subextensions of Ω/E form a directed family by compositum. Its union is Ω, since each algebraic element lies in a finite splitting field. The automorphism group is compact: the fields are countable, so take a nested cofinal sequence of such finite normal extensions. Compatible automorphisms on them extend by union, and the diagonal subsequence argument on the finite restriction groups gives compactness for the metric of agreement on the first stages. Thus continuous maps to discrete sets have finite image.

Choose a finite Galois K/E containing μ_{ℓ^M} such that G_K kills the reduction of ρ and all h_a. This follows by taking the compositum of the finitely many finite normal fields for the given open kernels and for $X^{\ell^M}-1$. Let S_K consist of the places over S. The intersection of the kernels of all characters in Lemma 601.2 for $K, S_K, n=\ell^M$ is open of finite index in G_K. Intersect its conjugates in G_E; there are finitely many since the subgroup has finite index in G_E. Use the quotient by this open normal subgroup for D. It refines the quotients of $\rho \bmod \ell^M$ and of each h_a.

For any eligible z, $z(1)=0$ by the identity at 1, and on G_K the identity becomes the additive homomorphism law since $A_y=1$ there. Its finite image is an abelian group of exponent dividing ℓ^M. Every nonzero element of this image is detected by a character to μ_{ℓ^M}, by Lemma 7.1 and the exponent bound. On composition with $z|_{G_K}$ each such character is trivial on inertia outside S_K, since an inertia element fixing K also belongs to the inertia group over E of the same extended valuation. Thus z vanishes on the normal subgroup defining D. For $g\in G_E$ and h in that subgroup, the defining identity gives $z(gh)=z(g)$, proving factorization even at points y for which A_y is only a matrix-valued map.

Conversely a function on D inflates to a continuous function. The cocycle equations descend because A_y and the product of two cosets are well defined on D. Inertia imposes vanishing at the images of its elements; all other linear relations descend by summing coefficients on equal cosets. Imposing these equations recovers exactly the original module. In the cocycle equations the only varying coefficients are those of the displayed monomial formula for A_y, whose constant matrices can be lifted from the finitely many values of ρ modulo ℓ^M. □

602. A sharp determinant for the length of a cocycle module

Lemma 602.1. *Let $F\subset K_\ell$ be a finite extension of $\mathbb{Q}_\ell$, with valuation $v_\ell(\ell)=1$ and valuation ring O. There is an integer $e\ge 1$ with $v_\ell(F^\times)=e^{-1}\mathbb{Z}$. Write k for the residue field of O. For a finite O-module T define its normalized length by*

$$\lambda_O(T)=\frac{\log|T|}{e\log|k|}.$$

It is additive in short exact sequences. For any square matrix $A\in \mathrm{Mat}_n(O)$ and integer $M\ge 1$,

$$\lambda_O\Big(\ker(A:(O/\ell^MO)^n\to(O/\ell^MO)^n)\Big)\le v_\ell(\det A),\qquad v_\ell(0)=+\infty.$$

If the rows of A span a submodule containing ℓ^MO^n, equality holds. For matrices over O the normalized kernel length in this formula is unchanged upon extension of the coefficient field to any finite $F'\supset F$ inside K_ℓ.

Proof. Put $d=[F:\mathbb{Q}_\ell]$. For $x\in F^\times$ all its conjugates have the same valuation by Lemmas 16.2 and 113.1. Their product, counting each embedding of F, is the determinant of multiplication by x over $\mathbb{Q}_\ell$: on $\mathbb{Q}_\ell(x)$ the power basis gives its monic minimal polynomial as characteristic polynomial, and a basis over that subfield repeats this matrix in diagonal blocks. Thus $d\,v_\ell(x)\in\mathbb{Z}$. A subgroup of $d^{-1}\mathbb{Z}$ containing 1 has the stated form, by integer division by its smallest positive member. Choose ϖ of valuation $1/e$. Every nonzero element

of O is a unit times a nonnegative power of ϖ. The norm comparison in the proof of Lemma 113.1 makes O a closed subset of a compact coordinate ball. Its maximal ideal ϖO is open, since it is precisely the set of elements of absolute value < 1. Its cosets form an open cover, so there are finitely many. The quotient is a field because each nonzero residue has a unit representative. For $h \geq 1$, successive multiplication by ϖ^j identifies $O/\varpi O$ additively with $\varpi^j O/\varpi^{j+1}O$, $0 \leq j < h$. Hence $|O/\varpi^h O| = |k|^h$. Additivity of λ_O follows by counting the equal-size fibers in an exact sequence.

Integral invertible row and column operations diagonalize A. Move a nonzero entry of smallest valuation to the top left. It divides every entry, so row and column subtractions clear its column and row. Repeat on the smaller block until only zero entries remain. Multiplication of a row by a unit normalizes every nonzero diagonal entry to ϖ^{h_i}. Write $h_i = +\infty$ for a zero entry. These operations preserve the size of the kernel modulo ℓ^M. In one coordinate its size is $|k|^{\min(h_i, eM)}$, by the preceding filtration and $\ell^M O = \varpi^{eM} O$. Consequently the normalized length is $\sum_i \min(h_i/e, M)$. Comparison with the product of the diagonal entries gives the inequality. If the row span contains $\ell^M O^n$, this containment persists under both kinds of operations, so each $h_i \leq eM$, giving equality. All operations remain invertible over $O_{F'}$. Each nonzero diagonal entry retains its valuation h_i/e there. The same minimum formula, including zero entries, proves invariance. □

Theorem 602.2. *Let O be as in Lemma 602.1, let $\mathbf{Y} = (Y_1, \ldots, Y_J)$, and let $R_\lambda(\mathbf{Y}) \in O[\mathbf{Y}]^n$ be any family of polynomial rows. At an integral point y of a finite extension with valuation ring $O_y \supset O$, write X_y for the solutions modulo ℓ^M of all $R_\lambda(y)x = 0$ in $(O_y/\ell^M O_y)^n$. For every prescribed such point u there is $F \in O[\mathbf{Y}]$ satisfying*

$$v_\ell(F(u)) = \lambda_{O_u}(X_u), \qquad \lambda_{O_y}(X_y) \leq v_\ell(F(y)) \quad \text{at every } y.$$

One can take F to be a single n-row determinant from the given rows augmented by ℓ^M times the coordinate rows. The choice works for every finite coefficient field containing the coordinates of the point of evaluation and O.

Proof. For $n = 0$ use the empty determinant 1. The added rows impose identities modulo ℓ^M. At u their determinant has valuation nM. All nonzero minors have nonnegative valuations in $e_u^{-1}\mathbb{Z}$, with e_u as in Lemma 602.1 for O_u. Choose n augmented rows giving a nonzero determinant of smallest valuation and let F be their polynomial determinant. Their evaluations at u span every evaluated row over O_u. Indeed over the fraction field the rows form a basis. The coefficient of row i in any augmented row is the determinant with row i replaced by that row divided by $F(u)$, by multilinearity and alternation of the determinant. Every numerator is zero or an augmented minor, so all the coefficients are integral. In particular the selected span contains $\ell^M O_u^n$, and its kernel modulo ℓ^M is X_u. The equality now follows from Lemma 602.1. The row expressions at u remain valid over any larger ring of integers. At any y, the full set of solutions injects in the kernel for the selected rows, to which the inequality of the same lemma applies. This also allows singular values of the selected matrix away from u. □

Corollary 602.3. *In Theorem 601.3, allow every additional homogeneous relation to have polynomial coefficients over O, with finitely many terms in each relation. At a point y denote the module in Definition 601.1 by Z_y, with coefficients reduced modulo ℓ^M. For each prescribed integral point u there is a polynomial F over O with $v_\ell(F(u)) = \lambda_{O_u}(Z_u)$ and $\lambda_{O_y}(Z_y) \leq v_\ell(F(y))$ at every integral point of every finite extension of the coefficient field.*

Proof. Use $n = r|D|$ coordinates in Theorem 601.3. The argument locating D uses the cocycle and inertia conditions alone, so it applies to every solution of the additional relations. Collecting terms at equal cosets adds their polynomial coefficients. The cocycle equations already have polynomial coefficients and inertia gives coordinate rows. Thus Z_y identifies with X_y of Theorem 602.2 for this fixed family of polynomial rows. Its determinant gives both assertions. □

603. Tame inertia and finite affine actions

Lemma 603.1. *Let $E \hookrightarrow \mathbb{Q}_p$ be a number field, let Ω be its algebraic closure inside $K_p = \overline{\mathbb{Q}}_p$, and let $D \subset G_E$ preserve the induced valuation. Restriction identifies $\mathrm{Aut}(K_p/\mathbb{Q}_p)$ with D and identifies their inertia subgroups. There exists $\phi \in D$ acting by $x \mapsto x^p$ on the residue field. We call any such element a Frobenius lift. All valuations on Ω above the same finite place of E, normalized at p, are conjugate under G_E.*

Proof. The elements of K_p algebraic over $\mathbb{Q}$ form Ω; this field is dense in K_p. Indeed for $a \in K_p$ approximate the coefficients of a monic equation for a over $\mathbb{Q}_p$ by rational numbers, keeping its degree d and leading coefficient. The resulting polynomial P_j has $|P_j(a)|_p \to 0$. At least one of its d roots, counted with multiplicity, has distance from a at most $|P_j(a)|_p^{1/d}$, by factoring $P_j(a)$. Every such root is in Ω. An isometry of Ω therefore extends, with its inverse, to the metric completion of K_p by taking limits of Cauchy sequences. The extension preserves the field operations and fixes the closure of $\mathbb{Q}$, namely $\mathbb{Q}_p$. It preserves K_p since the elements in this completion algebraic over $\mathbb{Q}_p$ are exactly K_p: their polynomials split there. Conversely every $\mathbb{Q}_p$-automorphism of K_p is an isometry by Lemma 113.1. It restricts to Ω and fixes E. Density also identifies the residue fields, since any integral element may be approximated within absolute distance less than one. This proves the inertia assertion.

In the unramified field L_a of Lemma 113.1, use the integral power basis and monic polynomial lifting the separable minimal polynomial of a residue generator constructed in its proof. Each residue conjugate is a simple root and lifts to a root in L_a by the same successive-digit argument. Sending the power-basis generator to the lift of its p-power residue gives an automorphism inducing p-power on residues. An automorphism acting trivially on residues fixes that generator by uniqueness of the simple lift. Thus these automorphisms agree on smaller L_a for a dividing a larger index. They give an automorphism of the union U in that lemma, with inverse given by the compatible inverses. Extend it to K_p by Lemma 16.2. Its residue action is the prescribed one since U represents all residues.

For conjugacy, in any finite Galois B/E two primes over the same prime of E must be conjugate. Otherwise choose an integer of B zero at the first and one modulo every conjugate of the second, by the Chinese remainder argument of Theorem 596.2. The product of all its Galois conjugates belongs to E, is divisible by the first prime and is one modulo the second, contradicting equality of their contractions. A normalized valuation at each such prime is determined by its prime ideal through Proposition 82.3. Take a nested cofinal sequence of finite normal subfields as in Theorem 601.3. At each stage extend an automorphism identifying the two restrictions to Ω by Lemma 16.2. A subsequence stabilizes on each stage by the diagonal selection in the same theorem. Its compatible restrictions give one global conjugacy of the two valuations. □

Theorem 603.2. *With D as in Lemma 603.1, write I for its inertia subgroup and fix a prime $\ell \neq p$. If a continuous homomorphism $f : D \to H$ to a finite group has ℓ-group image on I, then $f(I)$ is cyclic and*

$$f(\phi)f(\gamma)f(\phi)^{-1} = f(\gamma)^p \qquad (\gamma \in I)$$

for any Frobenius lift ϕ. Suppose in addition that $h : D \to \mathbb{Z}/\ell^N\mathbb{Z}$, $N \geq 1$, is a continuous homomorphism and $h(\gamma_0) = 1$ for some $\gamma_0 \in I$. Then $f(\gamma_0)$ generates $f(I)$ for every such f.

Proof. On the local group the restriction to global finite normal fields is continuous, since each such field is contained in a finite normal extension of $\mathbb{Q}_p$ after compositing with $\mathbb{Q}_p$. By Lemma 603.1 the map f therefore factors through a finite Galois extension $B/\mathbb{Q}_p$, using that an open kernel contains the group fixing some such field. Let ϖ generate the maximal ideal of its valuation ring O_B and k_B be the residue field, as constructed in Lemma 602.1 with coefficient prime p. For the inertia subgroup I_B of $\mathrm{Gal}(B/\mathbb{Q}_p)$ the rule

$$d(\sigma) = \overline{\sigma(\varpi)/\varpi} \in k_B^\times$$

is independent of the uniformizer, because inertia fixes the residue of every unit. It is a homomorphism by the same fact. Conjugating by an automorphism of B induces on d its residue automorphism, by using the inverse image of ϖ as uniformizer.

The kernel of d has p-power order. For $j \geq 1$ set $I_j = \{\sigma \in I_B : \sigma(\varpi) - \varpi \in \varpi^{j+1} O_B\}$, so $I_1 = \ker d$. On I_j the residue of $(\sigma(\varpi) - \varpi)/\varpi^{j+1}$ is an additive homomorphism with kernel I_{j+1}: apply σ to $\tau(\varpi) - \varpi$, use $\sigma(\varpi)/\varpi \equiv 1$ and use that σ acts trivially on k_B. This proves the subgroup assertion inductively and embeds each successive quotient in $(k_B, +)$. Their intersection is trivial. Indeed its elements fix ϖ. Every nonzero residue lifts uniquely to a root of $X^{|k_B|-1} - 1$ by the simple-root digit lifting argument of Lemma 113.1, here with successive powers of ϖ. Completeness follows from the coordinate norm comparison there. Inertia fixes these lifts, as well as zero. Successive division by ϖ after subtraction of a lifted residue gives a convergent expansion of every integer using these digits. An element fixing the digits and ϖ fixes B. The descending chain in the finite group therefore reaches the identity, proving the order assertion.

The image I' of I in $\mathrm{Gal}(B/\mathbb{Q}_p)$ lies in I_B. The p-subgroup $I' \cap \ker d$ maps trivially under f because $|f(I)|$ is an ℓ-power. Thus $f(I)$ is a quotient of a subgroup of $k_B^\times$, which is cyclic by Lemma 73.2. The conjugation formula follows from that for d and the p-power action of ϕ. Apply this reasoning also to the finite product map (f, h), whose inertia image is again an ℓ-group. That image is cyclic. In a cyclic ℓ-group mapping onto $\mathbb{Z}/\ell^N\mathbb{Z}$, an element mapping to 1 is a generator: its exponent relative to any generator is prime to ℓ. Projecting to H proves the last assertion. □

Corollary 603.3. *Let $\phi, I, p, \ell, h, \gamma_0$ be as in Theorem 603.2. Let $R = O'/\ell^M O'$ for a finite coefficient extension as in Lemma 602.1. Suppose an action $A : D \to \mathrm{GL}_r(R)$ is continuous and maps I to scalars of ℓ-power order dividing ℓ^a. For any continuous cocycle $z : D \to R^r$, $z(\gamma_0) = 0$ implies $z(I) = 0$. If moreover $\ell^{a+M} \mid p - 1$, the affine maps associated to ϕ and any $\gamma \in I$ commute.*

Proof. The affine map associated to g is $x \mapsto z(g) + A(g)x$. The cocycle law makes this a continuous homomorphism into the permutations of the finite set R^r. Its inertia image is an ℓ-group: its linear parts form a finite abelian group of exponent dividing ℓ^a, and the kernel of the linear-part map consists of translations of ℓ-power order. For the former assertion, a finite abelian group of exponent dividing ℓ^a is a quotient of a product of copies of $\mathbb{Z}/\ell^a\mathbb{Z}$, by mapping basis elements to its generators. Translations lie in the additive group of R^r, of ℓ-power order by the residue filtration in Lemma 602.1. Thus γ_0 generates the entire affine inertia image by Theorem 603.2. If its affine map fixes zero, all inertia maps fix zero. Raising each affine inertia map to ℓ^a leaves a translation, and raising that to ℓ^M gives the identity. The p-th power is consequently the original map when $\ell^{a+M} \mid p - 1$. The conjugation formula of the same theorem gives commutation. □

604. The cost of removing relations between inertia and Frobenius

Lemma 604.1. *Fix a prime ℓ, an integer $k \geq 0$, an integer $M \geq 1$ and $u \in K_\ell$ with $u^{\ell^k} = 1$. For any integral $t \in O_\ell$, put $Y = 1 + t(u - 1)$. Then*

$$v_\ell(Y - 1) \geq \ell^{-k}, \qquad Y^{\ell^{k+M-1}} \equiv 1 \pmod{\ell^M}.$$

Proof. In any ring of characteristic ℓ, raising to the ℓ-th power is additive since the intermediate binomial coefficients are divisible by ℓ. Iteration, applied modulo ℓ, gives $(u-1)^{\ell^k} \equiv 0$. Multiplying by t^{ℓ^k} proves the valuation bound and gives $Y^{\ell^k} \equiv 1 \bmod \ell$. For $j \geq 1$ each term of $(1 + \ell^j z)^\ell - 1$ is divisible by ℓ^{j+1}: the linear term has this factor and each higher term contains ℓ^{2j}. Iterate $M - 1$ times for the stated congruence. □

Definition 604.2. Let P be a finite set of places of E with distinguished $w_0 \in P$. Each w and its conjugate $\bar{w}$ are distinct with completions $\mathbb{Q}_{p_w}$, $p_w \neq \ell$. The pairs are disjoint.

Choose a decomposition group at each member of each pair, labelled by $b = +, -$, and elements $\gamma_{w,b}$ in its inertia and $\phi_{w,b}$ equal to a Frobenius lift or its inverse as in Lemma 603.1. Fix continuous homomorphisms $h_j : G_E \to \mathbb{Z}/\ell^N\mathbb{Z}$, $j \in P$, $N \geq 1$, satisfying

$$h_j(\gamma_{w,b}) = \mathbf{1}_{j=w}, \qquad h_j(\phi_{w,b}) = \mathbf{1}_{j=w_0} \quad (w \neq w_0).$$

The second condition places no restriction at w_0. For a representation $\rho : G_E \to \mathrm{GL}_r(O)$ as in Theorem 601.3, suppose its reduction modulo ℓ^M is trivial on the inertia at all these places. Let $t_w \in O$ satisfy

$$t_w(\rho(\phi_{w,b}) - I_r) \equiv 0 \pmod{\ell^M} \quad (w \in P,\ b = +, -).$$

At each w impose on a cocycle the two linear relations $z(\gamma_{w,b}) = t_w z(\phi_{w,b})$. For a fixed family of additional homogeneous polynomial relations and a fixed finite set of allowed places, let $Z_y(T)$ be the module of Corollary 602.3 with these two further relations for every $w \in T \subseteq P$. Denote its normalized length by $\lambda(Z_y(T))$ over the chosen finite valuation ring containing O and the coordinates of y.

Theorem 604.3. *Use Definition 604.2. Let $u^{\ell^k} = 1$, $k \geq 0$, and $N \geq k + 2M - 1$. Put $y_{w_0} = u$ and $y_w = 1 + t_w(u-1)$ for $w \neq w_0$. If $I \subseteq T \subseteq P$ and $u = 1$ whenever $w_0 \in I$, then*

$$\lambda(Z_y(T \setminus I)) - \lambda(Z_y(T)) \ \leq \sum_{\substack{w \in I \\ b=+,-}} \lambda(\ker(uU_{w,b} - I_r : (O'/\ell^M O')^r \to (O'/\ell^M O')^r)).$$

Here O' is any such valuation ring and $U_{w,b}$ is any integral matrix over O' congruent to $\rho(\phi_{w,b})$ modulo ℓ^M. Each summand is at most $v_\ell(\det(uU_{w,b} - I_r))$.

Proof. Write $R = O'/\ell^M O'$ and $a = k + M - 1$. By Lemma 604.1, including the slope 1 for w_0, all y_j satisfy $y_j^{\ell^a} = 1$ in R. Since $N \geq a$, the scalar $\tau_y(g) = \prod_j y_j^{[h_j(g)]}$ in R is a homomorphism: adding exponents modulo ℓ^N leaves the product unchanged. Thus $A(g) = \tau_y(g)\rho(g) \bmod \ell^M$ is an action. On inertia at either member of the pair at w it takes scalar values of order dividing ℓ^a. Theorem 603.2, applied to h_w, gives $(p_w - 1)h_w(\gamma_{w,b}) = 0$, because the target is abelian. Hence $\ell^N \mid p_w - 1$. Since $a + M = k + 2M - 1 \leq N$, Corollary 603.3 shows that the affine maps of $\gamma = \gamma_{w,b}$ and $\phi = \phi_{w,b}$ commute for every cocycle considered. Commutation with a Frobenius lift also gives commutation with its inverse. Evaluating both composites at zero gives

$$(A(\phi) - I_r)z(\gamma) = (A(\gamma) - I_r)z(\phi).$$

For $w \neq w_0$, the coordinate prescriptions give $A(\gamma) = y_w I_r$, $A(\phi) = u\rho(\phi)$ in R. For $w = w_0 \in I$, the hypothesis on u gives $y_j = 1$ for every j, so the same formulas hold and $y_w - 1 = t_w(u-1) = 0$. At every removed place, subtracting $t_w(u\rho(\phi) - I_r)z(\phi)$ in the displayed identity therefore yields

$$(u\rho(\phi) - I_r)\big(z(\gamma) - t_w z(\phi)\big) = t_w u(I_r - \rho(\phi))z(\phi) = 0$$

by the congruence in Definition 604.2. The map sending a cocycle to all these differences has kernel exactly $Z_y(T)$ on $Z_y(T \setminus I)$. Its image embeds in the direct sum of the displayed matrix kernels. Additivity and the determinant bound of Lemma 602.1 finish the proof, including the case of a singular matrix. □

605. Simultaneous specialization from untwisted cocycles

Theorem 605.1. *Fix the arithmetic data, $\ell, j, g_i \neq \ell$ as in Theorem 598.2, and write $e_\ell = v_\ell(g_i - 1)$. Fix positive integers m, J, H, M, a real $\eta > 0$, a finite set $\mathcal{A}$, ranks $r_i \geq 1$ and bounds $B_i \geq 0$ for $i \in \mathcal{A}$. For all sufficiently large N the following assertion holds.*

Let $1 \leq |P| \leq J$, and suppose we have idelic coordinates with bound H as in Definition 598.1, with the distinguished w_0 satisfying the Euler-factor conclusion of Theorem 596.2. On the same indexed set let $h_w, \gamma_{w,b}, \phi_{w,b}$ satisfy Definition 604.2. For each $i \in \mathcal{A}$ take

a representation ρ_i of rank r_i, slopes $t_{i,w}$, a finite set of allowed places, and additional homogeneous polynomial relations as in that definition. All data may be taken over a single finite coefficient field with valuation ring O. Denote its modules for row i by $Z^i_y(T)$. Suppose

$$\lambda(Z^i_{\mathbf{1}}(P)) + \sum_{b=+,-} \lambda_O\Big(\ker(\rho_i(\phi_{w_0,b}) - I_{r_i} : (O/\ell^M O)^{r_i} \to (O/\ell^M O)^{r_i})\Big) \le B_i.$$

There is a tuple $y_w^{\ell^N} = 1$ giving an idelic character $\vartheta = j^{-1} \circ \tau_y$ as in Definition 598.1 and a Galois character $\psi(g) = \prod_w y_w^{[h_w(g)]}$ such that, for all $1 \le d \le m$,

$$\lambda(Z^i_{y^d}(\varnothing)) \le B_i + \eta \quad (i \in \mathcal{A}),$$

$$v_\ell(j(\rho_p(P, \vartheta^{\epsilon d}))) \le 2e_\ell + \eta \quad (\epsilon = \pm 1, \text{ every ramified pair } p),$$

$$\sum_{w \in P} v_\ell(1 - j(\vartheta^d(u_{t_w}))) \le \eta.$$

The t_w in the last line are the residue generators of Definition 598.1. All the indicated idelic powers are primitive at P. The first bound is for the genuine twist $\psi^d \rho_i$ modulo ℓ^M with the specified additional relations evaluated at y^d; it holds over every finite coefficient extension containing O and the y_w. If $g \in G_E$ and $c \in C(P)$ satisfy $h_w(g) = e_w(c)$ for all w, then $\psi(g) = j(\vartheta(c))$.

Proof. Put $\epsilon_0 = \eta/(4(1 + 2J))$, $P' = P \setminus \{w_0\}$. At $u = 1$, Theorem 604.3 with $k = 0$, $I = \{w_0\}$, $T = P$ gives $\lambda(Z^i_{\mathbf{1}}(P')) \le B_i$ for $N \ge 2M - 1$. The sharper kernel form of the theorem applies even for singular $\rho_i(\phi_{w_0,b}) - I_{r_i}$. Use Corollary 602.3 at $\mathbf{1}$ with precisely the relations for P' added. This gives an integral polynomial G_i satisfying

$$v_\ell(G_i(\mathbf{1})) \le B_i, \qquad \lambda(Z^i_x(P')) \le v_\ell(G_i(x)).$$

Let $x_i(X)_{w_0} = X$ and $x_i(X)_w = 1 + t_{i,w}(X - 1)$ for $w \in P'$. The integral one-variable polynomial $G_i(x_i(X))$ has the same bound at $X = 1$. For every $w \in P'$, $b = +, -$, $i \in \mathcal{A}$ take also the monic characteristic polynomial $\det(XI_{r_i} - \rho_i(\phi_{w,b})^{-1})$. The sum of their degrees is at most $2J \sum_i r_i$.

Apply Theorem 594.1 in one variable with power one, depth one and tolerance ϵ_0. Its bound supplies $k \ge 1$, depending only on the fixed parameters, such that some $u^{\ell^k} = 1$ satisfies

$$v_\ell(G_i(x_i(u))) \le B_i + \epsilon_0,$$

$$v_\ell(\det(u\rho_i(\phi_{w,b}) - I_{r_i})) \le \epsilon_0 \qquad (w \in P').$$

Indeed $\det \rho_i(\phi_{w,b})$ is a unit, and multiplication by it turns the monic polynomial just used into the second determinant. The bound for k precedes the choice of all the polynomials, matrices and slopes.

Choose $N \ge k + 2M - 1$. At $x_i = x_i(u)$ apply Theorem 604.3 with $I = T = P'$. This gives

$$\lambda(Z^i_{x_i}(\varnothing)) \le B_i + (1 + 2J)\epsilon_0 = B_i + \eta/4.$$

Every coordinate of x_i has difference from one of valuation at least ℓ^{-k}, by Lemma 604.1. Choose a new determinant F_i using Corollary 602.3 with no relations from P and with witness x_i. Its valuation at x_i equals the length just bounded. At any integral point its valuation bounds the length for these same equations.

Use $G_{i,d} = F_i$ and $u_{i,d} = x_i$ for every $1 \le d \le m$ in Theorem 598.2, with depth integer ℓ^k, bounds $B_i + \eta/4$, and tolerance $\eta/2$. Its required lower bound for N again depends only on fixed parameters. Taking the maximum with the two earlier bounds verifies the quantifier on N. The resulting tuple gives $v_\ell(F_i(y^d)) \le B_i + 3\eta/4$, along with the two asserted idelic bounds and primitivity. The determinant inequality holds in all the finite coefficient extensions in the statement. Since every $y_w^{\ell^N} = 1$ exactly, addition of exponents modulo ℓ^N proves multiplicativity of ψ in characteristic zero. The coordinate monomials for

y^d therefore twist by its d-th power. Equality of the idelic and Galois coordinates at the stated pair g, c identifies the two products term by term. □

606. Choosing a local relation at two precisions

Theorem 606.1. *Let O have uniformizer ϖ, residue field k and ramification index e as in Lemma 602.1. For finite O-modules put $l(T) = e\lambda_O(T) = \log|T|/\log|k|$. Let Y be a finite O-module killed by a power of ϖ and $U, V : Y \to (O/\varpi^s O)^m$ linear, $s, m \geq 0$. There are at most m exceptional elements of k such that for every $\alpha \in O$ whose residue is outside them,*

$$l(\ker(\alpha U - V)) \leq l\big(\ker U \cap V^{-1}(U(Y))\big).$$

Proof. Let W be the sum in Y of all $\ker(\alpha U - V)$ for $\alpha \in O$, and $C = U(W)$. Then $V(W) \subseteq C$: on each summand one has $Vx = \alpha Ux$. Put $d = \dim_k(C/\varpi C)$. Multiplication by ϖ on C has kernel $C[\varpi]$ and image ϖC, hence $|C/\varpi C| = |C[\varpi]|$. For $s \geq 1$ the latter embeds in the ϖ-torsion of $(O/\varpi^s O)^m$, of size $|k|^m$ by the residue filtration in Lemma 602.1. For $s = 0$ one has $d = 0$. Thus $d \leq m$.

Choose $x_1, \ldots, x_d \in W$ so that the images of Ux_j are a basis of $C/\varpi C$, and write the images of Vx_j in that basis as the columns of a matrix D over k. The monic polynomial $\det(XI_d - D)$ has at most d roots in k, by successive division by its linear factors. For any other residue, $(\alpha U - V)x_j$ span $C/\varpi C$. Writing C' for the image of $(\alpha U - V)|_W$ gives $C = C' + \varpi C$, hence $C = C' + \varpi^a C$ by iteration for all $a \geq 1$. Since a power of ϖ kills C, we have $C' = C$. Thus the kernel in W has the same size as the kernel of $U|_W$, by counting fibers of the two surjections. The former is all of $\ker(\alpha U - V)$ by definition of W. The latter lies in $\ker U \cap V^{-1}(U(Y))$ since $V(W) \subseteq C$. This proves the inequality, including $d = 0$. □

Lemma 606.2. *Let $\rho : G_E \to \mathrm{GL}_r(O)$ have finite continuous reduction modulo ℓ^M. Take a finite set S of allowed places and fixed integral homogeneous linear relations in cocycle values. Let $w, \bar{w} \notin S$ be a new pair as in Definition 604.2, with $\rho \bmod \ell^M$ trivial on both inertia subgroups. For each side take γ_b, ϕ_b as there and a finite continuous cyclic character to $\mathbb{Z}/\ell^N\mathbb{Z}$, $N \geq 1$, taking γ_b to 1. For $0 \leq s \leq eM$ let X_s denote the untwisted cocycles with these relations modulo ϖ^s and allowed ramification S, and Y_s the ones with allowed set $S \cup \{w, \bar{w}\}$. On Y_s put*

$$Uz = (z(\gamma_+), z(\gamma_-)), \qquad Vz = (z(\phi_+), z(\phi_-)).$$

Then the modules are finite and $X_s = \ker U$.

Proof. Multiplication by ϖ^{eM-s} injects $O/\varpi^s O$ in $O/\ell^M O$, since $\ell^M O = \varpi^{eM} O$. Composing values with this injection preserves all the homogeneous relations, the cocycle law and the vanishing conditions. Finiteness follows from Theorem 601.3 with the trivial scalar twist and the larger allowed set. For $z \in \ker U$, apply Corollary 603.3 to the injected cocycle and the untwisted action at each of the two chosen valuations. It vanishes on both inertia groups. For any conjugate inertia element gag^{-1}, the cocycle law twice gives

$$z(gag^{-1}) = (I_r - \rho(gag^{-1}))z(g) + \rho(g)z(a) \pmod{\varpi^s}.$$

The identity $z(g^{-1}) = -\rho(g^{-1})z(g)$ used here follows from the law at gg^{-1}. Both terms vanish since the reduction of ρ is unramified at the new places. Lemma 603.1 gives all extended valuations over each place this way. Injectivity of the coefficient map proves $z \in X_s$. Conversely any $z \in X_s$ vanishes at both elements by its inertia condition. □

Corollary 606.3. *In Lemma 606.2, suppose $|k| > 4r + 1$ and $1 \leq s \leq eM$. There is a unit $t \in O$ such that at both precisions $j = s - 1, s$ the new module $X'_j = \{z \in Y_j : Uz = tVz\}$ satisfies*

$$l(X'_j) \leq l(X_j \cap V^{-1}(U(Y_j))) \leq l(X_j).$$

Whenever $V(X_j) \not\subseteq U(Y_j)$, the right-hand bound improves to $l(X_j) - 1$. If the slope for a different representation as in that lemma is chosen to be zero, its new module equals its former module at every precision.

Proof. Apply Theorem 606.1 with $m = 2r$ to both modules Y_j. Exclude also the zero residue. At most $4r + 1$ residues have been excluded, so choose a lift α of a remaining residue and take $t = \alpha^{-1}$. Then $Uz = tVz$ is precisely $(\alpha U - V)z = 0$. Substitute the kernel identity of Lemma 606.2. For the strict assertion the quotient of X_j by $X_j \cap V^{-1}(U(Y_j))$ is a nonzero finite O-module killed by ϖ^j. In any such module a nonzero element can be multiplied by a suitable power of ϖ to give a nonzero element killed by ϖ. Its cyclic submodule has size $|k|$, so the quotient has length at least one. Additivity proves the improvement. For slope zero the new module is $\ker U$, giving the last claim by the same lemma. □

607. A bound on the number of auxiliary primes

Definition 607.1. Fix the data of Theorem 605.1 over a fixed ring O, with $|k| > 4r_i + 1$ for every i. For row i fix a finite base set S_i and a fixed family of integral homogeneous linear relations $\mathcal{L}_i$ on the values of a cocycle. For each N under consideration let $\mathcal{P}_N$ be a collection of systems of paired primes, idelic and Galois coordinates, local evaluations and slopes satisfying the structural hypotheses of that theorem, with the size of P unrestricted. All chosen prime pairs are disjoint from the S_i. At $0 \le s \le eM$ write $X_{i,s}(P)$ for the untwisted cocycles modulo ϖ^s obeying $\mathcal{L}_i$, with allowed places $S_i \cup P \cup \bar{P}$ and with all the local linear relations of P. Use the same meaning for $P = \varnothing$, where there are no such relations and only S_i is allowed. Here and below $P, \bar{P}$ denote the chosen places and their conjugates.

An extension at row i has one new pair $w, \bar{w}$, with elements γ_b, ϕ_b fixed at that pair. It keeps all old local evaluations and slopes and the distinguished place. For each unit $t \in O^\times$ the system can be taken in $\mathcal{P}_N$ with new slope t at row i and zero at the other rows, keeping the same elements at the new pair. Write $Y_{i,s}$ for the cocycles obtained from $X_{i,s}(P)$ by allowing ramification also at the new pair, before imposing its two relations. Put U, V on $Y_{i,s}$ as in Lemma 606.2. An extension detects a class at precision s if $V(X_{i,s}(P)) \not\subseteq U(Y_{i,s})$.

Theorem 607.2. *Suppose each collection $\mathcal{P}_N$ in Definition 607.1 contains a singleton system. Fix real $\beta_i \ge 0$. Suppose for each of its systems P, for each i and each $1 \le s \le eM$, a detecting extension at row i exists whenever $l(X_{i,s}(P)) > \beta_i$ and some member has a unit coordinate at some element of G_E. Put $d_{i,s} = l(X_{i,s}(\varnothing))$. Define integers by*

$$J_0 = 1, \qquad J_s = J_{s-1} + \sum_{i \in \mathcal{A}} (d_{i,s} + 2sr_i J_{s-1}) \quad (1 \le s \le eM).$$

For each N there is a system $P \in \mathcal{P}_N$ with $|P| \le J_{eM}$ and $l(X_{i,eM}(P)) \le \beta_i$ for every i. All the J_s are independent of N.

Proof. Finiteness at every precision follows by embedding coefficients in $O/\ell^M O$ as in Lemma 606.2 and then using Theorem 601.3. The quotient between consecutive powers in the filtration $T \supseteq \varpi T \supseteq \cdots$ of a finite module killed by ϖ^s is a finite-dimensional k-space. By cardinality multiplication $l(T)$ is the sum of these dimensions, so all the lengths and all the $d_{i,s}$ are nonnegative integers.

For every eligible P, evaluation at the two inertia elements for each $w \in P$ gives

$$l(X_{i,s}(P)) \le d_{i,s} + 2sr_i|P|.$$

Its kernel is contained in $X_{i,s}(\varnothing)$. Indeed the proof of Lemma 606.2 applies at each pair and all its extended valuations, so all ramification outside S_i disappears in this kernel. The image is in $2r_i|P|$ copies of $O/\varpi^s O$, each of length s. Counting fibers proves the bound.

Start at $s = 0$ with any singleton system. The zero coefficient ring makes every length zero. Suppose we have reached $|P| \le J_{s-1}$ with every bound true modulo ϖ^{s-1}. Maintain the bounds at this earlier precision throughout the next stage. If $l(X_{i,s}(P)) > \beta_i$ and all coordinates of all its cocycles were divisible by ϖ, its length would equal $l(X_{i,s-1}(P))$. In fact multiplication by ϖ identifies $O/\varpi^{s-1}O$ with $\varpi O/\varpi^s O$. Division under this identification takes such cocycles to continuous functions satisfying all the same homogeneous equations

modulo ϖ^{s-1}, and multiplication reverses this operation. Both preserve the inertia conditions. Thus a failed bound gives a unit coordinate and the stated detection hypothesis applies.

Choose a detecting extension and the unit slope supplied by Corollary 606.3. For row i the length drops by at least one at s and does not increase at $s-1$. For the other rows the new slope is zero, and the modules at both precisions equal their old modules by that corollary. Thus each additional pair decreases $\sum_i l(X_{i,s}(P))$ by at least one and preserves the bounds at $s-1$. At the start of the stage the sum is at most $\sum_i (d_{i,s} + 2sr_iJ_{s-1})$. After at most that many additions every bound holds at s and the new size is at most J_s. This induction terminates at $s = eM$. The numbers $d_{i,s}$ concern only the fixed base sets, representations and relations. This proves uniformity in N. □

Corollary 607.3. *Assume the collections and bounds of Theorem 607.2 exist for every sufficiently large N. Suppose also for every system in them that the sum of the two normalized Frobenius kernel lengths at w_0 in Theorem 605.1 is bounded by $D_i \geq 0$ for row i, independently of N. Then that theorem gives both of its idelic estimates and, simultaneously for every i and $1 \leq d \leq m$,*

$$\lambda(Z^i_{y^d}(\varnothing)) \leq \beta_i/e + D_i + \eta.$$

Here the allowed set is $S_i \cup P \cup \bar{P}$ and the additional relations are exactly $\mathcal{L}_i$.

Proof. Choose $J = J_{eM}$ and $B_i = \beta_i/e + D_i$, then choose N large enough for these constants in Theorem 605.1. Apply Theorem 607.2 at that N. Since $\varpi^{eM}O = \ell^M O$, its final modules are exactly $Z^i_{\mathbf{1}}(P)$ with ordinary lengths $e\lambda(Z^i_{\mathbf{1}}(P))$. Adding the assumed bound at the distinguished place verifies the remaining inequality of the simultaneous specialization theorem. □

608. Dual relations on two conjugate cosets

Definition 608.1. Let R be a finite commutative ring, $V = R^r$, $r \geq 1$, and $V^\vee = \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Q}/\mathbb{Z})$, where V is given its additive structure. Let G be a group with an automorphism c satisfying $c^2 = 1$, and let $\rho : G \to \mathrm{GL}_r(R)$, $\chi : G \to R^\times$ be homomorphisms. Suppose $U \triangleleft G$ acts trivially under ρ. An equivariant relation is a group homomorphism $x : U \to V^\vee$, with additive target, satisfying

$$x(gag^{-1})(v) = x(a)(\chi(g)\rho(g)^{-1}v) \qquad (a \in U,\ g \in G,\ v \in V).$$

Let $H \triangleleft G$ be contained in U with $c(H) = H$. Fix $z_0 \in U$ with $c(z_0) \in U$, and let $y : G \to V$ satisfy $y(gh) = y(g) + \rho(g)y(h)$. The isotropy condition used here is the family of identities

$$x(z_0h)(ty(z_0h)) = x(c(z_0h))(ty(c(z_0h))) \quad (h \in H,\ t \in R).$$

Lemma 608.2. *In Definition 608.1, suppose $\rho(\gamma) = sI_r$, $\rho(c(\gamma)) = s^{-1}I_r$, $\chi(\gamma) = \chi(c(\gamma)) = 1$, where $s, s-1, s+1$ are units of R. Each equivariant relation satisfying the isotropy condition fulfills, for every $a, b \in H$, $t \in R$, $k \in G$,*

$$x(b)(t\rho(k)y(a)) = x(c(a))\big(t\chi(c(k))\rho(c(k))^{-1}y(c(b))\big).$$

Proof. Both x and y restrict to additive homomorphisms on U, the latter because U acts trivially on V. For $a', b \in H$ take the isotropy identities for $h = a'b, a', b, 1$ with signs $1, -1, -1, 1$. Additivity cancels the constant terms and terms in just one of a', b, on the conjugate side also because $c(z_0), c(a'), c(b) \in U$. The resulting identity is $A(t) + B(t) = C(t) + D(t)$, where

$$\begin{aligned} A(t) &= x(b)(ty(a')), & B(t) &= x(a')(ty(b)), \\ C(t) &= x(c(b))(ty(c(a'))), & D(t) &= x(c(a'))(ty(c(b))). \end{aligned}$$

All four functions are additive in t. For any $g \in G$ and $a' \in H$ the cocycle law gives $y(ga'g^{-1}) = \rho(g)y(a')$: the remaining two terms sum to $(I_r - \rho(ga'g^{-1}))y(g) = 0$. Apply the

preceding identity with a' replaced by $\gamma a'\gamma^{-1}$, which still belongs to H. The transformation rule for y and the equivariance of x give

$$A(st) + B(s^{-1}t) = C(s^{-1}t) + D(st).$$

The first identity says $A - D = C - B$ as additive functions on R. The second therefore gives $(A - D)((s - s^{-1})t) = 0$ for all t. The scalar $s - s^{-1} = (s - 1)(s + 1)s^{-1}$ is a unit, so $A = D$. Take $a' = kak^{-1}$, use the transformation rule for y on the left and the equivariance of x under $c(k)$ on the right. This proves the formula. □

Theorem 608.3. *Under Lemma 608.2, suppose the R-linear span of $\rho(G)$ is $\mathrm{Mat}_r(R)$, every commutator $[g, \gamma] = g\gamma g^{-1}\gamma^{-1}$ belongs to H, $y(\gamma) = 0$, and some coordinate of $y(g_0)$ is a unit for some $g_0 \in G$. Suppose also that there is $g_* \in G$ with $\rho(g_*) = \alpha I_r$, $\rho(c(g_*))^{-1} = \alpha' I_r$. Put $\delta = \alpha - \chi(c(g_*))\alpha'$. The number of distinct equivariant relations satisfying isotropy is at most*

$$|\ker(\delta : V \longrightarrow V)|.$$

In particular if $R = O/\varpi^j O$, $j \geq 1$, its logarithm to base $|k|$ is at most $r \min(j, e\, v_\ell(\widetilde{\delta}))$ for any lift $\widetilde{\delta} \in O$.

Proof. Set $\tau = [g_0, \gamma] \in H$. The cocycle law, centrality of sI_r and $y(\gamma) = y(\gamma^{-1}) = 0$ give $y(\tau) = (1 - s)y(g_0)$, which has a unit coordinate. Consequently the vectors $\rho(k)y(\tau)$, $k \in G$, span V over R. Indeed any vector is the image of $y(\tau)$ by a matrix supported on the column of its unit coordinate, dividing the desired column by that unit. Express the matrix in the assumed span.

Take $a = \tau,\ b = c(\tau)$ in Lemma 608.2. Write $X = x(c(\tau)) \in V^\vee$. Since $c^2 = 1$, for every t, k one has

$$X(t\rho(k)y(\tau)) = X(t\chi(c(k))\rho(c(k))^{-1}y(\tau)).$$

Replace k by kg_*. The new left side has the scalar factor α inside X, while on the right the extra scalar is $\chi(c(g_*))\alpha'$. Apply the original identity with t multiplied by this latter scalar and subtract. This gives $X(t\delta\rho(k)y(\tau)) = 0$. By the spanning property and additivity, X vanishes on δV.

The map $x \mapsto X$ is injective. For any $b \in H$, the lemma at $a = \tau$ expresses the values of $x(b)$ on all $t\rho(k)y(\tau)$ in terms of X. These values determine $x(b)$ by the same spanning property. For $a \in U$ the commutator $[a, \gamma] \in H$ gives, by additivity and equivariance,

$$x([a, \gamma])(v) = x(a)((1 - s^{-1})v) \qquad (v \in V).$$

As $1 - s^{-1}$ is a unit, this determines $x(a)$ on V.

It follows that the relations inject into $\mathrm{Hom}_{\mathbb{Z}}(V/\delta V, \mathbb{Q}/\mathbb{Z})$. The latter has the same order as $V/\delta V$ by Lemma 7.1, since exponentiation $t \mapsto \exp(2\pi i t)$ identifies $\mathbb{Q}/\mathbb{Z}$ with the complex roots of unity and every character of a finite group has finite image. Counting fibers of multiplication by δ on V gives the claimed kernel order. For $O/\varpi^j O$, the one-coordinate kernel has size $|k|^{\min(j, e\, v_\ell(\widetilde{\delta}))}$ by the residue filtration of Lemma 602.1. Multiplying for the r coordinates proves the final bound. □

609. A cyclic Galois character at a chosen split pair

Lemma 609.1. *Let G be a profinite group, H an open normal subgroup, and $\kappa : H \to \mathbb{Z}/n\mathbb{Z}$ a continuous homomorphism. For representatives s_C of the left cosets $C \in G/H$ put*

$$h^0(g) = \sum_{C \in G/H} \kappa(s_{gC}^{-1} g s_C).$$

This is a continuous homomorphism independent of the representatives. For fixed g, an orbit of length a starting at C contributes $\kappa(s_C^{-1} g^a s_C)$ to the sum.

Proof. Set $b(g,C) = s_{gC}^{-1} g s_C \in H$. Then $b(gg',C) = b(g,g'C)b(g',C)$. Sum under κ and use the permutation $C \mapsto g'C$ to get additivity. Replacing s_C by $s_C u_C$, $u_C \in H$, changes each term by $-\kappa(u_{gC}) + \kappa(u_C)$, with total zero. On the finitely many open cosets where g has a fixed permutation of G/H, the summands are continuous. On one orbit multiply the $b(g, g^j C)$, $0 \le j < a$, in reverse order. The intermediate representatives cancel and the product is $s_C^{-1} g^a s_C$. Applying κ gives the claim. □

Theorem 609.2. *Let E be the CM field, ℓ prime, $n = \ell^N$, $N \ge 1$, and $D = E(\mu_n)$. Fix an embedding of Ω in $\mathbb{C}$ and write c for complex conjugation on Ω. Suppose a prime $w \ne \bar{w}$ of E has completion $\mathbb{Q}_p$, $p \ne \ell$, and splits completely in D/E. Suppose one prime of D over w is $\mathfrak{W} = (\beta)$ and $\beta \in O_D$ satisfies $\beta \equiv 1 \bmod \ell^{2N+1} O_D$. Let T be a conjugation-stable finite set of places away from $w, \bar{w}, \ell$ such that $\beta \equiv 1$ at all primes of D above T. There is a continuous surjection $h : G_E \to \mathbb{Z}/n\mathbb{Z}$ satisfying*

$$h(cgc^{-1}) = -h(g) \qquad (g \in G_E).$$

It is trivial on inertia away from $w, \bar{w}$, and trivial on the full decomposition groups at T and above ℓ. At w it takes some inertia element γ to 1. At $\bar{w}$ it takes $(c\gamma c^{-1})^{-1}$ to 1.

Proof. Choose a primitive n-th root $\zeta \in D$ and $\xi \in \Omega$ with $\xi^n = \beta$. Define $\kappa : G_D \to \mathbb{Z}/n\mathbb{Z}$ by $\sigma(\xi)/\xi = \zeta^{\kappa(\sigma)}$. Since G_D fixes ζ, this is a homomorphism. It factors through the finite splitting field of $X^n - \beta$ over D, hence is continuous. For an extended place of D over a rational prime distinct from ℓ, the different roots of $X^n - 1$ have distinct residues. Indeed a nontrivial root ϵ reducing to one would make $1 + \epsilon + \cdots + \epsilon^{n-1} = 0$ reduce to the unit n. Thus at a place where β is a unit the n roots of $X^n - \beta$ have distinct residues, and inertia fixes each one. The same argument on $X^n - 1$ says that inertia over E is contained in G_D at all these residue characteristics, whether or not β is a unit.

At primes above T the equation $X^n = \beta$ has a root in the completed base field, by simple-root lifting at one as in Lemma 113.1. At primes above ℓ use the valuation normalized by $v_\ell(\ell) = 1$ and Newton iteration for $f(X) = X^n - \beta$ from $x = 1$. If x is a unit and $v_\ell(f(x)) = 2N + b$, $b > 0$, then the correction $-f(x)/(nx^{n-1})$ has valuation $N + b$. The linear error cancels and the binomial formula bounds the new error below in valuation by $2N + 2b$. The new argument remains a unit. Starting with valuation at least $2N + 1$, this either finds a root or gives a Cauchy sequence with a root as limit in the completion. Since all n-th roots of unity lie in D, all roots lie in the completion in either case. More precisely, on embedding into the completion of the chosen valued Ω, the polynomial already splits in Ω; any root found in that completion equals one of these roots. A decomposition element fixing D fixes its completion by continuity. Thus κ is zero on every decomposition group of G_D at these places.

Use Lemma 609.1 with $G = G_E$, $H = G_D$ to obtain h^0. For a decomposition element g over T or ℓ, every orbit term has $s_C^{-1} g^a s_C \in G_D$ in a decomposition group over the same place of E. All terms vanish by the preceding paragraph. At any other place outside w, an inertia element is already in G_D and every orbit is a singleton. Each conjugate inertia element is at a prime of D distinct from $\mathfrak{W}$, so κ vanishes there.

Use at w an extended valuation above $\mathfrak{W}$. The primes of D over w are permuted simply transitively by $\mathrm{Gal}(D/E)$: transitivity was proved in Lemma 603.1, and complete splitting gives $[D : E]$ primes. The local embedding of E into $\mathbb{Q}_p$ extends to D at $\mathfrak{W}$ in $\mathbb{Q}_p$, by the equality of the completions at a completely split place. For this embedding $v_p(\beta) = 1$. Take $s_{G_D} = 1$. For every inertia element over $\mathfrak{W}$ only this coset can contribute to h^0, since the other primes over w have unit β. In the field U of Lemma 113.1 the value group is $\mathbb{Z}$. The powers $1, \xi, \ldots, \xi^{n-1}$ under this embedding are independent over U: their nonzero multiples by coefficients in U have distinct valuations modulo $\mathbb{Z}$. Thus $X^n - \beta$ is irreducible there. Sending ξ to $\zeta\xi$ fixes U and gives an automorphism of $U(\xi)$. Extend it to K_p by

Lemma 16.2. It acts trivially on residues since U contains representatives of all residues. Its global restriction γ therefore has $h^0(\gamma) = \kappa(\gamma) = 1$.

Finally put $h(g) = h^0(g) - h^0(cgc^{-1})$. Here c preserves E, and $c^2 = 1$, so this is a homomorphism with the asserted sign. At γ the conjugate term is zero by unramifiedness of h^0 at $\bar{w}$. Stability of T gives all the stated decomposition conditions and the sign gives the value at the other inertia element. □

Corollary 609.3. *Use the marked prime $\mathfrak{s} \mid g_i \neq \ell$ and a finite list $\mathcal{B}$ as in Lemma 600.1. Fix T as in Theorem 609.2, disjoint also from $\mathfrak{s}, \bar{\mathfrak{s}}$, before choosing w. One can choose infinitely many new w, avoiding any finite set of rational primes, admitting both the coordinate e_w of that lemma and a Galois character h of the preceding theorem. The prime w satisfies the helper Euler-factor bounds of Theorem 596.2. It may be required to split completely in any specified finite Galois extension of E unramified at $\mathfrak{s}$.*

Proof. Use the compositum B of the specified field, L_0 from Lemma 600.1, and $D = E(\mu_n)$. These fields together are unramified at $\mathfrak{s}$. To justify this also for their compositum, start with the specified unramified extension and adjoin all the roots defining L_0, D. Their radicands are units at $\mathfrak{s}$, and Lemma 599.1 applies. In the proof of Theorem 599.2 use B/E, prescribe at $\mathfrak{s}$ the residue generator used in Theorem 596.2, and impose the congruence one modulo ℓ^K, $K \geq 2N + 1$, $K > v_\ell(g_i - 1)$, and at all places of T and the other marked places and their conjugates. Exclude the rational primes ramified in a normal closure of $B/\mathbb{Q}$, as permitted by Lemma 219.3. That proof chooses a principal prime (a) of B of rational prime norm with a itself congruent to one modulo all these latter ideals extended to O_B. Put $\beta = N_{B/D}(a)$, $\pi = N_{B/E}(a)$. The prime norm and contraction argument in that theorem gives $(\beta) = \mathfrak{W}$, $(\pi) = w$, and the congruences to one descend by multiplication over embeddings and ideal contraction. Complete splitting gives all the local degrees one. In particular the embeddings at $w, \mathfrak{W}$ into $\mathbb{Q}_p$ used above can also be read from the simple-root lifting in Lemma 601.2 at these good primes, with residue degree one. The prime w is split from its conjugate by the prime norm and the exclusion of ramification, as in Theorem 600.2. The congruences of π give the helper bound, splitting in L_0 gives e_w, and the congruences of β give h by Theorem 609.2. Infinitude follows from the norm construction. □

610. Matching the local coordinates at several primes

Lemma 610.1. *Let w_0 be a place of E with completion $\mathbb{Q}_{p_0}$. Suppose $n = \ell^N \mid p_0 - 1$, $N \geq 1$, and fix a residue generator $t_0 \in \mathbf{F}_{p_0}^\times$. Let $d_0 : G_E \to \mathbb{Z}/n\mathbb{Z}$ be the discrete logarithm to base t_0, reduced modulo n, of the exponent of the action on a primitive p_0-th root of unity. For any continuous $h : G_E \to \mathbb{Z}/n\mathbb{Z}$ surjective on inertia at w_0, one can multiply h by a unit modulo n so that $h + d_0$ is unramified there. Its fixed field, meaning the finite field fixed by its kernel, then has ramification index one at w_0. At a degree-one place of residue characteristic $p \neq p_0$, every arithmetic Frobenius lift has d_0-value $\log_{t_0}(p \bmod p_0) \bmod n$.*

Proof. Over K_{p_0} a primitive p_0-th root ϵ reduces to one. The ratios $(1 - \epsilon^j)/(1 - \epsilon)$ reduce to j for $1 \leq j < p_0$. As $\prod_{j=1}^{p_0-1}(1 - \epsilon^j) = p_0$, this gives $v_{p_0}(\epsilon - 1) = 1/(p_0 - 1)$. Its first $p_0 - 1$ powers, starting at exponent zero, have distinct valuations modulo $\mathbb{Z}$, so are independent over the field U of Lemma 113.1. The polynomial for ϵ over U has precisely the $p_0 - 1$ primitive roots as conjugates. Extension of embeddings as in Lemma 16.2 realizes every exponent in $\mathbf{F}_{p_0}^\times$ by an automorphism fixing U. By Lemma 603.1 these yield global inertia elements at w_0. Thus d_0 is surjective on inertia. The theorem on cyclic inertia, Theorem 603.2, applies to (h, d_0). Its inertia image is cyclic of exponent dividing n and surjects onto $\mathbb{Z}/n\mathbb{Z}$ in both coordinates. Hence both projections are isomorphisms from that image. The restriction of h is a unit multiple of d_0, and a rescaling gives $h = -d_0$ there.

In the chosen local embedding, the field fixed by the kernel of $h + d_0$ is pointwise fixed by $\mathrm{Aut}(K_{p_0}/U)$. The fixed field of this group is U, by the separability and extension argument

of Lemma 601.2. Thus the local image has integral valuation group. The ramification index is one by prime ideal factorization as in that lemma, and all conjugate places give the same index. Finally at $p \neq p_0$ the distinct p_0-th roots have distinct residues by the geometric-sum argument of Theorem 609.2. The residue action $x \mapsto x^p$ gives exponent p mod p_0. □

Theorem 610.2. *In Theorem 600.2 fix also a conjugation-stable finite set T disjoint from $\mathfrak{s}, \bar{\mathfrak{s}}$ and from the places above ℓ. For each prescribed $J \geq 1$ one can choose the asserted P, w_0 and idelic coordinates e_v together with surjective continuous characters $h_v : G_E \to \mathbb{Z}/\ell^N\mathbb{Z}$ satisfying $h_v(cgc^{-1}) = -h_v(g)$. Each h_v is ramified at precisely $v, \bar{v}$, trivial on the decomposition groups above ℓ and at T. There exist γ_w in inertia and arithmetic Frobenius lifts ϕ_w at w such that*

$$h_v(\gamma_w) = e_v(u_{t_w}) = \mathbf{1}_{v=w} \quad (v, w \in P),$$
$$h_v(\phi_w) = e_v(b_w) = \mathbf{1}_{v=w_0} \quad (v \in P,\ w \in P \setminus \{w_0\}).$$

The same values on the Galois side hold for $(c\gamma_w c^{-1})^{-1}, (c\phi_w c^{-1})^{-1}$ at $\bar{w}$. All ordinary $w \neq w_0$ split completely in the prescribed L/E and can also be required to do so at their conjugates.

Proof. Write $n = \ell^N$. Choose w_0 from Corollary 609.3, with its e_{w_0}, h_{w_0}, avoiding the supports of all prescribed ideles, T, and the exceptional rational primes for a normal closure of $L/\mathbb{Q}$ and of $E/\mathbb{Q}$ in Lemma 219.3. Call this normal closure containing both fields L'. At $p_0 = Nw_0$ the field L' embeds in the unramified field U of Lemma 113.1 for each local embedding. For example the simple-root argument used for the field K' in Lemma 601.2 applies on also excluding its displayed discriminant and index primes. The residue field at w_0 contains all n-th roots of unity by the splitting in $D = E(\mu_n)$ in the construction. Thus $n \mid p_0 - 1$. Use the same residue generator $t_0 = t_{w_0}$ on both sides of Lemma 610.1. Rescale h_{w_0} as specified there, preserving its sign, ramification and local triviality. Write B_0/E for the field cut out by $h_{w_0} + d_0$.

Proceed inductively after any list of ordinary places has been chosen. Take the field construction of Lemma 600.1 with the new buffer place w_0, and with the prescribed list enlarged by b_v for every previously chosen ordinary v. These ideles are integral at the buffer pair. Form the finite Galois extension B/E obtained by adjoining this radical field, D, L', B_0 and the fields of all previously chosen ordinary h_v. It is unramified at w_0. Indeed the radical fields just used have radicands which are units at w_0 and order prime to p_0; all their roots are fixed by inertia, by the distinct-residues argument of Theorem 609.2. The same holds for the n-th roots of unity. The old ordinary characters are unramified there, B_0 is fixed by inertia by construction, and L' has local image in U. Their compositum is fixed by $\mathrm{Aut}(K_{p_0}/U)$ and has local image in U, giving index one as in the preceding lemma.

Apply Theorem 599.2 in B/E at w_0 with prescribed residue t_0^{-1}. Impose the residue one at every other prime of E above p_0, at T and at all old ordinary pairs. Impose also one modulo ℓ^K, $K \geq 2N + 1$, and avoid all the finitely many excluded rational primes in the old data and the finite extension residue formulas. As in the proof of Corollary 609.3 the construction gives $w = (\pi)$, $p = Nw$ prime, and a principal prime (β) of D above it. The congruences to one hold for β at the imposed ideals of D. Theorem 609.2 supplies h_w trivial on decomposition at all old ordinary pairs, at T and above ℓ. Splitting in the radical field gives an e_w killing all the prescribed ideles and the old ordinary b_v.

The positive absolute norm $N_{E/\mathbb{Q}}\pi = p$, since E is totally imaginary and (π) has prime norm. Modulo p_0 it is t_0^{-1} in $\mathbf{F}_{p_0}^\times$. In fact the determinant of multiplication by π modulo p_0 is the product of its residue norms over the primes over p_0. There is one copy of each residue field because p_0 is unramified, by factorization and the Chinese remainder theorem as used in Theorem 599.2. All components but w_0 are one, and that component has residue degree one. At any arithmetic lift ϕ at w this gives $d_0(\phi) = -1$. Since w splits completely in B_0, $h_{w_0}(\phi) = -d_0(\phi) = 1$. Splitting of w in the old ordinary character fields gives $h_v(\phi) = 0$

for the other old indices. For these finite Galois conclusions, restriction of a decomposition element at a completely split prime is trivial: the stabilizer of a prime is trivial by transitivity and the number of primes. Replace ϕ by its product with an inertia element to obtain $h_w(\phi_w) = 0$, using surjectivity of h_w on inertia. This preserves all old values and the residue action. Choose γ_w with $h_w(\gamma_w) = 1$, and do the same at w_0. The other characters are unramified there, giving the diagonal inertia values.

For the old idelic coordinates, divide the idele b_w by the principal torus element $\pi/\bar{\pi}$. The quotient is integral. At the helper its residue is t_0, since π is one at $\bar{w}_0$. At the old ordinary places its residue is one. Hence the old $e_v(b_w)$ have exactly the same values as the old $h_v(\phi_w)$ by Lemma 600.1. As in Theorem 600.2, multiplication of the local uniformizer at w by a unit can set $e_w(b_w) = 0$ without changing the old evaluations. The new e_w, h_w kill the old ordinary elements b_v, ϕ_v by their respective constructions. This proves the two displayed arrays by induction, with the conjugate evaluations following from the sign of h_v. Since L' was taken normal over $\mathbb{Q}$, splitting in L'/E gives the same splitting over the conjugate places, by transport of primes under c. □

611. The cyclic norm quotient of a local unit group

Lemma 611.1. *For an abelian group A with an action of a cyclic group of order n and generator σ, put $d = 1 - \sigma$, $N = 1 + \sigma + \cdots + \sigma^{n-1}$ in additive notation. Write $h(A) = |\ker d/NA| \,/\, |\ker N/dA|$ when both orders are finite. This quotient is multiplicative in equivariant short exact sequences whenever two of the three groups have both orders finite. It is one for every finite A. For a free lattice A of finite rank over $O = \mathbb{Z}$ or $O = \mathbb{Z}_p$, put $r_+ = \dim \ker d$ and $D_- = \det(d|_{\ker N})$ over its fraction field. Then both orders are finite and*

$$h(A) = \begin{cases} n^{r_+}/|D_-| & (O = \mathbb{Z}), \\ p^{v_p(n)r_+ - v_p(D_-)} & (O = \mathbb{Z}_p). \end{cases}$$

On a zero-dimensional space the determinant is one.

Proof. Both composites of d, N vanish. The six-term sequence of Lemma 87.2 and its proof apply verbatim to these two maps: a connecting map lifts a cycle and applies the endomorphism which killed its image. The lift ambiguity is its image on the subgroup, and changing the cycle by a boundary changes this by zero since the two composites vanish. Exactness follows by subtracting a lifted boundary, as at each of the six positions in that lemma. Alternating orders give multiplicativity. For a finite group use $|\ker T|\,|TA| = |A|$ for each $T = d, N$.

Over the fraction field N/n is the projection onto $\ker d$ with kernel $\ker N$. The two intersections of these subspaces with the lattice therefore span a subgroup containing nA, of finite index in A. Passing to it does not change h by the finite-group assertion. On the first intersection $N = n$ and $d = 0$. On the second $N = 0$ and d is an injection of full rank. The indices of these two endomorphisms are the absolute determinant over $\mathbb{Z}$ and $p^{v_p(\det)}$ over $\mathbb{Z}_p$. The first assertion follows by lattice volume as in Lemma 12.1, the second by integral diagonal elimination as in Lemma 602.1. The two intersections are full lattices in their subspaces by clearing denominators in a basis. Over $\mathbb{Z}_p$ a submodule of a finite free module is free: project to the first coordinate, choose one vector giving a generator for the image ideal, and subtract its multiples to reduce to the remaining coordinates. The same argument works over $\mathbb{Z}$. The two contributions give the formulas. □

Lemma 611.2. *For a cyclic Galois extension B/F of degree n, with generator σ, every $a \in B^\times$ of norm one equals $b/\sigma b$ for some $b \in B^\times$.*

Proof. Set $a_0 = 1$, $a_i = \prod_{j=0}^{i-1} \sigma^j(a)$, so $a_n = 1$. Choose $t \in B$ such that $b = \sum_{i=0}^{n-1} a_i \sigma^i(t) \neq 0$. The independence of distinct embeddings as functions proved in Lemma 601.2 ensures such a choice, since the first coefficient is one. Multiplication of σb by a shifts the summation cyclically, proving $b = a\sigma b$. □

Theorem 611.3. *Let B/F be a cyclic extension of nonarchimedean local fields finite over $\mathbb{Q}_p$, of degree n. For $A = O_B^\times$, both quotients in Lemma 611.1 are finite and $h(A) = 1$. When B/F is unramified, the unit norm maps onto $O_F^\times$ and every norm-one unit has the form $b/\sigma b$ for a unit b.*

Proof. The ideal $p^2 O_B$ is a full $\mathbb{Z}_p$-lattice in B: the coordinate norm comparison of Lemma 113.1 places it between two scaled coordinate lattices, and freeness follows from Lemma 611.1. The maps

$$\log(1+x) = \sum_{j\ge 1} (-1)^{j+1} x^j/j, \qquad \exp(x) = \sum_{j\ge 0} x^j/j!$$

are inverse equivariant group isomorphisms between $1 + p^2 O_B$ and $p^2 O_B$. Indeed $v_p(j!) = \sum_{a\ge 1} \lfloor j/p^a \rfloor \le j - 1$, by counting factors of p (or summing the binary-digit geometric bound for $p = 2$). At $v_p(x) \ge 2$ the nonconstant terms tend to zero in both series and all terms after the first have strictly larger valuation than x. The formal identities $\exp(X+Y) = \exp(X)\exp(Y)$ and $\log(\exp X) = X$, $\exp(\log(1+X)) = 1 + X$ follow by the binomial theorem, formal differentiation and constant terms over $\mathbb{Q}$. Substitution of the indicated arguments preserves the identities: in a composite a term of total degree m has denominator valuation at most $m - 1$, by the bound $v_p(j!), v_p(j) \le j - 1$ for its inner and outer factors. Its valuation is at least $m + 1$. Each degree has finitely many terms and regrouping converges. The coefficients are rational, giving equivariance. The index of $1 + p^2 O_B$ in the unit group is finite by reduction modulo $p^2 O_B$.

On the F-space B, σ has the characteristic polynomial of a cycle of length n. Indeed after tensoring with an algebraic closure of F, evaluation at the n embeddings identifies B with a product of n copies of that closure after scalar extension. Choose a primitive generator: in characteristic zero a finite separable extension has one by the finite-embedding separation argument of Lemma 82.2, which uses only that the base field is infinite. The evaluation matrix on its power basis is a Vandermonde matrix of distinct conjugates. The action on its embeddings is a single cycle. The invariant dimension is one and $\det((1-\sigma)|_{\ker N}) = n$, by evaluating $(X^n - 1)/(X - 1)$ at one. Restriction of scalars from F to $\mathbb{Q}_p$ multiplies this dimension by $[F : \mathbb{Q}_p]$ and raises the determinant to that power: after extending scalars take the blocks for the embeddings of F, by the same evaluation argument. Lemma 611.1 on $p^2 O_B$ gives $h = 1$. The finite index comparison proves the first assertion.

If the ramification index is one, a uniformizer of F is a uniformizer π of B. If the residue degree is a, then $[B : F] = a$: lift a residue basis and use successive π-digits to expand every integer over O_F in those lifts, with independence by reducing a putative relation with one unit coefficient. Each residue conjugate of a generator lifts uniquely to a root of its monic separable polynomial lifted over O_F, by the simple-root argument of Lemma 113.1. This generator generates the entire field by the same residue-basis argument. Thus the Galois action on residues is the complete cyclic residue Galois group. Its norm on residue units is surjective since the residue multiplicative group is cyclic and the norm is raising to $1 + q + \cdots + q^{a-1}$, $q = |k_F|$. Its trace on the residue field is onto: the additive polynomial $X + X^q + \cdots + X^{q^{a-1}}$ is nonzero of degree less than q^a, so its k_F-linear image in k_F is nonzero. Lift a unit of the desired residue norm, and at stage $j \ge 1$ multiply it by $1 + \pi^j t$. Its norm is thereby multiplied by $1 + \pi^j \mathrm{Tr}(\bar{t})$ modulo π^{j+1}, with the residue trace interpreted in the coefficient of π^j. This corrects any next digit and converges in the complete field. For a norm-one unit the preceding lemma gives $b/\sigma b$ in $B^\times$. Multiplication of b by a power of the base uniformizer makes it a unit and preserves the quotient. □

612. The cyclic norm index on finite idele classes

Lemma 612.1. *For a finite Galois extension B/E, the stabilizer of a prime $w \mid v$ acts on B_w as its full Galois group over E_v. If $G = \mathrm{Gal}(B/E) = \langle \sigma \rangle$ is cyclic, the two cyclic*

quotients of $\prod_{w|v} O_{B,w}^\times$ under G are those of $O_{B,w}^\times$ under its stabilizer, with the generator obtained by the first return of σ to w.

Proof. Transitivity on primes follows from the norm and Chinese remainder argument of Lemma 603.1, which applies to any finite Galois extension and every residue degree. The ring of integers of B localized at the nonzero integers of E outside v is free of rank $[B:E]$ over $O_{E,(v)}$. Indeed it is finite and torsion free, by the integral lattice of Lemma 82.2 and the discrete valuation ring freeness argument of Theorem 599.2. The factorization $vO_B = \prod_{w|v} w^{e_w}$ of Proposition 82.3 and Chinese remaindering modulo all its powers identify the completed ring with $\prod_{w|v} O_{B,w}$. Here localization at w before taking the quotients by w^a does not change them, since every element outside w is invertible modulo w^a, and the exponents $a = e_w j$ are cofinal. Inverting a base uniformizer yields the product of the B_w, of total E_v-dimension $[B:E]$. All its factors have the same dimension by transitivity, hence their dimension is the order of the stabilizer. Its elements extend to distinct automorphisms of the completion by density and act faithfully on B. Equality of the number of automorphisms with the degree proves the Galois assertion, by separability.

Write a for the number of primes over v. Identify the unit groups at the successive primes with copies of the first using σ. Then σ cycles the factors, using σ^a on the return. Invariants are determined by the first coordinate fixed by σ^a. Modulo the image of $1-\sigma$, in additive notation, every coordinate can be moved to the first one; making a complete turn imposes $1-\sigma^a$ there. More explicitly the inverse to inclusion from the first factor modulo $1-\sigma^a$ transports the coordinates back by the chosen powers of σ and adds them modulo $1-\sigma^a$. Under these identifications the norm from coinvariants to invariants is the norm for σ^a, by starting with a tuple supported in the first factor and summing its translates. The cokernel and kernel of this norm are precisely the two quotients stated. □

Theorem 612.2. *Let B/E be a finite cyclic extension of totally imaginary number fields, of degree n. Write $J_{F,f}$ for the restricted product of $F_v^\times$ over finite places with respect to $O_{F,v}^\times$, and $C_{F,f} = J_{F,f}/F^\times$. On ideles the norm $N_{B/E}$ takes at v the product of the local field norms at the places over v. Put $Q_B = C_{E,f}/N_{B/E}C_{B,f}$. Then*

$$|Q_B| = n.$$

For some integer $m_0 \geq 2$, divisible by every rational prime below a ramified place, Q_B is a quotient of the ray class group $C_{m_0}(E)$ of Definition 219.1. A prime ideal not dividing m_0 maps to its uniformizer idele class in Q_B. The sum of $(Nv)^{-\sigma}$ over such primes having trivial class in Q_B and failing to split completely in B stays bounded as $\sigma \downarrow 1$.

Proof. Use the cyclic quotient h of Lemma 611.1. Let $J_{B,f}^0 = \prod_{w \text{ finite}} O_{B,w}^\times$. By Lemma 612.1 and Theorem 611.3, the two cyclic quotients of each product over v are finite and have equal orders. Both are trivial away from ramification. Only finitely many places ramify by Lemma 219.3. The norm and $1-\sigma_B$, with σ_B a generator of the Galois group, operate independently on the products over the distinct v. Images, kernels and both quotients are products, since component preimages can be chosen independently. Thus $h(J_{B,f}^0) = 1$.

By Lemma 83.1, the global unit group of B modulo finite torsion is a full lattice in the real logarithm hyperplane on its complex places, one coordinate per conjugate pair. The permutation action there consists of cycles of size n, one for each complex place of E. Indeed the extensions of a specified complex embedding of E form a free transitive orbit of the n automorphisms, and none restricts to its conjugate since E is totally imaginary. On the whole real permutation space the fixed dimension is the number $f_E = [E:\mathbb{Q}]/2$ and the determinant of $1-\sigma_B$ on the complement of the fixed space is n^{f_E}, by the cycle computation in Theorem 611.3. Passing to the sum-zero hyperplane removes one fixed dimension and preserves the complement. The rational dimensions and determinant are the same by scalar extension from the unit lattice. Hence $h(O_B^\times) = n^{f_E-1}/n^{f_E} = 1/n$ by Lemma 611.1. The

ideal valuation map gives

$$1 \longrightarrow J^0_{B,f}/O_B^\times \longrightarrow C_{B,f} \longrightarrow \mathrm{Cl}(B) \longrightarrow 1$$

by Proposition 82.3. The last group is finite by Lemma 82.5. Multiplicativity in the two exact sequences gives $h(C_{B,f}) = n$, with both quotients finite.

A class $[x] \in C_{B,f}$ fixed by σ_B satisfies $x/\sigma_B x \in B^\times$ of norm one. By Lemma 611.2 this element equals $b/\sigma_B b$, $b \in B^\times$. The idele x/b is fixed. It comes from $J_{E,f}$ since a fixed tuple at v is determined by one value fixed by the full local Galois group of Lemma 612.1, hence in $E_v^\times$. Also $J_{E,f} \cap B^\times = E^\times$ by global invariance. This identifies the fixed classes with $C_{E,f}$, and the cyclic group norm with $N_{B/E}$, by grouping its factors over each local orbit. The numerator of $h(C_{B,f})$ is therefore $|Q_B|$, so $|Q_B| \geq n$.

All local units away from ramification are norms by Theorem 611.3. At a ramified place over p, every base unit sufficiently close to one is an n-th power in $E_v^\times$: use the Newton argument of Theorem 609.2 with $v_p(n)$ in place of N. An n-th power of a base element is the norm of its diagonal embedding. Choose m_0 so units congruent to one modulo m_0 at ramified places suffice. Sending ideals prime to m_0 to products of uniformizer ideles gives a well-defined homomorphism to Q_B, independently of the uniformizers. A principal ideal generated by an element locally congruent to one modulo m_0 maps to zero: after division by that principal idele the remaining components are units and at ramified places are among the norms just described. These local norms combine to an idele norm, choosing unit preimages at almost every place. The induced map from $C_{m_0}(E)$ is onto: by Lemma 89.1 divide an arbitrary idele by an element of $E^\times$ so that its components above m_0 are sufficiently close to one. Discard local unit norms at the other places, leaving precisely the indicated uniformizer class.

Write $H_\sigma = \log(1/(\sigma-1))$ for real $\sigma > 1$. The ray-class counting formula in Theorem 595.2, summed over the kernel of this surjection, gives $H_\sigma/|Q_B| + O(1)$ for the primes of trivial class. The sum over completely split primes is $H_\sigma/n + O(1)$. Indeed $\log \zeta_B(\sigma) = H_\sigma + O(1)$ by Corollary 83.4. In its Euler logarithm the higher powers have bounded sum. Outside the finite exceptional set of Lemma 219.3, a split prime of E gives n primes of the same norm and a nonsplit prime gives at most n primes each of norm at least $(Nv)^2$. The latter sum is bounded by absolute convergence at two. Every split prime away from m_0 has trivial class in Q_B, since a uniformizer supported on one split component has that uniformizer as norm. Comparing the sums gives $|Q_B| \leq n$, hence equality. Subtracting the two sums now gives the asserted boundedness. □

613. Recurrence of a Frobenius element in a specified ray class

Definition 613.1. For a totally imaginary E and integer $m \geq 2$, write $C_m(E)$ for Definition 219.1. For $v_* \mid m$, let $J_*(m) \subseteq C_m(E)$ be the subgroup of classes of (a) with $a \in E^\times$ a unit at every $v \mid m$, congruent to one modulo m at those different from v_*. Only the residue of a in $(O_{E,v_*}/mO_{E,v_*})^\times$ matters. Every invertible residue there occurs by Chinese remaindering, as in Lemma 89.1. We call the resulting class its residue class in $C_m(E)$.

Theorem 613.2. *Let L/E be finite Galois, E totally imaginary, $m \geq 2$ and ℓ prime. Let $\mathfrak{q} \nmid m\ell$ be unramified in L/E. Suppose that an arithmetic Frobenius of $\mathfrak{q}$ in L/E is g of ℓ-power order. One may take $B_* = 1$ in $C_m(E)$, or take any $B_* \in J_*(m)$ for a place $v_* \mid m$ away from ℓ unramified in L/E. Then the sum of $(Nw)^{-\sigma}$ over degree-one primes w with class $[\mathfrak{q}]B_*$ and with some arithmetic Frobenius equal to g is at least*

$$c \log(1/(\sigma-1)) - O(1) \qquad (\sigma \downarrow 1)$$

for some $c > 0$. A fixed finite set of rational primes can be omitted. Neither c nor the bounded term needs to be uniform in the data.

Proof. Write $Q = N\mathfrak{q} > 1$ and choose $k \geq 1$ so that the order of Q mod ℓ^k is divisible by the order of g. Indeed the order divides $(\ell-1)\ell^{k-1}$, the number of units modulo ℓ^k. If its

ℓ-part stayed bounded by ℓ^d for all k, the nonzero integer $Q^{(\ell-1)\ell^d}-1$ would be divisible by every ℓ^k. This also covers $\ell=2$.

Put $A=LE(\mu_{\ell^k})$. This extension is unramified at $\mathfrak{q}$ and at v_* if the latter is used, by Lemma 599.1. Choose a prime of A over the one giving g. Its Frobenius $\widetilde{g}$ projects to g and acts on μ_{ℓ^k} by exponent Q, since these roots have distinct residues. Here the Frobenius at any unramified prime is the unique member of its stabilizer inducing the residue power map. This follows from Lemma 612.1 and the unramified local degree and simple-lift proof in Theorem 611.3: a lifted residue generator gives the whole unramified extension and its automorphisms are exactly the residue automorphisms. Restrictions respect the power map. The restriction to the roots of unity is injective on $\Delta=\langle\widetilde{g}\rangle$, because the order on these roots is divisible by the order of g and the two restrictions determine an automorphism of A. Put $B=A^{\Delta}$. Then A/B is cyclic of group Δ by the primitive-generator fixed-field argument in Lemma 220.1. At the prime $\mathfrak{Q}$ of B just selected above $\mathfrak{q}$, the residue field has order Q: its elements are fixed by the Q-power action of $\widetilde{g}$ and contain the residue field of $\mathfrak{q}$. In particular the relative ideal norm of $\mathfrak{Q}$ is $\mathfrak{q}$.

Take $u=1$ when $B_*=1$. Otherwise choose an integer $u\in O_B$ congruent to 1 modulo ℓ^kO_B and at the places over m prescribing $N_{B/E}u$ to give the residues defining B_*. More explicitly, at v_* choose a unit lifting the desired residue at that place of E. At one place of B above it choose a unit of this norm, and at the other places use one. Such a lift exists since the local extensions are unramified, hence cyclic as just proved, and their unit norms are onto by Theorem 611.3. At all remaining places over $m\ell^k$ use one. Integral Chinese remaindering to the precision of $m\ell^k$ at these local units suffices: the norm is the determinant of multiplication on the local integral lattice, so two integral elements congruent modulo a base ideal have congruent norms there. The local tensor-product calculation of Lemma 612.1 before its use of transitivity identifies the determinant with the product of the local determinants also for B/E which may be nonnormal. Thus $(N_{B/E}u)$ is in the desired ray class and $N_{B/\mathbb{Q}}u\equiv 1 \bmod \ell^k$.

In B use the ray class of $\mathfrak{Q}(u)$ modulo $m\ell^k$. The prime sum proved in Theorem 595.2 for this class and degree-one primes $\mathfrak{R}$ is $\log(1/(\sigma-1))/|C_{m\ell^k}(B)|+O(1)$. Omit finitely many rational primes, in particular all ramified ones. Write $\mathfrak{R}=\mathfrak{Q}(u)(b)$ with $b\equiv 1 \bmod m\ell^k$ locally at the divisors of that integer. Then $N\mathfrak{R}\equiv Q \bmod \ell^k$. This follows by multiplicativity of norms, positivity of field norms over $\mathbb{Q}$ for totally imaginary fields, and the local determinant congruence for b and u. For prime ideals relative norm means the power of the contracted prime with exponent the residue degree. This ideal norm takes a principal ideal to the ideal of the field norm: locally the index of the multiplication lattice for an integral element is computed both by its determinant and by the residue filtrations in the product of completed rings of Lemma 612.1. Division treats fractional ideals. Since $\mathfrak{R}$ has rational prime norm, its contraction w has the same norm. The relative ideal norm calculation gives its ray class $[\mathfrak{q}]B_*$.

The Frobenius at a prime over $\mathfrak{R}$ for A/B is in Δ and acts on the chosen roots of unity by $N\mathfrak{R}\equiv Q \bmod \ell^k$. Thus it equals $\widetilde{g}$. Since the residue degree of $\mathfrak{R}$ over w is one, this very element gives a Frobenius over E as well, hence induces g on L. At most $[B:\mathbb{Q}]$ of the $\mathfrak{R}$ lie over each resulting w, by the ideal factorization of the rational prime there. Dividing the preceding lower bound by $[B:\mathbb{Q}]$ proves the assertion. □

Corollary 613.3. *In Theorem 612.2 suppose n is an ℓ-power and m is a multiple of the integer m_0 there. For every prime $\mathfrak{q}\nmid m\ell$ its class in Q_B is trivial if and only if it splits completely in B/E.*

Proof. The split implication was proved in Theorem 612.2. Otherwise a nonsplit unramified prime with trivial norm class would have a nonidentity Frobenius. Apply Theorem 613.2 to B/E with $B_*=1$. The resulting primes would all have the same trivial class in Q_B

and be nonsplit, and their sum would tend to infinity. This contradicts the boundedness in Theorem 612.2. □

614. Reciprocity for cyclic characters of prime-power order

Lemma 614.1. *For a totally imaginary number field E, let*

$$K_m(E) = \prod_{v\nmid m} O_{E,v}^\times \prod_{v\mid m}(1+mO_{E,v}), \qquad m \geq 2.$$

Sending an ideal prime to m to a product of uniformizer ideles at its prime factors gives an isomorphism

$$C_m(E) \;\simeq\; J_{E,f}/(E^\times K_m(E)).$$

At $v_* \mid m$ a residue class represented by a in Definition 613.1 corresponds to the single local unit a^{-1} at v_* in this quotient. Two continuous homomorphisms from $J_{E,f}$ to a finite discrete abelian group, trivial on $E^\times$, which agree on uniformizers at all but finitely many primes and are unramified there are equal. Here unramified means trivial on the local units.

Proof. Changing uniformizers multiplies by units outside m. A principal generator congruent to one at m gives its principal idele up to $K_m(E)$. Conversely, equality to a principal idele times $K_m(E)$ supplies such a generator on comparing valuations everywhere and residues above m. For surjectivity divide an idele by a rational element of $E^\times$ sufficiently close to it above m, using Lemma 89.1. The quotient represents the ideal of its valuations away from m. For a residue class as stated, division of the representative uniformizer idele by a leaves a^{-1} at v_*, up to $K_m(E)$. Finally an open kernel contains $K_{m'}(E)$ for some rational integer m', by the restricted product topology and the neighborhoods of one in the finitely many exceptional local unit groups. Take a common multiple for the two kernels. Every resulting ray class has a prime of rational prime norm outside any fixed finite set by the prime sum in Theorem 595.2. Equality on those primes proves the last assertion. □

Theorem 614.2. *Let E be totally imaginary, ℓ prime and $n = \ell^N$ with $N \geq 1$. For each continuous homomorphism $h : G_E \to \mathbb{Z}/n\mathbb{Z}$ there is a unique continuous homomorphism $\lambda_h : J_{E,f} \to \mathbb{Z}/n\mathbb{Z}$ trivial on $E^\times$ such that at all but finitely many primes v it is unramified and has value $h(\phi_v)$ on a local uniformizer, with ϕ_v arithmetic Frobenius in the finite quotient of h. This assignment is additive and commutes with automorphisms of E extended to its algebraic closure. If H/E is the finite cyclic extension cut out by h, then $\lambda_h(N_{H/E}J_{H,f}) = 0$. In particular λ_h kills all units at every unramified place of H/E and kills all of $E_v^\times$ at every completely split place.*

Proof. The extension H exists since the open kernel contains the group fixing a finite normal field. On taking fixed fields in its finite image quotient, Galois correspondence follows from the primitive-generator argument in Lemma 220.1. Choose a rational prime $P \neq \ell$ split completely in a normal extension of $\mathbb{Q}$ containing H and μ_n, using Theorem 220.2 for the identity. It may avoid any given finite set. Thus $P \equiv 1 \bmod n$, by the distinct residues of μ_n. Fix a place $v_* \mid P$ of E and a generator $t_* \in \mathbf{F}_P^\times$. Let $d : G_E \to \mathbb{Z}/n\mathbb{Z}$ be the discrete logarithm to this base of the exponent on μ_P, reduced modulo n. By Lemma 610.1 it is surjective on inertia at v_*. The character h is zero on its decomposition group by complete splitting. Consequently $h - d$ is surjective, of exact order n. Call its cyclic field L.

Take a rational integer m divisible by ℓP, by the rational primes ramified in H/E, and by the modulus of Theorem 612.2 for L/E. Let K be the kernel of $C_m(E) \to Q_L$, a subgroup of index n. On $C_m(E)$ put

$$d_{\mathrm{id}}([\mathfrak{a}]) = \log_{t_*}(N\mathfrak{a} \bmod P) \pmod n.$$

The norm of a fractional ideal prime to m is a positive rational number with numerator and denominator prime to P. For a principal generator congruent to one at P the positive absolute ideal norm equals its field norm, congruent to one by the determinant of multiplication

modulo P. Thus d_{id} is well defined. At each prime away from m its value is $d(\phi_v)$, since the reduction of a P-th root of unity retains its order and arithmetic Frobenius acts by exponent Nv. On $J_*(m)$ of Definition 613.1 the value is simply the log of its one residue at v_* modulo P: in the residue norm product all other entries are one and $E_{v_*} = \mathbb{Q}_P$. It is surjective on $J_*(m)$.

There is a prime $\mathfrak{q}_0$ in the trivial ray class of E split completely in H/E. To see this take a principal prime of H of rational prime norm generated by an integer congruent to one modulo mO_H, away from ramification, by Theorem 595.2. Its contraction splits completely by the residue degree one, and is generated by the norm of its generator, still one modulo m. Now for any $B_* \in J_*(m)$ apply Theorem 613.2 to $H/E, \mathfrak{q}_0, B_*$. There is a prime in that class with $h(\phi) = 0$. It splits in L/E precisely when $d_{\mathrm{id}}(B_*) = 0$. By Corollary 613.3 this is equivalent to $B_* \in K$. Thus $K \cap J_*(m) = \ker(d_{\mathrm{id}}|_{J_*(m)})$. Its index in $J_*(m)$ is $n = [C_m(E) : K]$, so $J_*(m)$ maps onto the entire quotient by K. There is accordingly a homomorphism $f : C_m(E) \to \mathbb{Z}/n\mathbb{Z}$ with kernel K whose restriction to $J_*(m)$ is $-d_{\mathrm{id}}$.

Let now $v \nmid m$, with ray class A. Choose $B_* \in J_*(m)$ with $d_{\mathrm{id}}(B_*) = h(\phi_v) - d_{\mathrm{id}}(A)$. The same recurrence theorem, this time for v, gives a prime with class AB_* and the same Frobenius in H/E as v. Its d-value is $d_{\mathrm{id}}(AB_*) = h(\phi_v)$, so it splits in L/E. The splitting test gives $0 = f(AB_*) = f(A) - d_{\mathrm{id}}(B_*)$. Hence

$$h(\phi_v) = (f + d_{\mathrm{id}})(A).$$

The character induced by $f + d_{\mathrm{id}}$ on the finite ideles via Lemma 614.1 has the claimed values. Uniqueness follows from that lemma. It also gives additivity by checking on common good primes. For an automorphism of E, transporting the primes, their residue maps and the Frobenius action gives the same values there after conjugation of G_E, hence equality of the idele characters. Different extensions of the automorphism differ by inner conjugation in G_E, invisible to h.

Compose λ_h with $N_{H/E}$. This is continuous by the local determinant formula of Lemma 612.1 and the restriction to units at almost all places. It is trivial on $H^\times$. Away from the finitely many exceptional primes of E, a uniformizer at a prime of H above v has norm of valuation the residue degree a, by the local norm calculation in Theorem 613.2. Its value under the composite is $ah(\phi_v) = 0$, since the Frobenius in the cyclic group $\mathrm{Gal}(H/E)$ has order a. By the uniqueness lemma over H the composite is zero everywhere. At an unramified place local unit norms are surjective by Theorem 611.3, and at a completely split place every local element is a norm from one of the factors. This proves the last claims. □

615. Global comparison for the finite-order torus twists

Lemma 615.1. *In Theorem 614.2, at every unramified $v \nmid \ell$ the value of λ_h on a uniformizer at v equals $h(\phi_v)$ for arithmetic Frobenius, including when v divides an auxiliary modulus used to construct λ_h.*

Proof. Write H for the cyclic field of h and $Q = Nv$. As in Theorem 613.2, choose k such that $Q \bmod \ell^k$ has order divisible by the order of the Frobenius in H/E. In $A = HE(\mu_{\ell^k})$ fix a prime over v, let Δ be generated by its Frobenius and $B = A^\Delta$. The same proof gives a place $w_1 \mid v$ of B with residue degree one and ramification index one, so $B_{w_1} = E_v$. Choose a uniformizer idele x at w_1; its norm to E is a uniformizer idele at v. Take m so that λ_h factors modulo $K_m(E)$, and increase m to include all its exceptional primes in the Frobenius comparison. By Lemma 614.1 the idele x represents a ray class modulo $m\ell^k$ in B, even when w_1 divides m. Choose a degree-one prime $\mathfrak{R}$ of B in its class avoiding $m\ell$ and all relevant discriminants, by the ray prime sum of Theorem 595.2. On passing to the ray group modulo ℓ^k, the class of x is that of the ideal w_1, since $w_1 \nmid \ell$. Thus $N\mathfrak{R} \equiv Q \bmod \ell^k$ by the positive absolute norm calculation in Theorem 613.2. As in that theorem the contraction w of $\mathfrak{R}$ then has the same Frobenius in H/E as v. The norm of $K_{m\ell^k}(B)$ lies in $K_m(E)$ by the local determinant congruence in the same proof. Hence the uniformizer idele at w

obtained by norm has the same λ_h-value as $N_{B/E}x$. At w the good-prime equality is known, and at v unit independence holds by Theorem 614.2. This proves the claim. □

Definition 615.2. For a complex character ϑ of a finite torus class group $C(P)$ of Definition 589.1, its finite base change is the character $\vartheta^{\mathrm{bc}}(x) = \vartheta(x/\bar{x})$ on $J_{E,f}$. Fix $j : \mathbb{C} \to K_\ell$. An inverse avatar of ϑ means here a continuous finite-order character $\psi : G_E \to K_\ell^\times$ taking the value $j(\vartheta^{\mathrm{bc}}(\mathrm{loc}_v \pi_v))^{-1}$ at geometric Frobenius for all but finitely many v, with π_v a uniformizer. Here loc_v means the idele supported at v and geometric Frobenius is the inverse of arithmetic Frobenius. At these primes both characters are unramified.

Theorem 615.3. *Take a system of Galois characters and places constructed in Theorem 610.2. The h_v may be normalized by units modulo $n = \ell^N$ and the inertia generators chosen so that there are idelic coordinates $e'_v : C(\{v\}) \to \mathbb{Z}/n\mathbb{Z}$ with the same two evaluation arrays as in that theorem and with*

$$\lambda_{h_v}(x) = e'_v(x/\bar{x}) \qquad (x \in J_{E,f}).$$

For every tuple $y_v^n = 1$, define

$$j(\vartheta(c')) = \prod_{v \in P} y_v^{[e'_v(c')]}, \qquad \psi(g) = \prod_{v \in P} y_v^{[h_v(g)]}, \qquad c' \in C(P),\ g \in G_E.$$

Then ψ is the unique inverse avatar of ϑ, and ψ^d is the unique inverse avatar of ϑ^d for every integer d. Each coordinate character is trivial on the decomposition groups above ℓ and on the specified decomposition groups at T.

Proof. At each v of rational prime norm p_v, take the residue generator t_v in the construction. Let d_v be the cyclotomic logarithm at p_v of Lemma 610.1 with this base. Normalize h_v so that $h_v + d_v$ is unramified at v. The helper is already so normalized. For an ordinary v, scaling keeps its zero Frobenius values at other ordinary places, its zero value at its own chosen Frobenius, and its triviality on the other inertia groups. Replace its inertia generator by a unit power of the old one to recover the value one. Each resulting h_v has the form $k_v - k_v \circ c$, where $c(g) = cgc^{-1}$ here denotes conjugation on G_E by complex conjugation on Ω, and k_v is the correspondingly scaled transfer h^0 of Theorem 609.2. That proof shows k_v is unramified off v and is zero on decomposition at T and above ℓ.

Use additivity and conjugation in Theorem 614.2. For every finite idele x we obtain

$$\lambda_{h_v}(x) = \lambda_{k_v}(x) - \lambda_{k_v}(\bar{x}) = \lambda_{k_v}(x/\bar{x}).$$

The restriction of λ_{k_v} to T_f kills $T(E)$ because it kills $E^\times$. It is trivial on the units outside v by the norm assertion of that theorem. At v, λ_{h_v} on the local units is $-\lambda_{d_v}$, by unramifiedness of $h_v + d_v$. The latter cyclotomic character before negation has value $-\log_{t_v}(\bar{u})$ on a local unit u at v. Indeed its ray formula on ideals prime to p_v is the logarithm of the positive rational norm modulo p_v, by the unramified Frobenius action on μ_{p_v} and the uniqueness of Lemma 614.1. For a local unit at v, that lemma inverts the one residue in the ray class of a principal ideal. The absolute norm on that residue is its own value since $E_v = \mathbb{Q}_{p_v}$ and the rational prime was chosen unramified in E. Thus

$$\lambda_{h_v}(\mathrm{loc}_v u) = \log_{t_v}(\bar{u}) \pmod{n},$$

with $\bar{u}$ in this formula denoting its residue.

At a split pair the unit torus element of components $u, \bar{u}^{-1}$, with the second component transported by field conjugation, equals $x/\bar{x}$ for $x = \mathrm{loc}_v u$. This proves that the restriction of λ_{k_v} to T_f is trivial on $K_T(\{v\})$ and equals the indicated residue log on K_T. It defines e'_v on $C(\{v\})$, normalized to one at u_{t_v}. For each ordinary $w \neq v$, the element b_w equals $x/\bar{x}$ for a local uniformizer idele at w. Lemma 615.1 gives $e'_v(b_w) = h_v(\phi_w)$. For ordinary $w = v$ multiply its local uniformizer by a unit so that $e'_w(b_w) = 0$, which is possible by its surjectivity on residue units. This preserves every other e'_v-value at w. The Galois values hold on the conjugate inverses as in Theorem 610.2. We have recovered both evaluation

arrays. Triviality at T and above ℓ holds on the Galois side by construction and on the corresponding local idele groups for λ_{k_v} by complete splitting in the cyclic field and the norm assertion.

For all $x \in J_{E,f}$ the products defining ϑ^{bc} evaluate through the $\lambda_{h_v}(x)$ by the displayed character identity. On uniformizers at common good places their exponents are $h_v(\phi)$ for arithmetic Frobenius, so inversion gives the value of ψ at geometric Frobenius in Definition 615.2. The same calculation commutes with integer powers. Finally two finite characters agreeing away from a finite set of Frobenius places agree on every element: pass to the finite normal extension through which both factor and apply Theorem 220.2 to each element outside the finite exclusion. This gives uniqueness. □

616. Realization of finite characters over a field containing roots of unity

Lemma 616.1. *Let F be totally imaginary with $f = [F : \mathbb{Q}]/2$ and $\mu_n \subset F$, $n = \ell^N,\ N \geq 1$. For a finite set S of places containing all those above ℓ, write*

$$U_S = \{a \in F^\times : v(a) = 0 \text{ for every finite } v \notin S\},$$

using integer-valued prime valuations. Then

$$|U_S/U_S^n| = n^{f+|S|}, \qquad \prod_{v \in S} |F_v^\times/F_v^{\times n}| = n^{[F:\mathbb{Q}]+2|S|}.$$

Proof. The valuation map from U_S to $\mathbb{Z}^S$ has kernel $O_F^\times$ by Proposition 82.3. Its image contains $h\mathbb{Z}^S$, $h = |\mathrm{Cl}(F)|$, by the finiteness in Lemma 82.5: the h-th power of each prime in S is principal. The image is free of rank $|S|$, by the sublattice argument of Lemma 12.1. Lifts of a basis split the valuation map onto its image. By Lemma 83.1, modulo its finite torsion the ordinary unit group is a full lattice of rank $f - 1$. Its torsion consists of the roots of unity of F, cyclic by Lemma 73.2; it has order divisible by n. Lifting a free basis again splits off the free part. Taking the quotient by n-th powers now gives the first count.

At $v \mid p$, put $A = O_{F,v}^\times,\ D = 1 + p^2 O_{F,v}$. The logarithm in Theorem 611.3 identifies D equivariantly with the additive lattice $p^2 O_{F,v}$, so D is torsion free of finite index in A and $[D : D^n] = p^{[F_v:\mathbb{Q}_p]v_p(n)}$. The kernel of $A/D \to A^n/D^n$ is $DA[n]/D$, of order $|A[n]| = n$. Dividing $[A : D^n] = [A : D][D : D^n]$ by $[A^n : D^n]$ gives $[A : A^n] = n[D : D^n]$. The powers of a uniformizer contribute another factor n. Summing $[F_v : \mathbb{Q}_\ell]$ over $v \mid \ell$ gives $[F : \mathbb{Q}]$ by the tensor-product decomposition in the proof of Lemma 612.1. This proves the second assertion. □

Lemma 616.2. *With F, n as in Lemma 616.1, every finite set of places is contained in a finite set S including all places above ℓ, generating $\mathrm{Cl}(F)$ by its prime ideals, and satisfying*

$$U_S/U_S^n \ \hookrightarrow\ \prod_{v \in S} F_v^\times/F_v^{\times n}.$$

Here the arrow is simultaneous localization.

Proof. Start with S_0 containing the given set and the places above ℓ. Enlarge it by the prime divisors of a finite list of ideals representing the class group, using Lemma 82.5 and Proposition 82.3. Its primes now generate the class group. The first count of the preceding lemma makes $U_{S_0}/U_{S_0}^n$ finite. Each nontrivial class has a representative a which is not an n-th power in F: any root in F would lie in U_{S_0} by valuations. The field $F(\sqrt[n]{a})/F$ is finite cyclic and nontrivial, since all the n-th roots of unity lie in F and the multipliers on a root embed its Galois group in μ_n. Use Theorem 220.2 to choose a new unramified prime with nonidentity Frobenius in that field, outside S_0 and all previous choices. The element a is not a local n-th power there. A local root would give all the roots in the base completion, contradicting the nontrivial Frobenius action. Multiplication by an n-th power preserves this test.

Let S include these detecting primes. For every added prime w, choose $b_w \in F^\times$ with valuation one at w and zero outside $S_0 \cup \{w\}$: multiply the ideal w by a fractional ideal supported in S_0 to make it principal, using the generating property. If $u \in U_S$ is a local power at every member of S, its valuation at each new w is a multiple of n. Divide by the corresponding n-th powers of the b_w to obtain $u_0 \in U_{S_0}$. This element remains a local power at all the detecting primes, so its class in $U_{S_0}/U_{S_0}^n$ is trivial. Reinstating the removed powers makes u an n-th power in U_S. □

Theorem 616.3. *Let F be totally imaginary containing μ_n, $n = \ell^N$, $N \geq 1$. Every continuous homomorphism $\theta : J_{F,f} \to \mathbb{Z}/n\mathbb{Z}$ trivial on $F^\times$ is λ_h of Theorem 614.2 for a unique continuous $h : G_F \to \mathbb{Z}/n\mathbb{Z}$.*

Proof. Take S from Lemma 616.2 containing all primes where θ is nontrivial on the units. There are finitely many by Lemma 614.1. The finite idele classes modulo the units at all places outside S identify with $(\prod_{v\in S} F_v^\times)/U_S$. In fact the ideal of the valuations outside S can be made principal by multiplying by an ideal supported in S. Division by a generator removes every valuation outside S. The ambiguity in its choice is an element with valuations zero outside S, precisely U_S. This proves the identification.

Modulo n-th powers the order of this last group is

$$\frac{\prod_{v\in S} |F_v^\times / F_v^{\times n}|}{|U_S/U_S^n|} = n^{f+|S|},$$

by the injection and counts of Lemmas 616.2 and 616.1. All characters into $\mathbb{Z}/n\mathbb{Z}$ on this quotient are continuous: its kernel upstairs contains the outside units and sufficiently deep units at the finitely many remaining places, which are n-th powers by the Newton argument of Theorem 612.2. The finite abelian quotient is killed by n, and the number of its characters into $\mathbb{Z}/n\mathbb{Z}$ equals its order by Lemma 7.1, identifying the target with μ_n.

Fix a primitive root $\zeta \in F$. Each $a \in U_S$ gives a Galois character h_a by $g(\sqrt[n]{a})/\sqrt[n]{a} = \zeta^{h_a(g)}$. The choice of root has no effect and the distinct classes in U_S/U_S^n give distinct characters, by the fixed-field argument in Lemma 601.2. At every place outside S, a and n are units, and Lemma 599.1 shows its cyclic field is unramified there. By the norm assertion of Theorem 614.2, λ_{h_a} is accordingly trivial on all the outside units. These idele characters are distinct: any difference of the Galois characters is detected by a Frobenius at a common good prime, by Theorem 220.2 in the finite field cut out by both. The preceding count gives exactly as many distinct λ_{h_a} as possible idele characters with this ramification and exponent. Thus they include θ. The same good-prime argument proves uniqueness among all continuous Galois characters of the stated order. □

617. Descent of finite reciprocity characters

Lemma 617.1. *Write $T_\ell = \mathbb{Z}[1/\ell]/\mathbb{Z}$ with its discrete topology. For finite-image characters to T_ℓ, define λ_h by Theorem 614.2 using $\mathbb{Z}/\ell^a\mathbb{Z} \simeq T_\ell[\ell^a]$, $b \mapsto b/\ell^a$. This is independent of the containing finite subgroup, is injective, and commutes with addition. For any finite extension L/F of totally imaginary number fields,*

$$\lambda_{h|_{G_L}} = \lambda_h \circ N_{L/F}.$$

Proof. The definitions via different finite target groups agree on all their common good primes, hence agree everywhere by Lemma 614.1 in a finite cyclic group containing both. The same argument gives additivity. Injectivity follows by applying Theorem 220.2 to a finite normal extension through which the Galois character factors. At a good prime $w \mid v$ of residue degree a the Frobenius over L acts on the residue of the field cut out by h by $(Nv)^a$. Thus its h-value is a times the value at v. More explicitly take a common finite normal field over F containing L and the cyclic field, and use the defining residue-power equations of Lemma 219.3 outside its finite exceptional set. The local norm of a uniformizer

has valuation a, by the calculation in Theorem 613.2. Therefore the two sides agree at the good primes of L, and both are continuous and trivial on $L^\times$. The same uniqueness lemma over L proves the identity. □

Lemma 617.2. *Let L/F be cyclic of prime degree p, F totally imaginary. Suppose $\theta : J_{F,f} \to T_\ell$ is continuous with finite image and trivial on $F^\times$. If $\theta \circ N_{L/F} = \lambda_k$ for a continuous finite-image $k : G_L \to T_\ell$, then $\theta = \lambda_h$ for a continuous finite-image $h : G_F \to T_\ell$.*

Proof. The pullback of θ by the norm is invariant under $\mathrm{Gal}(L/F)$. Equivariance in Theorem 614.2 and injectivity in Lemma 617.1 give $k(gag^{-1}) = k(a)$ for $g \in G_F,\ a \in G_L$. Choose $g_0 \in G_F$ lifting a generator of the quotient. Choose $t \in T_\ell$ with $pt = k(g_0^p)$: multiplication by $p \neq \ell$ is invertible on each finite ℓ-primary subgroup, and for $p = \ell$ an additional power of ℓ in the denominator supplies a preimage. Define $h'(g_0^j a) = jt + k(a)$ for $0 \le j < p,\ a \in G_L$. In a product, conjugation past g_0^j preserves k and a carry of p in the exponent contributes exactly $k(g_0^p) = pt$. Thus h' is a homomorphism, continuous on each of the finitely many open cosets and of finite image.

By the norm identity of the preceding lemma, $\theta - \lambda_{h'}$ kills $N_{L/F} J_{L,f}$. It therefore factors through Q_L, of order p by Theorem 612.2. For $p \neq \ell$ such a character to T_ℓ is zero. For $p = \ell$, take any isomorphism $\mathrm{Gal}(L/F) \simeq T_\ell[\ell]$ and let χ be its inflation to G_F. The character λ_χ kills the same norms by the norm identity and takes a generator value at a good Frobenius by Theorem 220.2. Thus it identifies Q_L with $T_\ell[\ell]$. Every character from Q_L to T_ℓ is a multiple of this one. Adding the corresponding multiple of χ to h' gives the requested h. For $p \neq \ell$ take $h = h'$. □

Theorem 617.3. *For every totally imaginary F and every $n = \ell^N$, $N \ge 1$, $h \mapsto \lambda_h$ is an isomorphism from the continuous characters $G_F \to \mathbb{Z}/n\mathbb{Z}$ to the continuous characters $J_{F,f}/F^\times \to \mathbb{Z}/n\mathbb{Z}$. It respects restriction of Galois characters and norm pullback of idele characters in all finite extensions of F.*

Proof. The finite field extension $K = F(\mu_n)$ has abelian Galois group, by its faithful action on the cyclic μ_n. There is a tower from F to K with cyclic prime-degree steps. Indeed in a nontrivial finite abelian group, an element of some order $m > 1$ has a power of prime order. Take the subgroup it generates and repeat on the quotient. All subgroups are normal, and taking fixed fields gives the asserted tower by the fixed-field degree argument of Lemma 220.1. Every intermediate field is totally imaginary since it contains F.

For a given θ with target $T_\ell[n]$ pull it back by the norm from K. Theorem 616.3 realizes this pullback there. Descend by Lemma 617.2 one step at a time, using transitivity of field and local idele norms. The latter follows by multiplication over embeddings, or by grouping the local factors in Lemma 612.1. The resulting $h : G_F \to T_\ell$ has $\lambda_{nh} = n\theta = 0$, so $nh = 0$ by injectivity. Identify its image and that of θ inside $T_\ell[n]$ with $\mathbb{Z}/n\mathbb{Z}$. Additivity, uniqueness and the stated compatibility follow from Lemma 617.1. □

618. Decomposition and inertia at all finite places

Lemma 618.1. *Let F be totally imaginary, v a finite place of residue cardinality Q, and $h : G_F \to T_\ell$ a continuous character of finite image, with T_ℓ as in Lemma 617.1. If h is unramified at v, then $\lambda_h(\pi_v) = h(\phi_v)$ for a uniformizer idele π_v and arithmetic Frobenius ϕ_v. For each integer $d > 1$ the extension $F(\mu_{Q^d-1})/F$ at v is unramified of local degree d.*

Proof. Set $M = Q^d - 1$ for $d > 1$. This integer is prime to the residue characteristic. Lemma 599.1 gives unramifiedness. The residues of the M-th roots are distinct since $X^M - 1$ has simple reduction and each simple residue root has at most one lift, by Lemma 113.1. Frobenius acts on these roots by exponent Q. Its order is d, since M divides $Q^d - 1$ and exceeds $Q^j - 1 > 0$ for $0 < j < d$. The unramified residue action proved in Theorem 611.3 gives the local degree.

For the first assertion take d to be the order of $h(\phi_v)$. If $d = 1$, the cyclic field of h splits at v and Theorem 614.2 gives the assertion. For $d > 1$ send $Q \bmod M$ to $h(\phi_v)$ on the subgroup it generates in $(\mathbb{Z}/M\mathbb{Z})^\times$. Extend this homomorphism to a character u of the entire group into T_ℓ. Indeed, to extend from a subgroup B across an element x of order a modulo B, choose $t \in T_\ell$ with $at = u(x^a)$ and put $u(x^j b) = jt + u(b)$. Reduction of j modulo a proves well-definedness and multiplicativity in the argument. The preimage t exists by factoring a into its ℓ-part and its factor prime to ℓ, as in the proof of Lemma 617.2. Finitely many extensions suffice.

Let k apply u to the exponent of the Galois action on μ_M. Its idele character is induced by the assignment $\mathfrak{a} \mapsto u(N\mathfrak{a} \bmod M)$ on fractional ideals prime to M. This assignment factors through the ray class group modulo M, since a principal generator locally one modulo M has positive absolute norm one modulo M, by the determinant calculation of Theorem 613.2. Lemma 614.1 identifies the resulting character with λ_k by their values at common good primes, where Frobenius raises roots to the residue cardinality. At v it gives $u(Q)$ on a uniformizer. The character $h - k$ is unramified at v and vanishes on its Frobenius, hence vanishes on its decomposition group. The norm assertion of Theorem 614.2 makes λ_{h-k} zero on $F_v^\times$. Additivity gives the claimed evaluation. □

Theorem 618.2. *With F, h, v as in Lemma 618.1, write D_v, I_v for a decomposition group and its inertia subgroup. Regard $F_v^\times$ as a subgroup of the finite ideles by taking every other coordinate equal to one. Then*

$$\lambda_h(F_v^\times) = h(D_v), \qquad \lambda_h(O_{F,v}^\times) = h(I_v) \quad \text{as subgroups of } T_\ell.$$

In particular the two tests on the left characterize complete splitting and unramifiedness in the cyclic field of h by triviality of the respective images.

Proof. Write H for the field of h and $K = F_v$. First suppose that λ_h vanishes on $K^\times$. For a chosen completion B of H over K, put $d = [B : K]$. Suppose $d > 1$. In a common local closure let $U = K(\mu_{Q^d-1})$. Its Galois group is cyclic of order d with generator φ the arithmetic Frobenius, by Lemma 618.1. The group of B/K is cyclic by Lemma 612.1. Choose a generator g inducing φ on $B \cap U$. Indeed both cyclic groups surject onto the group of this intersection. If its order is f, a generator upstairs initially maps to a generator downstairs. Choose a power congruent to the desired unit exponent modulo f and to one modulo each prime dividing d but not f, by integer Chinese remaindering. The power is still a generator. For $f = 1$ any generator suffices.

There is an automorphism s of BU/K restricting to g, φ. To verify the extension assertion, lift φ to BU by extension of an embedding to an algebraic closure as in Lemma 16.2 and normality of both fields. The image of $\mathrm{Gal}(BU/U)$ on B has fixed field $B \cap U$, by finite Galois correspondence. It therefore contains the correction required on B. The corrected lift s has order d since its two restrictions have order d and determine it.

Let $A = HF(\mu_{Q^d-1})$. Its completion at the chosen prime is BU: the closure contains the two completed subfields and their compositum is complete by the coordinate norm comparison of Lemma 113.1. Lemma 612.1 thus realizes s in the stabilizer of the chosen prime of A. Put $L = A^{\langle s \rangle}$ and let w be the place below this prime in L. The group of A/L is the whole stabilizer at w. It acts faithfully on μ_{Q^d-1}; hence $A = L(\mu_{Q^d-1})$ by finite Galois correspondence. It is unramified at w by Lemma 599.1 applied over L. Its Frobenius there is consequently a generator, whose restriction to B has order d. The restriction of h to G_L takes a nonzero value at this Frobenius, since h is faithful on $\mathrm{Gal}(H/F)$. On the other hand $\lambda_{h|_{G_L}}$ vanishes on $L_w^\times$ by the norm identity in Lemma 617.1. This contradicts Lemma 618.1 over L. Thus $d = 1$. The converse follows from the norm assertion at a split place in Theorem 614.2.

Next suppose λ_h vanishes on $O_{F,v}^\times$. Let $n > 1$ annihilate h, so that it annihilates λ_h by additivity. Set $C = F(\mu_{Q^n-1})$. At a place w above v the local norm has valuations in $n\mathbb{Z}$

by the local degree just proved and the norm valuation calculation of Theorem 613.2. It follows that $\lambda_{h|_{G_C}}$ kills $C_w^\times$. The complete-splitting test already proved shows that HC/C splits there. Hence $B \subset C_w$ inside the chosen local closure. An intermediate field of this unramified extension has ramification index one, by multiplication of ramification indices, which follows by restricting normalized valuations of uniformizers. Thus $h(I_v) = 0$. The converse is the local unit norm assertion of Theorem 614.2.

Apply each triviality equivalence to $\ell^j h$ for $j = 0, 1, \ldots$. Additivity shows that the subgroups on either side of each asserted equality have the same exponent. Finite subgroups of T_ℓ with a given exponent are equal, since $T_\ell[\ell^j]$ is cyclic of order ℓ^j. This proves both equalities. □

619. Auxiliary characters with a prescribed Frobenius

Lemma 619.1. *Let E be totally imaginary, $n = \ell^N$, $N \geq 1$, and T a finite set of finite places. Write*

$$A_T = \{a \in E^\times : v(a) \in n\mathbb{Z} \text{ for all } v \notin T\}/E^{\times n}.$$

This is a finite group. Adjoining μ_n and one n-th root of a representative of each class gives a finite Galois extension V/E independent of those choices. It is unramified outside T and the places over ℓ. At every w splitting completely in V/E, every representative of every class of A_T is an n-th power in $E_w^\times$.

Proof. Enlarge T to a finite set T' generating the class group by its primes, using Lemma 82.5 and Proposition 82.3. For a representative a, the fractional ideal $\prod_{v \notin T'} v^{v(a)/n}$ can be made principal by multiplication by an ideal supported on T'. If b generates the product, $a/b^n \in U_{T'}$, in the notation of Lemma 616.1. The proof of that lemma gives finite generation of $U_{T'}$ also without roots of unity in E: its valuation image is a subgroup of $\mathbb{Z}^{T'}$, free and finitely generated by successive coordinate projection, and its kernel has finite torsion and a lattice of finite rank by Lemma 83.1. Thus $U_{T'}/U_{T'}^n$ is finite. Every class in A_T lies in its image in $E^\times/E^{\times n}$, proving finiteness.

The indicated field is the splitting field of the finitely many equations $X^n = a$ together with $X^n = 1$. Changing a representative or a root multiplies a chosen root by a base-field element or an n-th root of unity. At a prime $v \notin T$ of characteristic different from ℓ, take $\pi \in E^\times$ generating the maximal ideal in the local discrete valuation ring of Proposition 82.3. For purposes of testing ramification at this prime, replace each a by $a/\pi^{v(a)}$. This is the same class and is a unit there. Lemma 599.1 now proves unramifiedness. At a completely split w, a completion of V equals E_w by Lemma 612.1. All the chosen roots and their changes therefore lie in E_w. □

Theorem 619.2. *Let E be the CM field, c its complex conjugation extended by complex conjugation on $\Omega \subset \mathbb{C}$, and T a conjugation-stable finite set containing the places above ℓ. Fix $n = \ell^N$, $N \geq 1$. Let $z \in G_E$ lie in the closed normal subgroup generated by inertia at the finite places outside T, and let W be any neighborhood of z in G_E. Infinitely many degree-one primes $w \neq \bar{w}$ of E, away from any prescribed finite set of rational primes, admit the following data. Their norms satisfy $Nw \equiv 1 \bmod n$. There are a residue generator t_w, an additive coordinate $e : C(\{w\}) \to \mathbb{Z}/n\mathbb{Z}$, and a continuous character $h : G_E \to \mathbb{Z}/n\mathbb{Z}$ such that*

$$e(u_{t_w}) = 1, \qquad \lambda_h(x) = e(x/\bar{x}) \quad (x \in J_{E,f}).$$

Here the finite cyclic target of λ_h is identified as in Theorem 617.3. The character h is anticyclotomic, is unramified away from $w, \bar{w}$, and vanishes on the full decomposition groups at T. For one valuation above w there are an inertia element γ and an arithmetic Frobenius lift $\phi \in W$ with $h(\gamma) = 1$, $h(\phi) = 0$. The same conclusions hold for any specified generator t_w of the residue unit group at the selected prime.

Proof. Choose a finite normal extension B/E such that the coset of z modulo G_B lies in W. Such cosets form a neighborhood basis by the topology on the absolute Galois group in Definition 601.1. Use V of Lemma 619.1. Every inertia group outside T fixes V, by that lemma. Its kernel is closed and normal, so z fixes V. Apply Theorem 220.2 to BV/E and the restriction of z. Exclude T, the given finite set and the rational primes ramified in a normal closure of $BV/\mathbb{Q}$, as permitted by Lemma 219.3. For each resulting prime w choose the valuation giving exactly this Frobenius on BV. It splits in V, so the distinct reductions of μ_n give $n \mid Nw - 1$. Its rational prime norm and unramifiedness give $w \neq \bar{w}$, as in Theorem 596.2.

Form the finite quotient of $J_{E,f}$ by the subgroup generated by

$$E^{\times}, \quad J_{E,f}^{n}, \quad \prod_{v \in T} E_v^{\times}, \quad (1 + \mathfrak{p}_w) \prod_{v \neq w} O_{E,v}^{\times}.$$

Here $\mathfrak{p}_w$ denotes the maximal ideal of $O_{E,w}$. This quotient is killed by n and is a quotient of the finite ray class group modulo $p = Nw$, by Lemma 614.1 and the finiteness in Theorem 595.2. It has the discrete quotient topology. Fix any generator t_w of $k_w^{\times}$ and lift it to a local unit t. Its idele supported at w has order exactly n in the quotient. Indeed, if its j-th power is trivial there, a principal factor $a \in E^{\times}$ of a corresponding factorization has all valuations outside T divisible by n, also at w, by taking valuations of this idele. By complete splitting in V, a is an n-th power in $E_w^{\times}$. At w the factorization now reads $t^j = y^n u$ with $u \in 1 + \mathfrak{p}_w$. The valuation gives $y \in O_{E,w}^{\times}$. Thus t_w^j is an n-th power in the cyclic group $k_w^{\times}$ of order divisible by n, forcing $n \mid j$.

Send the class of t to $1/n \in T_\ell$. Extend the character to the whole quotient by adjoining generators as in Lemma 618.1. The image is killed by n, so the identification $b/n \mapsto b \bmod n$ gives an idele character θ into $\mathbb{Z}/n\mathbb{Z}$. It is trivial at T and its only nontrivial local unit character is at w, where it is $\log_{t_w}$ on residues modulo n. Restrict θ to T_f to obtain e: the defining subgroup $T(E)K_T(\{w\})$ of Definition 589.1 lies in its kernel. The torus element u_{t_w} has value one. On finite ideles put $\theta'(x) = \theta(x/\bar{x})$. It vanishes at T by conjugation stability. Its unit restriction at w is again $\log_{t_w}$, since θ is trivial on units at $\bar{w}$. It is trivial on units away from the pair.

By Theorem 617.3 there is a unique h with $\lambda_h = \theta'$. Naturality under conjugation and injectivity give $h(cgc^{-1}) = -h(g)$ from $\theta'(\bar{x}) = -\theta'(x)$. Theorem 618.2 gives all the stated decomposition and inertia conditions, in particular surjectivity on the chosen inertia group. Choose γ with $h(\gamma) = 1$. Lift arithmetic Frobenius above w as in Lemma 603.1. Its image on B is the restriction of z by choice of the valuation. Multiply it by an inertia element canceling its h-value. The whole inertia group fixes B here, so its corrected lift ϕ still belongs to W. This yields all the assertions. □

620. Preserving the marked residue when prescribing Frobenius

Lemma 620.1. *Let E be totally imaginary, p a rational prime unramified in E, and $v \mid p$ a place with $E_v = \mathbb{Q}_p$. Let $m \geq 2$ be divisible by p exactly once and $\theta : C_m(E) \to T_r$ a character for any prime r. Via Lemma 614.1 also regard θ as a finite idele character. Its restriction to $O_{E,v}^{\times}$ induces an additive character $\delta : \mathbf{F}_p^{\times} \to T_r$. Let h_θ correspond to θ by Theorem 617.3. For any z in inertia above v, let $a_p(z) \in \mathbf{F}_p^{\times}$ denote its exponent on μ_p. Then*

$$h_\theta(z) = -\delta(a_p(z)).$$

Proof. Since $mO_{E,v} = pO_{E,v}$, the local principal units lie in the kernel. Set $k(g) = \delta(a_p(g))$, where the exponent on μ_p is defined on all of G_E. At good primes its arithmetic Frobenius values come from the ideal character $\mathfrak{a} \mapsto \delta(N\mathfrak{a} \bmod p)$, as in Lemma 618.1. The positive norm congruence and ray class uniqueness in that proof give this same formula for λ_k on $C_p(E)$. For a local unit u at v, choose $b \in O_E$ congruent to u^{-1} modulo v and one

at all other primes over p, by the ideal Chinese remainder argument of Theorem 596.2. Then (b) is prime to p and represents the idele supported at v with value u in $C_p(E)$, by Lemma 614.1. Since p is unramified, reduction of the multiplication determinant for b is the product of the residue norms. It is the residue of u^{-1}, by $E_v = \mathbb{Q}_p$. Consequently $\lambda_k(u) = -\delta(u \bmod v)$. The sum $\lambda_{h_\theta+k}$ kills all local units. Theorem 618.2, with coefficient prime r, gives $(h_\theta + k)(z) = 0$. □

Theorem 620.2. *In Theorem 596.2 choose $p = g_i \neq \ell$ and $v = w_*$. Let z be in inertia above v with $a_p(z) = t_*$ a generator of $\mathbf{F}_p^\times$. Let W be any neighborhood of z in G_E. Take a conjugation-stable finite T containing the places above ℓ, with $v, \bar{v} \notin T$, and take $N \geq 1$. For every integer $K > v_\ell(p-1)$, infinitely many primes w_0 simultaneously satisfy the Euler-factor conclusions of that theorem for K, t_* and all the conclusions of Theorem 619.2 for T, N, W. They may avoid any given finite set of rational primes. Such z exists at the marked place.*

Proof. At a degree-one place, the inertia action on μ_p surjects onto $\mathbf{F}_p^\times$: with ϵ a primitive p-th root, $v_p(1-\epsilon) = 1/(p-1)$ as computed in Lemma 610.1. Over the union of unramified local fields in Lemma 113.1 the powers $(1-\epsilon)^j$, $0 \leq j < p-1$, have distinct valuations modulo $\mathbb{Z}$ and hence are linearly independent. The degree is at most $p-1$ by the cyclotomic polynomial, hence is exactly $p-1$ there. All conjugates are in the same field, and extending its automorphisms to the local closure while fixing the unramified union gives inertia elements, as in Lemma 603.1. This proves existence.

Use $m = \operatorname{lcm}(\ell^K, g_0, g_1)$. The primes g_0, g_1 are distinct and completely split in E by Lemma 13.2, so p occurs with multiplicity one. Put $C = C_m(E)$. For every character $\theta : C \to T_r$ of prime-power order dividing $|C|$, with $r \mid |C|$, use the Galois character h_θ of the preceding lemma. There are finitely many such characters, as their images are in the finite groups $T_r[|C|]$. They separate points of C. Indeed Lemma 7.1 supplies a complex character separating any nonidentity element. Identify its finite cyclic image with a subgroup of $\mathbb{Q}/\mathbb{Z}$ by exponentiation, and project to a nonzero prime-primary component, using the integer Chinese remainder theorem on the image order. The case $C = 1$ requires no characters.

Replace W by the intersection of W with the conditions $h_\theta(g) = h_\theta(z)$ for these finitely many θ. This is still a neighborhood of z. Apply Theorem 619.2. Its normal subgroup hypothesis holds because $v \notin T$. Exclude the primes over m, all the prescribed exclusions, and the finitely many ramified primes of the fields of the h_θ. For each resulting w_0, ϕ the uniformizer evaluation of Lemma 618.1 and Lemma 620.1 give

$$\theta([w_0]) = h_\theta(\phi) = h_\theta(z) = -\delta_\theta(t_*) = \theta(\mathrm{loc}_v(\widetilde{t_*^{-1}})).$$

Here $\widetilde{t_*}$ is any unit lift at v and loc_v inserts it at the indicated finite place. Separation gives equality of the two classes in C. By Lemma 614.1 this ray class has a representative generated by an element congruent to t_* at v, and to one modulo $mO_{E,u}$ at all other $u \mid m$. Such an element exists by the ideal Chinese remainder calculation in Theorem 596.2. Equality in the ray group therefore writes $w_0 = (\pi)$ with π satisfying the same congruences: the ideal quotient has a generator locally one modulo m. Since all finite valuations of π are nonnegative, $\pi \in O_E$ by Proposition 82.3. The norm and archimedean weight calculations for this π in Theorem 596.2 give $Nw_0 \equiv 1 \bmod \ell^K$ and the indicated Euler-factor valuations for every critical pair, integer exponent, sign and embedding. All auxiliary-character conclusions, including $\phi \in W$, survive the restriction of the neighborhood. □

621. Extending a system of auxiliary primes with prescribed evaluations

Proposition 621.1. *Use the hypotheses of Theorem 620.2. Let R_i be finite rings and $\rho_i : G_E \to \mathrm{GL}_{r_i}(R_i)$ continuous for a finite set of indices. Put $g_+ = g$, $g_- = (cgc^{-1})^{-1}$. The prime w_0 in that theorem can be chosen with $\rho_i((\phi_0)_b) = \rho_i(z_b)$ for every i and both $b = +, -$. The ranks, cardinalities and, for modules over a finite quotient of a discrete*

valuation ring, the lengths of the kernels of the corresponding endomorphisms $\rho_i((\phi_0)_b) - 1$ are then equal to those for z_b.

Proof. Inversion and conjugation by c are continuous in the finite-extension topology since a finite normal extension has a finite normal conjugate. Thus all the finitely many equalities on matrix entries define an open neighborhood of z. Intersect it with the prescribed neighborhood and apply Theorem 620.2. The endomorphisms in each comparison are equal on the same module, so their kernels coincide. □

Theorem 621.2. *Fix $n = \ell^N,\ N \geq 1$, and a system of paired primes P with distinguished w_0 as in Definition 604.2. Use the conventions $g_+ = g,\ g_- = (cgc^{-1})^{-1}$, $\gamma_{u,b} = (\gamma_u)_b,\ \phi_{u,b} = (\phi_u)_b$. Suppose each coordinate h_u is anticyclotomic, unramified outside $u, \bar{u}$, and $\lambda_{h_u}(x) = e_u(x/\bar{x})$ for a coordinate e_u of Definition 598.1 with $e_u(u_{t_u}) = 1$. Suppose the evaluations at old ordinary primes have $e_u(b_v) = h_u(\phi_v)$ for $v \in P \setminus \{w_0\}$. Here b_v denotes the uniformizer at v divided by its conjugate, as in that definition. Let T be a conjugation-stable finite set containing the places over ℓ and all old ordinary pairs, disjoint from $w_0, \bar{w}_0$. Write D_T for the closed normal subgroup generated by all inertia groups outside T. Put $z_0 = \gamma_{w_0,+}$, and let*

$$H \ \subseteq \ D_T \cap \overline{[G_E, G_E]}$$

be a subgroup. For any $a \in H$ and any finite list of locally constant functions on G_E with finite targets, one can add a new pair $w, \bar{w}$ with $\phi_{w,b}$ having exactly the listed values of $(z_0 a)_b$, for both b. One may omit any specified finite set of rational primes. All old coordinate evaluations continue to hold and the enlarged system fulfills

$$h_u(\gamma_{v,b}) = \mathbf{1}_{u=v},$$
$$h_u(\phi_{v,b}) = e_u(b_v) = \mathbf{1}_{u=w_0} \qquad (v \neq w_0).$$

The first line holds for every index and every prime in the enlarged system. Every coordinate still satisfies the idele character identity above. In particular, for a row i with $\rho_i((z_0 a)_b) = 1$ at the coefficient precision ℓ^M for both signs, one can give the new prime any slope at this row and slope zero at the other rows as far as the matrix congruences of Definition 604.2 are concerned. Here $\rho_i \bmod \ell^M$ is continuous with finite image, and the new pair is chosen outside its ramification.

Proof. The element $z_0 a$ belongs to D_T since z_0 is inertia at a prime outside T. Every old h_u kills a, by continuity and its abelian target, so $h_u(z_0 a) = h_u(z_0) = \mathbf{1}_{u=w_0}$. In a neighborhood of $z_0 a$ impose these old character values and all desired observations, composing each observation also with $g \mapsto g_-$. Both signs thus give finitely many open conditions. Include the matrix entries of the row at the required precision, when present. Theorem 619.2 supplies a new pair, $e_w, h_w, \gamma_w, \phi_w$. Exclude the old primes, their conjugates, all base sets in the cocycle conditions and the stated ramification. The new character has $h_w(\gamma_w) = 1,\ h_w(\phi_w) = 0$. Its full decomposition restrictions at the old ordinary places vanish. All old characters are unramified at the new pair, and the new character is unramified at all old pairs. This proves the two Galois arrays on the plus side. On the minus side the two sign changes cancel, since each character is anticyclotomic.

For $u \neq v$ with v ordinary, the old or new character is unramified at v. Applied to its uniformizer idele, the identity $\lambda_{h_u}(x) = e_u(x/\bar{x})$ and Lemma 618.1 give $e_u(b_v) = h_u(\phi_v)$. At $v = w$ choose the local uniformizer defining b_w so that $e_w(b_w) = 0$: multiplying it by a unit of residue t_w increments the value by one as $e_w(u_{t_w}) = 1$. This change leaves all other values there fixed by unramifiedness. Every old b_v stays fixed. All signs and equalities in the display follow. If the two matrices at the chosen row are identities, their deviations from the identity vanish at the required precision after multiplication by any slope. At the other rows multiplication by zero gives the required congruence. □

622. Power residues and prescribed inertia values

Lemma 622.1. *Let L be totally imaginary, $m = \ell^a$, $a \geq 1$, and fix a primitive m-th root $\zeta \in L$. Let P be a degree-one prime of L over $p \neq \ell$, with p unramified in L. Thus $L_P = \mathbb{Q}_p$ and $m \mid p-1$. Inertia above P contains an element δ_P satisfying $\delta_P(p^{1/m})/p^{1/m} = \zeta$. For $x \in O_{L,P}^{\times}$ define $b_P(x) \in \mathbb{Z}/m\mathbb{Z}$ by*

$$\overline{\zeta}^{\,b_P(x)} = \overline{x}^{\,(p-1)/m}.$$

Bars here mean residues in the chosen local closure. For every continuous $k : G_L \to \mathbb{Z}/m\mathbb{Z}$,

$$\lambda_k(\mathrm{loc}_P x) = -b_P(x)\, k(\delta_P).$$

If $x \in L^{\times}$ is a unit at P, every arithmetic Frobenius lift ϕ_P satisfies $\phi_P(x^{1/m})/x^{1/m} = \zeta^{b_P(x)}$.

Proof. Reduction is injective on μ_m by the distinct-residues argument of Lemma 618.1. It gives all the m-th roots in the cyclic residue group. Over the union of the unramified local fields in Lemma 113.1, the first m powers of $p^{1/m}$, starting at exponent zero, have distinct valuations modulo $\mathbb{Z}$ and are linearly independent. Its extension there has degree m, contains every conjugate since μ_m is in the base, and has the automorphism sending the chosen root to ζ times itself. Extend it to the local algebraic closure fixing that union. It is an inertia element of G_L by Lemma 603.1. The ratio is independent of the root because inertia fixes μ_m.

Write η for a primitive p-th root. The valuations of $(1-\eta)^{(p-1)/m}$ and $p^{1/m}$ are both $1/m$, by Lemma 610.1. Their quotient is a unit with residue fixed by δ_P. For $j = a_p(\delta_P)$, the exponent on μ_p, the ratio $(1-\eta^j)/(1-\eta)$ reduces to j by the geometric sum. Consequently $j^{(p-1)/m} = \overline{\zeta}$. The kernel of $t \mapsto t^{(p-1)/m}$ on $\mathbf{F}_p^{\times}$ is its subgroup of m-th powers by cyclicity. Thus $\bar{x}/j^{b_P(x)}$ is an m-th power.

Every principal unit at every prime above p is an m-th power by the simple-root lifting at one of Lemma 113.1, since $p \nmid m$. Thus it is killed by λ_k. Choose a rational ray modulus for λ_k by Lemma 614.1 and replace its p-part by p. The character still factors through this modulus since p is unramified and its local principal units have just been treated. By Lemma 620.1, λ_k on a unit lifting j has value $-k(\delta_P)$. Its value on x follows from the preceding m-th power congruence and triviality on principal units. Finally $x^{1/m}$ is a local unit. The ratio of its Frobenius image to itself is in μ_m and reduces to $(\overline{x^{1/m}})^{p-1} = \bar{x}^{(p-1)/m}$. Injectivity on μ_m gives the last assertion. □

Theorem 622.2. *Let L, m, ζ be as in Lemma 622.1. Let T be a finite set of finite places generating the ideal class group of L, and let $\mathcal{P}$ be a finite set disjoint from T of primes as in that lemma. Choose δ_P as there. A tuple $d = (d_P)_{P \in \mathcal{P}}$ in $(\mathbb{Z}/m\mathbb{Z})^{\mathcal{P}}$ is realized by a continuous character $k : G_L \to \mathbb{Z}/m\mathbb{Z}$ with the properties*

$$k(D_v) = 0 \quad (v \in T), \qquad k(I_v) = 0 \quad (v \notin \mathcal{P}), \qquad k(\delta_P) = d_P \quad (P \in \mathcal{P})$$

if and only if

$$\sum_{P \in \mathcal{P}} b_P(u) d_P = 0 \quad \text{for every } u \in U_T.$$

The character with these properties is unique.

Proof. Write $J = J_{L,f}$ and $B = \prod_{v \in T} L_v^{\times} \prod_{v \notin T} O_{L,v}^{\times}$. One has $J = L^{\times} B$, $L^{\times} \cap B = U_T$. Indeed the valuations outside T of an idele specify a fractional ideal which can be made principal by multiplying by an ideal supported on T. Division by its generator makes all those valuations zero. The intersection formula is the definition of U_T in Lemma 616.1.

By Theorem 618.2 and Lemma 622.1, any character in the assertion has

$$\lambda_k(x) = -\sum_{P \in \mathcal{P}} b_P(x_P) d_P \quad (x \in B).$$

In obtaining this identity for the full product of units, finite-support tuples are dense in the product: each basic open set restricts only finitely many coordinates. Equality on it extends to all of B by continuity. The principal ideles give the necessary condition.

Conversely, the displayed expression on B is a continuous homomorphism because each b_P is additive on products and depends only on the residue. If it vanishes on U_T, it extends uniquely to $J = L^\times B$ by giving value zero on $L^\times$. This extension is continuous since B is an open subgroup of J. Theorem 617.3 gives a unique k. Theorem 618.2 gives all required vanishing conditions. At each P, the first lemma computes the local unit values as $-b_P(x_P)k(\delta_P)$, and b_P is surjective by cyclicity of $\mathbf{F}_p^\times$. Comparison recovers $k(\delta_P) = d_P$. $\square$

623. Power residues for cocycles with coefficients in a finite module

Lemma 623.1. *Let G be profinite, U open normal, $\Delta = G/U$, and V a finite additive module with action ρ of Δ, also regarded as a G-action. For a continuous homomorphism $f : U \to V$ and representatives $r_d \in G$ of $d \in \Delta$ put*

$$C_f(g) = \sum_{d\in\Delta} \rho(\bar{g}d) f(r_{\bar{g}d}^{-1} g r_d), \qquad \bar{g} = gU.$$

This is a continuous cocycle. On U its value is $\sum_d \rho(d) f(r_d^{-1} g r_d)$. Changing the representatives changes the cocycle by a coboundary, a function $g \mapsto (\rho(g) - 1)v$ with $v \in V$. If $D \le G$ and f kills $U \cap r_d^{-1} D r_d$ for every d, then C_f restricts to a coboundary on D.

Proof. The element $b(g, d) = r_{\bar{g}d}^{-1} g r_d$ lies in U. The formula $b(gg', d) = b(g, \bar{g}'d)b(g', d)$ and reindexing the sum give $C_f(gg') = C_f(g) + \rho(g)C_f(g')$. All terms are continuous on each of the finitely many open cosets of U, and the stated restriction follows by putting $\bar{g} = 1$. Replacing r_d by $r_d u_d$ changes $f(b(g, d))$ by $f(u_d) - f(u_{\bar{g}d})$. The total difference is $(\rho(g) - 1)\sum_d \rho(d) f(u_d)$. On any orbit of D on Δ, choose one representative r_{d_0}. For each d in this orbit take the new representative $s_d r_{d_0}$ with $s_d \in D$; it exists by definition of the orbit. For $g \in D$ the element inside f for this choice lies in $U \cap r_{d_0}^{-1} D r_{d_0}$. All summands vanish. The change-of-representatives formula gives the assertion for the original representatives. $\square$

Theorem 623.2. *Let L/E be finite Galois with E totally imaginary, $\mu_m \subset L$, $m = \ell^a$, $a \ge 1$, and ζ a fixed primitive m-th root in L. Write $\Delta = \mathrm{Gal}(L/E)$ and $U = G_L$. Let V be a finite abelian group killed by m, with action ρ of Δ. Choose a finite set T of places of E containing the ramified places of L/E, such that the primes $\widetilde{T}$ above T generate $\mathrm{Cl}(L)$. Such sets exist by the ideal factorization of representatives for the finite class group. Let w_i, $i \in I$, be distinct degree-one primes of E outside T which split completely in L/E, over rational primes $p_i \ne \ell$ unramified in L. Fix a valuation above each w_i, its prime P_i in L, an arithmetic Frobenius lift ϕ_i and δ_i as in Lemma 622.1. Put $b_i(u) = b_{P_i}(u)$ on $U_{\widetilde{T}}$. Let $\mathcal{Z}$ consist of the continuous V-valued cocycles on G_E unramified outside $T \cup \{w_i\}$ whose restrictions to decomposition groups at T are coboundaries. Here unramified means vanishing on inertia. Then each $x \in \mathcal{Z}$ satisfies*

$$\sum_{\substack{i\in I\\ d\in\Delta}} b_i(du)\rho(d)^{-1} x(\delta_i) = 0 \quad (u \in U_{\widetilde{T}}),$$

where in each double sum i ranges over I and d over Δ. If all $x(\delta_i) = 0$, then $x|_U = 0$. Conversely, suppose $F_{i,d} \in \mathbb{Z}/m\mathbb{Z}$ satisfy

$$\sum_{i,d} b_i(du) F_{i,d} = 0 \quad (u \in U_{\widetilde{T}}).$$

For every $v_0 \in V$ there exists $x \in \mathcal{Z}$ with

$$x(\delta_i) = \sum_{d\in\Delta} F_{i,d}\rho(d)v_0 \quad (i \in I).$$

Proof. Relative to lifts $r_d \in G_E$ of d, the primes of L over w_i are the distinct $P_{i,d} = r_d^{-1}P_i$, by complete splitting and Lemma 612.1. Use at $P_{i,d}$ the inertia element $r_d^{-1}\delta_i r_d$ and the primitive root $d^{-1}\zeta$. The former multiplies a chosen m-th root of p_i by the latter; transporting the choices of root has no effect since inertia fixes μ_m. The corresponding symbol on u is $b_i(du)$ by transporting the defining residue equation. Theorem 622.2 applies with these choices: its local formula is proved place by place for any primitive root, and its global condition is the sum of those local formulas on each unit. Thus the test for these transported inertia values is precisely the indicated sum with $b_i(du)$.

For $x \in \mathcal{Z}$, its restriction to U is an additive homomorphism since U acts trivially on V. The cocycle law twice gives $x(gag^{-1}) = \rho(g)x(a)$ for $a \in U$. This restriction kills decomposition at every prime above T: a coboundary on the base decomposition group vanishes on its intersection with U, and the conjugation formula handles every choice above the base prime. At all other primes outside the displayed list it is unramified. For any additive $\omega : V \to \mathbb{Z}/m\mathbb{Z}$, the scalar test of the preceding paragraph on $k = \omega \circ x|_U$ gives $\sum_{i,d} b_i(du)\omega(\rho(d)^{-1}x(\delta_i)) = 0$. These ω separate points by Lemma 7.1, identifying the m-th roots with $\mathbb{Z}/m\mathbb{Z}$ and using $mV = 0$. This proves the vector identity. If all the displayed values of x at inertia vanish, uniqueness in the same scalar test gives $k = 0$ for every ω, whence $x|_U = 0$.

Conversely, the scalar test realizes $F_{i,d}$ by a single $k : U \to \mathbb{Z}/m\mathbb{Z}$, trivial on decomposition at $\widetilde{T}$. Apply Lemma 623.1 to $f(a) = k(a)v_0$. At a decomposition group D over T the subgroups $U \cap r_d^{-1}Dr_d$ are the corresponding decomposition groups over L, since valuation preservation is the condition in either base field. Thus $C_f|_D$ is a coboundary. At a base prime outside T and the chosen w_i, its inertia lies in U by unramifiedness of L/E. All $k(r_d^{-1}gr_d)$ vanish there by the scalar construction, hence C_f is unramified. At $\delta_i \in U$ the restriction formula of the transfer lemma gives precisely the asserted sum. □

624. Relations at additional split primes

Definition 624.1. Let $E, L, U, \Delta, V, m, \zeta$ be as in Theorem 623.2. Fix a finite set S outside which the action on V is unramified. At each $v \in S$ choose a decomposition group and an additive subgroup $\mathcal{L}_v$ of its continuous cocycles containing every coboundary. For a finite set P away from S, let $Y(P)$ denote the continuous global cocycles on V unramified outside $S \cup P$ with their restrictions at S in these chosen subgroups. Take $T \supset S$ as in the preceding theorem, including the places over ℓ. Write $\widetilde{T}$ for all places of L above T and set

$$K = L(u^{1/m} : u \in U_{\widetilde{T}}), \qquad B = \mathrm{Gal}(K/L), \qquad V^\vee = \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Q}/\mathbb{Z}).$$

Here K/E is finite normal: $U_{\widetilde{T}}$ is finitely generated by Lemma 616.1, it is Δ-stable, and L contains all μ_m. Write $\chi : G_E \to (\mathbb{Z}/m\mathbb{Z})^\times$ for the exponent of its action on ζ.

Theorem 624.2. *For each $b \in B$ one can choose a distinct degree-one prime w_b of E split completely in L and an extended valuation at which the arithmetic Frobenius in K/E is b. They can avoid T and any given finite set of rational primes, including the ramified primes of a normal closure of K over $\mathbb{Q}$. Write $P = \{w_b : b \in B\}$ and choose normalized inertia elements δ_b at these places using ζ as in Lemma 622.1. Suppose $\nu_b \in V^\vee$ satisfy*

$$\sum_{b \in B} \nu_b(y(\delta_b)) = 0 \qquad (y \in Y(P)).$$

Then $x(a) = \nu_{a|_K}$ defines a continuous homomorphism $x : U \to V^\vee$, satisfying

$$x(gag^{-1})(t) = x(a)(\chi(g)\rho(g)^{-1}t) \quad (g \in G_E,\ a \in U,\ t \in V).$$

The assignment $\nu \mapsto x$ is injective. If $v_1, \dots, v_s$ are additional distinct degree-one primes outside $P \cup T$ split completely in L and unramified in K, over rational primes unramified in

L, take arithmetic lifts $\varphi_j \in U$ and normalized inertia elements δ'_j above them. For every $y \in Y(\{v_1, \dots, v_s\})$,

$$\sum_{j=1}^{s} x(\varphi_j)(y(\delta'_j)) = 0.$$

Proof. Apply Theorem 220.2 over E to each $b \in \mathrm{Gal}(K/E)$ in the subgroup B, successively omitting every previously used rational prime. Its restriction to L is the identity, hence the selected prime splits there. For $a \in U$ and $u \in U_{\widetilde{T}}$ write $a(u^{1/m})/u^{1/m} = \zeta^{B_a(u)}$. This exponent is defined independently of the root and depends on a only through $a|_K$. Multiplication of roots and the fact that U fixes ζ show its additivity in the multiplicative variables a, u. For $g \in G_E$ one has

$$B_{gag^{-1}}(u) = \chi(g) B_a(g^{-1}u)$$

by applying g to the root-ratio identity for $g^{-1}u$. Lemma 622.1 identifies the symbol at w_b on these units with B_b.

Take $b_1, b_2 \in B$, $g \in G_E$, and $b_0 = b_1 g b_2 g^{-1}$ in $\mathrm{Gal}(K/E)$. For each u the preceding identities give

$$B_{b_0}(u) - B_{b_1}(u) - \chi(g) B_{b_2}(g^{-1}u) = 0.$$

Use the converse of Theorem 623.2 with coefficients $+1$ at $(b_0, 1)$, -1 at $(b_1, 1)$ and $-\chi(g)$ at $(b_2, \bar{g}^{-1})$, adding the coefficients when indices coincide. The resulting cocycle belongs to $Y(P)$. Indeed its restrictions at T are coboundaries, which gives the conditions at S and vanishing on inertia at $T \setminus S$ since the action is unramified there. Every other required inertia vanishing is part of the theorem. Evaluating the assumed relation on this cocycle for each $t \in V$ gives

$$\nu_{b_0}(t) = \nu_{b_1}(t) + \nu_{b_2}(\chi(g)\rho(g)^{-1}t).$$

With $g = 1$ this proves additivity in b, including $\nu_1 = 0$. With $b_1 = 1$ it gives the equivariance. Composition with $U \to B$ gives continuity and injectivity of the assignment, since every b has a lift.

At an additional v_j, its symbol on the units is B_{b_j}, $b_j = \varphi_j|_K$, by the same local Frobenius calculation. On the enlarged prime set apply the converse of Theorem 623.2 with $+1$ at $(v_j, 1)$, -1 at $(w_{b_j}, 1)$, and with vector $t_j = y(\delta'_j)$. The scalar condition holds because the two symbols coincide. This yields y_j with inertia values $t_j, -t_j$ at the indicated generators and zero at all the others. Put $y' = y - \sum_j y_j$. It satisfies the conditions at S and $T \setminus S$ by the preceding coboundary argument. It vanishes at every δ'_j, hence on all inertia there. For the last implication the function on inertia is additive since inertia acts trivially on V. The local action on a root of p_j of order m, written as multiplication by ζ^u, gives a character of the local decomposition group into $\mathbb{Z}/m\mathbb{Z}$, since the completion contains μ_m. It takes δ'_j to one. The affine representation $g \mapsto (t \mapsto \rho(g)t + y'(g))$ has finite ℓ-group image on inertia, consisting of translations. By Theorem 603.2 its inertia image is generated by the image of δ'_j, which is the identity. The conjugacy and cocycle calculation of Lemma 606.2 gives this vanishing at all extensions of the same prime.

Thus $y' \in Y(P)$. At each old generator y is zero by its ramification condition, so $y'(\delta_b)$ is the sum of the t_j with $b_j = b$. Substituting this in the defining relation for ν proves the final identity. □

625. Detection at a conjugate pair of primes

Lemma 625.1. *Let $w, \bar{w}$ be a degree-one pair over $p \neq \ell$, $m = \ell^a \mid p-1$, $a \geq 1$. Fix $\zeta \in \mu_m$ primitive. If δ at w acts on $p^{1/m}$ by multiplication by ζ, then $c\delta^{-1}c^{-1}$ at the conjugate valuation acts by the same multiplier. Let $h : G_E \to \mathbb{Z}/\ell^N\mathbb{Z}$, $N \geq 1$, take γ in inertia at w with $h(\gamma) = 1$, and use $g_+ = g$, $g_- = (cgc^{-1})^{-1}$. Suppose y' is a cocycle on a finite module*

V killed by m, on which both decomposition groups act trivially. There exists an integer k prime to ℓ with

$$y'(\delta_b) = ky'(\gamma_b) \qquad (b = +, -).$$

Proof. Inertia fixes μ_m and $c(\zeta) = \zeta^{-1}$, so the multiplier on the transported root for $c\delta^{-1}c^{-1}$ is ζ. Changing the chosen radical by a root of unity has no effect. Since the completion contains μ_m, the exponent on $p^{1/m}$ to base ζ defines a local character $\kappa : D_w \to \mathbb{Z}/m\mathbb{Z}$, with $\kappa(\delta) = 1$ as in Lemma 622.1. The map $D_w \to (\mathbb{Z}/m\mathbb{Z}) \times V^2$ given by $\tau \mapsto (\kappa(\tau), y'(\tau_+), y'(\tau_-))$ is a continuous homomorphism. Indeed each restriction of y' is additive by triviality of the actions, and inversion gives negation. Theorem 603.2, applied with $h|_{D_w}$, makes the whole inertia image cyclic generated by the image of γ. If the image of δ is k times this generator, the first coordinate gives $k\kappa(\gamma) = 1 \bmod m$, so $\ell \nmid k$. The other coordinates give the required equations. □

Theorem 625.2. *Use the old auxiliary system and T_*, z_0, H of Theorem 621.2, writing T_* for its set of prescribed local triviality. Take $R = O/\varpi^j O$, $V = R^r$, and a power $m = \ell^a > 1$ annihilating V. Use also the data of Definition 624.1, allowing its set T to be larger and to contain all old prime pairs. Assume L is stable under c, $H \triangleleft G_E$, $cHc^{-1} = H$, $H \subset U = G_L$, $z_0 \in U$. Use χ from that definition as scalars on V. Let $y : G_E \to V$, $\gamma, g_*, s, \alpha, \alpha'$ satisfy the cocycle, scalar, spanning, unit-coordinate and commutator hypotheses of Lemma 608.2 and Theorem 608.3, with $c(g) = cgc^{-1}$. In particular $y(\gamma) = 0$. Choose the split test set P of Theorem 624.2 outside the old system. Assume the following condition at new auxiliary pairs. For each $a_0 \in H$, every extension of the old system given by Theorem 621.2 arbitrarily near z_0a_0, outside $P \cup T$ and its conjugates, with $w, \bar{w}$ split in L and unramified in K, satisfies*

$$\forall t \in R \quad \exists y' \in Y(\{w, \bar{w}\}) \quad y'(\gamma_{w,b}) = ty(\phi_{w,b}) \qquad (b = +, -).$$

It suffices that such extensions satisfying the displayed condition exist in every neighborhood of each z_0a_0. Then the number of tuples ν annihilating the split test values in Theorem 624.2 is at most

$$|\ker((\alpha - \chi(c(g_*))\alpha') : V \to V)|.$$

Proof. For a given tuple let $x : U \to V^\vee$ be its equivariant extension in Theorem 624.2. Fix $a_0 \in H$ and put $z = z_0a_0$. Both z and $c(z)$ belong to U. Impose at the new Frobenius the same values as at z for y, x and the functions obtained by conjugation by c. For x use any extension by zero outside the open and closed U. These are locally constant observations. Also prescribe the restrictions to L so that $\phi_w, c(\phi_w) \in U$. A continuous homomorphism into a finite group can ramify at only finitely many rational primes, by factoring through a finite normal field and applying Lemma 219.3; omit the ramified primes of K and of its conjugate. Theorem 621.2 allows these observations and exclusions. Under the stated condition take such a pair and y' for the given t.

Take δ normalized by ζ at w and $c\delta^{-1}c^{-1}$ at $\bar{w}$. Use the arithmetic lifts $\phi_w, c(\phi_w)$, respectively. Since both places split in L, their entire decomposition groups act trivially on V. Apply the final assertion of Theorem 624.2 to y'. By Lemma 625.1 the resulting identity is

$$k\big(x(\phi_w)(ty(\phi_w)) + x(c(\phi_w))(ty(c(\phi_w)^{-1}))\big) = 0.$$

All these values lie in $(\mathbb{Q}/\mathbb{Z})[m]$, so multiplication by the integer k prime to ℓ can be canceled. The cocycle inverse formula gives $y(c(\phi_w)^{-1}) = -y(c(\phi_w))$ by trivial action. The prescribed observations therefore give

$$x(z)(ty(z)) = x(c(z))(ty(c(z))).$$

This holds for every a_0, t. The action formula in the split test theorem is exactly the equivariance of Definition 608.1. Thus x satisfies that definition's isotropy condition. Theorem 608.3 bounds the number of these x by the asserted kernel, and the map from the tuples to x is injective. □

626. Counting the choices of local coboundaries

Definition 626.1. Use Definition 624.1. Put $G = G_E$, $f_E = [E : \mathbb{Q}]/2$. Choose a decomposition subgroup D_v at each $v \in T$, using the given one for $v \in S$. Form the disjoint union of G-sets

$$D = \coprod_{v \in T} G/D_v \ \sqcup \ \coprod_{j=1}^{f_E} G.$$

For split primes P outside T, write $\mathcal{X}(V)$ for the additive group of pairs (y, b). Here $y : G \to V$ is a continuous cocycle unramified outside $T \cup P$, and $b : D \to V$ satisfies

$$b(gd) = \rho(g)b(d) - y(g) \qquad (g \in G,\ d \in D).$$

The notation retains T, D, P as fixed. For any orbit with base point d, its function is determined by $b(d)$. A prescribed $b(d) = t$ works exactly if $y(g) = (\rho(g) - 1)t$ on the stabilizer of d: this condition makes the formula independent of the representative for its orbit point, by the cocycle law. Such a function is continuous in the quotient topology on each orbit.

Proposition 626.2. *Let P be the split test set of Theorem 624.2 and $\mathcal{N}$ its group of annihilating tuples. Then the groups counted below are finite and*

$$|Y(\varnothing)|\,|V|^{|S|+f_E} \ \le \ |\mathcal{N}|\,\frac{|\mathcal{X}(V)|}{|V|^{|P|}} \prod_{v \in S} |\mathcal{L}_v|.$$

If $g_1 \in G$ has $\rho(g_1) - 1$ invertible on V and $X \subseteq Y(\varnothing)$ consists of cocycles zero at g_1, then

$$|X|\,|V|^{|S|+f_E} \ \le \ |\mathcal{N}|\,\frac{|\mathcal{X}(V)|}{|V|^{|P|+1}} \prod_{v \in S} |\mathcal{L}_v|.$$

Proof. There are finitely many continuous cocycles on V unramified off a given finite set. Their restrictions to G_L are homomorphisms unramified off the primes above that set and the ramification of L/E. Finitely many additive characters $\omega : V \to \mathbb{Z}/m\mathbb{Z}$ separate points by Lemma 7.1. After identifying the cyclic target with μ_m in L, each ω gives only finitely many such restrictions by Lemma 601.2. Values at finitely many representatives for $\mathrm{Gal}(L/E)$ then determine the cocycle by its law. The choices of b are finite by the orbit criterion in Definition 626.1.

For the local finiteness assertion, the action on V at v factors through the finite local Galois extension L_P/E_v for $P \mid v$, by Lemma 612.1. Over L_P each additive character of a cocycle takes values in μ_m with trivial action and is determined by a class in $L_P^\times / L_P^{\times m}$. Indeed the independence-of-embeddings proof in Lemma 601.2 applies over any characteristic-zero field containing μ_m to each finite cyclic quotient. The local quotient is finite by the uniformizer and logarithm calculation in Lemma 616.1. The absolute decomposition group agrees with the local absolute group: the density argument of Lemma 603.1 applies after replacing $\mathbb{Q}_p$ there by E_v, since E is dense in E_v and the approximation of monic polynomial coefficients uses only that density. Finitely many coset representatives and the separating ω give the local finiteness as well.

Consider the homomorphism on $Y(P) \times V^{|T|+f_E}$ obtained at $v \in S$ by taking $y|_{D_v} - d_0 t_v$ and at $v \in T \setminus S$ by taking $y(\phi_v) - d_0 t_v(\phi_v)$, where $d_0 t_v(g) = (\rho(g) - 1)t_v$. The target has order $|V|^{|T|-|S|} \prod_{v \in S} |\mathcal{L}_v|$ since these subgroups contain the coboundaries. Here ϕ_v lifts arithmetic Frobenius. One may choose it in a single finite quotient containing the actions ρ and all the finitely many affine actions $g \mapsto (t \mapsto \rho(g)t + y(g))$ for $y \in Y(P)$. These are unramified at $T \setminus S$. The decomposition group surjects onto the finite stabilizer, by valuation conjugacy in the proof of Lemma 603.1 and extension of embeddings. The Frobenius generates the finite unramified local group by Theorem 611.3. For $v \in T \setminus S$, vanishing of the displayed difference at that element therefore gives vanishing on all of D_v, by the cocycle law. The orbit criterion identifies the kernel of the homomorphism with

$\mathcal{X}(V)$. In the reverse direction any member of $\mathcal{X}(V)$ satisfies the conditions at S and is unramified at $T \setminus S$ because a coboundary vanishes on inertia acting trivially. Consequently

$$|Y(P)|\,|V|^{|S|+f_E} \le |\mathcal{X}(V)| \prod_{v\in S} |\mathcal{L}_v|.$$

Evaluation on the selected δ_b maps $Y(P)$ to V^P with kernel $Y(\varnothing)$, by the normalized inertia argument at the end of Theorem 624.2. Its image has order $|V|^{|P|}/|\mathcal{N}|$, since the annihilators are all characters of the quotient by that image. Use Lemma 7.1 twice, identifying $\mathbb{Q}/\mathbb{Z}$ with the roots of unity under exponentiation. Substitution in the preceding inequality proves the first claim. Every global coboundary belongs to $Y(\varnothing)$, since the base action is unramified away from S. Its values at g_1 fill V by invertibility. Thus evaluation at g_1 has kernel of order $|Y(\varnothing)|/|V|$. This gives the second inequality. □

Lemma 626.3. *At a degree-one prime away from ℓ, suppose the decomposition action on V is unramified, and write γ_b, ϕ_b as in Definition 604.2 for one side of a pair. Let t be a scalar such that $t(\rho(\phi_b) - 1) = 0$ on V. The local continuous cocycles satisfying $y(\gamma_b) = ty(\phi_b)$ contain all local coboundaries and have order at most $|V|$.*

Proof. A coboundary vanishes on the inertia element and its value on ϕ_b is killed by t. Evaluation at ϕ_b is injective on the indicated subgroup: a zero there gives zero at γ_b. The affine inertia action consists of translations of ℓ-power order, and its image is generated by the image of γ_b by Theorem 603.2 applied to the surjective inertia coordinate. Thus the cocycle vanishes on inertia. Its finite affine image on the decomposition group now comes from an unramified finite quotient, generated by the image of the chosen Frobenius or its inverse by Theorem 611.3. Since that image fixes zero, the cocycle vanishes throughout. The cardinality bound follows by injection in V. □

627. An equivariant count for the unit lattice

Lemma 627.1. *Let $k = \mathbf{F}_\ell$, let Δ be a finite group and let V be a finite-dimensional k-space with Δ-action ρ. Suppose that for a k-linear endomorphism J of V,*

$$\sum_{d\in\Delta} \rho(d) J \rho(d)^{-1} = 1_V.$$

For every finite $k\Delta$-module A, the sums $\sum_d \rho(d)F(d^{-1}u)$ for linear $F : A \to V$ give all equivariant linear maps $A \to V$. An equivariant map from a submodule of A extends equivariantly to A. If $C_1 \hookrightarrow C_2$ is an equivariant embedding of free abelian lattices of equal finite rank and finite index, then

$$|\operatorname{Hom}_\Delta(C_1/\ell C_1, V)| = |\operatorname{Hom}_\Delta(C_2/\ell C_2, V)|.$$

Here and below $\operatorname{Hom}_\Delta$ denotes additive equivariant maps, equivalently k-linear ones on spaces killed by ℓ. The trace hypothesis holds for direct summands of a finite free $k\Delta$-module and for every module if $\ell \nmid |\Delta|$.

Proof. Given an equivariant map f, take $F = Jf$ and use the trace identity. For the extension assertion, extend Jf to an arbitrary k-linear map on A by extending a basis, and take the same sum. The sum is equivariant by reindexing. It restricts to f by the trace identity. In particular the function

$$d(A) = \dim_k \operatorname{Hom}_\Delta(A, V)$$

is additive in exact sequences of finite $k\Delta$-modules: restriction is onto, and its kernel consists of maps factoring through the quotient.

Put $D = C_2/C_1$. There is an exact sequence

$$0 \to D[\ell] \to C_1/\ell C_1 \to C_2/\ell C_2 \to D/\ell D \to 0.$$

The first map represents a class by $b \in C_2$ and sends it to $\ell b \bmod \ell C_1$. Its injectivity uses the absence of torsion in C_2. The kernel and image statements follow by lifting classes from the indicated quotients. For the finite group D only its ℓ-primary component contributes at either end, by integer Chinese remaindering on its exponent. On that component set $D_j = \ell^j D/\ell^{j+1}D$. The map $\ell : D_j \to D_{j+1}$ is onto. Its kernel is the image of $D[\ell] \cap \ell^j D$: if $\ell b \in \ell^{j+2}D$, subtract from b an element of $\ell^{j+1}D$ with the same ℓ-multiple. Thus these kernels are the successive quotients of the filtration of $D[\ell]$ by its intersections with $\ell^j D$. Additivity of d and telescoping give $d(D[\ell]) = \sum_j (d(D_j) - d(D_{j+1})) = d(D/\ell D)$. The exact sequence and cardinality $|k|^{d(-)}$ prove the lattice assertion.

On the regular module $k\Delta$, projection onto the coefficient at the identity, with image its span, has conjugates the coordinate projections. Their sum is the identity. Take these projections on each summand of a free module. Composing with an equivariant inclusion and retraction gives the formula on a direct summand. When $|\Delta|$ is invertible, $J = |\Delta|^{-1} 1_V$ works. □

Theorem 627.2. *Let $L/E, \Delta, \widetilde{T}$ be as in Definition 624.1, with $\mu_\ell \subset L$. Write D_L for the disjoint union of $\widetilde{T}$ and the archimedean places of L, and use its permutation module $k[D_L]$, $k = \mathbf{F}_\ell$. Suppose V satisfies Lemma 627.1 and some $z \in \Delta$ fixes μ_ℓ while $\rho(z) - 1$ is invertible on V. Then*

$$|\operatorname{Hom}_\Delta(U_{\widetilde{T}}/U_{\widetilde{T}}^{\ell}, V)| = |\operatorname{Hom}_\Delta(k[D_L], V)|.$$

Proof. Write μ_L for the finite group of roots of unity of L and $C = U_{\widetilde{T}}/\mu_L$, a free abelian group of finite rank by the valuation and unit calculation in Lemma 616.1. The archimedean places mean embeddings in $\mathbb{C}$ modulo complex conjugation. Since L is totally imaginary all their degree weights are two. Lemma 83.1 identifies the real span of the ordinary units modulo μ_L with the coordinate-sum-zero subspace on these places. The map is equivariant by permutation of embeddings. Its kernel on ordinary units is exactly μ_L: it is finite by that lemma, and every root of unity has all absolute values one. The finite-place valuation map on C has this group of ordinary units modulo torsion as kernel. Its image contains $|\mathrm{Cl}(L)|\mathbb{Z}^{\widetilde{T}}$, since each indicated prime to that power is principal, by Lemma 82.5. It spans the real permutation space on $\widetilde{T}$. A linear right inverse over $\mathbb{R}$ can be averaged over Δ by dividing the sum of its conjugates by $|\Delta|$, giving an equivariant right inverse. Adding a real line of constant archimedean coordinates therefore gives an equivariant real isomorphism

$$(C \oplus \mathbb{Z}) \otimes \mathbb{R} \simeq \mathbb{R}[D_L].$$

There is an equivariant rational isomorphism as well. In integral bases the intertwining equations form a homogeneous rational linear system. Its real solution space has a rational basis by row elimination; rational combinations are dense there. Invertibility is open by the determinant, so some rational solution is invertible. Clearing its denominators gives an equivariant integral embedding of equal rank, of finite index by the nonzero integer determinant and the adjugate identity. Lemma 627.1 applies to its reduction. Maps from the added trivial line are zero since $\rho(z) - 1$ is invertible.

Finally the kernel of $U_{\widetilde{T}}/U_{\widetilde{T}}^{\ell} \to C/\ell C$ is the image of μ_L/μ_L^{ℓ}, by lifting the ℓ-th power in the quotient C. This cyclic group's action agrees with that on μ_ℓ: raising to the power $|\mu_L|/\ell$ gives an equivariant isomorphism to μ_ℓ, by cyclicity. Thus z fixes this kernel. Every equivariant map into V kills it, so all maps on the unit quotient factor through $C/\ell C$. These facts prove the equality. □

628. The count for modules admitting a trace identity

Theorem 628.1. *Use the prime set $P = \{w_i : i \in I\}$ of Theorem 623.2 with $m = \ell$, $k = \mathbf{F}_\ell$. In Definition 626.1 use these P, T. Let V be a finite-dimensional $k\Delta$-module. Assume the*

trace identity of Lemma 627.1. Let $z \in \Delta$ be central, fix μ_ℓ, and have $\rho(z) - 1$ invertible on V. On $A = U_{\widetilde{T}}/U_{\widetilde{T}}^{\ell}$, in additive notation, assume

$$\bigcap_{\substack{i \in I \\ d \in \Delta}} \ker(b_i \circ d) \ \subseteq \ \ker(z - 1 : A \to A).$$

Then

$$|\mathcal{X}(V)| = |V|^{|P|+1}.$$

For the split test set of Theorem 624.2 chosen using any $m = \ell^a$, $a \geq 1$, the displayed condition holds for the symbols of order ℓ, with zero intersection on the left.

Proof. For $(v_i)_{i \in I} \in V^I$, define

$$\beta((v_i))(u) = \sum_{i,d} b_i(du)\rho(d)^{-1} v_i \quad \text{in } \operatorname{Hom}_\Delta(A, V).$$

Reindexing d by right multiplication proves equivariance. Let $A_0 = \bigcap_{i,d} \ker(b_i \circ d)$, a Δ-submodule. Every equivariant $f : A \to V$ kills A_0 since z fixes A_0 and $\rho(z) - 1$ is invertible. The coordinate map $A/A_0 \to k^{I \times \Delta}$ using these linear forms is injective. Extending a basis shows that each linear functional on this quotient is a linear combination of these forms. Consequently Jf is a sum of maps $u \mapsto b_i(d_0 u)t$. Taking the trace of such a map, in the convention $\sum_d \rho(d)^{-1} F(du)$, gives the i-th summand of β with $v_i = \rho(d_0)t$: put $d' = d_0 d$. The trace of Jf equals f by the assumption on J. This proves surjectivity of β.

Evaluate a pair $(y, b) \in \mathcal{X}(V)$ at the inertia elements by $E_P(y, b) = (y(\delta_i))_i$. Its image is $\ker \beta$. Inclusion follows from the first identity of Theorem 623.2 and the orbit criterion in Definition 626.1. For the converse suppose $\beta((v_i)) = 0$. Fix a basis e_j of V and write l_j for its coordinate maps. For each j the coefficients

$$F_{i,d}^{j} = l_j(J\rho(d)^{-1} v_i)$$

annihilate all the unit relations by applying $l_j J$ to $\beta((v_i))(u) = 0$. That theorem constructs a cocycle with inertia values $\sum_d F_{i,d}^{j} \rho(d) e_j$, with local coboundaries at T. Sum these cocycles over j. The value at δ_i is $\sum_d \rho(d) J \rho(d)^{-1} v_i = v_i$. The sum admits the required function b, by choosing one local coboundary value on each finite orbit and any value on each free orbit. This proves equality of the image with the kernel.

If $E_P(y, b) = 0$, the same theorem shows $y|_U = 0$. Hence y descends to Δ by the cocycle law. A cocycle y there satisfies, on comparing $y(zg) = y(gz)$,

$$(\rho(z) - 1)y(g) = (\rho(g) - 1)y(z).$$

It is the coboundary of the unique $v = (\rho(z) - 1)^{-1} y(z)$, since the inverse commutes with $\rho(g)$. Subtract from b the constant function of value v. The resulting function is equivariant by the affine transformation law, and any equivariant function can occur. It factors through $U \backslash D$ since U acts trivially on V. As a Δ-set $U \backslash D$ identifies with D_L of Theorem 627.2. For a finite orbit this follows from the stabilizer computation of Lemma 612.1. Over each archimedean place of the totally imaginary E, choose one complex embedding. Its extensions form a free transitive Δ-set by finite Galois correspondence, and none are complex conjugates of one another since their restrictions to E would then be conjugate. Thus the free orbits give exactly the archimedean places of L. We obtain

$$|\ker E_P| = |V| \, |\operatorname{Hom}_\Delta(k[D_L], V)|.$$

Multiplying by $|\ker \beta| = |V|^{|P|} / |\operatorname{Hom}_\Delta(A, V)|$ and applying Theorem 627.2 gives the count, including $V = 0$.

Finally take $K = L(u^{1/m} : u \in U_{\widetilde{T}})$ with $m = \ell^a$. It contains every ℓ-th root of these units. The ϕ_i realize each element of $\operatorname{Gal}(K/L)$. If all the order-ℓ symbols $b_i(u)$ vanish, the Frobenius formula of Lemma 622.1 shows that a chosen ℓ-th root is fixed by every element

of this group. It lies in L; its valuations away from $\widetilde{T}$ vanish by those of u. Thus u is an ℓ-th power of a member of $U_{\widetilde{T}}$. □

629. The count over a subgroup with normal Sylow subgroup

Lemma 629.1. *Fix G, U, Δ, T, P, D as in Definition 626.1, and let z be central in Δ. Over $k = \mathbf{F}_\ell$, restrict attention to finite $k\Delta$-modules on which $z - 1$ is invertible. This condition passes to submodules and quotients. For a short exact sequence $0 \to A \to B \to C \to 0$ of such modules,*

$$0 \longrightarrow \mathcal{X}(A) \longrightarrow \mathcal{X}(B) \longrightarrow \mathcal{X}(C)$$

is exact at its first two positions. Each C is a quotient of a module B in the same class satisfying the trace identity of Lemma 627.1.

Proof. On a stable subspace injectivity of $z - 1$ gives surjectivity by finiteness. This also gives invertibility on the quotient. The maps on pairs apply the coefficient map to both functions. A pair mapping to zero takes all its values in A, and the cocycle, ramification and transformation equations are the same equations in that submodule. This proves exactness.

Write the order of z as $\ell^d b$ with $\ell \nmid b$. Put $z' = z^{\ell^d}$, $Z = \langle z' \rangle$, a central subgroup of order b. On C, $z' - 1 = (z-1)^{\ell^d}$ by the binomial theorem in characteristic ℓ, so it is invertible. Take B to be the functions $\Delta/Z \to C$ with action $(gF)(t) = \rho(g)F(g^{-1}t)$. Summation over the finite set maps onto C, by the functions supported at one point. On B, $z' - 1$ is invertible pointwise since z' acts trivially on Δ/Z. Hence $z - 1$ is invertible. The projection onto functions supported at Z, divided by b, has conjugates summing to 1_B: each coordinate occurs b times. This proves the trace identity. □

Theorem 629.2. *Let $L/E, \Delta, \mu_{\ell^a}, T, K$ and the split test set P be as in Theorem 624.2. Choose T large enough that its primes in every intermediate field of L/E generate that field's ideal class group. Such a choice can be made before choosing K, P. Let $z \in \Delta$ be central and fix μ_ℓ. Let $\Gamma \leq \Delta$ contain z and have a normal subgroup D_ℓ of ℓ-power order and index prime to ℓ. Put $F = L^\Gamma$. Use all places above T, P for T_F, P_F in Definition 626.1 over F, and denote the resulting group by $\mathcal{X}_F$. For every finite $k\Gamma$-module W, $k = \mathbf{F}_\ell$, on which $z - 1$ is invertible,*

$$|\mathcal{X}_F(W)| = |W|^{|P_F|+1}.$$

Every coefficient surjection in a short exact sequence in this class induces a surjection of the corresponding $\mathcal{X}_F$.

Proof. There are finitely many intermediate fields by finite Galois correspondence. In each, choose ideal class representatives and include the contractions to E of all their prime divisors in T, using Lemma 82.5. The stated generating property follows by factoring the ideals. All primes of P split completely through the tower, so $|P_F| = [F : E]|P|$.

First suppose D_ℓ acts trivially on W, and set $L_0 = L^{D_\ell}$. Then $\mu_\ell \subset L_0$, since the ℓ-group acts on it through a group of order $\ell - 1$. The action on W factors through $\mathrm{Gal}(L_0/F) = \Gamma/D_\ell$, of order prime to ℓ, so it has the trace identity. We check the detection hypothesis of Theorem 628.1 for $L_0/F, P_F$. Fix a unit u of L_0 outside the primes above T, in the sense of membership in $U_{\widetilde{T}_0}$. If its order-ℓ symbols vanish at all primes of P_F, using the valuations in L_0 induced by the original valuations for one prime over each $w_i \in P$, a root r with $r^\ell = u$ lies in L. Indeed u is also a unit outside $\widetilde{T}$ in L, hence $r \in K$. The ϕ_i in the original family fix L and are arithmetic Frobenius also over F and L_0 at the selected places, because their residue degrees over E are one. By Lemma 622.1 they fix r, and their restrictions exhaust $\mathrm{Gal}(K/L)$. Every representative of the joint kernel in the detection hypothesis satisfies the vanishing just used.

For $\sigma \in D_\ell$, the multiplier $\sigma r/r$ is in μ_ℓ. Centrality of z on L and its identity on these roots imply $\sigma(zr/r) = zr/r$. Hence zr/r lies in L_0, and its ℓ-th power is zu/u. Its valuations

outside $\widetilde{T}_0$ vanish, since those of this power vanish. Thus z fixes the class of u modulo $U^{\ell}_{\widetilde{T}_0}$. The detection condition holds. We may use any remaining choices of valuations over P_F after imposing the ones just used. Applying Theorem 628.1 over L_0/F proves the equality for these W.

Now every nonzero $k\Gamma$-module has a nonzero D_ℓ-fixed vector. Indeed D_ℓ acts on its underlying finite set with orbit sizes powers of ℓ. The number of fixed vectors is congruent to the size of the space modulo ℓ; it includes zero and hence is at least ℓ. The fixed subspace is Γ-stable by normality. Taking this subspace and repeating on the quotient gives a finite filtration by modules on which D_ℓ acts trivially. The invertibility condition passes to each. The exactness at the first two positions in Lemma 629.1 makes cardinalities submultiplicative on short exact sequences. It follows that $|\mathcal{X}_F(W)| \le |W|^{|P_F|+1}$ for every such W.

For equality, take the cover $B \to W$ in that lemma and let A be its kernel. For B the trace identity holds over Γ itself. Apply Theorem 628.1 over L/F: total unit symbols over L detect all classes, because the same original ϕ_i occur over F and cover $\mathrm{Gal}(K/L)$. The proof of the last assertion in that theorem applies to this subset of P_F. Thus equality holds for B. Submultiplicativity and the upper bounds on A, W give

$$|B|^{|P_F|+1} = |\mathcal{X}_F(B)| \le |\mathcal{X}_F(A)|\,|\mathcal{X}_F(W)| \le (|A|\,|W|)^{|P_F|+1}.$$

The ends agree, forcing equality for W. Apply the resulting equalities to any short exact sequence in this class. The image order of its last map on pairs then equals the order of its target, by exactness at the middle term. □

630. Descent of surjectivity for cocycles with local choices

Lemma 630.1. *Use $G = G_E$, $U = G_L$, T, P, D of Definition 626.1 with P split in L. For an intermediate field F put $G' = G_F$, $j = [F : E]$. Define $\mathcal{X}_F$ using all primes above T, P. For any finite Δ-module V, restriction and transfer are additive maps*

$$\mathcal{X}(V) \longrightarrow \mathcal{X}_F(V) \longrightarrow \mathcal{X}(V).$$

They commute with coefficient maps. For any choice of representatives $s_C \in G$ for $C \in G/G'$, their composite on (y, b) is

$$j(y, b) + (d_0 t, \mathrm{const}_t), \qquad t = \sum_{C \in G/G'} y(s_C).$$

Here const_t denotes the constant function of value t on D.

Proof. Restrict the action on D to G'. Its orbits above a finite v correspond to the places of F above v: a coset gD_v gives the restriction to F of the valuation transported by g. Every place above v extends to the closure, and two extended valuations above the same F-place are conjugate by G', by the finite prime transitivity and compact inverse-limit proof of Lemma 603.1. The stabilizer is $G' \cap gD_v g^{-1}$, its decomposition group over F. A free G-orbit restricts to j free G'-orbits, giving $jf_E = f_F$ in all. Thus we can take the same D over F. Restrict y to G' and keep b. The inertia conditions over F follow by restricting valuations.

For a pair (u, b') over F, set

$$v(g) = \sum_C \rho(s_{gC}) u(s_{gC}^{-1} g s_C),$$
$$b''(d) = \sum_C \rho(s_C) b'(s_C^{-1} d).$$

The factors inside u lie in G'. In a product gg' they split as $(s_{gg'C}^{-1} g s_{g'C})(s_{g'C}^{-1} g' s_C)$. Using the cocycle law on these two factors and reindexing by $C \mapsto g'C$ gives $v(gg') = v(g) + \rho(g)v(g')$.

The sums are continuous on the open sets where each coset permutation is fixed. Substituting $d \mapsto gd$ into b'' and writing its index as gC gives

$$b''(gd) = \sum_C \rho(s_{gC}) b'\big((s_{gC}^{-1} g s_C)(s_C^{-1} d)\big) = \rho(g) b''(d) - v(g).$$

At an inertia group outside $T \cup P$, g fixes L by the unramifiedness of L/E. Hence $gC = C$ at every coset since G_L is normal and contained in G'. Each conjugate $s_C^{-1} g s_C$ is an inertia element over F outside its allowed set, so $v(g) = 0$. This proves that the formulas give transfer on pairs. Both maps commute with coefficient maps by their formulas.

For $u = y|_{G'}$, the cocycle law expresses each term of $v(g)$ as $-y(s_{gC}) + y(g) + \rho(g) y(s_C)$. Their sum is $j y(g) + d_0 t(g)$. For $b' = b$, each term of $b''(d)$ equals $b(d) + y(s_C)$ by the defining transformation law. This gives the stated composite. □

Theorem 630.2. *Assume the fields, test primes, enlarged T and central z in Theorem 629.2. For $k = \mathbf{F}_\ell$, on finite $k\Delta$-modules with $\rho(z) - 1$ invertible, the functor $V \mapsto \mathcal{X}(V)$ takes every short exact sequence to a short exact sequence. In particular the integer*

$$D(V) = \dim_k \mathcal{X}(V) - (|P| + 1) \dim_k V$$

is additive in these sequences and invariant under equivariant isomorphism.

Proof. Write $|\Delta| = \ell^q d$, $\ell \nmid d$. There is a subgroup P_ℓ of order ℓ^q. Indeed translation permutes the subsets of Δ of size ℓ^q. Their number is congruent to d mod ℓ by the coefficient of X^{ℓ^q} in $(1+X)^{\ell^q d} = (1+X^{\ell^q})^d$ over k. Thus some orbit has order prime to ℓ; its stabilizer has order divisible by ℓ^q. The stabilizer acts freely on that subset by translations, so its order divides ℓ^q as well. Set $\Gamma = P_\ell \langle z \rangle$. Centrality of z makes P_ℓ normal there, and the indices of P_ℓ in Γ and of Γ in Δ are prime to ℓ. Take $F = L^\Gamma$ and $j = [F : E]$.

For a coefficient surjection $B \to C$ with admissible kernel and a pair $(y, b) \in \mathcal{X}(C)$, restrict to F. By Theorem 629.2 the restricted pair has a lift in $\mathcal{X}_F(B)$. Transfer it back by Lemma 630.1. Its image is $j(y, b) + (d_0 t, \mathrm{const}_t)$. Choose a lift $\tilde{t} \in B$. The pair $(d_0 \tilde{t}, \mathrm{const}_{\tilde{t}})$ belongs to $\mathcal{X}(B)$ since G_L acts trivially on B and L/E is unramified outside T. Subtract it and multiply by $j^{-1} \in k$. This produces a lift of the original pair. Exactness at the other terms was established in Lemma 629.1. Each finite group of pairs here is a k-space by pointwise scalar multiplication, so its dimension, and hence the displayed difference, is additive. Coefficient isomorphisms induce inverse maps of pairs. □

631. The split-prime count at arbitrary prime-power exponent

Lemma 631.1. *Let Δ be finite, $k = \mathbf{F}_\ell$, and z central in Δ. Let D be an integer-valued function on finite $k\Delta$-modules on which $z - 1$ is invertible. Assume invariance under equivariant isomorphisms, additivity on short exact sequences, and $D(B) = 0$ on modules in this class satisfying the trace identity of Lemma 627.1. Then $D(V) = 0$ for every module in this class.*

Proof. Fix V. As in Lemma 629.1 take $z' = z^{\ell^q}$ of order prime to ℓ, with $z' - 1$ invertible on V, and put $A = \Delta / \langle z' \rangle$. For a finite A-set Y use the $k\Delta$-module V^Y of functions with action $(gf)(y) = \rho(g) f(g^{-1} y)$. Each is in the stated class, since $z' - 1$ acts invertibly pointwise. On the functions k^A, with action of A permuting the coordinates by left translation, put

$$X_i = \{x \in k^A : \sum_{a \in A} x(a) = i\}, \quad i = 0, 1.$$

Every point stabilizer in A of a member of X_1 has order prime to ℓ. If its order were divisible by ℓ, the subset-translation argument in Theorem 630.2 applied to that stabilizer gives a nontrivial subgroup of ℓ-power order. It acts freely on the coordinate set A. A function constant on its orbits has coordinate sum zero in k, contradicting the defining sum for X_1. The stabilizers in Δ for $X_1 \times Y$ also have order prime to ℓ, since the kernel $\langle z' \rangle$ has such order. On each orbit project onto the coordinate at one representative, multiplied by the

inverse order of its stabilizer. The conjugates of the sum of these projections add to the identity on $V^{X_1\times Y}$, by counting translates taking each representative to each point of its orbit. Thus $D(V^{X_1\times Y})=0$.

We prove $D(V^{X_0\times Y})=D(V^{X_1\times Y})$. Fix $x_1\in X_1$ and identify X_0 with X_1 as sets by addition of x_1. Under this identification the second action on X_0 is $x\mapsto gx+(gx_1-x_1)$. Let $H=X_0\rtimes\Delta$, where Δ acts through A on the additive group X_0. It acts on X_0 by $(t,g)x=t+gx$, on Y through A, and on V through Δ. On the single space $V^{X_0\times Y}$ take this function action of H. Its restrictions under $g\mapsto(0,g)$ and $g\mapsto(gx_1-x_1,g)$ are the two modules being compared; the second map is a homomorphism by its displayed translation rule.

The normal additive subgroup X_0 has ℓ-power order. The fixed-vector filtration argument in Theorem 629.2, applied to this finite kH-module, gives a finite filtration by H-submodules with X_0 trivial on each successive quotient. Consequently each quotient has identical Δ-actions on the two restrictions. Also z' has the same lifts under both maps, since its image in A is trivial. Its action minus one is invertible on the whole space, hence on all these subquotients. Additivity and isomorphism invariance of D give the desired equality.

For each subgroup $B\le A$, apply this equality to $Y=A/B$. Thus $D(V^{X_0\times A/B})=0$. Decompose the set into A-orbits and use additivity on direct sums. An orbit contains a point (x,B) after translation, whose stabilizer is a subgroup $C\le B$, so its contribution is $D(V^{A/C})$. The orbit of $(0,B)$ contributes $D(V^{A/B})$. Induct on $|B|$. All terms with proper $C<B$ vanish by the induction hypothesis, leaving a positive integer times $D(V^{A/B})$ equal to zero. This proves its vanishing for all B, including the initial trivial subgroup. Taking $B=A$ gives $D(V)=0$. □

Theorem 631.2. *Use the data of Theorem 630.2, with test primes chosen using the exponent $m=\ell^a,\ a\ge1$. For every finite abelian Δ-module V killed by m on which $z-1$ is invertible,*

$$|\mathcal{X}(V)|\le|V|^{|P|+1}.$$

There is equality whenever $\ell V=0$.

Proof. For $\ell V=0$, put $k=\mathbf{F}_\ell$. The deficit $D(V)$ of Theorem 630.2 is additive and invariant under isomorphism. It vanishes on all modules satisfying the trace identity, by Theorem 628.1 and the order-ℓ detection proved there for the chosen test primes. Lemma 631.1 therefore gives $D(V)=0$. Taking cardinalities yields equality, including the zero module.

For the general assertion, filter V by the stable subgroups ℓ^jV until $\ell^aV=0$. Invertibility of $z-1$ passes to subgroups and quotients by finiteness, as in Lemma 629.1. Each successive quotient is killed by ℓ. The argument for exactness at the first two positions in that lemma uses only the defining equations for pairs and works for finite abelian groups of any exponent. Thus their cardinalities are submultiplicative on the filtration. Multiplying the prime-exponent equalities for its quotients gives the asserted bound. □

632. Detection from the local length balance

Theorem 632.1. *Let O be a finite coefficient valuation ring with uniformizer ϖ, residue field k of characteristic ℓ, and $e=v_\varpi(\ell)$. Here a finite coefficient valuation ring means the ring of integers of a finite extension of $\mathbb{Q}_\ell$. Take $M\ge1,\ 1\le j\le eM,\ R=O/\varpi^jO$, $V=R^r,\ r\ge1$, and $l(A)=\log_{|k|}|A|$ for finite R-modules. Let $\rho_M:G_E\to\mathrm{GL}_r(O/\ell^MO)$ be continuous, unramified off a finite set S_0, with reduction ρ on V whose values span $\mathrm{Mat}_r(R)$ over R. Use an old auxiliary system J as in Theorem 621.2, with all pairs away from S_0, with the prescribed tie slopes at this row satisfying Definition 604.2. Write T_* for the union of the old ordinary pairs and the places over ℓ.*

Let γ be an inertia element at a place of S_0 away from ℓ whose pair is disjoint from T_*. Suppose $\rho_M(\gamma)=sI,\ \rho_M(c(\gamma))=s^{-1}I$ with $s,s+1,s-1$ units, writing $c(g)=cgc^{-1}$. Take g_* with $\rho(g_*)=\alpha I,\ \rho(c(g_*))^{-1}=\alpha' I$ in $\mathrm{GL}_r(R)$. Put $\delta=\alpha-\chi(c(g_*))\alpha'$, where χ

is the exponent on μ_{ℓ^M}, acting on R by scalars. For $v \in S_0$ choose an R-submodule $\mathcal{B}_v$ of continuous local cocycles on V containing the coboundaries, on a specified decomposition group.

Let X be an R-module of the untwisted cocycles with the old ties, unramified off $S_0 \cup J \cup \bar{J}$, zero at γ, and with restrictions at S_0 in $\mathcal{B}_v$. It may have additional homogeneous constraints. Suppose it has a member y with a unit coordinate at some group element. At each admissible new pair let Y_w consist of relaxed cocycles with the same old ties and local restrictions, now permitted ramification also at $w, \bar{w}$. They may retain the additional constraints. Then at least one of the following assertions holds.

(1) *There is a new pair as in Theorem 621.2, with $\rho_M(\phi_{w,b}) = 1$ for both signs, at which*

$$\{(u(\phi_{w,+}), u(\phi_{w,-})) : u \in X\} \not\subseteq \{(u'(\gamma_{w,+}), u'(\gamma_{w,-})) : u' \in Y_w\}.$$

Any previously prescribed finite set of rational primes may be excluded.

(2) *One has the upper bound*

$$l(X) \le \sum_{v \in S_0} l(\mathcal{B}_v) + l(\ker(\delta : V \to V)) - (|S_0| + f_E) l(V), \qquad f_E = [E : \mathbb{Q}]/2.$$

Proof. Let $m = \ell^M$, and take for L the field of the common kernel of $\rho_M, \rho_M \circ c, \chi$. This is a finite normal extension, contains μ_m, and is c-stable because $c^2 = 1$ and $\chi \circ c = \chi$, the latter by the abelian cyclotomic action. Write $U = G_L$. The image of γ in $\Delta = \mathrm{Gal}(L/E)$ is central: each of its two matrix coordinates is scalar and the cyclotomic coordinate is abelian. It fixes μ_m since inertia at a prime away from ℓ fixes these roots by injectivity of reduction. It acts on V with $\rho(\gamma) - 1$ invertible. The element $z_0 = \gamma_{w_0,+}$ at the distinguished old prime belongs to U: the action is unramified at both members of this pair, as is the cyclotomic character. Its conjugate belongs to U as well.

Let A be the closed normal subgroup generated by $\gamma, c(\gamma)$ and H the closed subgroup generated by all $[a, g]$, $a \in A,\ g \in G_E$. It is normal and c-stable because its defining sets are. The image of A in Δ is central by both scalar identities, so $H \subseteq U$. Also H lies in the closed commutator subgroup and in the closed normal subgroup generated by inertia outside T_*: the latter contains A by the choice of the marked pair. It contains each $[g, \gamma]$ since these are inverses of $[\gamma, g]$. These are all the subgroup requirements in Theorem 625.2.

Put $S = S_0 \cup J \cup \bar{J}$ and take local conditions $\mathcal{L}_v = \mathcal{B}_v$ at S_0. At each old auxiliary place take the full local cocycle subgroup obeying its single tie with the indicated sign. It contains the coboundaries and has order at most $|V|$ by Lemma 626.3. Form $Y(\varnothing)$ from these local conditions as in Definition 624.1. Then $X \subseteq Y(\varnothing)$ and $Y_w \subseteq Y(\{w, \bar{w}\})$ by their defining conditions. Choose T containing S, the primes above ℓ and the required ramification and ideal class representatives in every intermediate field of L/E, as in Theorem 629.2. Construct its split test set P by Theorem 624.2, omitting all old pairs and the given finite exclusions.

Suppose the first assertion fails. For each $a_0 \in H$ the two elements $(z_0 a_0)_b$ lie in U. By Theorem 621.2 one can realize the required observations at a new pair near $z_0 a_0$, keeping the matrices trivial on both sides and meeting every finite exclusion of Theorem 625.2. For each $t \in R$ the values of $ty \in X$ at the new Frobenius pair are then inertia values of some $u' \in Y_w$. It belongs to the larger local module at that pair. The scalar, spanning and unit-coordinate assumptions together with $y(\gamma) = 0$ supply the remaining hypotheses of that theorem. The number $|\mathcal{N}|$ of split-prime relations is consequently at most $|\ker(\delta : V \to V)|$.

By Theorem 631.2 for the central image of γ, $|\mathcal{X}(V)| \le |V|^{|P|+1}$. Use the second inequality of Proposition 626.2 with $g_1 = \gamma$. Taking logarithms to base $|k|$ gives

$$l(X) \le l(\ker \delta) + \sum_{v \in S} l(\mathcal{L}_v) - (|S| + f_E) l(V).$$

There are $|S| - |S_0|$ old auxiliary places, each contributing at most $l(V)$ in the sum. Canceling these contributions gives the stated bound. □

633. Joint specialization from bounds at the original places

Theorem 633.1. *Use the arithmetic data of Theorem 598.2 with $p = g_i \neq \ell$, $e_\ell = v_\ell(p-1)$, and fix an isomorphism $j_\ell : \mathbb{C} \to K_\ell$. At one of the marked places above p choose an inertia element z_* whose exponent on μ_p is a residue generator, as in Theorem 620.2. Fix integers $m, M \geq 1$, a real $\eta > 0$, and a finite list of integral representations ρ_i as in Theorem 605.1. Use a common coefficient valuation ring O with ramification index e, uniformizer ϖ and residue field k with $|k| > 4r_i + 1$. For row i take a finite base set S_i, a scalar inertia element γ_i above S_i away from ℓ, and fixed homogeneous integral relations $\mathcal{L}_i$ including $y(\gamma_i) = 0$, as in Definition 607.1. Suppose that at every $1 \leq s \leq eM$ the reduction $V_{i,s} = (O/\varpi^s O)^{r_i}$, its representation, γ_i and some $g_{*,i}, \alpha_{i,s}, \alpha'_{i,s}$ satisfy the matrix and scalar hypotheses of Theorem 632.1. Assume local modules $\mathcal{B}_{v,i,s}$ there contain the restrictions of all untwisted cocycles obeying $\mathcal{L}_i$ and allowed any of the new auxiliary pairs. Write $\delta_{i,s} = \alpha_{i,s} - \chi(c(g_{*,i}))\alpha'_{i,s}$. Assume the finite bound*

$$\sum_{v \in S_i} l(\mathcal{B}_{v,i,s}) + l(\ker(\delta_{i,s} : V_{i,s} \to V_{i,s})) \leq (|S_i| + f_E) l(V_{i,s}) + \beta_i \qquad (1 \leq s \leq eM)$$

with $\beta_i \geq 0$, and put

$$D_i = \frac{1}{e} \sum_{b=+,-} l\big(\ker(\rho_i((z_*)_b) - 1 : V_{i,eM} \to V_{i,eM})\big).$$

There exist a finite auxiliary set Q of prime pairs represented by one member each, a tame torus character ϑ at Q, and its inverse avatar ψ. The pairs can avoid any fixed finite set of rational primes. The character ψ is trivial on the full decomposition groups over ℓ and unramified off these pairs. For each $1 \leq d \leq m$, ϑ^d is primitive at Q, ψ^d is its inverse avatar, and

$$\lambda(Z^i_{\psi^d}) \leq \beta_i/e + D_i + \eta \quad \text{for every } i,$$
$$v_\ell(j_\ell(\rho_a(Q, \vartheta^{\epsilon d}))) \leq 2e_\ell + \eta \qquad \text{for every ramified pair } a,\ \epsilon = \pm 1,$$
$$\sum_{w \in Q} v_\ell(1 - j_\ell(\vartheta^d(u_{t_w}))) \leq \eta.$$

Here $Z^i_{\psi^d}$ is the module of Theorem 605.1 for $\psi^d \rho_i \bmod \ell^M$, the allowed places $S_i \cup Q \cup \bar{Q}$, and the given additional relations, with all auxiliary ties removed. Its normalized length bound holds over every finite coefficient extension containing the specialized values and O.

Proof. Exclude all rational primes below the original S_i and their conjugates from the new primes. For every $N \geq 1$ take a singleton auxiliary system at w_0 by Proposition 621.1. Use T consisting of the places above ℓ, choose $K > e_\ell$ in the marked prime theorem, and prescribe in the helper neighborhood both matrices at precision ℓ^M for all rows. Set all helper slopes equal to zero. The Frobenius kernels now have exactly the normalized sums D_i, independently of N. Its idelic coordinate takes value one at the chosen residue generator, and the Euler bounds hold there for every ramified pair.

Consider the systems obtained by adjoining pairs using Theorem 621.2 with T_* the old ordinary pairs and the places over ℓ. For each row and every system the field and subgroup in Theorem 632.1 can be chosen anew. All original scalar inertia primes are disjoint from T_* by the rational exclusions. That theorem and the local inequality yield a detecting extension at any precision where the old tied module has length greater than β_i and contains a class with a unit coordinate. Take Y_w there to be exactly the module with the new pair allowed and the old relations retained. The first alternative must then occur. At its two Frobenius elements the matrix is the identity modulo ℓ^M, so a unit slope at the selected row and zero at every other row give allowable extensions for every choice of the unit. All previous coordinate evaluations persist. The two idelic conditions in Definition 598.1 hold with $H = 1$.

The hypotheses of Corollary 607.3 now follow. In particular Theorem 607.2 bounds the number of primes independently of N, as required before choosing N for the specialization. That corollary and Theorem 605.1 give the three bounds at one tuple of roots of unity, along with primitivity. Each coordinate satisfies $\lambda_{h_w}(x) = e_w(x/\bar{x})$ by Theorems 619.2 and 621.2. Multiplying these identities at the chosen roots gives the inverse avatar and the identity for all powers, by the arithmetic-to-geometric Frobenius calculation and uniqueness in Theorem 615.3. The ramification and local triviality follow for the product from the coordinate properties. In particular the twists are trivial on each γ_i, preserving its scalar action. □

634. Local lengths for finite cocycles and integral relations

Lemma 634.1. *Let D be the decomposition group at a degree-one prime with completion $\mathbb{Q}_p$, I its inertia, and $V = (O/\varpi^j O)^r$ in the notation of Theorem 632.1, where $p \neq \ell$. Suppose a continuous representation $\rho : D \to \mathrm{GL}(V)$ kills the b-th powers of all inertia elements for some positive b prime to p. The module of continuous local cocycles has length bounded by*

$$l(Z^1(D,V)) \le l(V) + l(V^I) \le l(V) + l(\ker(\rho(\tau) - 1)) \quad (\tau \in I).$$

If ρ is the reduction of an integral representation ρ_O, then $l(\ker(\rho(\tau)-1)) \le e\, v_\ell(\det(\rho_O(\tau)-1))$, reading the right-hand side as $+\infty$ at a zero determinant.

Proof. The group of continuous local cocycles is finite as in the local finiteness proof of Proposition 626.2. Indeed one may first adjoin the m-th roots of unity for m annihilating V and the finite field of the local action. Form the product of the finitely many affine actions $g \mapsto (t \mapsto \rho(g)t + y(g))$ over all these cocycles. Its inertia image has order prime to p. Its linear part is the single group $\rho(I)$ of exponent dividing b; if p divided its order it would have a nontrivial p-subgroup by the translation counting argument in Theorem 630.2 with prime p. A nonidentity element of this group would have p-power order, impossible here. The kernel on linear parts consists of translations in a product of copies of V, of ℓ-power order. Thus the asserted coprimality holds.

The inertia image in this product is cyclic. In the proof of Theorem 603.2 the character $\sigma \mapsto \overline{\sigma(\pi)/\pi}$ on a finite local inertia group has p-group kernel and cyclic image. Its restricted kernel maps trivially to any finite group of order prime to p, by divisibility of the orders. This proves cyclicity for the present whole product. Choose $\delta \in I$ whose image generates it. If $y(\delta) = 0$, its affine map fixes zero, hence every inertia map does. Thus the kernel of evaluation at δ consists exactly of cocycles zero on all I.

For $g \in D$ and $a \in I$, write $a' = gag^{-1}$. Since $y(a) = y(a') = 0$, the equality $ga = a'g$ gives $y(g) = \rho(a')y(g)$. As a' runs through I, this shows $y(g) \in V^I$. Its values are constant on the cosets of I. It descends to a cocycle on the residue Galois group with values in V^I, on which inertia is trivial. Evaluation at an arithmetic Frobenius is injective there: the induced finite affine action factors through a finite unramified extension, whose Frobenius generates its Galois group by Theorem 611.3. A generator fixing zero makes the whole group fix zero. The evaluation at δ has image of order at most $|V|$; its kernel has order at most $|V^I|$. This proves both length bounds. For an integral matrix, the diagonal elimination of Lemma 602.1 gives truncated kernel length $\sum_i \min(j, v_\varpi(d_i))$ on its nonzero invariant factors d_i, with j for each zero pivot. This is at most the stated determinant valuation. □

Lemma 634.2. *Let O be a discrete valuation ring with finite residue field, D a set and $r \ge 1$. Let B be a finitely generated O-submodule of the functions $D \to O^r$ of rank d. A coordinate relation of B means any finite O-linear combination of coordinate evaluations on D vanishing on every member of B. For $R = O/\varpi^j O$, $j \ge 1$, the module of all functions $D \to R^r$ satisfying every such relation is isomorphic to R^d. Relations with matrix coefficients acting on vector values impose exactly these same conditions. If the functions of B are*

integral cocycles including all integral coboundaries, the local continuous cocycles satisfying the relations have length at most jd and include all coboundaries modulo ϖ^j.

Proof. The torsion-free module B is free. For clarity, a finite torsion-free module over O embeds in its fraction-field span since a fraction is zero only after multiplication by a nonzero denominator. Express its generators in a field basis and clear one common denominator to embed it in O^d. A submodule of O^d is free of rank at most d by successive coordinate projection onto principal ideals, as in Lemma 611.1.

Let $\Lambda \subset B^\vee = \mathrm{Hom}_O(B, O)$ be generated by all the coordinate evaluations. It has rank d: if their field span were proper in the dual of the field span of B, there would be a nonzero vector annihilated by all of them. Clearing a denominator gives a nonzero function of B with all coordinates zero, a contradiction. Freeness by the same projection argument gives $\Lambda \simeq O^d$. A function to R^r assigns one value to each of these generators. It defines an O-linear map on Λ exactly when every finite relation among them holds on those values, by the presentation as a quotient of the free module on the indexed generators. Thus the solutions form $\mathrm{Hom}_O(\Lambda, R) \simeq R^d$. Each matrix-coefficient equation decomposes into scalar row equations. Conversely any scalar equation occurs as one row with every other row zero. This proves the assertion on matrix relations. Restricting the solution space to continuous cocycles gives the stated upper bound. Every coboundary over R comes from an integral coboundary by lifting its defining vector. Such a reduction satisfies every relation by hypothesis. □

635. Scalar actions and a common choice at the marked place

Lemma 635.1. *Let O be a coefficient valuation ring, $r \geq 1$, and let $\rho : G_E \to \mathrm{GL}_r(O)$ have image spanning $\mathrm{Mat}_r(\mathrm{Frac}\, O)$ over the fraction field. Let κ be the ℓ-adic cyclotomic character, so its reduction modulo ℓ^M gives the exponent on μ_{ℓ^M}. Write $c(g) = cgc^{-1}$. Suppose that for an integer $a \geq 0$ one has*

$$\mathrm{tr}\rho(c(g)) = \kappa(g)^{-a}\mathrm{tr}\rho(g^{-1}) \qquad (g \in G_E).$$

Whenever $\rho(g) = \alpha I$, one has $\alpha \in O^\times$ and

$$\rho(c(g))^{-1} = \kappa(g)^a \alpha I, \qquad \alpha - \kappa(c(g))\kappa(g)^a\alpha = \alpha(1 - \kappa(g)^{a+1}).$$

In particular a scalar at an inertia element away from ℓ has the same value at the inverse of its conjugate. If its residue has order greater than $4r + 2$, then its differences from 1 and -1 are units and the residue field has more than $4r + 1$ elements.

Proof. The determinant of an invertible scalar matrix is the unit α^r, so α is a unit. Apply the trace identity to gh and to h. Since $\rho((gh)^{-1}) = \alpha^{-1}\rho(h^{-1})$, it gives

$$\mathrm{tr}\big((\rho(c(g)) - \kappa(g)^{-a}\alpha^{-1}I)\rho(c(h))\big) = 0 \quad (h \in G_E).$$

The $\rho(c(h))$ span the full matrix algebra. Pairing with the matrix unit having its one in position (j, i) reads the (i, j) entry under the trace of a product. Thus the parenthesized matrix is zero. Invert the identity and use $\kappa(c(g)) = \kappa(g)$, by its abelian target. An inertia element away from ℓ fixes every μ_{ℓ^s} by distinct reduction, as in Lemma 618.1. The residue order assumption excludes residues $1, -1$; it also gives the size requirement since the order divides $|k| - 1$. □

Theorem 635.2. *Fix a marked degree-one place w_* above $p = g_i \neq \ell$, and an extended valuation. For a finite list of integral representations $\rho_i : G_E \to \mathrm{GL}_{r_i}(O)$ fix a precision $M \geq 1$. Suppose for each row at both members of the marked pair that its inertia image modulo ℓ^M has exponent prime to p. There is a single z_* in the chosen inertia with $a_p(z_*)$ a generator of $\mathbf{F}_p^\times$, such that, at every $1 \leq j \leq eM$ and on each $V_{i,j} = (O/\varpi^j O)^{r_i}$,*

$$\ker(\rho_i((z_*)_b) - 1) = V_{i,j}^{I_{w_*,b}} \subseteq \ker(\rho_i(\tau) - 1) \qquad (\tau \in I_{w_*,b},\ b = +, -).$$

Here $I_{w_*,-} = cI_{w_*,+}c^{-1}$ and the minus element is the inverse conjugate in that group. Thus any independent upper bounds for the kernels at elements of the two inertia groups also bound the two kernels at the distinguished prime in Proposition 621.1.

Proof. On the decomposition group at w_* take the finite product of the matrix actions on $V_{i,eM}$, of their conjugates $g \mapsto \rho_i(c(g))$ at the same precision, and of a_p. Each of its inertia images has order prime to p by the exponent argument in Lemma 634.1. This also holds for a_p, whose target has order $p-1$. The product inertia image is a subgroup of the product of these finite groups, hence also has order prime to p. The tame inertia proof in that lemma gives cyclicity for the product. Take any inertia lift z_* of a generator. It generates each projected inertia image, and a_p is surjective on inertia by the local cyclotomic computation in Theorem 620.2. Consequently $a_p(z_*)$ is a residue generator.

Reduction to each smaller j preserves the generator property. In a cyclic group, an element of a module is fixed by the group exactly if it is fixed by a generator or its inverse. This proves the equality on either side and the inclusion for each τ. The helper transfer in the cited proposition prescribes the same two matrices by placing its Frobenius in a sufficiently small neighborhood of z_*. □

636. The contribution of the original finite places

Lemma 636.1. *Let E be a CM field of degree $2f$, $f \geq 1$, in which the distinct primes g_0, g_1 split completely. From each conjugate pair above either prime choose one place. Write $\mathcal{M}$ for these $2f$ choices and k_w for the residue field. For positive integers r_*, t_*, set*

$$N = r_* t_*, \quad A_N = \prod_{w \in \mathcal{M}} |\mathrm{GL}_N(k_w)|, \qquad c_\ell = v_\ell(A_N) + 2^{-\ell}$$

for each rational prime ℓ, with $v_\ell(\ell) = 1$. For $p = g_0, g_1$ and $1 \leq h \leq N$ one has $v_\ell(p^h - 1) \leq v_\ell(A_N) < c_\ell$.

Proof. Over the field of p elements the j-dimensional span of the first j independent columns has p^j elements by uniqueness of coordinates. There are $p^N - p^j$ choices for the next column. Thus the order of the general linear group is

$$\prod_{j=0}^{N-1} (p^N - p^j) = p^{N(N-1)/2} \prod_{h=1}^{N} (p^h - 1).$$

For each p at least one of the chosen residue fields has size p, since p splits completely and $f \geq 1$. The positive integer $p^h - 1$ divides its group order and hence A_N. Taking valuations proves the bound. □

Theorem 636.2. *Use the arithmetic setting and the ramified ratios of Theorem 633.1, and the two marked primes of Lemma 636.1. Fix $m, M \geq 1$, $\eta > 0$, and a finite list of continuous representations $\rho_i : G_E \to \mathrm{GL}_{r_i}(O)$ over one coefficient valuation ring. Write S_ℓ for all places over ℓ, S_g for all places over g_0, g_1, and $S = S_\ell \cup S_g$. Suppose the following conditions hold for each i.*

1. *One has $1 \leq r_i \leq r_*$, $0 \leq t_i \leq t_*$ with t_i integral, and the matrices of ρ_i span $\mathrm{Mat}_{r_i}(O)$ over O. The action is unramified off S. The trace identity of Lemma 635.1 holds with $a_i = \max(t_i - 1, 0)$.*
2. *Some $g_{*,i}$ acts by a scalar and satisfies $\kappa(g_{*,i}) = p_i^{r_i}$ for $p_i \in \{g_0, g_1\}$. At some place of $S_g \setminus S_\ell$, an inertia element γ_i acts by a scalar of residue order greater than $4r_i + 2$.*
3. *At every $v \in S_g \setminus S_\ell$, inertia at the chosen valuation acts modulo ℓ^M with exponent prime to the residue characteristic. There is τ_v in that inertia with*

$$v_\ell\big(\det(\rho_i(\tau_v) - 1)\big) \leq r_i c_\ell.$$

Conjugate groups at a pair are taken at conjugate valuations.

(4) *At $v \mid \ell$ let $B_{v,i}$ be a finitely generated module of integral local cocycles containing all coboundaries. Write $d_{v,i} = \operatorname{rank}_O B_{v,i}$. Suppose that*

$$\sum_{v|\ell}(d_{v,i} - r_i) \le f r_i.$$

Impose its coordinate relations on the local restriction and $y(\gamma_i) = 0$ on the global cocycle. Additional fixed homogeneous integral relations as in Theorem 633.1 are allowed.

Then there is one Q, ϑ, ψ as in that theorem. For every $1 \le d \le m$ its normalized lengths, valuations of ramified ratios, and primitivity sums satisfy

$$\begin{aligned} \lambda(Z^i_{\psi^d}) &\le (|S_g \setminus S_\ell| + 3)\, r_i c_\ell + \eta, \\ v_\ell(j_\ell(\rho_a(Q, \vartheta^{\epsilon d}))) &\le 2c_\ell + \eta \qquad (\epsilon = \pm 1), \\ \sum_{w \in Q} v_\ell(1 - j_\ell(\vartheta^d(u_{t_w}))) &\le \eta. \end{aligned}$$

In particular $\eta = c_\ell$ yields $\lambda(Z^i_{\psi^d}) \le (10 f r_i + 1) c_\ell$.

Proof. Let $1 \le s \le eM$, $V = (O/\varpi^s O)^{r_i}$ and $l(V) = s r_i$, with $e = v_\varpi(\ell)$. Spanning descends to the quotient by reducing any expression for each elementary matrix. The trace computation of Lemma 635.1 gives both scalar conditions at γ_i, the required units, and the residue field size. It also gives $\delta = \alpha(1 - p_i^h)$ reduced on V, where α is the scalar at $g_{*,i}$ and $h = r_i(a_i + 1) = r_i \max(1, t_i)$. Since $1 \le h \le N$, Lemma 636.1 bounds $v_\ell(1 - p_i^h)$ by c_ℓ. A scalar of ϖ-valuation b has kernel length $r_i \min(b, s)$ on V, by taking each coordinate. Thus the scalar kernel costs at most $e r_i c_\ell$.

At $v \mid \ell$ take the module of continuous cocycles satisfying the stated local relations. Its length is at most $s d_{v,i}$ by Lemma 634.2. Elsewhere in S take all continuous local cocycles. Lemma 634.1 bounds their length by $s r_i + e r_i c_\ell$. All these conditions contain the coboundaries. Adding over S, using the rank hypothesis above ℓ, and adding the scalar kernel gives at most

$$(|S| + f) l(V) + e(|S_g \setminus S_\ell| + 1) r_i c_\ell.$$

These estimates persist when an untwisted global cocycle obeys additional relations, since the possible restrictions can only decrease.

Choose a marked $p \ne \ell$, possible since the primes are distinct. At a pair over p, use Theorem 635.2 for all rows together. At each member of this pair the kernel is contained in the kernel at τ_v, of length at most $e r_i c_\ell$ by diagonal elimination as in Lemma 634.1. The sum D_i of the two normalized lengths is at most $2 r_i c_\ell$. Apply Theorem 633.1 with the displayed balance and with $v_\ell(p - 1) \le c_\ell$ from Lemma 636.1. This gives all three assertions for the same twist, with the stated range of powers. Finally, there are at most $4f$ places above the two split primes. Since $4f + 3 \le 10 f$ for $f \ge 1$, the last numerical inequality follows. □

637. A rank inequality from the filtration by a uniformizer

Lemma 637.1. *Let $F = \mathbb{Q}_p$, K/F a finite extension in the algebraic closure K_p of Definition 233.1, and $G = \operatorname{Gal}(K_p/K)$. Let A be a discrete valuation ring over F with fraction field H, uniformizer t, and residue field C_p. Suppose G acts on H by F-algebra automorphisms preserving A, inducing the usual residue action, and*

$$g(t)/t \bmod t = \chi_p(g) \qquad (g \in G).$$

The map $F \to C_p$ on residue fields is its given inclusion. Let V be an n-dimensional F-representation of G. Suppose $C_p \otimes_F V$ has a basis $b_1, \dots, b_n$ and integers $k_1, \dots, k_n$ such

that on a common open subgroup of G the diagonal action satisfies $gb_i = \chi_p(g)^{-k_i} b_i$. Put $W = H \otimes_F V$, $L = A \otimes_F V$, with the diagonal action. Then

$$\dim_F W^G \le [K : F]n, \qquad \dim_F (W/L)^G \le [K : F]\#\{i : k_i < 0\}.$$

Proof. The field C_p^G equals K. Its algebraic fixed field is K by the extension of embeddings of Lemma 16.2. Theorem 233.2 and completeness of K, as in Theorem 223.4, give the claim. In a semilinear C_p-space, K-independent G-fixed vectors are independent over C_p. Otherwise select a dependent subfamily of least size, normalize the last coefficient of a dependence to one, apply any g and subtract. The smaller subfamily is independent, so every coefficient is fixed and belongs to K, a contradiction.

For an integer j, write $L_j = t^j L$. Multiplication by t^j identifies L_j/L_{j+1} with $C_p \otimes_F V$ as a vector space. In these coordinates g acts as $\chi_p(g)^j$ times the original residue action, by the uniformizer hypothesis, also for $j < 0$. Write a fixed vector as $\sum_i a_i b_i$. On the common open subgroup, $g(a_i) = \chi_p(g)^{k_i - j} a_i$. Since G is open in G_p, this is an identity near one in G_p. Theorem 233.4 therefore gives $a_i = 0$ if $k_i \ne j$. The independence argument bounds the K-dimension of $(L_j/L_{j+1})^G$ by $\#\{i : k_i = j\}$.

Every nonzero vector in W has a first nonzero graded class, by taking the least valuation among its coordinates in an F-basis of V. For an invariant vector this class is invariant. Thus its degree lies among the finitely many k_i. Filtering W^G by $W^G \cap L_j$, each quotient injects into $(L_j/L_{j+1})^G$. The terms are zero above the largest weight and exhaust W^G at the smallest. For W/L the same argument uses only $j < 0$: a nonzero coset has a representative with least coordinate valuation $j < 0$, and representatives differ in $L \subset L_{j+1}$. An invariant coset therefore has a fixed leading class. All its successive invariant quotients inject into the same graded spaces in negative degrees. Summing their F-dimensions gives both bounds. For $n = 0$ they hold directly. □

Theorem 637.2. *Use F, K, G, A, H, V of Lemma 637.1. Let $B \subset H$ be a G-stable F-subalgebra with an F-linear endomorphism ϕ fixing F pointwise and commuting with G. Suppose*

$$B^{\phi=1} \cap A = F.$$

Let O be the ring of integers of a finite extension F_0/F, T a free O-lattice of rank r whose underlying F-span is V, with G acting O-linearly. Let $Z_B(T)$ be the module of functions $c : G \to T$ such that for some $u \in B \otimes_F V$,

$$c(g) = gu - u \qquad (g \in G).$$

These functions are cocycles, include the integral coboundaries, and form a finite free O-module of rank satisfying

$$[F_0 : F]\big(\operatorname{rank}_O Z_B(T) - r\big) \le [K : F]\#\{i : k_i < 0\}.$$

No continuity hypothesis on the functions is needed for this bound.

Proof. View $B \otimes_F V$ inside W by choosing an F-basis. Put $D = (B \otimes_F V)^G$ and $U = \{u \in B \otimes_F V : gu - u \in V \text{ for all } g\}$. Then $D \subset W^G$ is finite dimensional by Lemma 637.1. On U consider the F-linear map

$$u \longmapsto \big((\phi \otimes 1 - 1)u,\ u \bmod L\big) \quad \text{into } D \oplus (W/L)^G.$$

Indeed every $gu - u$ lies in $V \subseteq L$, where $\phi \otimes 1$ is the identity. The kernel is V. In an F-basis this follows coordinatewise from $B^{\phi=1} \cap A = F$. Consequently U is finite dimensional and $\dim_F U \le n + \dim_F D + \dim_F (W/L)^G$. The map $u \mapsto (g \mapsto gu - u)$ on U has kernel D. Its image thus has dimension at most $n + \dim_F (W/L)^G$. Subtracting the finite dimension of D is the step that removes its contribution. Its image satisfies the cocycle identity by $ghu - u = g(hu - u) + (gu - u)$.

The image and the submodule of T-valued functions are stable under O, which acts on the V tensor factor and commutes with G. The F_0-span of the latter submodule has some finite

dimension d. There are d evaluations at group elements followed by individual coordinates in an O-basis of T that separate this span. Indeed if all evaluations lay in a proper subspace of its dual, a nonzero function would vanish everywhere. Choose a basis of the dual from the evaluations. On integral functions they give an injection into O^d. The coordinate projection argument of Lemma 634.2 shows that a submodule of O^d is finite free. Its rank here is d by the definition of the span. Taking $u \in T$ gives every integral coboundary. Finally $n = r[F_0 : F]$ by taking successive field bases, and $[F_0 : F]d \le n + \dim_F(W/L)^G$. The preceding lemma now gives the claim. □

638. Local embeddings and the sum of the negative weights

Lemma 638.1. *Let $F = \mathbb{Q}_\ell$, let E be a number field, and for each $v \mid \ell$ choose an embedding $e_v : E \to K_\ell$ inducing v, extended to an algebraic closure of E. Write $K_v = Fe_v(E)$ for the subfield generated by the two fields. Restriction identifies $\mathrm{Gal}(K_\ell/K_v)$ as a topological group with the decomposition group of this extended place in G_E. Let F_0/F be a finite normal extension in K_ℓ containing all K_v, put $d = [F_0 : F]$, and fix an isomorphism $j_\ell : \mathbb{C} \to K_\ell$. For fixed v the maps*

$$j_\ell^{-1} h^{-1} e_v, \qquad h \in \mathrm{Gal}(F_0/F),$$

give $[K_v : F]$ distinct embeddings $E \to \mathbb{C}$, each $d/[K_v : F]$ times. The sets so obtained are disjoint for different v.

Proof. The image of the algebraic closure of E is precisely the set of elements of K_ℓ algebraic over $\mathbb{Q}$, since $E/\mathbb{Q}$ is finite and the image is algebraically closed. It is dense in C_ℓ. Indeed for $a \in K_\ell$ approximate the coefficients of its monic minimal polynomial over F by rational coefficients, retaining the leading term. The resulting degree b polynomials P have $|P(a)|_\ell \to 0$ and some root has distance at most $|P(a)|_\ell^{1/b}$ from a, by multiplication of the linear factors. Density in C_ℓ follows from Definition 233.1. Each isometric field automorphism of this dense algebraic field fixing $e_v(E)$ extends by Cauchy sequences to a field automorphism of C_ℓ with isometric inverse. It fixes F by continuity and $\mathbb{Q}$-linearity. It preserves K_ℓ, the set of elements algebraic over F, hence fixes K_v on restriction as claimed. Conversely members of $\mathrm{Gal}(K_\ell/K_v)$ act isometrically by Definition 233.1, preserve the dense algebraic field, and fix $e_v(E)$. Density gives that the restrictions are inverse. They respect the topology of fixing finitely many algebraic elements. In the direction requiring a local element $a \in K_\ell$ to be fixed, choose a dense algebraic element closer to a than the distance to any other F-conjugate of a. Fixing this approximant forces a to be fixed by the ultrametric inequality. When $a \in F$ the condition holds throughout.

Two restrictions $h^{-1}e_v$ agree precisely when the two embeddings h^{-1} agree on K_v, as they already fix F. There are $[K_v : F]$ such restrictions by separability. Each extends to F_0 by the extension argument of Lemma 16.2 and normality. The fibers have equal size by composition with an automorphism. All the restrictions induce v: each F-automorphism of F_0 preserves the absolute value by Lemma 113.1. Distinct v therefore give disjoint sets. □

Lemma 638.2. *In the notation of Theorem 637.2, let O' be the integers of a finite extension of F_0. Extend T to $T' = O' \otimes_O T$ with scalar-linear action. Then $Z_B(T') = O' \otimes_O Z_B(T)$ inside the T'-valued functions, in the sense of the natural extension and multiplication of functions. In a fixed O-basis of T, every finite coordinate relation on $Z_B(T')$ is an O'-linear combination of the relations on $Z_B(T)$. Both assertions persist under a scalar twist whose restriction to G is trivial.*

Proof. The ring O' is finite free as an O-module. To see finiteness, choose an F_0-basis of its fraction field. The coordinate norm and the field norm are equivalent by the compact unit sphere argument of Lemma 113.1. Hence all coordinates of elements of O' lie in a single fractional O-ideal. A submodule of a finite free O-module is finite free by Lemma 634.2.

Choose an O-basis of O'; it is also a field basis after inverting a uniformizer, since sufficiently large powers carry each field element into O'.

In this basis T' is a direct sum of copies of T, the rational span is the same sum of copies of V, and the equation $gu - u = c(g)$ splits coordinatewise in $B \otimes_F (F \otimes_{\mathbb{Z}_\ell} T')$. Thus every component of c has a primitive precisely when c does. For any finite relation with O'-coefficients, decompose its coefficients in that same basis. Evaluating it on $1 \otimes Z_B(T)$ shows that every component vanishes there because the basis is independent over O. Conversely the first equality shows that each base relation vanishes on the full extended module. Triviality of a twist on G leaves the equation for its primitive identical. □

Proposition 638.3. *Use a CM field E as in Theorem 636.2, and F, F_0, e_v, j_ℓ as in Lemma 638.1. Let T be an integral row of rank r over the integers of F_0, and let J_σ be a multiset of integers for each complex embedding $\sigma : E \to \mathbb{C}$. Denote the number of negative entries by b_σ. Suppose $b_\sigma + b_{\bar\sigma} \le r$ for every σ. At each $v \mid \ell$ assume A, B, H, ϕ satisfy the ring assumptions of Theorem 637.2 for $K = K_v$. For each $h \in \mathrm{Gal}(F_0/F)$ assume the component $C_\ell \otimes_{F_0,h} (F_0 \otimes_O T)$ has a basis transforming near one with weights $J_{j_\ell^{-1} h^{-1} e_v}$ in the convention of Lemma 637.1. Write $B_v = Z_B(T|_{G_{K_v}})$. Then*

$$\sum_{v|\ell} (\mathrm{rank}_O B_v - r) \le \sum_{\sigma : E \to \mathbb{C}} b_\sigma \le fr, \qquad f = [E : \mathbb{Q}]/2.$$

These modules supply the coordinate relations in Theorem 636.2. For a twist trivial at ℓ they impose exactly the corresponding period relations after every finite extension of the coefficient field.

Proof. Since F_0/F is finite separable, the evaluation map $C_\ell \otimes_F F_0 \to \prod_h C_\ell$ is an isomorphism. Indeed a generator of the extension can be chosen with distinct values under all embeddings by taking a linear combination of a basis outside the finitely many hyperplanes where two values coincide. Such a choice exists over the infinite field F, by the nonvanishing of the product of the nonzero linear forms, or by successive substitution in a nonzero polynomial. The minimal polynomial of that element has d distinct roots, one for each embedding. Evaluation on them is bijective on polynomials of degree less than d by interpolation. This gives the stated isomorphism. Consequently $C_\ell \otimes_F (F_0 \otimes_O T)$ is the direct sum of the displayed components. Restricting to a common open subgroup fixing F_0 makes all the chosen basis identities simultaneous, as needed for the graded lemma.

Apply Theorem 637.2 at v and divide by d. Its right side becomes $[K_v : F] d^{-1} \sum_h b_{j_\ell^{-1} h^{-1} e_v}$, the sum of b_σ over the distinct embeddings in that list by Lemma 638.1. The lists are disjoint for different v, and the omitted terms, if any, are nonnegative. Complex conjugation pairs all $2f$ embeddings without a fixed point. Adding the hypothesis once per pair gives the final bound. The functions in B_v contain all integral coboundaries and are finite free by Theorem 637.2. Their relation length estimate for continuous local cocycles follows directly from Lemma 634.2; that lemma applies to a module of functions even before continuity is imposed on the solutions. The last assertion is Lemma 638.2. □

639. The tilt of the valuation ring and the Witt lift

Lemma 639.1. *Write $C = C_p$ as in Definition 233.1, $O_C = \{x \in C : |x|_p \le 1\}$, and $v = -\log_p |\cdot|_p$ with $v(0) = +\infty$. The ring*

$$R = \{(x_n)_{n \ge 0} : x_n \in O_C/pO_C,\ x_{n+1}^p = x_n\}$$

with componentwise operations is a perfect valuation domain of characteristic p. For every $x \in R$ there is a unique sequence $x^{(n)} \in O_C$ reducing to x_n and satisfying $(x^{(n+1)})^p = x^{(n)}$. Write $x^\sharp = x^{(0)}$. These lifts are multiplicative. The valuation $v^\flat(x) = v(x^\sharp)$ has

$$x \mid y \quad \Longleftrightarrow \quad v^\flat(x) \le v^\flat(y).$$

The ring is complete for this valuation, and each projection $x \mapsto x_n$ is onto O_C/pO_C. All constructions commute with the isometric action of G_p.

Proof. Frobenius on O_C/pO_C is a surjective ring map. For surjectivity choose an algebraic element of K_p within distance $< |p|_p$ of any proposed integral lift, using Definition 233.1. It is integral and has a p-th root in K_p of norm at most one. Thus successive roots in the quotient give arbitrary lifts to the inverse limit, also starting at a prescribed index by taking powers for the earlier indices. Frobenius on R is bijective, with inverse the shift $x_n \mapsto x_{n+1}$.

For $a, b \in O_C$, $a - b \in p^s O_C$, $s \geq 1$ integral, the binomial expansion gives $a^p - b^p \in p^{s+1} O_C$. Indeed the intermediate binomial coefficients are divisible by p, and $ps \geq s + 1$ for the last term. For arbitrary lifts $\widetilde{x}_j \in O_C$, it follows that

$$x^{(n)} = \lim_{m \to \infty} \widetilde{x}_{n+m}^{p^m}$$

exists, is independent of the lifts and reduces to x_n. For successive approximants their difference is in $p^{m+1} O_C$. Completeness follows from the definition of C. Taking p-th powers in the formula gives compatibility. Any exact sequence of lifts must agree with the formula, by using its own terms as lifts. Multiplication and the Galois action preserve the defining properties, proving the respective compatibilities.

The condition $x^\sharp = 0$ forces all the exact roots to vanish, hence $x = 0$. Multiplicativity gives that R is a domain. In general $v(x^{(n)}) = p^{-n} v^\flat(x)$. Modulo p the exact lifts of a sum agree with the sums of exact lifts. It follows for every n that

$$v^\flat(x + y) \geq \min(v^\flat(x), v^\flat(y), p^n).$$

Letting n increase gives the valuation inequality. When $x \neq 0$ and $v^\flat(y) \geq v^\flat(x)$, the quotients $y^{(n)}/x^{(n)}$ are integral compatible roots. Reducing them gives $z \in R$ with $y = xz$. Conversely such a factorization gives the valuation inequality. With $x = 0$ both conditions mean $y = 0$. This proves the asserted divisibility and the valuation ring property, since in its fraction field either a nonzero fraction or its inverse is integral. The kernel of the n-th projection is exactly $v^\flat(x) \geq p^n$. A Cauchy sequence therefore has eventually constant residues at every n. Their compatible tuple is its limit in R, as claimed. □

Proposition 639.2. *For a perfect domain R of characteristic p, let $W(R)$ be the sequences in R with sum and product defined by the universal polynomials making*

$$w_n(X) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \qquad (n \geq 0)$$

add and multiply componentwise over the integer polynomial ring. These polynomials have integer coefficients and make $W(R)$ a p-adically complete domain, with reduction $W(R)/pW(R) = R$ by the initial coordinate. Write $[x] = (x, 0, 0, \ldots)$. One has $[x][y] = [xy]$, and every element is uniquely a convergent sum $\sum_{j \geq 0} p^j [x_j]$ for $x_j \in R$. The map ϕ raising every Witt coordinate to its p-th power is an automorphism. Its fixed ring is $\mathbb{Z}_p$, embedded via the unit.

Proof. In a polynomial ring P over $\mathbb{Z}$ in all the variables, let σ raise variables to their p-th powers and fix the coefficients. For every $b \in P$, $\sigma b \equiv b^p \bmod pP$, by the binomial theorem and $a^p \equiv a \bmod p$ for integers a. The latter follows by induction on a from the same binomial identity, then reduction modulo p. Congruence modulo $p^s P$, $s \geq 1$, implies congruence of p-th powers modulo $p^{s+1} P$ by the expansion used in Lemma 639.1.

A sequence $b_n \in P$ has a unique expression $b_n = w_n(a_0, \ldots, a_n)$ with $a_i \in P$ if $b_n \equiv \sigma b_{n-1} \bmod p^n P$ for $n \geq 1$. In fact $a_0 = b_0$. At stage n, for $i < n$, one has

$$a_i^{p^{n-i}} \equiv (\sigma a_i)^{p^{n-1-i}} \pmod{p^{n-i} P}.$$

Thus subtracting the already determined terms from b_n leaves an element of $p^n P$. Division by p^n gives a_n uniquely since P is torsion free. This computation also proves the congruence

for every sequence $w_n(a)$. The congruences persist under addition, multiplication and negation of the b_n, giving the claimed integral polynomials and additive inverses. All ring identities follow by checking their ghost sequences over the universal polynomial ring, where the triangular equations are injective, and then specializing their coordinate identities in R. The identity is [1], and $w_n([x]) = x^{p^n}$ gives multiplicativity of $[\cdot]$ by the same argument.

Apply the congruence criterion to the shifted sequence $w_{n+1}(X)$, and write $\mathcal{F}_i(X)$ for the resulting coordinates. They satisfy $\mathcal{F}_i(X) \equiv X_i^p \bmod pP$. For $i = 0$ this is $w_1(X) = X_0^p + pX_1$. Inductively, in the equation $w_i(\mathcal{F}) = w_{i+1}(X)$ the lower-coordinate terms agree modulo p^{i+1} by the preceding power congruence. The remaining terms give the assertion upon division by p^i. Let $\mathcal{V}X = (0, X_0, X_1, \ldots)$. The identity $w_{n+1}(\mathcal{V}X) = pw_n(X)$ shows $\mathcal{F}\mathcal{V}X = pX$ over the universal ring. Consequently in characteristic p

$$pX = (0, X_0^p, X_1^p, \ldots).$$

Perfectness now identifies $p^aW(R)$ with the vectors whose first a coordinates vanish, and multiplication by p is injective. The coordinate formulas are triangular, so agreement in the first a coordinates is equivalently congruence modulo p^a. Existence of limits follows coordinatewise and separation follows by intersection. Successively subtracting the Teichmüller vector of the first coordinate and dividing by p gives the asserted expansion and its uniqueness. The product of two vectors with nonzero initial coordinate has nonzero initial coordinate because R is a domain. Extracting the maximal powers of p from two arbitrary nonzero elements proves that $W(R)$ is a domain.

All operations are functorial by the integral polynomial formulas, so coordinatewise Frobenius is an automorphism. A fixed vector has all coordinates solving $x^p = x$. This polynomial already has its p distinct roots in the prime field, so all such coordinates belong to $\mathbf{F}_p$. The embedded ring $W(\mathbf{F}_p)$ is $\mathbb{Z}_p$. Indeed the homomorphism from $\mathbb{Z}$ given by the unit extends by completeness. Its reduction modulo p is the identity of the prime field. Lifting one residue digit at a time, using injectivity of multiplication by p, shows surjectivity and injectivity of the extended homomorphism. This proves the fixed-ring assertion. □

640. Fontaine evaluation and its simultaneous vanishing ideal

Theorem 640.1. *Use the tilt R and $D = W(R)$ of Lemma 639.1 and Proposition 639.2. There is a surjective ring map $\theta : D \to O_C$ with $\theta([x]) = x^\sharp$, equivariant for G_p. Choose $\zeta_0 = 1$ and compatible primitive p^n-th roots ζ_n as in Definition 233.1. Let $\epsilon \in R$ be their reductions in successive coordinates, $\epsilon' = \epsilon^{1/p}$, and put*

$$\tau = [\epsilon'] - 1, \qquad \mu = [\epsilon] - 1, \qquad \xi = \sum_{a=0}^{p-1} [\epsilon']^a.$$

Then $\xi\tau = \mu$, $\phi(\tau) = \mu$, $\theta(\tau) = \zeta_1 - 1 \neq 0$, and

$$\ker\theta = \xi D, \qquad \bigcap_{j\geq 0} \ker(\theta\phi^j) = \mu D.$$

Proof. For Witt coordinates $a_i \in R$, lift their n-th tilt coordinates arbitrarily to $u_{i,n} \in O_C$. Set

$$H_n = \sum_{i=0}^{n} p^i u_{i,n}^{p^{n-i}}.$$

Its class modulo $p^{n+1}O_C$ is independent of the lifts by the power congruence of Lemma 639.1. Moreover $u_{i,n+1}^p \equiv u_{i,n} \bmod p$, so $H_{n+1} \equiv H_n \bmod p^{n+1}$. The limit defines θ. To check additivity and multiplicativity use the integral Witt polynomials of Proposition 639.2 on the chosen lifts at index n. Their outputs are lifts of the correct coordinates modulo p. The ghost identities at index n therefore give the ring identities modulo p^{n+1} for every n. The unit maps to one, and $\theta([x]) = x^\sharp$ follows from the limit formula for the exact root sequence. All constructions commute with G_p. The reduction of θ modulo p is the

surjection $R \to O_C/pO_C$ given by the initial tilt coordinate, after reducing D to R. Lift a given element of O_C successively modulo $p, p^2, \ldots$: after subtracting the image of the first lift, divide the remainder by p and repeat. The series of lifts converges in D, and its image is the original element by continuity of a ring map preserving p.

Use a bar for reduction in $D/pD = R$. The exact lifts of ϵ' are ζ_{n+1}. For large n, $v(\zeta_{n+1} - 1) = 1/(p^n(p-1)) < 1$ by the root valuation computed in Lemma 233.3. The n-th exact lift of $\epsilon' - 1$ has the same residue modulo p and hence the same valuation. Multiplying this valuation by p^n gives $v^\flat(\bar{\tau}) = 1/(p-1)$. The geometric sum identity and Frobenius give $\xi\tau = \mu$ and $\phi(\tau) = \mu$. In R the geometric sum is $(\epsilon' - 1)^{p-1}$, since $Y^p - 1 = (Y-1)^p$ in characteristic p. Thus

$$v^\flat(\bar{\xi}) = 1, \qquad v^\flat(\bar{\mu}) = p/(p-1).$$

Evaluation gives $\theta(\xi) = 0$ by the geometric sum at ζ_1, $\theta(\tau) = \zeta_1 - 1$, and $\theta\phi^j(\mu) = 0$ for every $j \geq 0$.

For a p-saturated ideal $I \subset D$, suppose $c \in I$ and $\bar{c}$ divides $\bar{a}$ for every $a \in I$. Here saturation means $pa \in I \Rightarrow a \in I$. Lifting the quotient modulo p writes $a = cb_0 + pa_1$, where $a_1 \in I$. Repeating gives $a = c\sum_{j\geq 0} p^j b_j$ by completeness and separation in Proposition 639.2. Thus $I = cD$. The kernel of θ is saturated since O_C has no p-torsion. For a in that kernel, $\bar{a}$ has initial tilt coordinate zero and hence $v^\flat(\bar{a}) \geq 1$ by Lemma 639.1. The same lemma gives $\bar{\xi} \mid \bar{a}$. This proves the first kernel identity.

For the intersection of kernels, put $\xi_i = \phi^{-i}\xi$. The kernel of $\theta\phi^i$ is $\xi_i D$ by the first identity. For $j > i$ one has $\theta\phi^j(\xi_i) = p$, as each of the p summands evaluates to one. If a is in the intersection it is divisible by $\prod_{i=0}^{b-1} \xi_i$ for every b. Indeed, after factoring this product, application of $\theta\phi^b$ shows that p^b times the image of the remaining factor vanishes. The integral domain O_C then puts that factor in $\xi_b D$. Consequently

$$v^\flat(\bar{a}) \geq \sum_{i=0}^{b-1} p^{-i} \quad \text{for every } b, \qquad v^\flat(\bar{a}) \geq p/(p-1).$$

This proves $\bar{\mu} \mid \bar{a}$ in the valuation ring. The intersection is saturated, again since O_C has no p-torsion. Applying the lifting argument with $c = \mu$ gives the second kernel identity. □

641. The integral period algebra

Lemma 641.1. *Let D be a p-adically complete separated ring where p is a nonzero divisor. Let $x \in D$ have reduction $\bar{x}$ a nonzero divisor in $R = D/pD$. If R is complete and separated for the $\bar{x}$-adic topology, then D is complete and separated for $I = (p, x)$. In particular this holds for $D = W(R)$, $x = \xi$ in Theorem 640.1.*

Proof. If $x^b c \in p^a D$ for $a, b \geq 1$, reduction modulo p gives $c \in pD$. Divide by p and continue to obtain $c \in p^a D$. For $a \geq 2$ this proves exactness of the sequence of additive groups

$$0 \longrightarrow R/\bar{x}^b R \xrightarrow{p^{a-1}} D/(p^a, x^b) \longrightarrow D/(p^{a-1}, x^b) \longrightarrow 0.$$

The first map uses any lift of a representative. If $p^{a-1}d = p^a u + x^b c$, the preceding divisibility makes $c = p^{a-1}t$. Cancellation gives $d = pu + x^b t$, proving injectivity. The image gives the kernel of reduction by the generators of the indicated ideals.

We show that a system of compatible classes in $D/(p^a, x^b)$, over every $b \geq 1$ with a fixed, comes from exactly one element of $D/p^a D$. The case $a = 1$ is the hypothesis on R. Inductively lift the already determined class modulo p^{a-1} to an element of D and subtract its reductions from the system. The differences give a compatible system in $R/\bar{x}^b R$, by exactness and injectivity at each b. Lift its limit in R to D and add p^{a-1} times that lift. This gives existence and the same kernel computation gives uniqueness. Finally $I^{a+b-1} \subseteq (p^a, x^b)$ by counting factors in its monomials, and $(p^m, x^m) \subseteq I^m$. Thus an I-adic system is equivalently one for all pairs (a, b). The unique classes just constructed for every a are compatible by uniqueness. Completeness and separation for p give the result. For the Witt ring, use

Proposition 639.2. Its residue R is complete for $v^\flat$ by Lemma 639.1, and $v^\flat(\bar\xi) = 1$ by Theorem 640.1. The valuation divisibility there identifies $\bar\xi^b R$ with the ball $v^\flat \geq b$, as required. □

Proposition 641.2. *Use the notation of Theorem 640.1. Define*

$$S = D[X]/(pX - \xi), \qquad A_{\max} = \varprojlim_{a\geq 1} S/p^a S, \qquad B^+ = A_{\max}[1/p].$$

Write X also for its image. Multiplication by p in $A_{\max}$ is injective, and reduction modulo the ideal it generates gives

$$A_{\max}/pA_{\max} = (R/\bar\xi R)[X].$$

Evaluation θ extends to a surjection $\theta_+ : B^+ \to C_p$, with $X \mapsto 0$, mapping $A_{\max}$ onto O_C. The map $D \to A_{\max}$ is injective, has closed image in the p-adic topology, and its inverse image of $p^a A_{\max}$ is $(p,\xi)^a D$. The actions of G_p and ϕ on D extend to commuting Galois and Frobenius actions on B^+, preserving $A_{\max}$. Here Frobenius is a ring endomorphism, and $\phi(\xi)/p \in 1 + pA_{\max}$.

Proof. Suppose $pP = (pX - \xi)Q$ in $D[X]$. Reducing gives $\bar\xi\bar Q = 0$ and thus $Q = pQ'$. Cancel p in $D[X]$, using Proposition 639.2, to get $P = (pX - \xi)Q'$. Thus p is regular in S. In its completion the kernel of projection to $S/p^b S$ is $p^b A_{\max}$: for a compatible sequence in the kernel divide its class at precision $a + b$ by p^b to get a unique class modulo p^a, using regularity in S. These classes are compatible. The same argument at precision $a + 1$ shows $pu = 0$ in the limit implies $u = 0$. Projection is surjective since an element of S lifts all its reductions. The presentation of S now gives the stated reduction modulo p. The map $D \to O_C$ sending ξ to zero extends to S by $X \mapsto 0$ and to $A_{\max}$ by p-adic limits in O_C. It is already onto on D. Inverting p gives θ_+.

Multiplication by X in $(R/\bar\xi R)[X]$ is injective, since it shifts polynomial coefficients. Reduction and successive division by p show that $Xc \in p^a A_{\max} \Rightarrow c \in p^a A_{\max}$. Let $a \geq 1$ and take $z \in D$ mapping into $p^a A_{\max}$. Then $\theta(z) \in p^a O_C$; choose $w \in D$ with $\theta(w) = p^{-a}\theta(z)$. The kernel formula of Theorem 640.1 writes $z = p^a w + \xi u$ in D. In $A_{\max}$ one has $pXu \in p^a A_{\max}$ and hence $u \in p^{a-1} A_{\max}$. Induction gives $u \in (p,\xi)^{a-1} D$ and then $z \in (p,\xi)^a D$. The reverse inclusion follows from $\xi = pX$. Separation in Lemma 641.1 proves injectivity, and its completeness proves closedness: a p-adic limit of elements of D is the image of their (p,ξ)-adic limit.

Each $g \in G_p$ preserves the kernel ξD of θ. Hence $g\xi = u_g \xi$ for a unit u_g of the domain D. Extend g to S by $gX = u_g X$. For Frobenius, $\phi(\xi) = \sum_{j=0}^{p-1}(1+\mu)^j$. Use $\mu = pX\tau$. After division by p the constant term is one and the terms $\binom{p}{i+1}\mu^i/p$, $1 \leq i \leq p-2$, lie in pS. The last term becomes $p^{p-2}X^{p-1}\tau^{p-1}$. Its reduction modulo p vanishes also for $p = 2$, since $\bar\tau^{p-1} = \bar\xi = 0$ in S/pS. This gives an element of $1 + pS$ as $\phi(\xi)/p$ and defines ϕX there. All maps preserve p, so extend to $A_{\max}$ and B^+. Uniqueness on S with the indicated restriction to D follows by multiplying the image of X by p and canceling. Hence the group identities, commutation with ϕ, and the inverse identities hold already on S, then in the completion by density. Equivariance of θ_+ under G_p follows in the same way from its values on D and X. □

642. The de Rham valuation and the Frobenius kernel

Lemma 642.1. *Use $D, X, A_{\max}, B^+$ of Proposition 641.2. Then*

$$\ker(\theta_+ : B^+ \to C_p) = \xi B^+, \qquad \bigcap_{j\geq 0} \ker(\theta_+\phi^j : B^+ \to C_p) = \mu B^+, \qquad (B^+)^{\phi=1} = \mathbb{Q}_p.$$

The ring B^+ is separated for the ideal ξB^+, and ξ is a nonzero divisor there.

Proof. For each monic polynomial $M(X) \in D[X]$ of positive degree and each $y \in A_{\max}$ one may write

$$y = P(X) + M(X)z, \qquad z \in A_{\max}, \quad P \in D[X], \quad \deg P < \deg M$$

with zero remainder polynomial permitted. Indeed lift each successive p-digit of y by a polynomial in $D[X]$ using the description modulo p in Proposition 641.2. This writes $y = \sum_{j\geq 0} p^j P_j(X)$. Divide each P_j by the monic polynomial M by successive removal of its leading term. The finitely many remainder coefficients converge after summing with weights p^j, since D is p-adically complete. The quotients with the same weights converge after evaluation in $A_{\max}$. This proves the formula. Use $M = X$. If $\theta_+(y) = 0$, its constant remainder lies in ξD by Theorem 640.1. Since $\xi = pX$, inverting p proves the first assertion.

In D the polynomial identity $\sum_{i=0}^{p-1}(1+\tau)^i = p + \tau h$ defines some $h \in D$ and gives $\xi = p + \tau h$. Thus

$$X(X-1) = \mu h/p^2 \quad \text{in } B^+.$$

Take $M = X(X-1)$ in the division formula. Every element of B^+ is the sum of an element of $D[1/p]$ and one in μB^+, since $X = \xi/p$ belongs to $D[1/p]$ inside B^+. The maps $\theta_+\phi^j$ kill μ by Theorem 640.1. For an element of $D[1/p]$ killed by all of them, the same theorem, after clearing a power of p, gives divisibility by μ. This proves the second assertion.

Take $y \in A_{\max}$ fixed by ϕ. Division by $X - 1$ and $p(X-1) = \tau h$ give $py = a_0 + \tau z_0$ with $a_0 \in D$, $z_0 \in A_{\max}$. If $py = a_j + p^j \tau z_j$, applying ϕ gives

$$py = \phi(a_j) + p^{j+1}\tau X \phi(z_j),$$

since $\phi(\tau) = \mu = pX\tau$. Hence py is a p-adic limit of elements of D. That subring is closed and injects by Proposition 641.2. The limit is fixed by ϕ in D, so lies in $\mathbb{Z}_p$ by Proposition 639.2. Clearing denominators now gives $(B^+)^{\phi=1} = \mathbb{Q}_p$.

Finally X acts injectively modulo p on $A_{\max}$ by Proposition 641.2. It follows by successive reduction that $X^n u \in p^b A_{\max}$ implies $u \in p^b A_{\max}$ for $u \in A_{\max}$, and by separation that X is regular. Regularity passes to B^+ and to $\xi = pX$. For a nonzero element of B^+ multiply by a power of p with integer exponent to put it in $A_{\max}$ with nonzero reduction. Call the result y. If $y = X^n z$ in B^+, clear a denominator of z and use the preceding implication to get $z \in A_{\max}$. The reduction of y would then be divisible by X^n in $(R/\bar{\xi}R)[X]$. A nonzero polynomial cannot have this property for arbitrarily large n, by its degree. This proves separation. □

Theorem 642.2. *Set*

$$A_{\mathrm{dR}} = \varprojlim_{n\geq 1} B^+/\xi^n B^+, \qquad H = \mathrm{Frac}(A_{\mathrm{dR}}), \qquad B_{\max} = B^+[1/\mu].$$

The ring A_{dR} is a discrete valuation ring over $\mathbb{Q}_p$ with residue field C_p, and B^+ embeds into it. The Galois action extends to H, preserving A_{dR} and inducing the usual action on the residue. Its uniformizer $t = \mu$ satisfies

$$(gt)/t \bmod t = \chi_p(g) \qquad (g \in G_p).$$

The ring $B_{\max}$ embeds in H, is stable under G_p, and has a commuting Frobenius endomorphism fixing $\mathbb{Q}_p$. It satisfies

$$B_{\max}^{\phi=1} \cap A_{\mathrm{dR}} = \mathbb{Q}_p.$$

Proof. By Lemma 642.1, ξ is regular in B^+, its quotient is C_p, and its powers have intersection zero. Thus the map to the completion is injective. Division of compatible classes by ξ^n, exactly as for p^n in Proposition 641.2, identifies the kernel of projection at n with $\xi^n A_{\mathrm{dR}}$ and shows regularity of ξ in the completion. An element with nonzero residue is a unit: lift the inverse residue and invert the resulting element $1 - u$, $u \in \xi A_{\mathrm{dR}}$, by its convergent geometric series. Each nonzero element has finite order of divisibility by ξ, by separation, and hence is its corresponding power times a unit. Regularity gives that this is a domain.

The least order of any nonzero ideal supplies a generator, proving the valuation ring claim. Since $\theta_+(\tau) = \zeta_1 - 1 \neq 0$, $t = \xi\tau$ is a uniformizer. The G_p-action extends by the compatible quotients since it preserves ξD, and extends to the fraction field. Equivariance of θ_+ in Proposition 641.2 gives the residue action.

For $a \in \mathbb{Z}_p$ form $\epsilon^a \in R$ by the exact root sequence $\zeta_n^{a \bmod p^n}$, and put $U(a) = [\epsilon^a]$. This is a multiplicative character of the additive group, by uniqueness of exact lifts in Lemma 639.1. Since $\theta(U(a)) = 1$, its difference from one belongs to $\xi D \subset pA_{\max}$. Thus $U(a + p^m b)/U(a) = U(b)^{p^m} \in 1 + p^{m+1} A_{\max}$ by the binomial congruence of that lemma. For nonnegative integers a one has $U(a) = (1+\mu)^a$. For every fixed j, the rational polynomial $\binom{a}{j}$ is continuous on $\mathbb{Z}_p$ and integral-valued there, by density of the nonnegative integers. As $\mu \in pA_{\max}$, the binomial series converges uniformly in a and agrees by continuity with $U(a)$. Taking $a = \chi_p(g)$ gives $g\mu = \mu(\chi_p(g) + \mu v_g)$ with $v_g \in A_{\max}$. The parenthesis is a unit by p-adic completeness, and its residue in A_{dR} is $\chi_p(g)$. This proves the uniformizer formula and the stability of $B_{\max}$. It embeds in H because B^+ injects and $\mu \neq 0$. Moreover $\phi(\mu) = \mu\phi(\xi)$, and $\phi(\xi)$ is invertible in B^+ by Proposition 641.2. Therefore ϕ extends to the localization, where it commutes with G_p and fixes $\mathbb{Q}_p$.

Suppose y is in the indicated intersection, and write $y = b/\mu^m$, $b \in B^+$. If $m > 0$, then $\phi^j(b) = \phi^j(\mu)^m y$ belongs to tA_{dR} for every $j \geq 0$. Indeed $\phi^j(y) = y \in A_{\mathrm{dR}}$ and $\phi^j(\mu) = (1+\mu)^{p^j} - 1$ is divisible by t. Consequently $\theta_+\phi^j(b) = 0$ for all j, whence $b \in \mu B^+$ by Lemma 642.1. Cancel one power and repeat to get $y \in B^+$. The fixed-ring assertion of that lemma gives $y \in \mathbb{Q}_p$. The reverse inclusion follows from the constructions. □

643. Descent of the period lattice and the weights of a row

Lemma 643.1. *Use $A = A_{\mathrm{dR}}, H, t$ of Theorem 642.2. There is a G_p-equivariant embedding of K_p into A whose composite with reduction equals its inclusion in C_p. For a finite extension $K/\mathbb{Q}_p$ in K_p, write $G = G_K$. Then $H^G = K$, and for every $j \in \mathbb{Z}$*

$$(H/t^j A)^G = \mathrm{image}(K \to H/t^j A).$$

If $L \subset H^n$ is a free A-module spanning H^n, of rank n, stable under componentwise G-action, there is a K-basis $e_1, \ldots, e_n$ of K^n and integers a_i with $L = \bigoplus_i A t^{a_i} e_i$.

Proof. For $z \in K_p$ its minimal polynomial P over $\mathbb{Q}_p$ has $P'(z) \neq 0$ in C_p. Starting at an arbitrary lift of z to A, an approximate root modulo t^m can be corrected by a multiple of t^m to a root modulo t^{m+1}, since the coefficient of this correction modulo t is the unit $P'(z)$. Completeness gives a root. If two roots have the same residue, factoring the difference of their polynomial values gives their difference times a unit, whose residue is $P'(z)$. Thus the lift is unique. For a finite extension of $\mathbb{Q}_p$ in K_p, choose a primitive element by the distinct-embedding argument of Proposition 638.3. Sending it to this root embeds the finite extension in A. Every element of the extension maps to its unique lift by the minimal polynomial description. The embeddings are therefore compatible, giving the asserted field embedding, and uniqueness also gives equivariance.

For a representative $x \notin t^j A$ of an invariant coset, write its t-order as $d < j$. The nonzero residue of $t^{-d}x$ transforms under G by χ_p^{-d}, using the uniformizer formula of Theorem 642.2. Theorem 233.4 forces $d = 0$. Its residue is then in K by Lemma 637.1 and its fixed-field proof. Subtract the algebraic lift of this residue. If its coset were still nonzero, its order would be positive and less than j, contradicting the same argument. For x fixed in H the leading-class argument applies at every finite order, and the difference after subtracting its lifted residue must vanish. This proves both invariant statements.

For the lattice proceed by induction on n, starting at $n = 0$. Projection onto the first coordinate gives a nonzero fractional ideal $t^a A$: the finitely many coordinates of a generating family have a least valuation. The kernel $L' \subset H^{n-1}$ is a full lattice. Indeed a lift of t^a splits off one free summand, and L' is a finite free module by the submodule argument of

Lemma 634.2, which uses only the discrete valuation property. The ranks over H show that it spans the coordinate hyperplane. The kernel is G-stable. By induction write it in the prescribed form with exponents a_i, $2 \le i \le n$. In the new fixed coordinates take a lift $v = (t^a, y_2, \dots, y_n)$. For each g the difference $gv - (gt/t)^a v$ belongs to the kernel, as gt/t is a unit. Hence

$$g(y_i/t^a) - y_i/t^a \in t^{a_i - a} A.$$

By the invariant-coset identity choose $u_i \in K$ with $y_i/t^a - u_i \in t^{a_i-a}A$. Subtracting an element of L' replaces the lift by $t^a(1, u_2, \dots, u_n)$. This fixed direction together with the kernel basis gives the desired K-basis and decomposition. □

Proposition 643.2. *Let $F = \mathbb{Q}_p$, K/F finite, $G = G_K$, and V a finite-dimensional F-linear representation. Call it potentially crystalline if $B_{\max} \otimes_F V$ is spanned over $B_{\max}$ by vectors each fixed by an open subgroup of G. Then $H \otimes_F V$ admits an H-basis fixed by G. In some such basis e_i, $A_{\mathrm{dR}} \otimes_F V = \bigoplus_i A_{\mathrm{dR}} t^{a_i} e_i$. The reductions of $t^{a_i} e_i$ give weights $k_i = -a_i$ in Lemma 637.1, and*

$$\dim_F \big((H \otimes_F V)/(A_{\mathrm{dR}} \otimes_F V)\big)^G = [K : F] \#\{i : k_i < 0\}.$$

If V is also linear over a finite field extension F_0/F, then for each embedding $h : F_0 \to K_p$ its component $C_p \otimes_{F_0, h} V$ admits a frame near one. For an integer u, the multiplicity of u in that frame equals the K_p-dimension of the h-semilinear maps $f : V \to C_p$ satisfying

$$f(gx) = \chi_p(g)^{-u} g(f(x)) \qquad (x \in V,\ g \text{ sufficiently near } 1).$$

The neighborhood may depend on x.

Proof. Choose a basis over H among the given spanning vectors after scalar extension, and a finite normal field K'/K whose fixing subgroup $G_{K'}$ fixes them all. Such a subgroup exists by the topology of Definition 233.1. In that basis the $G_{K'}$-invariants form $(K')^n$ by Lemma 643.1, where $n = \dim_F V$. This space carries a semilinear action of $\Delta = \mathrm{Gal}(K'/K)$. Its fixed vectors span over K'. Otherwise a nonzero K'-linear functional λ annihilates their span. For each vector z and scalar $a \in K'$, the sum $\sum_{\sigma \in \Delta} \sigma(az)$ is fixed. Consequently $\sum_\sigma \lambda(\sigma z)\sigma(a) = 0$ for every a. Distinct field automorphisms are linearly independent as functions. In a dependence of minimal support, comparing the identity at ab with its value at a times one selected $\sigma(b)$ removes one term, and a second term survives for a choice of b distinguishing the two automorphisms. A singleton dependence is impossible by evaluation at one. This proves independence and gives $\lambda(z) = 0$ for all z, a contradiction. Choose a basis among the fixed vectors. In it G acts on H^n componentwise. Apply Lemma 643.1 to the stable lattice. Its reduction vectors transform by $\chi_p^{a_i}$ by the uniformizer formula of Theorem 642.2. Moreover the invariants in each quotient $H/t^{a_i} A_{\mathrm{dR}}$ are the image of K by the lattice lemma. This image is zero for $a_i \le 0$ and has F-dimension $[K : F]$ for $a_i > 0$, since every nonzero element of K has order zero in A_{dR}. This gives the equality.

For F_0-coefficients first restrict to an open subgroup fixing all coefficient embeddings. The interpolation proof in Proposition 638.3 splits the action of F_0 over A_{dR}, using the lifted embeddings from Lemma 643.1. Indeed the differences of the roots of a primitive polynomial are units there, since their residues are distinct. Each component projection preserves near-fixed vectors and their H-span. Choose a basis for each component among their projections and repeat the lattice argument over an open subgroup fixing that basis. This produces its frame.

An h-semilinear map extends uniquely to a C_p-linear functional on the h-component. Test its transformation law on a finite F_0-basis, shrinking the open subgroup to fix the image of F_0. On this subgroup the extended identity is $f(gb) = \chi_p(g)^{-u} g(f(b))$ on the whole component, by semilinearity of the action. On a frame vector of weight k_i it gives $g(f(b_i)) = \chi_p(g)^{u - k_i} f(b_i)$. Theorem 233.4 forces zero for unequal weights. For equal weights the value is fixed near one and hence belongs to K_p by Theorem 233.2, as in Lemma 234.1.

Conversely any list of elements of K_p at the positions $k_i = u$, with zero elsewhere, defines such a map, since finitely many algebraic elements are fixed on a common open subgroup. This proves the multiplicity formula. □

Corollary 643.3. *In Theorem 636.2 retain its first three conditions. For each row at every $v \mid \ell$ suppose its restriction to the local group is potentially crystalline, with the dimensions of the semilinear maps in Proposition 643.2 given by the multiplicities in $J_{j_\ell^{-1}h^{-1}e_v}$ at each h, with the coefficient field as in Proposition 638.3. Suppose the negative multiplicities at each conjugate pair of complex embeddings sum to at most the row rank. Impose at the places above ℓ all integral coordinate relations of the crystalline cocycles, namely the functions with $c(g) = gu - u$ for $u \in B_{\max} \otimes_{\mathbb{Q}_\ell} V$, and require $c(\gamma_i) = 0$. Then the common twist and all three bounds in Theorem 636.2 hold for these relations over every finite coefficient extension containing the specialized values.*

Proof. The period ring hypotheses of Theorem 637.2 hold for $B_{\max}, A_{\mathrm{dR}}$ by Theorem 642.2. The local frames and their multiplicities follow from Proposition 643.2. Thus Proposition 638.3 gives the aggregate rank inequality with the specified relations. Apply Theorem 636.2. Its twist is trivial above ℓ. Lemma 638.2 therefore identifies the defining relations after specialization and change of coefficients. □

644. The fundamental sequence and the exact rank of crystalline cocycles

Lemma 644.1. *Every monic polynomial of positive degree over the tilt R of Lemma 639.1 has a root in R. For $s, b \in D = W(R)$ the equation*

$$\phi x - sx = b$$

has a solution in D.

Proof. Project a monic degree d polynomial P over R to the n-th copy of O_C/pO_C. Its coefficients have lifts in the integral algebraic elements of K_p, by density in Definition 233.1. The resulting monic polynomial has a root in K_p, necessarily integral since at a root of norm greater than one the leading term would strictly dominate all others. Reduce this root modulo p and lift it to $z_n \in R$ by Lemma 639.1. Then $v^\flat(P(z_n)) \geq p^n$.

For any positive integer j, among points with $v^\flat(P(z)) \geq dj$ there are at most d residue classes modulo the ball $v^\flat \geq j$. Otherwise take $d+1$ points $z'_0, \ldots, z'_d$ in different classes. Interpolation over the fraction field at those points gives for the leading coefficient

$$1 = \sum_{i=0}^{d} \frac{P(z'_i)}{\prod_{h \neq i}(z'_i - z'_h)}.$$

Indeed the interpolating polynomials $\prod_{h \neq i}(X - z'_h)/(z'_i - z'_h)$ have the desired values, and the difference of the two degree at most d polynomials has $d+1$ roots. Each term on the right has strictly positive valuation, a contradiction. Starting from the infinite list of indices n, retain an infinite subset in one class at $j = 1$, and then for each successive j retain an infinite subset of the previous indices in one such class. Only finitely many indices at each stage fail $p^n \geq dj$. A choice of one increasing index from each nested subset gives a Cauchy sequence whose polynomial values tend to zero. Completeness of R gives a root by continuity of its ring operations. Now reduce $\phi x - sx = b$ modulo p. The monic polynomial $Y^p - \bar{s}Y - \bar{b}$ has a root by what was proved. After any lift of this root the error is divisible by p. Divide the error by p and repeat with that right-hand side. The resulting series of corrections converges by Proposition 639.2; ϕ preserves p, so the limit solves the equation. □

Theorem 644.2. *With $B = B_{\max}$, $A = A_{\mathrm{dR}}$, $H = \mathrm{Frac}(A)$ of Theorem 642.2, the following sequence of $\mathbb{Q}_p$-spaces with Galois action is exact:*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B \xrightarrow{\;y \mapsto (\phi y - y,\ y \bmod A)\;} B \oplus H/A \longrightarrow 0.$$

Proof. The kernel is Theorem 642.2. For surjectivity first take any $\beta \in H$, choose $b \geq 0$ with $\mu^b \beta \in A$, and choose $x \in B^+$ congruent to it modulo $\mu^b A$. This is possible since $\mu^b A = \xi^b A$ and projection of B^+ to each quotient of its completion is onto. For $b = 0$ take $x = 0$. Subtract the image of x/μ^b from the target pair. It remains to solve $(\phi - 1)y = a/\mu^m$, $y \in B \cap A$, where $a \in B^+$ and $m \geq 0$.

We first solve $(\phi - \xi^m)z = a$ in B^+. Use the monic polynomial $(X(X-1))^{m+1}$ in the division argument of Lemma 642.1. Since $X \in D[1/p]$ and $X(X-1) = \mu h/p^2$ for $h \in D$ there, this gives $a = a_0 + \mu^{m+1} w$ with $a_0 \in D[1/p]$, $w \in B^+$. The equation with right-hand side a_0 is solvable in $D[1/p]$ by clearing a power of p and using Lemma 644.1. For the other summand note the identity

$$(\phi - \xi^m)(\xi \tau^{m+1} d) = \mu^{m+1}(\phi(\xi)\phi(d) - d).$$

It uses $\phi(\tau) = \xi\tau = \mu$. The additive operator $U(d) = \phi(\xi)\phi(d)$ carries $p^j A_{\max}$ into $p^{j+1} A_{\max}$ by Proposition 641.2. Thus $d = -\sum_{i \geq 0} U^i(w)$ belongs to B^+, by clearing one denominator of w and summing in the complete ring $A_{\max}$. Telescoping gives $U(d) - d = w$. Adding the two solutions gives the required z. Finally $y = z/\tau^m$ belongs to B, since $\tau^{-1} = \xi/\mu$, and belongs to A, since $\theta_+(\tau) \neq 0$. The identity

$$(\phi - 1)(z/\tau^m) = (\phi z - \xi^m z)/\mu^m$$

proves surjectivity. The maps commute with Galois actions by Theorem 642.2. □

Theorem 644.3. *Use $F = \mathbb{Q}_p$, $G = G_K$ for a finite extension K/F, and a potentially crystalline F-representation V as in Proposition 643.2. Assume T is a G-stable free lattice of rank r over the integers O of a finite extension F_0/F in V, with $V = F_0 \otimes_O T$ and G acting F_0-linearly. Write $Z_f(T) = Z_{B_{\max}}(T)$ as in Theorem 637.2. For the multiset of weights k_i of the underlying F-representation one has*

$$[F_0 : F](\operatorname{rank}_O Z_f(T) - r) = [K : F]\#\{i : k_i < 0\}.$$

Proof. Tensor the exact sequence in Theorem 644.2 with V over F, which in a basis takes a finite direct sum of exact sequences. Use D, U, W, L as in the proof of Theorem 637.2, with $B = B_{\max}$. Every invariant target pair has a preimage u, for which the image of $gu - u$ is zero for every g. Thus $gu - u \in V$, so $u \in U$. The map from U onto $D \oplus (W/L)^G$ is surjective with kernel V. The image of the cocycle map on U therefore has dimension $\dim_F V + \dim_F (W/L)^G$, by subtracting $\dim_F D$ as in the same theorem. The second dimension is computed in Proposition 643.2.

Every cocycle in this rational image is an F-multiple of an integral one. Choose a $\mathbb{Z}_p$-basis of T; such a basis exists by the finite freeness proof in Lemma 638.2. For its primitive u in the rational period space choose $m, b \geq 0$ with $v = p^b \mu^m u$ having all coordinates in $A_{\max}$. Since the action preserves T and $A_{\max}$, and since $g\mu/\mu$ is a unit in $A_{\max}$ by the binomial calculation in Theorem 642.2, all coordinates of

$$p^b \mu^m (gu - u) = (g\mu/\mu)^{-m} gv - v$$

belong to $A_{\max}$, uniformly in g. The nonzero element μ^m has a finite order h for the p-adic filtration on $A_{\max}$, by separation and injectivity in Proposition 641.2. For a nonzero scalar $c \in F$, $p^b \mu^m c \in A_{\max}$ implies $v_p(c) \geq -b-h$: a p-adic unit acts invertibly, and μ^m is divisible by exactly p^h. Therefore $p^{b+h}(gu - u) \in T$ for every g. The scaled function has primitive $p^{b+h}u$ and belongs to $Z_f(T)$. The rational dimension computed above is consequently the F-dimension of the span of $Z_f(T)$; it equals $[F_0 : F] \operatorname{rank}_O Z_f(T)$ by its finite freeness in Theorem 637.2. Substitution gives the formula. □

645. Continuity and saturation of the local conditions

Lemma 645.1. *Give $A_{\max}$ its p-adic topology, and B^+ the topology with $p^n A_{\max}$ as neighborhoods of zero. For each $b \in B^+$ both $g \mapsto gb$ and $g \mapsto g\mu/\mu$ are continuous on G_p in this*

topology. For a continuous integral representation T as in Theorem 644.3, every member of $Z_f(T)$ is continuous in its ordinary p-adic topology.

Proof. For each $x \in R$ of Lemma 639.1, $gx \to x$ as $g \to 1$ in the $v^\flat$-topology. Indeed prescribe any tilt coordinate modulo p. Approximate its lift in O_C within distance $< |p|_p$ by an algebraic element. Its open stabilizer fixes the prescribed residue since the action is isometric. The kernels of the coordinate projections give a neighborhood basis in R.

For an element $u \in D = W(R)$ and fixed $m \geq 1$ the first m Witt coordinates of $gu - u$ tend to zero in R by the integral polynomial formulas of Proposition 639.2. We claim that they can be required small enough to put $gu - u$ in $I^m = (p, \xi)^m D$. Indeed multiplication by $[c]$ multiplies the i-th Witt coordinate by c^{p^i}. This identity follows by evaluating the ghost polynomials over the universal integer polynomial ring and comparing their terms, as in that proposition. If the first m coordinates of a vector are divisible in R by $\bar{\xi}^{mp^{m-1}}$, then modulo $p^m D$ it is a multiple of $[\bar{\xi}]^m$. Divide its i-th coordinate by $\bar{\xi}^{mp^i}$ for $0 \leq i < m$ and take all later coordinates zero to get a multiplier. The difference has first m coordinates zero. Since $[\bar{\xi}] - \xi \in pD$, this proves the claim. The required divisibilities define neighborhoods by the valuation formula of Theorem 640.1. By Proposition 641.2 this gives continuity for u in $A_{\max}$. In particular $gX - X = (g\xi - \xi)/p$ tends to zero. Thus all polynomial values in X with coefficients in D have continuous orbits at one. They are dense in $A_{\max}$ for the p-adic topology, and the action preserves $p^n A_{\max}$ in both directions, giving continuity for each element by approximation. Inverting p gives the assertion for B^+. Translation gives continuity at all g. The binomial series for $(g\mu)/\mu = \sum_{j \geq 1} \binom{\chi_p(g)}{j} \mu^{j-1}$ from Theorem 642.2 converges uniformly and its terms are continuous, since each cyclotomic residue exponent is determined on a finite extension. It also takes values in the units of $A_{\max}$; inversion there is continuous by the difference-of-inverses formula.

For a primitive u of a cocycle c, write $v = p^b \mu^m u$ with coordinates in $A_{\max}$ as in Theorem 644.3. The expression $p^b \mu^m c(g) = (g\mu/\mu)^{-m} gv - v$ is continuous coordinatewise in B^+. Indeed the matrices on the chosen $\mathbb{Z}_p$-basis of T are continuous and taking a product or a finite sum in $A_{\max}$ is continuous by its ideal filtration. For rational scalars a, multiplication by $p^b \mu^m$ induces the usual topology of $\mathbb{Q}_p$ on its image. In fact if μ^m has exact p-order h in $A_{\max}$, the order of $p^b \mu^m a$ is $b + h + v_p(a)$ when $a \neq 0$, by factoring off a unit of $\mathbb{Z}_p$. The displayed identity thus proves continuity of every coordinate of $c(g)$. □

Proposition 645.2. *Let T, free of rank r over O, satisfy the hypotheses of Theorem 644.3 with continuous action. Let ϖ be a uniformizer of O. For $j \geq 1$ the functions $G \to T/\varpi^j T$ obeying every integral coordinate relation of $Z_f(T)$ are precisely the reductions of $Z_f(T)$. They are continuous cocycles. The kernel of reduction is $\varpi^j Z_f(T)$. In particular the solution module at precision j is free over $O/\varpi^j O$ of rank* $\operatorname{rank}_O Z_f(T)$.

Proof. If $c \in Z_f(T)$ has values in ϖT, then $\varpi^{-1} c$ is T-valued and $\varpi^{-1} u$ is its primitive for any primitive u of c. Here ϖ^{-1} acts on the rational representation factor. Thus $Z_f(T)$ is saturated in the module of T-valued functions. Iterating gives the assertion on the kernel. In particular the evaluation map on $Z_f(T)/\varpi Z_f(T)$ has zero kernel. This is a finite-dimensional space over the residue field by Theorem 637.2. Hence all scalar coordinate evaluations span its full dual: otherwise their common annihilator would be nonzero. Select as many as its dimension giving a residue dual basis. Their matrix on any O-basis of $Z_f(T)$ has determinant a unit and therefore an inverse by the adjugate formula. They form a basis of the full O-dual. The module Λ of evaluations in Lemma 634.2 is consequently that full dual. Every map $\Lambda \to O/\varpi^j O$ lifts to one into O by lifting the values on its basis, and the latter is evaluation on an element of $Z_f(T)$ by finite freeness and the dual basis. That lemma identifies these maps with exactly the functions in the statement. Continuity follows from Lemma 645.1 and the cocycle identity descends by reduction. □

646. Complementary counts for local lattices in duality

Lemma 646.1. *In the rings of Theorem 642.2, there is an element $t_0 \in B_{\max}^{\times} \cap A_{\mathrm{dR}}$ associated to μ in A_{dR}, with $\phi(t_0) = pt_0$, $gt_0 = \chi_p(g)t_0$. If O is the ring of integers of $F_0/\mathbb{Q}_p$ finite, there is a $\mathbb{Z}_p$-linear $\lambda : O \to \mathbb{Z}_p$ such that for every finite free O-module T, composition with λ identifies $\mathrm{Hom}_O(T, O)$ with $\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$.*

Proof. Take $t_0 = \sum_{j\geq 1}(-1)^{j-1}\mu^j/j$. This converges in $A_{\max}$ since $\mu \in pA_{\max}$ and $j - v_p(j) \to \infty$. For $p = 2$ one even has $\tau = \xi - 2 = 2(X - 1)$ and $\mu \in 4A_{\max}$. Consequently $t_0/\mu \in 1 + pA_{\max}$: for $j \geq 2$ the bound $j - 1 - v_p(j) \geq 1$ holds except at $p = j = 2$, which the extra divisibility covers. Here $v_p(j) \leq \log_p j$ follows from $p^{v_p(j)} \leq j$, and $p^{j-1} > j$ for $j \geq 3$ by induction. The quotient is a unit by the geometric series, so t_0 is a unit after inverting μ. Over $\mathbb{Q}[[Z]]$, differentiation and the zero constant term give $\log((1 + Z)^a) = a \log(1 + Z)$ for every nonnegative integer a. At $Z = \mu$ the substitution is legitimate: each term of degree s in the outer term indexed by j has denominator dividing $j \leq s$, and has valuation in the p-adic ideal filtration at least $s - \log_p s$. Thus the regrouped series converges. Taking $a = p$ gives the Frobenius identity. For $a \to \chi_p(g)$ through nonnegative integers, the binomial series in Theorem 642.2 gives $(1 + \mu)^a \to 1 + g\mu$. The logarithm is continuous on $1 + pA_{\max}$ by uniform convergence of its terms there. This proves the Galois identity.

The trace pairing $F_0 \times F_0 \to \mathbb{Q}_p$, $(x, y) \mapsto \mathrm{Tr}_{F_0/\mathbb{Q}_p}(xy)$, is nondegenerate: for $x \neq 0$ take $y = x^{-1}$, which gives the nonzero degree. Hence every field-linear functional over $\mathbb{Q}_p$ is represented by a unique trace multiplier, by equality of the finite dimensions. The trace multipliers of the integral functionals $O \to \mathbb{Z}_p$ form a full fractional O-ideal. Indeed a $\mathbb{Z}_p$-basis of O from Lemma 638.2 identifies these multipliers with the $\mathbb{Z}_p$-span of its trace-dual basis. This span is stable under O because the integral functionals remain integral after multiplying their arguments by any element of O. It is thus generated over O by these finitely many elements, and a member of least valuation among the generators generates the ideal. Choose such a generator c, and take $\lambda(x) = \mathrm{Tr}_{F_0/\mathbb{Q}_p}(cx)$. For $T = O$ the assertion is the choice of c; for a finite free T apply it on the coordinates of a basis. □

Theorem 646.2. *Let $K/\mathbb{Q}_p$ be finite of degree d, $G = G_K$, and T, T' continuous $\mathbb{Z}_p$-lattices of rank n with a perfect G-equivariant pairing into $\mathbb{Z}_p(1)$. Here $\mathbb{Z}_p(1)$ is the inverse limit of μ_{p^j} under p-th powers, generated over $\mathbb{Z}_p$ by the compatible roots of Definition 233.1. Assume both rational representations are potentially crystalline as in Proposition 643.2. Fix $m \geq 1$. Denote by $A(T)$ and $A(T')$ the images of the integral crystalline cocycles modulo p^m. One has*

$$|A(T)|\,|A(T')| = p^{mn(2+d)}.$$

The conclusion includes the underlying $\mathbb{Z}_p$-lattices of a perfect O-linear Tate pairing, after composition with the functional of Lemma 646.1.

For $N \in \mathbb{Z}$, let $T(N)$ denote the same lattice with action multiplied by χ_p^N. Some positive N preserves both actions modulo p^m when the exponents on T, T' are $N, -N$. For that N, over G and over each open subgroup $G_{K'}$, the two reduced crystalline images for $T(N), T'(-N)$ are orthogonal in the following sense. For every pair of their cocycles a, c, the product

$$(g, h) \longmapsto \langle c(g), ga(h)\rangle_m$$

is a multiplicative coboundary in $K_p^{\times}$. Here $\langle\ ,\ \rangle_m$ is the pairing into μ_{p^m}, and a coboundary is $b(g)\, gb(h)/b(gh)$ for a continuous function b to the discrete group $K_p^{\times}$. The complementary count holds for the twisted images over $G_{K'}$ with d replaced by $[K' : \mathbb{Q}_p]$.

Proof. If the jumps of the underlying representation on T are k_i, the jumps on its Tate dual are $-k_i - 1$. Indeed take the dual frame to the residue frame supplied by Proposition 643.2. It transforms on its basis positions by $\chi_p^{k_i}$; tensoring by $\mathbb{Z}_p(1)$ multiplies by χ_p. The multiplicity characterization of that proposition gives the claimed jumps. Exactly one of $k_i, -k_i - 1$

is negative for every integer k_i. Theorem 644.3 therefore gives that the ranks of the two crystalline cocycle lattices sum to $n(2+d)$. Proposition 645.2 gives the product of the orders. For O-duality the functional in Lemma 646.1 identifies the two underlying notions of perfect pairing; on $\mathbb{Z}_p(1)$ it acts on the coefficient alone.

Choose N a positive multiple of $|(\mathbb{Z}/p^m\mathbb{Z})^\times|$ large enough that all jumps of $T'(-N)$ are positive. This is possible since a twist by χ_p^{-N} adds N to the jumps in a residue frame. The finite group order kills every unit under exponentiation by its order, by partition into cosets of its cyclic subgroup. Both reduced actions therefore agree with the originals. Twisting preserves potential crystallinity: multiply each near-fixed period vector by t_0^{-N} for the action twisted by χ_p^N, and use Lemma 646.1 and invertibility of t_0 in $B_{\max}$. Restriction to an open subgroup preserves potential crystallinity and the same residue frames near one. The multiplicity formula there gives exactly the same weights at each such subgroup. Thus the first part applies after twisting and restriction.

Over any of the displayed groups H, the rational space of crystalline cocycles on $T'(-N)$ consists of coboundaries. Its dimension is n by the rank and rational-span argument in Theorem 644.3. The fixed subspace of the representation is zero: in a residue frame the coordinates of a fixed vector satisfy $gx_i = \chi_p(g)^{k_i'} x_i$ near one with $k_i' > 0$. Theorem 233.4 forces every coordinate to vanish. Rational coboundaries therefore already have dimension n and are crystalline.

Choose integral crystalline lifts $\tilde{a}, \tilde{c}$ of a, c. Write $\tilde{c}(g) = gy - y$ for a rational vector y in the twisted dual. The projection $\mathbb{Z}_p(1) \to \mu_{p^m}$ extends to a G_p-equivariant homomorphism $q_m : \mathbb{Q}_p(1) \to \mu_{p^\infty}$: with respect to the chosen compatible generator it takes a coordinate z to $\zeta_{m+j}^{p^j z \bmod p^{m+j}}$ for any $j \geq 0$ with $p^j z \in \mathbb{Z}_p$. Compatibility of the roots gives independence of j, and the exponent definition of χ_p gives equivariance. Its kernel is $p^m\mathbb{Z}_p(1)$. The function $b(g) = q_m(\langle y, \tilde{a}(g)\rangle)$ is continuous into the discrete roots because its kernel threshold is open and the lift is continuous by Lemma 645.1. Its multiplicative coboundary equals $q_m(\langle gy - y, g\tilde{a}(h)\rangle)$, by the cocycle identity for $\tilde{a}$ and equivariance of the pairing. This equals the product asserted in the statement. □

647. Relative factor sets and cyclic norms

Definition 647.1. Let $K/\mathbb{Q}_p$ be finite, $\Omega = K_p$, $G = G_K$. Give $\Omega^\times$ the discrete topology for cochains. A normalized factor set is a continuous function $z : G^2 \to \Omega^\times$ equal to one when either argument is one and satisfying

$$z(g,h)z(gh,k) = g(z(h,k))z(g,hk).$$

Write $\mathcal{H}(K)$ for the group of such functions under multiplication modulo the coboundaries $(\partial b)(g,h) = b(g)g(b(h))/b(gh)$, for continuous $b : G \to \Omega^\times$ with $b(1) = 1$. For an open subgroup G_L restriction of functions gives restriction of classes. A class split by L means one with trivial restriction. For finite abelian G-modules M, N with a multiplicative-valued G-equivariant biadditive pairing $\langle\,,\,\rangle$ into $\Omega^\times$, the cup of additive continuous cocycles $x : G \to M,\ y : G \to N$ is represented by

$$(x \smile y)(g,h) = \langle x(g), gy(h)\rangle.$$

Lemma 647.2. *The displayed cup is a normalized factor set, biadditive on cocycles. Reversing the factors and transposing the pairing gives the inverse class. A cup with a coboundary in either variable represents the identity. Every continuous multiplicative cocycle $a : G \to \Omega^\times$, satisfying $a(gh) = a(g)g(a(h))$, has the form $a(g) = g(u)/u$ for some $u \in \Omega^\times$. Every normalized factor set is inflated from one on a finite Galois group* $\mathrm{Gal}(L/K)$ *with values in* $L^\times$.

Proof. Expand $x(gh) = x(g) + gx(h)$ and $y(hk) = y(h) + hy(k)$ on the two sides of the factor set identity. Both yield the product of the same three pairings, and $x(1) = y(1) = 0$ gives normalization. Biadditivity follows immediately. For $b(g) = \langle x(g), y(g)\rangle$, expanding $b(gh)$

shows ∂b equals the inverse of the product of the two orders of cup. When $x(g) = gu - u$, the cup is the coboundary of $g \mapsto \langle u, y(g) \rangle$ by expansion. Transposition handles the other variable.

For the finiteness assertions recall why the profinite group here is compact. By Lemma 638.1 and its rational-polynomial approximation proof, the p-adically dense field of elements algebraic over $\mathbb{Q}$ supplies a countable family of finite local Galois extensions cofinal among stabilizers. More explicitly, approximate each element of a finite list in Ω by a rational algebraic element closer than its distances to other K-conjugates. Fixing the approximants forces the list to be fixed. Enumerate the rational algebraic elements and take their successive finite normal closures over K. Their increasing union is Ω: the field fixed by each finite normal stabilizer is its field by Lemma 16.2, and cofinality puts each given element in one of these fields. Restrictions identify G with the compatible tuples in the resulting countable product of finite groups. The product is compact: an open cover without a finite subcover would give successively nested cylinders with no finite subcover by partitioning each cylinder into its finitely many children. Their unique point belongs to a covering open set containing one of its cylinders, a contradiction. Compatibility is closed and the same construction applies to finite powers of G. A continuous map from G^a to a discrete space is therefore constant on coset rectangles for one common open normal subgroup in each coordinate: take a finite cover by cylinders of constancy and their common level. Its finitely many values lie in a finite normal extension if they are algebraic. Enlarging this extension to include the selected common level gives the inflation assertion, also for the continuous maps appearing in a coboundary. This applies to any open local subgroup by using its own fixed field.

For a multiplicative cocycle take the resulting finite group $\Delta = \mathrm{Gal}(L/K)$. For some $t \in L$ the sum $b = \sum_{\sigma \in \Delta} a(\sigma)\sigma(t)$ is nonzero, by the independence of the distinct automorphisms proved in Proposition 643.2. The cocycle law gives $gb = a(g)^{-1}b$. Taking $u = b^{-1}$ proves the assertion. □

Theorem 647.3. *For a finite Galois extension L/K, inflation identifies the classes split by L in $\mathcal{H}(K)$ with the factor sets of $\Delta = \mathrm{Gal}(L/K)$ in $L^\times$ modulo coboundaries on Δ. If Δ is cyclic of order b with chosen generator σ, these classes are identified with $K^\times / N_{L/K} L^\times$, a group of order b. The parameter $a \in K^\times$ represents the factor set*

$$z_a(\sigma^i, \sigma^j) = a^{\lfloor (i+j)/b \rfloor} \qquad (0 \le i, j < b).$$

Proof. For a normalized factor set z, the set $P = \Omega^\times \times G$ is a group for the law $(x, g)(y, h) = (x\, g(y)\, z(g, h), gh)$. The factor set identity is associativity. Normalization gives the unit $(1, 1)$ and solving the product for the identity gives inverses. It is a topological group in the product topology, since Galois acts continuously on discrete algebraic elements. A continuous section of $P \to G$ has factor set cohomologous to z: multiplying its lifts by a cochain multiplies the factor set by the coboundary. The restriction is trivial on $J = G_L$ precisely when a continuous section there can be chosen as a group homomorphism $s : J \to P$.

Suppose this has been done. For any lift $x \in P$ of g, the sections $h \mapsto xs(g^{-1}hg)x^{-1}$ and $h \mapsto s(h)$ differ by a continuous multiplicative cocycle $d(h) \in \Omega^\times$ on the left, as their values act on the kernel by h. Write $d(h) = h(u)/u$ by Lemma 647.2. Then ux normalizes $s(J)$. The normalizer P_0 therefore maps onto G, and its kernel is $L^\times$, since conjugation by a scalar a multiplies $s(h)$ on the left by $a/h(a)$. Thus $P_0/s(J)$ is a group extension of Δ by $L^\times$ with its usual action. Choose lifts in P_0 of one representative r of each coset of J, using the unit for J. On rh extend the lift of r by right multiplication by $s(h)$; this is continuous. Its factor set is valued in $P_0 \cap \Omega^\times = L^\times$ and depends only on the cosets, as seen by projecting to $P_0/s(J)$, which is injective on $L^\times$. Conversely an inflated normalized factor set is trivial on J. If it equals ∂b on G, $b|_J$ is a multiplicative cocycle. Dividing b by a cochain $g(u)/u$ gives $b|_J = 1$ by the preceding lemma without altering ∂b. Then $b(gh) = b(g)$ and $b(hg) = h(b(g))$ for $h \in J$, so $b(g) \in L^\times$ by normality and b descends to Δ. This proves injectivity.

In the finite extension group choose a lift u of σ. Then $u^b = a \in L^\times$ commutes with u and hence lies in $K^\times$. Using the lifts u^i gives exactly the displayed carry factor. Replacing u by wu multiplies a by $\prod_{i=0}^{b-1} \sigma^i(w)$. The carry identity, or addition of the exponents with reduction modulo b, gives a factor set for each a. These facts prove the asserted isomorphism. Finally apply the multiplicativity of the cyclic quotient ratio in Lemma 611.1 to $1 \to O_L^\times \to L^\times \xrightarrow{v_L} \mathbb{Z} \to 0$ with the surjective integral valuation. The unit ratio is one by Theorem 611.3. The ratio for the trivial module $\mathbb{Z}$ is b, its norm being multiplication by b and its norm kernel zero. The norm-one quotient for $L^\times$ is trivial by Lemma 611.2. The fixed group is $K^\times$, so its norm quotient has order b, as required. □

648. The local pairing for modules with a finite p-group action

Theorem 648.1. *Let $K/\mathbb{Q}_p$ be finite of degree d, and suppose K contains a primitive p-th root ζ. Put $G = G_K$ and $V = \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{F}_p)$, written additively. Use the cup product of Definition 647.1 with $\langle i, j\rangle = \zeta^{ij}$. Then $\dim V = d + 2$. There is a subgroup $D \subset \mathcal{H}(K)$ of order p such that the cup induces an isomorphism*

$$V \longrightarrow \mathrm{Hom}_{\mathbb{F}_p}(V, D).$$

Here the multiplication in D is its vector addition. For every cyclic degree-p extension L/K inside the chosen algebraic closure, the entire group of classes split by L is D.

Proof. For $a \in K^\times$, choose $\alpha^p = a$. The relation $g\alpha/\alpha = \zeta^{y_a(g)}$ defines $y_a \in V$ independently of the root chosen. Multiplication of roots gives additivity in a. The kernel consists of p-th powers in K. Conversely, for $y \in V$ the multiplicative cocycle ζ^y has the form $g\alpha/\alpha$ by Lemma 647.2. Then α^p is fixed, hence belongs to K by Lemma 16.2. This identifies V with $K^\times/K^{\times p}$.

Let $A = O_K^\times$, $U = 1 + p^2 O_K$, and $u = [A : U]$, finite by finiteness of the residue quotients in Lemma 113.1. The logarithm in Theorem 611.3 identifies U with a free $\mathbb{Z}_p$-module of rank d. Hence $[U : U^p] = p^d$ and U has no p-torsion. The map $A/U \to A^p/U^p$ is onto and has kernel $A[p]U/U$, of size p, since $A[p] = \langle\zeta\rangle$. Thus $[A : A^p] = (up^d)/(u/p) = p^{d+1}$. The integral valuation supplies one more factor p, proving the dimension assertion.

Fix $0 \neq x \in V$. Its kernel fixes a cyclic extension L/K of degree p, and x identifies its Galois group with $\mathbb{F}_p$. This fixed-field description follows on a finite Galois quotient as in Lemma 647.2 and Lemma 16.2. Let $i_g \in \{0, \dots, p-1\}$ represent $x(g)$. For $a = \alpha^p$ the cochain $b(g) = \alpha^{i_g}$ satisfies

$$(\partial b)(g, h) = \zeta^{y_a(g)x(h)} a^{\lfloor (i_g + i_h)/p \rfloor}.$$

By the reversal identity in Lemma 647.2, the class $x \smile y_a$ is the class represented by the carry factor on the right. Theorem 647.3 therefore identifies the image of the row $y \mapsto x \smile y$ with the full group split by L. In particular every nonzero row has rank one in the $\mathbb{F}_p$-vector space $\mathcal{H}(K)[p]$. Reversal implies that the common kernel of all rows is zero.

Two rows with distinct kernels must have the same image. Indeed, if their images were independent, a vector in the kernel of each row but outside that of the other would give, on adding the rows, two independent values. Their sum is itself a row. There exist two distinct kernels, since a single hyperplane in V has positive dimension and the common kernel is zero. Fix such a pair. A row with a different image would have to have both of these distinct kernels. Thus the nonzero rows all have the same image D. The displayed map is injective and its source and target have the same finite dimension. Every cyclic degree-p extension defines a nonzero x by choosing a generator of its Galois group, giving the last assertion. □

Theorem 648.2. *Keep K, G, d as in Theorem 648.1. Let M be a finite abelian p-group on which G acts continuously through a finite p-group. Put $M^\vee = \mathrm{Hom}(M, \mu_{p^\infty})$, with the action $(g\lambda)(u) = g(\lambda(g^{-1}u))$. Write $Z^1(G, M)$ for the group of continuous $c : G \to M$ with*

$c(gh) = c(g) + gc(h)$. *Use the evaluation pairing to define cups. If subgroups* $X \subset Z^1(G, M)$, $Y \subset Z^1(G, M^\vee)$ *satisfy* $x \smile y = 1$ *in* $\mathcal{H}(K)$ *for every* $x \in X, y \in Y$, *then both are finite and*

$$|X|\,|Y| \le |M|^{2+d}.$$

Proof. For $|M| = 1$ this holds directly. For $|M| = p$, the action is trivial because $|\mathbb{F}_p^\times| = p-1$. Choose a generator of M and the dual generator defined by ζ. The groups of cocycles are copies of V. Under the perfect pairing of Theorem 648.1, the annihilator of a subspace of dimension k has dimension $d + 2 - k$, by extending a basis of that subspace to a basis of V. This proves the bound.

For $|M| > p$ choose an invariant subgroup U of order p fixed pointwise. Indeed $M[p]$ is a nonzero vector space (a suitable power of p times any nonzero element gives one of order p). The acting p-group partitions it into orbits. All nonsingleton orbits have size divisible by p, so the number of fixed vectors is divisible by p and includes the zero vector. Put $N = M/U$. The dual sequence $0 \to N^\vee \to M^\vee \to U^\vee \to 0$ is exact. For surjectivity, extend a homomorphism from a subgroup A by one generator b at a time. If p^j is the order of $b + A$, choose as the value on b a p^j-th root of the given value on $p^j b$. It lies in μ_{p^∞}, and the relation $p^j b \in A$ gives a well-defined extension on $A + \mathbb{Z}b$.

Map X to $Z^1(G, N)$, writing X_N for the image and $X_U \subset Z^1(G, U)$ for the kernel. Map Y to $Z^1(G, U^\vee)$, writing Y_U for the image and $Y_N \subset Z^1(G, N^\vee)$ for the kernel. Evaluation commutes with quotient and inclusion. Hence X_U, Y_U and X_N, Y_N are orthogonal by the cup formula in Definition 647.1. The bound for U and the induction bound for N imply finiteness of all four groups and give

$$|X|\,|Y| = (|X_U|\,|Y_U|)(|X_N|\,|Y_N|) \le |U|^{2+d}|N|^{2+d} = |M|^{2+d}.$$

□

649. Transfer of local factor sets and crystalline cocycles

Lemma 649.1. *Let* $G = G_K$ *and let* $J = G_L$ *for a finite extension* L/K *of index* $h = [G : J]$. *Use a fixed set of representatives* r_C *of the left cosets* $C \in G/J$, *with* $r_J = 1$, *and put* $u_{g,C} = r_{gC}^{-1} g r_C \in J$. *Use additive notation for any discrete abelian* G*-module* P, *including the multiplicative group of the algebraic closure. For normalized continuous cochains* $b : J \to P$ *and* $z : J^2 \to P$ *define*

$$(\operatorname{Cor} b)(g) = \sum_C r_{gC}\, b(u_{g,C}),$$

$$(\operatorname{Cor} z)(g, k) = \sum_C r_{gkC}\, z(u_{g,kC}, u_{k,C}).$$

These formulas carry cocycles and coboundaries to cocycles and coboundaries of the same degree. For a factor set class α *of* G, $\operatorname{Cor}(\operatorname{res}\alpha) = h\alpha$. *For a* 1*-cocycle* a *of* G, *one has the equality of functions*

$$\operatorname{Cor}(\operatorname{res} a)(g) = h\, a(g) + gv - v, \qquad v = \sum_C a(r_C).$$

Proof. Cosets are open and the index is finite, so the functions are continuous. They are normalized since $u_{1,C} = 1$. The identities $u_{gk,C} = u_{g,kC} u_{k,C}$ and $r_{gC} u_{g,C} = g r_C$ give the degree-one cocycle identity by expansion. The transfer of $b(j) = jw - w$ is $gv_0 - v_0$, with $v_0 = \sum_C r_C w$. Substituting $(\partial b)(i, j) = b(i) + ib(j) - b(ij)$ in the second formula gives $\partial(\operatorname{Cor} b)$, after reindexing the first sum by kC. For the degree-two cocycle identity at (g, k, l), apply the identity for z to $(u_{g,klC}, u_{k,lC}, u_{l,C})$, multiply by the action of r_{gklC}, and sum. Reindexing by lC yields

$$\operatorname{Cor} z(g, k) + \operatorname{Cor} z(gk, l) = g \operatorname{Cor} z(k, l) + \operatorname{Cor} z(g, kl).$$

For the restriction of a degree-one cocycle, use $r_{gC} a(u_{g,C}) = a(g) + g a(r_C) - a(r_{gC})$ and sum. For degree two, let z_0 be a factor set of G representing α. Use lifts $x_g = (0, g)$ in its

extension, with $x_g x_k = z_0(g,k) x_{gk}$, as in the proof of Theorem 647.3. Write a coefficient on the left of a lift, retaining addition for the coefficient group. Define $e(g,C) \in P$ by $x_g x_{r_C} = e(g,C) x_{r_{gC}} x_{u_{g,C}}$. Associating $x_g x_k x_{r_C}$ in the two ways gives

$$z_0(g,k) + e(gk,C) = g e(k,C) + e(g,kC) + r_{gkC} z_0(u_{g,kC}, u_{k,C}).$$

Thus $hz_0 - \mathrm{Cor}(\mathrm{res}\, z_0) = \partial(\sum_C e(\cdot, C))$. The cochain on the right is continuous by the formula for the group law and the open cosets. This proves the equality of classes. □

Proposition 649.2. *Use the representatives of Lemma 649.1. Let $\langle\,,\,\rangle : M \times N \to P$ be an equivariant biadditive pairing of discrete abelian G-modules, $a \in Z^1(G,M)$ and $b \in Z^1(J,N)$. In additive notation for the factor set class one has*

$$\mathrm{Cor}(\mathrm{res}\, a \smile b) = a \smile \mathrm{Cor}\, b.$$

For a lattice T as in Theorem 644.3, the degree-one formula carries reductions of integral crystalline cocycles on J to reductions of integral crystalline cocycles on G, at every precision p^m. This applies also to its potentially crystalline Tate twists.

Proof. Compute the difference of the two cocycles at (g,k) using Definition 647.1. Its summand at C is the pairing of

$$r_{gkC} a(u_{g,kC}) - a(g) = g a(r_{kC}) - a(r_{gkC}) \quad \text{with} \quad g r_{kC} b(u_{k,C}).$$

Put $D(g) = \sum_C \langle a(r_{gC}), r_{gC} b(u_{g,C}) \rangle$. On expanding $D(gk)$, the part involving $b(u_{g,kC})$ equals $D(g)$ by reindexing. The other part, together with $gD(k)$, shows that the difference is $\partial D(g,k)$.

For the claim about lattices, take an integral lift $b(j) = jw - w$, with crystalline primitive $w \in B_{\max} \otimes_{\mathbb{Q}_p} (T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ in the sense of Theorem 644.3. Substitution gives $(\mathrm{Cor}\, b)(g) = gw_0 - w_0$, where $w_0 = \sum_C r_C w$ is in the same period space since the Galois action preserves $B_{\max}$. The values of the sum of transferred differences belong to T, since each term does. Reduction commutes with both sums and the lattice action. The same proof applies to each twisted lattice of Theorem 646.2. □

650. The local invariant of a factor set of prime-power order

Lemma 650.1. *Let $K/\mathbb{Q}_p$ be finite, $v_K(K^\times) = \mathbb{Z}$, and let Q be the size of its residue field. There is an injection*

$$J_K : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathcal{H}(K)$$

whose image of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ is the group split by the unramified extension of degree n for every positive integer n. For any finite K'/K, restriction sends $J_K(u)$ to $J_{K'}([K' : K]u)$.

Proof. For $n > 1$, put $K_n = K(\mu_{Q^n - 1})$. Recall the local construction in the proof of Lemma 618.1. One can lift a monic irreducible polynomial generating the residue extension of size Q^n. A root gives an extension of degree at most n and residue degree at least n, by reduction and independence of residue basis lifts. Its degree is therefore n. Every nonzero residue element lifts uniquely to a root of $X^{Q^n - 1} - 1$, by the successive simple-root lifting in Lemma 113.1. These lifts generate the extension, since their residues generate its residue field. They give the same field in every unramified extension of degree n. Residue Frobenius $z \mapsto z^Q$ generates its Galois group: the field is a splitting field of the indicated separable polynomial, an automorphism fixing residues fixes all the lifts, and the residue Frobenius has order n by Lemma 618.1. Use $K_1 = K$.

Write π_K for an element of valuation one. For the generator given by arithmetic Frobenius, the parameter $a \in K^\times$ of Theorem 647.3 is classified modulo norms by $v_K(a)$ mod n. Indeed all unit norms are attained by Theorem 611.3 and $N_{K_n/K} \pi_K = \pi_K^n$. If $n \mid n'$, then $K_n \subset K_{n'}$ by divisibility of the orders of roots. Inflation of the carry factor sends parameter a to $a^{n'/n}$: a lifted Frobenius contributes a for each complete run of n exponents. Consequently $u = k/n$ mod $\mathbb{Z}$ has a well-defined class $J_K(u)$ given by π_K^k at degree n, independently of

uniformizer and denominator. Addition may be computed over a common multiple. The order and splitting assertions now follow from the norm quotient.

Put $e = v_{K'}(\pi_K)$, let f be the residue degree of K'/K and $h = [K' : K]$. To see $ef = h$, the coordinate norm comparison in Lemma 113.1 places $O_{K'}$ between two full O_K-lattices in a K-basis. Indeed the same compact unit-sphere argument applies over K, whose integer ring is compact by its finite coordinate lattice over $\mathbb{Z}_p$. Thus $O_{K'}$ is free of rank h over O_K, by successive projection onto one coordinate and the fact that ideals of a discrete valuation ring are principal, as in Lemma 611.1. Its quotient by π_K has Q^h elements. Filtering this quotient by powers of a uniformizer of K' gives e factors of size Q^f.

Over K', residue Frobenius acts on the roots of K_n as the f-th power of the original Frobenius. The compositum with K_n is unramified, since $Q^n - 1$ divides $Q^{fn} - 1$ and all these roots therefore belong to the degree-n unramified extension over K'. Its degree is n/g, where $g = \gcd(n, f)$, by the order of Frobenius on the roots. In the extension group of the carry factor, write u_0 for the lift of the old Frobenius, so $u_0^n = a$. A lift of the new generator is u_0^f, whose (n/g)-th power is $a^{f/g}$. This computes the restricted parameter: multiplying the carry factors along consecutive powers of the new generator gives that power, up to the norm of any change of lift. Its invariant over K' is $v_{K'}(a^{f/g})/(n/g) = ef\, v_K(a)/n \bmod \mathbb{Z}$, as asserted. □

Theorem 650.2. *Put $P_p = \mathbb{Z}[1/p]/\mathbb{Z}$. For every finite $K/\mathbb{Q}_p$, J_K identifies P_p with the entire p-primary subgroup $\mathcal{H}(K)[p^\infty]$. For every finite L/K corestriction maps $J_L(u)$ to $J_K(u)$ for $u \in P_p$; in particular it is injective on these classes.*

Proof. First suppose $\mu_p \subset K$ and that α of p-power order is split by a finite Galois F/K of p-power degree. Induct on $[F : K]$, the case $F = K$ being immediate. A nontrivial finite p-group has a normal subgroup of index p: the class equation shows that its center has order divisible by p, so it contains an element of order p by taking a power of a nonidentity element. For groups of order greater than p, induct on the quotient by that central subgroup and take an inverse image. Thus F contains a cyclic degree-p extension K_1/K. By induction $\operatorname{res}_{K_1}\alpha = J_{K_1}(u)$ with $u \in P_p$. Choose $w \in P_p$ with $pw = u$ by increasing a denominator. By Lemma 650.1, $\alpha - J_K(w)$ is split by K_1, using addition for classes. By Theorem 648.1 the group split by K_1 equals the group split by the unramified extension of degree p, which is $J_K(\frac{1}{p}\mathbb{Z}/\mathbb{Z})$. This proves the assertion in this case.

In general Lemma 647.2 supplies a finite normal splitting field F, which may be enlarged to contain μ_p. Write $|\mathrm{Gal}(F/K)| = p^a b$ with $p \nmid b$. There is a subgroup of order p^a. Indeed the coefficient of X^{p^a} in $(1 + X)^{p^a b} \equiv (1 + X^{p^a})^b \bmod p$ equals $b \bmod p$. Under left translation on the subsets of size p^a, one orbit therefore has size prime to p. Its stabilizer has order divisible by p^a and acts freely on the elements of that subset, so its order also divides p^a. Its fixed field L has degree b over K by Lemma 16.2. The subgroup acts trivially on μ_p since its action there has order dividing $p - 1$. The first case over L gives $\operatorname{res}_L\alpha = J_L(u)$ with $u \in P_p$. Choose $w \in P_p$ with $bw = u$, using an inverse of b modulo the order of u. The difference $\alpha - J_K(w)$ restricts to zero. It is killed by b by Lemma 649.1, and has p-power order. Hence it is zero.

For any finite L/K of degree h, multiplication by h is onto on P_p: invert its prime-to-p part and increase the denominator for its p-power part. Choose w with $hw = u$. The restriction formula of Lemma 650.1 and the degree-two restriction–corestriction identity of Lemma 649.1 give $\operatorname{Cor} J_L(u) = hJ_K(w) = J_K(u)$. □

651. The index of crystalline reductions in a group of cocycles

Theorem 651.1. *Let $K/\mathbb{Q}_p$ have degree d, $G = G_K$, and let T, T' be potentially crystalline lattices in perfect Tate duality as in Theorem 646.2. For $m \geq 1$, let $M = T/p^mT$ and $M' = T'/p^mT'$, identified by duality with $M^\vee$ through the pairing into μ_{p^m}. For any*

subgroups $I \subset Z^1(G, M)$, $I' \subset Z^1(G, M')$ orthogonal under the local cup product, both groups are finite and

$$|I|\,|I'| \le |M|^{2+d}.$$

Proof. Choose a finite Galois F/K through which both reduced actions factor and which contains μ_p. Such a field exists since the kernels of the two continuous actions on finite groups are open normal subgroups, as in Lemma 647.2. Let L be the fixed field of a subgroup of largest p-power order in $\mathrm{Gal}(F/K)$, and $J = G_L$. The subset argument in Theorem 650.2 shows that $h = [G : J]$ is prime to p and $\mu_p \subset L$. Both restrictions have p-group image, so Theorem 648.2 applies over L. Each full cocycle group there is finite by taking the zero subgroup as its orthogonal partner. The full groups $P = Z^1(G, M)$, $P' = Z^1(G, M')$ are finite as well, since a cocycle is determined by its restriction to J and its values on the finitely many coset representatives. They have exponent dividing p^m.

Take the same large twist $T(N), T'(-N)$ over G and J as in Theorem 646.2. Their reductions have the original actions. Write W, X for the images of the integral crystalline cocycles of the twisted lattices over G, and W_J, X_J for the corresponding images over J. Each pair is orthogonal. Over J its product of orders attains $|M|^{2+[L:\mathbb{Q}_p]}$. Therefore the annihilator of X_J in the full cocycle group is exactly W_J: it contains W_J and its product with $|X_J|$ is at most this same value by Theorem 648.2. The reversed annihilator is likewise X_J.

Suppose $a \in P$ annihilates X. For $c \in X_J$, Proposition 649.2 gives $\mathrm{Cor}\, c \in X$ and $\mathrm{Cor}(\mathrm{res}\, a \smile c) = a \smile \mathrm{Cor}\, c = 0$, writing classes additively. This cup is killed by p^m by biadditivity. Corestriction is injective on it by Theorem 650.2. It follows that $\mathrm{res}\, a \in W_J$. Transferring back and using Lemma 649.1 puts ha in W: the difference between ha and the transferred cocycle is an integral coboundary modulo p^m, whose constant lifts to the lattice and so gives a crystalline primitive. Since h is invertible modulo p^m, $a \in W$. The reverse inclusion follows from orthogonality. Interchanging the two lattices and using cup reversal from Lemma 647.2 gives $\mathrm{ann}(X) = W$, $\mathrm{ann}(W) = X$.

View the pairing on the finite groups P, P' as taking values in P_p by Theorem 650.2. Let P_0, P_0' be the kernels against the entire opposite group. For a finite abelian p-group D, every character of any subgroup into P_p extends to D. When adjoining an element of order p^i modulo that subgroup, choose a preimage under multiplication by p^i of the forced value on its p^i-multiple. There are exactly p^i choices, since $P_p[p^i]$ has that cardinality. Induction gives $|\mathrm{Hom}(D, P_p)| = |D|$. The groups P/P_0 and P'/P_0' embed in each other's character groups, so these embeddings have equal finite orders on source and target and are onto. Restricting characters to the image of $Y \subset P'$ in P'/P_0' is onto by extension. Its kernel in P is $\mathrm{ann}(Y)$, whence

$$|\mathrm{ann}(Y)|\,|Y| = |P|\,|Y \cap P_0'|.$$

As $P_0' \subset \mathrm{ann}(W) = X$, taking $Y = X$ gives $|P|\,|P_0'| = |W|\,|X| = |M|^{2+d}$ by the complementary count over K. Finally $I \subset \mathrm{ann}(I')$, giving the bound. □

Corollary 651.2. *With M, M', T, T' as in Theorem 651.1, let A, A' be the reductions of $Z_f(T), Z_f(T')$ for the original lattices. Let $S \subset Z^1(G, M)$ be a subgroup and $H \subset Z_f(T')$ an additive subgroup of finite index. If the reduction $\bar{H} \subset A'$ annihilates both S and A, then*

$$[S : S \cap A] \;\mid\; [A' : \bar{H}] \;\mid\; [Z_f(T') : H].$$

Proof. By biadditivity the pair $S + A, \bar{H}$ is orthogonal. Their cardinality product is at most $|M|^{2+d}$ by Theorem 651.1. By Theorem 646.2 the latter equals $|A|\,|A'|$ for the original lattices. Divide by $|A|\,|\bar{H}|$ and use $(S + A)/A \simeq S/(S \cap A)$. Both resulting indices are powers of p, so the inequality implies the first divisibility. The surjection from $Z_f(T')/H$ to $A'/\bar{H}$ gives the second. □

652. A common integral lift for cocycles satisfying column relations

Theorem 652.1. *Let O be the integer ring of a finite extension $E/\mathbb{Q}_p$, G_0 a group, and $u_i : G_0 \to \mathrm{GL}_O(V_i)$ a finite family on finite free O-modules. Put $P = \bigoplus_i \mathrm{End}_O(V_i)$ with left action $g(B_i) = (u_i(g)B_i)$, and put $e = (\mathrm{id}_{V_i})_i$, $d_g = ge - e$. For every $m \geq 1$ there is a G_0-stable full lattice $Y \subset P$ containing all d_g, with the following property for every finite free O-representation T of G_0. Write $M = T/p^mT$. The cocycles $s : G_0 \to M$ satisfying*

$$\sum_g a_g d_g = 0 \quad \Longrightarrow \quad \sum_g a_g s(g) = 0 \qquad ((a_g) \text{ finitely supported over } O)$$

are precisely the functions $s(g) = b(d_g)$ for $b \in \mathrm{Hom}_{O[G_0]}(Y, M)$. They form a finite O-module. The single Y-valued cocycle $g \mapsto d_g$ is a coboundary over E, and $E \otimes_O Y$ is a direct sum of copies of the given column representations over E.

Proof. Let $D = \sum_g Od_g \subset P$. Since $gd_h = d_{gh} - d_g$, D is stable. Its saturation $D_* = (ED) \cap P$ is finite free over O: each submodule of a finite free module over a discrete valuation ring is finite free by the coordinate projection argument in Lemma 611.1. Clearing the denominators for a finite basis of D_* gives $p^bD_* \subset D$ for some integer $b \geq 0$. Fix $Y = D + p^{m+b}P$. For $v \in D \cap p^{m+b}P$ we have $p^{-m-b}v \in D_*$, so $v \in p^mD$. If s satisfies the relations, there is a unique linear map $b_0 : D \to M$ with $b_0(d_g) = s(g)$. Its equivariance follows from the two cocycle identities at gh. Extend it by zero on $p^{m+b}P$; the intersection just computed makes the extension well defined. Both summands are stable, so the extension is equivariant. Conversely any such b preserves the relations and sends the cocycle d_g to a cocycle. The lattice Y is finite free, and an O-linear map $Y \to M$ is determined on a finite basis with values in finite M, giving finiteness. Finally $EY = EP$ and $d_g = ge - e$ there. Choosing a basis in each V_i identifies left multiplication on matrices with the action on their separate columns, proving the last assertion. □

Proposition 652.2. *Retain G_0, T, Y, M from Theorem 652.1. Let $K/\mathbb{Q}_p$ be finite, $\Gamma = G_K$, and $\eta : \Gamma \to G_0$ a homomorphism such that the restricted lattice representations are continuous. Suppose T and its Tate dual $T' = \mathrm{Hom}_O(T, O(1))$ are potentially crystalline over K, where $O(1) = O \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. Use the perfect pairing given by the functional on O in Lemma 646.1. Let C_m be the module of cocycles satisfying the column relations of the theorem and A the reduction of $Z_f(T)$ over Γ. Set $C_m^f = \{s \in C_m : s \circ \eta \in A\}$. Write $a(g) = d_{\eta(g)}$ and let $H_m \subset Z_f(T')$ consist of those cocycles whose reductions annihilate A and all $b \circ a$ for $b \in \mathrm{Hom}_{O[\Gamma]}(Y, M)$. Then*

$$\lambda_O(C_m/C_m^f) \leq \lambda_O(Z_f(T')/H_m).$$

Here λ_O is the normalized length of Lemma 602.1. It suffices in defining C_m^f to impose on $s \circ \eta$ all the integral coordinate relations of $Z_f(T)$.

Proof. The cocycle a lies in $Z_f(Y)$, since $e \in EY$ is a rational primitive. It is continuous in Y, which contains $p^{m+b}P$ for the integer b in the theorem and hence has the induced lattice topology. Every map $Y \to M$ under consideration is continuous since it kills p^mY. The image $S = \{s \circ \eta : s \in C_m\}$ is therefore a group of continuous cocycles, all of the form tested in the definition of H_m. This subgroup H_m contains $p^mZ_f(T')$; it is an O-module since multiplication by a scalar can be moved to the tested cocycle in A, or to the tested map b, in the pairing. The restriction map gives $C_m/C_m^f \simeq S/(S \cap A)$. Apply Corollary 651.2 and the logarithmic cardinality formula for λ_O. Proposition 645.2 gives the final assertion, since p^mO is a uniformizer power. □

Corollary 652.3. *Use Proposition 652.2. Let $\mathcal{C}$ be a class of local lattices containing the restricted T and all restricted V_i, closed under finite direct sums and replacement by any stable lattice in an isomorphic rational representation. Suppose H is an O-submodule of finite index in $Z_f(T')$ with the following property. For every $V \in \mathcal{C}$ and integral form*

$\psi : V \otimes_O T' \to O(1)$ whose reduction modulo p^m is equivariant, every member of H cups to zero against every member of $Z_f(V)$ under the reduction of ψ followed by the fixed functional on O. Then $\lambda_O(C_m/C_m^f) \le \lambda_O(Z_f(T')/H)$. For every such form the function $g \mapsto (g\psi - \psi)/p^m$ is an integral crystalline cocycle on $\mathrm{Hom}_O(V \otimes_O T', O(1))$.

Proof. With $V = T$ use the equivariant evaluation form, obtaining the tests against A. With $V = Y$, which belongs to $\mathcal{C}$ by Theorem 652.1, lift an equivariant $b : Y \to M$ to an O-linear $\widetilde{b} : Y \to T$ on a basis and put $\psi(y, l) = l(\widetilde{b}(y))$. Its reduction is equivariant since b is, and its pairing with $a(g) = d_{\eta(g)}$ gives the test against $b \circ a$. Thus $H \subset H_m$; apply the proposition and additivity of normalized length. For the last assertion the function has integral values by equivariance of the reduction, and is the coboundary of the rational form ψ/p^m, which is a crystalline primitive by the definition in Theorem 637.2. □

653. The scalar crystalline local condition at the coefficient prime

Lemma 653.1. *Let $K/\mathbb{Q}_p$ be finite with residue field of size p^f. For $g \in G_K$, let $\nu_K(g) \in \mathbb{Z}_p$ be the p-component of its unramified exponent, assigning one to arithmetic Frobenius on the unramified tower of K. There exists $w \in B_{\max}$ such that*

$$gw - w = \nu_K(g) \qquad (g \in G_K).$$

The character $\nu_K : G_K \to \mathbb{Z}_p$ is onto and trivial on inertia.

Proof. Use $R = \varprojlim_\phi O_{C_p}/p$ of Lemma 639.1. Its residue constants give an equivariant embedding of $k = \overline{\mathbb{F}}_p$ into R. In detail, for each $q = p^b$ every element of $\mathbb{F}_q$ lifts to a unique solution of $X^q = X$ in O_{C_p}/p with that residue. For nonzero residues existence follows from the $(q-1)$-st roots of unity in Lemma 113.1. Uniqueness follows because the difference of two lifts with the same residue is nilpotent (a sufficiently high power has valuation at least one) and equals its q-th power. Zero has lift zero. The roots are closed under addition and multiplication in this characteristic-p ring. Uniqueness gives compatibility of all the finite fields. Taking the lift of the successive unique p^i-th roots of a residue as its i-th components gives the asserted field map into R.

The Witt construction of Proposition 639.2 now gives $W(k) \subset W(R) \subset B_{\max}$, with Frobenius acting on the Teichmuller coordinates by p-th powers. The second inclusion is Proposition 641.2. There is $b \in W(k)$ with $\phi b - b = 1$. First solve $x^p - x = 1$ in k. Given a solution modulo p^j, correct it by $p^j[y]$, choosing $y^p - y$ equal to the residual error divided by p^j modulo p. This is possible since k is algebraically closed. Completeness gives b.

On the residue field, $g \in G_K$ acts as an element of the inverse limit of the cyclic Frobenius groups $z \mapsto z^{p^f}$ in Lemma 650.1. Its exponent has residues in every $\mathbb{Z}/n\mathbb{Z}$, compatible under divisibility. Reduction to the groups $\mathbb{Z}/p^j\mathbb{Z}$ defines ν_K; each reduction is onto by the finite Galois groups of the unramified extensions of degree p^j. The absolute group is compact by the proof of Lemma 647.2. Its image is consequently closed and equals $\mathbb{Z}_p$, as it meets every residue class. Inertia acts trivially on all residues.

For fixed j, the first j Witt digits of b lie in a finite residue extension. Choose a nonnegative integer n representing the exponent of g modulo a common multiple of that extension degree over $\mathbb{F}_{p^f}$ (after compositum with $\mathbb{F}_{p^f}$) and p^j. Thus $gb \equiv \phi^{fn} b \bmod p^j$, while $n \equiv \nu_K(g) \bmod p^j$. Iteration of $\phi b = b + 1$ gives $\phi^{fn} b = b + fn$. Separation for all j gives $gb - b = f\nu_K(g)$ in $W(k)$. Take $w = b/f$ in the rational period ring. □

Theorem 653.2. *Let O be the integers of a finite coefficient field $E/\mathbb{Q}_p$, let T be a finite free O-module and let G_K act on it by the scalar χ_p^{-a} for an integer $a \ge 0$. If $a = 0$, then*

$$Z_f(T) = \{g \mapsto \nu_K(g)v : v \in T\}.$$

If $a > 0$, let $I_a \subset O$ be the ideal generated by the $\chi_p(g)^{-a} - 1$ for $g \in G_K$, and put $I_a^{-1}T = \{v \in E \otimes_O T : I_a v \subset T\}$. Then

$$Z_f(T) = \{g \mapsto (\chi_p(g)^{-a} - 1)v : v \in I_a^{-1}T\}.$$

In particular, an element $s \in G_K$ annihilates every cocycle by evaluation whenever $\chi_p(s)^a = 1$ and, in the case $a = 0$, s belongs to inertia. The same vanishing holds for all functions satisfying the integral crystalline relations modulo every uniformizer power.

Proof. Write $n = \operatorname{rank}_{\mathbb{Z}_p} T$. The scalar rational representation has a crystalline frame, obtained by multiplying an underlying $\mathbb{Q}_p$-basis by t_0^a in Lemma 646.1. In the residue representation each basis vector transforms by χ_p^{-a}, so the weights in Proposition 643.2 all equal a. Hence the rational space spanned by $Z_f(T)$ has dimension n by Theorem 644.3.

For $a = 0$ the functions $\nu_K(\cdot)v$, $v \in T$, have primitives $w \otimes v$ by Lemma 653.1. Their rational span has dimension n by evaluation at any element with $\nu_K(g) = 1$. At that element any integral-valued function in the span has $v \in T$. For $a > 0$ there is $g \in G_K$ with $\chi_p(g)^a \neq 1$; otherwise the constant $1 \in C_p$ would transform by χ_p^a near one, contrary to Theorem 233.4. Thus the rational coboundary map is injective and its image has dimension n. Each rational coboundary is crystalline after multiplying by a denominator of its primitive, so this is exactly the rational space just computed. Such a function has values in T precisely when $I_a v \subset T$, and its rational primitive proves crystallinity in this case. Evaluation gives the asserted vanishing. The equality between solutions of all coordinate relations and actual reductions in Proposition 645.2 proves the assertion at finite precision. □

Appendix A. Elementary tools for the analytic reductions

Lemma A.1. *For positive integers a, b, their greatest common divisor is an integer linear combination of a, b. In particular congruences modulo pairwise coprime integers can be solved simultaneously, and prime factorizations of positive integers are unique up to order.*

Proof. The least positive member h of $a\mathbb{Z} + b\mathbb{Z}$ divides each member, since division by h leaves a remainder in the same set. Hence $h = (a, b)$. For coprime a, b, an integer inverse of a modulo b solves $u + ak \equiv v \bmod b$, giving the two-modulus assertion. Iteration proves the finite case. If a prime p divides ab and does not divide a, multiply $1 = pa' + ab'$ by b to see that $p \mid b$. Factor a composite by induction on its size. To see uniqueness, match a prime on one side against a factor on the other by the divisibility property just proved, then cancel and repeat. □

Lemma A.2. *Increasing nonnegative measurable functions converge in integral to the integral of their pointwise limit. A pointwise convergent sequence dominated in absolute value by one integrable function converges in integral. A series can be integrated term by term if the integrals of its absolute values are summable.*

Proof. Define the nonnegative integral as the supremum over simple functions below the integrand. If $f_n \uparrow f$, $0 \le u \le f$ is simple and $0 < c < 1$, the sets $\{f_n \ge cu\}$ increase to the whole space up to points where $u = 0$, which do not affect the integral of u. Countable additivity on disjoint differences gives convergence of their measures on every level set of u. Hence $\lim \int f_n \ge c \int u$. Take the supremum and let $c \uparrow 1$. The opposite inequality is monotonicity. For $|g_n|, |g| \le h$ with h integrable and $g_n \to g$, apply this result to $2h - \sup_{m \ge n} |g_m - g| \uparrow 2h$. The integral of the supremum tends to zero. For the series, apply the first assertion to the partial sums of absolute values, then the second to the partial sums of the original series. □

Lemma A.3. *A holomorphic function on a neighbourhood of $|z - z_0| \le r$ has a Taylor series about z_0 throughout the open disk. Its k-th coefficient has modulus at most $r^{-k} \max_{|z-z_0|=r} |f(z)|$. Holomorphic functions agreeing on a set with an interior accumulation point in a connected domain agree throughout it. The same uniqueness holds for*

meromorphic functions. At a zero of finite order a of f, the logarithmic derivative f'/f has principal part $a/(z - z_0)$. A locally integrably dominated integral of holomorphic functions is holomorphic, provided the parameter dependence is measurable and the domination holds on every compact subdisk.

Proof. For the disk formula, first integrate a holomorphic function around a triangle. Dividing into four similar triangles cancels the interior edges. If the original integral has modulus A, a nested sequence of triangles has integrals of modulus at least $4^{-n}A$ at stage n. At their common point w, subtract the affine function $f(w) + f'(w)(z - w)$, whose contour integral is zero. The remaining integral is $o(2^{-n})$ times a perimeter of order 2^{-n}. Thus $A = 0$. Triangulating polygonal annuli and approximating their circular boundaries by inscribed polygons applies this result to $f(w)/(w - z)$ away from z. The integral along the inner circle shrunk about z tends to $2\pi i f(z)$, by continuity and explicit integration of $1/(w - z)$. On any fixed circle the polygonal integrals converge by uniform continuity on a surrounding compact annulus and ordinary parametrisation of the circle. This proves the disk integral formula. Expanding $1/(w - z)$ in its uniformly convergent geometric series on that boundary gives the expansion and coefficient bound. If the series is nonzero, its first nonzero coefficient exhibits a factorisation $f(z) = (z - z_0)^a h(z)$ with $h(z_0) \neq 0$. This gives isolated zeros and the asserted principal part. For a function vanishing on an accumulating set, the points at which its germ vanishes form a nonempty open and relatively closed set. Connectedness gives uniqueness. Clearing local denominators proves the meromorphic version.

For the integral assertion, on any closed disk of domination the coefficient bound just proved dominates the k-th coefficient by the integrable majorant times r^{-k}. On a smaller disk these bounds sum to an integrable geometric series. Lemma A.2 therefore allows integration of the coefficients. The integral is given locally by the resulting power series. □

References

[1] L. Sernau, collaborators at Anthropic, and Claude, *GRH for Dirichlet L-functions, in Lean 4*, compact edition. The repository credits the underlying argument to L. Alpöge and Claude. Source snapshot 72fead1.

[2] NIST Digital Library of Mathematical Functions, *Dirichlet L-functions*, §25.15, equations (25.15.4) and (25.15.5).